A Numerical and Electronic Exploration of a Simple Chaotic System

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(Dated: May 7, 2013)

In this experiment chaos is demonstrated on a simple electronic circuit that models a third order differential 'jerk' equation. The circuit consists of three integrators and a non-linear element composed of a simple set of inexpensive components. The experiment is repeated using classical numerical simulation. Chaos is demonstrated on both systems by performing a number of analysis techniques on both the simulation and the real data. We found that the simulation to be in reasonable agreement with the experimental data throughout these tests.

I. INTRODUCTION

The phenomenon of chaotic dynamics arises in wide variety of situations. These range everywhere from the vibrations of three dimensional structures, to weather, to fluid dynamics. [1]. The fine control and manipulation of these types of systems is essential to the engineering of many everyday mechanical and electrical systems as well numerous modern scientific measurement techniques.

The common conception of chaos as a random process entirely misses the nature of the phenomena. A chaotic system is one that can not be solved exactly by analytical means, but still has a deterministic behavior. This is well summarized by the progenitor of modern chaotic dynamics Edward Lorentz: "Chaos is [sic] when the present determines the future, but the approximate present does not approximately determine the future." [2]

If the start conditions are completely known then the behavior of the system will be able to be determined analytically, but generally they are only known them to an approximation. These initial conditions are typically very hard to measure precisely enough to make a good forecast.

To wit, the weather in a closed environment, say a test tube, is easy to predict because the forces acting on the system are well known, and the conditions of that system are relatively easy to measure to high precision. However, in open systems such as the weather in a region, instruments are typically only able to precisely measure the conditions precisely enough to make good predictions about three days in advance. [3] This type of problem and many others constitute the field of chaotic dynamics.

A reasonably common way of creating chaos in the laboratory is by the construction of non-linear circuits. This type of system has the key advantage that it allow for precise measurement and has the propensity for good numerical analysis. Following that cue we construct a circuit as discussed by Kiers, Schmidt, and Sprott [4], see Figure 1. It combines a third order differential 'jerk' equation with a nonlinear element. This type of combination is extremely common in undergraduate demonstrations of chaos,[5] due to its inexpensive components and relative ease of construction, compared to other types of chaotic systems.

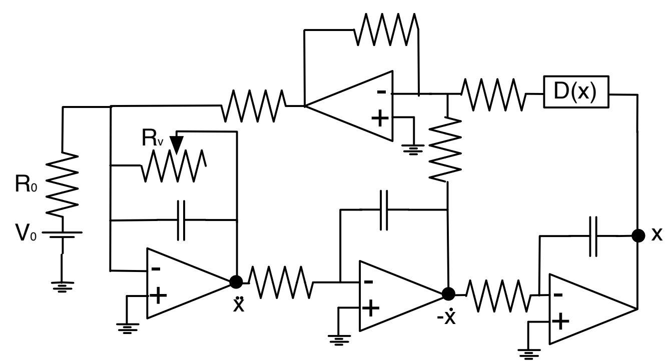

FIG. 1: Diagram of circuit used in the experiment. All unlabeled resitors are $R = 45.9k\Omega$, all unlabeled capacitors have $C = 1\mu F$, and $V_0 = 0.250V$. The op-amps are LF411s. The variable resistor R_v consisted of a 6 linked digital potentiometer AD5206 chip in series with a $46.5k\Omega$ resistor. This allows for an approximate 2200 step resolution across a $80k\Omega$ range, which in our case means a minimum step size of $0.039k\Omega$.

One of the essential tricks to chaotic dynamics is proving rigorously that the behavior of the system being observed is in fact behaving chaotically, and not just noisily. There are numerous methods that do this: some rely on through brute force, while others take advantage of the recursive, and therein fractal, nature of chaos. These methods are easy to model computationally, and can be be used to make strong predictions about the behavior of a system.

This paper takes the view that numerical simulations do not provide a theoretical prediction of the system, that they instead constitute an intrinsically different, though similar, system. As we've noted numerics can make strong predictions, but there is a limit to the potential accuracy of a modeling system governed by the divergence between simulation and experiment. In an ideal world a computer would have infinite accuracy, but it is a system with it's own flaws and problems. These tendencies tend to arise in highly sensitive numerical experiments such as this one. Thus we will treat the computationally constructed data as a second experiment.

We will first describe out physical experimental design in Section II, then conduct our numerical experiment in Section III, and then finally compare those to the electronic results in Section IV. We will conclude with a discussion of ways to expand the experiment in Section V.

II. EXPERIMENTAL DESIGN CIRCUIT ARCHITECTURE

In this experiment we used the same circuit as designed by Kiers, Schmidt, and Sprott et al. The circuit consists of three successive inverting op amp integration circuits and a non-linear element coupled to a simple inverting amplifier, as in Figure 1. We selected our parameter values for this experiment following the roughly following the choices of Wiener et al[6], who have also written a paper on this circuit. The circuit functions as a differential equation solver, that performs its task by modifying the voltages in the locations designated $x, \dot{x}$ and $\ddot{x}$.

Following a Kirchoff and golden rule analysis of the circuit yields the differential equation:

$$\dddot{x} = -A\ddot{x} - \dot{x} + D(x) - \alpha \tag{1}$$

Where the value being measured at each of the nodes is the voltage and the derivatives are being performed with respect to non-dimensionalized time.

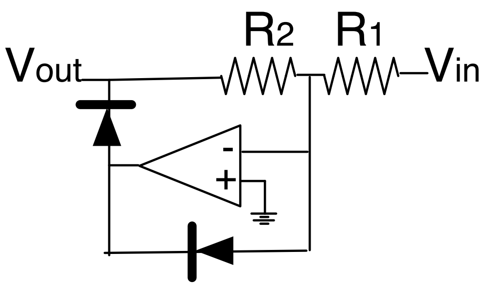

FIG. 2: Non-linear sub-circuit used in the experiment. The resistors have values $R_1 = 15.8k\Omega$, $R_2 = 88.1k\Omega$, the diodes are IN914's

We substitute the real variables of the system in for the parameters in the equation:

$$\dddot{x} = -\left(\frac{R}{R_v}\right)\ddot{x} - \dot{x} + D(x) - \left(\frac{R}{R_0}\right)v_0 \tag{2}$$

The non-linear element, represented by D(x) in the figure, can take a wide variety of formats [5], but the one utilized in this circuit is shown in Figure 2. A similar analysis of the non-linear component in conjunction with the Shockley equation for analyzing the I-V curves of the diodes, yields the form:

$$V_{out} = D(V_{in}) = -\left(\frac{R_2}{R_1}\right)\min(V_{in}, 0) \tag{3}$$

This non-linearity manifests itself as a knee, as seen in Figure 3. In an ideal world the sub circuit would simply take absolute value of the input voltage, however that type of circuit is often hard to build and can be highly sensitive to environmental conditions. [4]

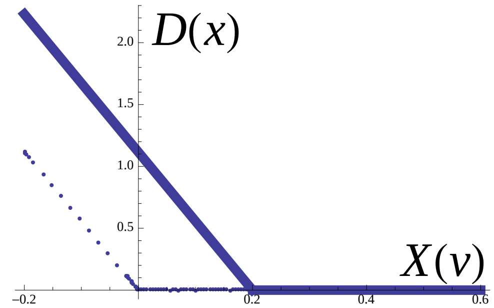

FIG. 3: This figure shows the experimentally measured response curve (dots) and the theoretical one which has been offset by 0.2 to the right. Clearly experiment agrees quite well theory.

III. COMPUTATIONAL METHODS & THEORY

Typically chaotic numerics boils down to solving ordinary differential equations and this system is no exception. Following Wiener et al's lead, we shift our our equation for appropriate and easy numerical analysis:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -\left(\frac{R}{R_v}\right)z - y + D(x) - \left(\frac{R}{R_0}\right)V_0 \end{aligned} \tag{4}$$

This allows us to implement a simple fourth order Runge-Kutta integration method.[7][1] We briefly employed a parallelized Runge-Kutta Method [8], which had a reasonable computation time speed up. However we recommend against it's use in the future due to the difficulty associated with implementing it.

This method produces a list of simulated data points. We can then perform the same analysis techniques for verifying chaos that would be performed on real data. We will generate numerical versions of return maps, phase space diagrams, and a bifurcation diagram. The first two tend to deal with individual values of variable parameter. The final one looks at the spectra of stability of system as a whole.

One of the most common ways to examine the behavior of an oscillator is return maps. They are a map between one value of a parameter and the next: they acts as a discrete phase space, providing a description of the overall set of values for a parameter value. The intersection between a unit sloped line and the return map describes points of stability. Signal paths tend to fluctuate around these intersections creating stable or or unstable orbits.

[1] While there are fancier versions of this algorithm that include adaptive step size or higher order approximations, this one best serves our purposes because the resolution that it has matches our that of our experiment reasonably well and has a fast computation time.

A semi-stable orbit might make a closed loop around the intersection point without ever making contact with the point itself. For instance in Figure 4 a semi-stable period-two orbit has been drawn onto the return map in black. A chaotic orbit will bounce through the entire diagram, in a similar manner as depicted in the blue lines on figure.

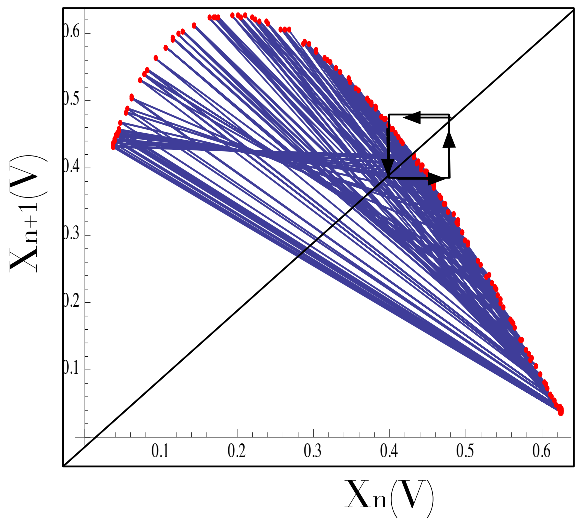

FIG. 4: Computationally generated simulation of a first order return map (red dots) for parameter value $R_v = 72.1$. The Cobweb diagram (black) was added by hand, it indicates a forced semi-stable two period orbit, while the chaotic orbit (blue) was generated computationally.

A natural extension to this technique is the continuous mapping of points in phase space. When systems behave chaotically they exhibit a type of motion known as an attractor. An attractor is a point, or area which is unchanged overtime, and tends to act as a basin for the motion of the system to return to, and then deviate away from. [9] They are the continuous equivalent to the stable points on the return maps. Phase space diagrams have an interesting topological feature that they are partially-space filling, that is they aren't quite a line and they aren't quite a plane, they are somewhere in between. A traditional method for discussing the behavior of this type of diagram is the fractal dimension, or how space filling the shape is. However this is somewhat beyond the means of this experiment, for reasons that will be discussed in the Section V. In Figure 5 we examine a pair of three dimensional phase space diagrams.

And finally we consider the bifurcation diagram. In the most general sense this type of diagram is a measure of the stability of one variable compared to another. It is so named because of the behavior that is exhibited by a channel as it experiences a number of period doubling bifurcations that eventually lead into wide band chaos. This allows us to use period doubling as strong evidence

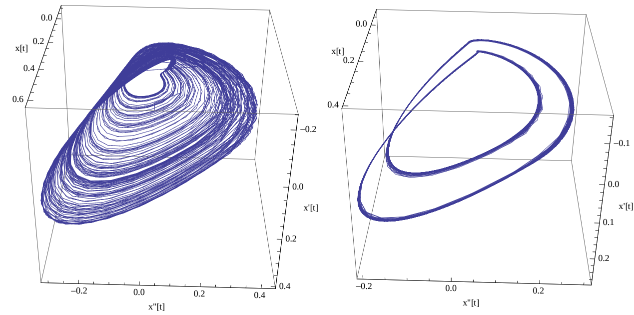

FIG. 5: Computationally constructed 3D Phase space diagrams for $R_v = 72.1k\Omega$ and $R_v = 55k\Omega$. Each of these diagrams demonstrate a strange attractor behavior at the center of their orbits.

for chaos.[2]

In our case the measured parameter is the stability of the local voltage peaks compared to the resistance in the variable resistor R_v. This means that for a given resistance we plot all of the different values for the peaks of the output the x voltage, as in Figure 6.

The signal begins to experience period doubling from about $55k\Omega$ until it becomes chaotic at around $70k\Omega$. this firmly suggests that the system is in fact chaotic. The signal stays chaotic until just above $100k\Omega$ after which is starts to move through a series of different types of periodicity, which follows the findings of Kiers, Schmidt, and Sprott.

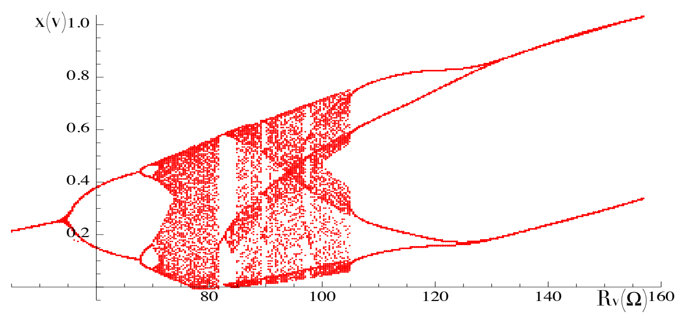

FIG. 6: Computational bifurcation diagram for this system, displaying path from periodicity to chaos to periodicity.

[2] An interesting alternative to the brute force simulation approach that we have adopted for the computational methods, would be to generate a bifurcation diagram through Monte Carlo simulation. This would most likely offer good qualitative results and poor quantitative results, however the big gain would be in the most likely exceptionally fast computation time, possibly to the point of real time parameter control.

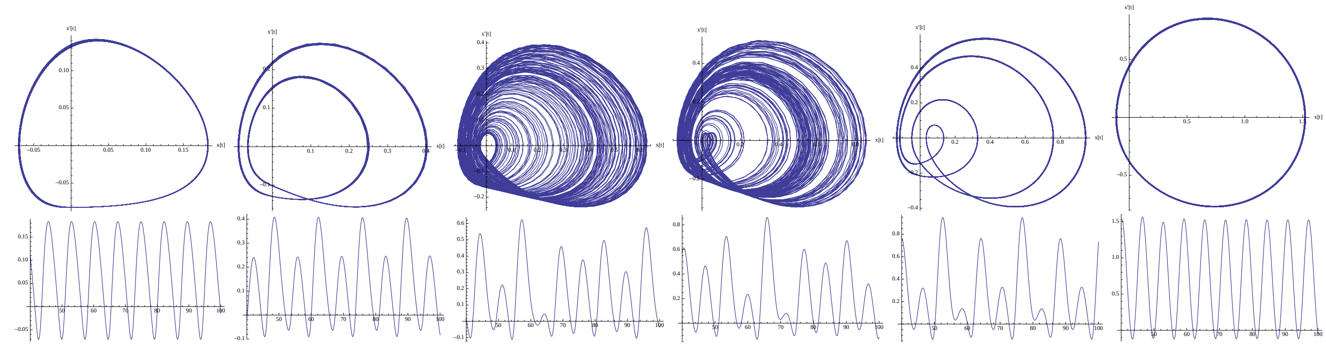

FIG. 7: In this diagram we display a number of two dimensional phase portraits for R_v values $30k\Omega, 55k\Omega, 72.1k\Omega, 95k\Omega, 105k\Omega$ $200k\Omega$ a sample of the signal during that period. In this first three it the signal gradually picks up nodes eventually reaching 72.1 and and having almost complete chaos. After this point rings that represent stability start to creep in eventually taking over as the signal returns to completely sinusoidal.

IV. RESULTS

We now present the experimental findings of the techniques discussed in the previous section. We first present a 'time series'-style graphic for the progression of chaos in phase space diagrams as resistance is increased in Figure 7. These plots behave just as expected and offer a good envisionment of the story of chaos in the system.

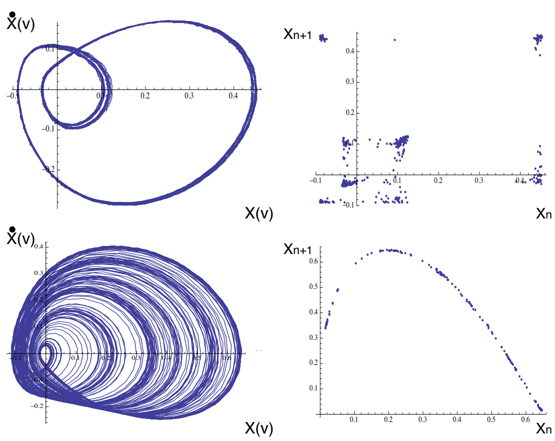

FIG. 8: Diagram of experimental return plot and corresponding two dimensional phase diagram for $R_v = 57k\Omega$ (top) and $R_v = 75k\Omega$ (bottom).

Next we consider some return maps for our system Figure 8. On the top left of the figure, we can see the phase diagram for $57k\Omega$, which demonstrates a regular periodic orbit. On the right we can see the return map for that value, which consists of a series of discrete islands. This is what we expect for quasi-stable periodic orbits.[1] On the bottom we can see a chaotic attractor and a chaotic, unstable, return map, just as expected.

Finally we present the bifurcation diagram, in Figure 9. There was an persistent error in the data in which during some data runs the system would get stuck on on a particular resistance value. We found that these occurrences were caused by a digital error and not a physical feature of the system, so we cut them out of the final results. We have left an artifact of one of these cuts for reference in Figure 9 at points IIa and IIb, it is the horizontal line that occurs at approximately $64k\Omega$.

TABLE I: Comparison of numerical and experiment bifurcation points, as indexed in Figure 9.

Point	Exp (kΩ)	Num (kΩ)	Avg. Diff %
I	54.74	54.23	0.93
III (a&b)	67.98	67.78	2.9
IV (a&b)	84.71	83.38	1.6
V	94.44	92.610	2.1

The experimental bifurcation has somewhat cleaner lines than the numerical bifurcation, this suggests that there is a some sort of transient behavior present in the numerical analysis of the system. That is, the numerics count a wider band of peaks than the experimental. This either could be an artifact of our integration method or our peak counting method.

The easiest way to characterize error for the bifurcation diagram is by selecting a set of nodes on the diagram and comparing the resistance parameters that leads to that node. This can be seen in Table I. This technique provides a measurement of the accuracy of the tuning of the start parameters. We can see from both this table and from the actual bifurcation result that the results are slightly skewed, this suggests firmly that our understanding of the start conditions is flawed, however interestingly they are not flawed directly in a particular direction.

Our experimental electronic results have a smaller range than our numerical result because when designing our experiment we selected the AD5206, of six $10k\Omega$ digital pots. We selected it because of it's high resolution, however it had the noted disadvantage that the full range is not entirely swept out by one chip. We decided that it was more worthwhile to have a high resolution small range than a lower resolution large range.

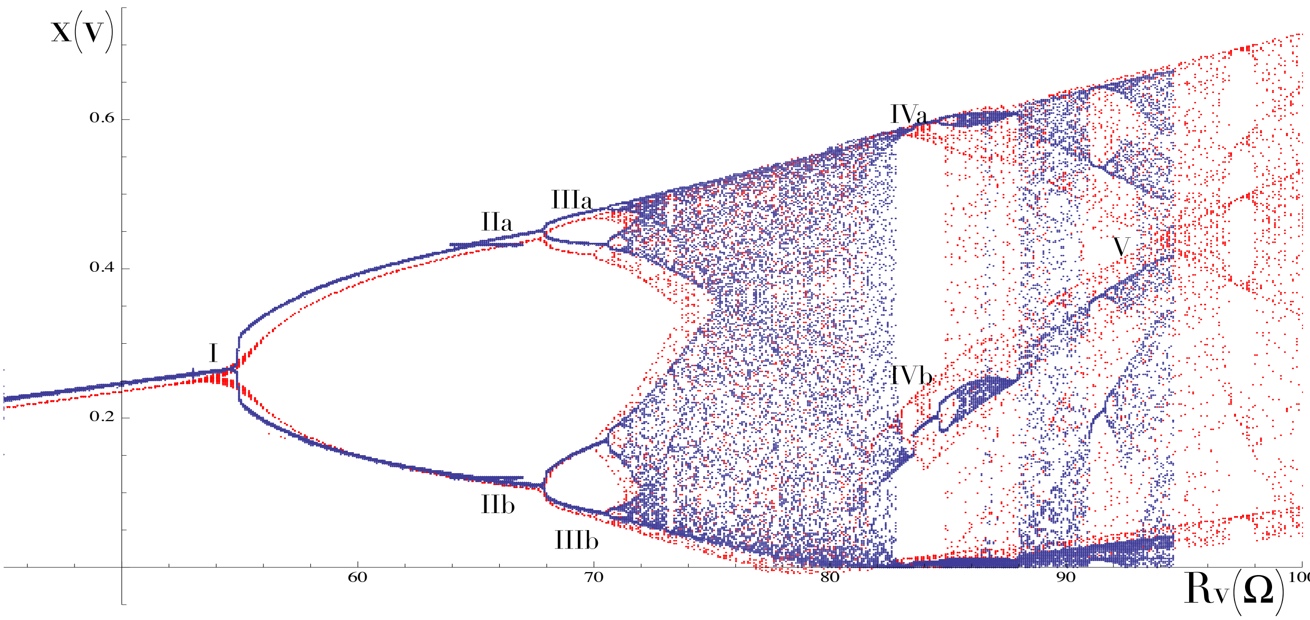

FIG. 9: In this diagram the experimental (blue) and numerical (red) bifurcations are plotted on top of each other. They have essentially the same form, and have a similar set of nodes. However, the difference in behavior can be seen from the very beginning, and it propagates and gets generally more significantly skewed as resistance is increased.

V. DISCUSSION

In this paper we explored numerical and experimental chaos. We found that the chaos was relatively easy to induce on both systems, however there is an extreme difficulty tuning the start parameters of the system to make them perform the same. We could have improved the results by improving the electronic system. This system is very sensitive to the conditions of its constituent components. Using parts that have been cared for badly or that have been mishandled can lead to the burning out op-amps semi-frequently, and therein weak results

The results could be enlarged by considering additional techniques. Power spectral density plots could be made along side other other Fourier methods in order to better discuss the period doubling double phenomena. The fractal dimension of the phase space plots could measured, which can done by either buying a fractal analyzer or creating a home-brew version. The commercial examples tend to be prohibitively expensive, while the home-brew example tend to be very difficult to implement. However if we could measure it in some way, we could find the points of maximal chaos, which would be the points with the greatest fractal dimension.

This type of study could be repeated and expanded in a variety of ways. The circuit offers a wide starting point for the analysis of non-linear systems. This circuit could be made to produce additional instances of chaos simply by changing the non-linear sub circuit to a different one.

One of the first things taught in the study of chaotic dynamics is the study of the logistic equation, which maps population growth under certain constraints. This equation can be expanded in a two linked first order differential equations known as Lotka-Volterra equations, which model predator prey-dynamics. This system offers a strong grounds for both experimental study and electronic simulation.

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