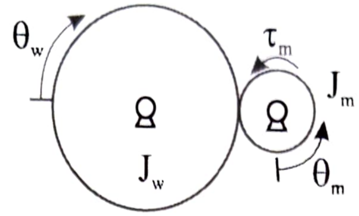
System Dynamics

Midterm Exam 1

2:50 pm - 4:05 pm, Thursday, September 23, 2021

- Sign the following honor statement by printing your name below:
- I __Will____Buziak_______ have neither given nor received unauthorized assistance on this exam.
- The exam is closed book, closed notes, but you may use one 8.5×11-inch page of notes (both sides).
- Communication with other students during the exam is not allowed.
- Use of any outside resources is not allowed.
- Calculators are permitted.
- Write your work on the printed exam.
- An important aspect of engineering is communicating your methods clearly to others. Therefore, show your full work for all problems with a logical flow of steps in legible handwriting.
- Work will be graded based on a correct process that shows understanding and execution of the relevant engineering principles. It is possible to have the right final answer and not receive full credit because of inconsistent, unclear, or insufficient work.
- It is in your interest not to leave anything blank, since blank answers receive zero credit. When in doubt, write down true mathematical statements that are relevant to the problem.
- Draw a box around your final answers.
- If necessary, write on the back or attach extra pages for scratch work.
- When you are done, bring the exam to the front of the room. Scan each page of the exam (including this one and your notes sheet). Turn in the hard copy at the front desk.
- Upload the scanned exam as a single PDF to Canvas->Assignments->Exam 1 by 6:00 pm. It would be best to do this before you leave.

1. In this system, an electromagnetic motor drives a wheel using a gear transmission to reduce speed and increase torque. The motor applies a torque τ_m to the small gear of radius r_m and moment of inertia J_m. The moment of inertia of the large gear (wheel) is J_w, and it has a radius r_w. There is no slip between the gears, and no friction or damping at the axles.

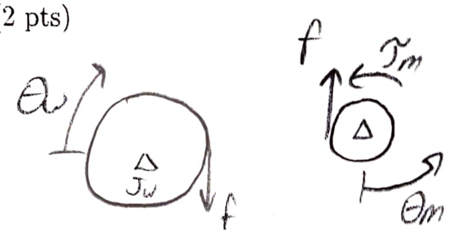


a. How many degrees of freedom does this system have? (1 pt) 1

b. What is the kinematic relationship between θ_w and θ_m? (1 pt) $\theta_m r_m = \theta_w r_w$

c. Draw free-body diagrams for the two inertias in the system. Assume no damping is present. (2 pts)

$\theta_m = \frac{r_w}{r_m}\theta_w$



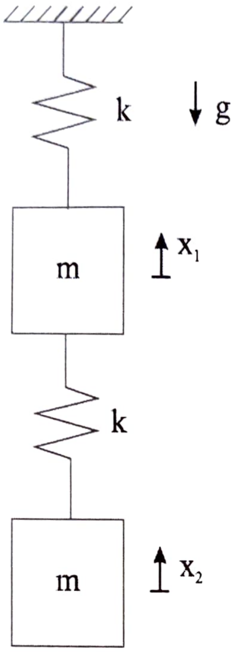
d. Consistently with your diagrams in (c), write the appropriate form of Newtons law for each inertia in the system - (3 pts). Then eliminate any unknown forces and sub in the kinematic relationship to obtain the equation of motion in terms of θ_w. (3 pts)

θ_w: $\sum \tau_w = -f r_w = J\ddot{\theta}_w \Rightarrow f = \frac{-J_w\ddot{\theta}_w}{r_w}$

θ_m: $\sum \tau_m = \tau_m - f r_m = J_m \ddot{\theta}_m \Rightarrow \tau_m = J_m \frac{r_w}{r_m}\ddot{\theta}_w - J_w \frac{r_m}{r_w}\ddot{\theta}_w$

$$\tau_m = \left(\frac{J_m r_w}{r_m} - \frac{J r_m}{r_w}\right)\ddot{\theta}_w$$

2. A schematic for a mechanical system is shown below. Assume that $x_1 = 0$ and $x_2 = 0$ when the springs are unstretched. Gravity exists in the direction shown. There are no external forces applied to the system and no damping elements present.



a. (1 pt) True or False: The total energy (including gravitational potential energy) in this system will be constant.

True

b. (1 pt) True or False: We can use principle of conservation of energy to obtain all the equations of motion for this system. True

c. (2 pts) Write down the total kinetic energy in the system, T.

$$T = 1/2\, m\dot{x}_1^2 + 1/2\, m\dot{x}_2^2$$

d. (2 pts) Write down the total potential energy in the system, V. Include the potential energy due to gravity.

$$U = mgx_1 + mgx_2 + 1/2\, kx_1^2 + 1/2\, k(x_1 - x_2)^2$$

e. (1 pt) Write down the system's Lagrangian, L.

$$L = T - V = 1/2\, m\dot{x}_1^2 + 1/2\, m\dot{x}_2^2 - mgx_1 - mgx_2 - 1/2\, kx_1^2 - 1/2\, k(x_1 - x_2)^2$$

f. (3 pts) Use the Euler-Lagrange equations and your Lagrangian as defined above to obtain the differential equations governing x_1 and x_2. $R = 0$

$P_i = 0$

x_1: $$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\ddot{x}_1$$

$$\frac{\partial L}{\partial x_1} = -mg - kx_1 - k(x_1 - x_2)$$

$$\Rightarrow \boxed{m\ddot{x}_1 + kx_1 + k(x_1 - x_2) + mg = 0}$$

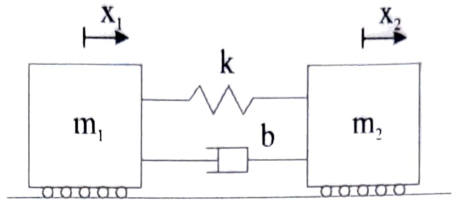
$\rightarrow$ x_2 on back

x_2: $\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m\ddot{x}_2$

$$\frac{\partial L}{\partial x_2} = -mg + k(x_1 - x_2)$$

$$\Rightarrow \boxed{m\ddot{x}_2 - k(x_1 - x_2) + mg = 0}$$

3. A schematic for a mechanical system is shown below. The circles denote frictionless rolling contact. Assume that the spring is relaxed (unstretched) when $x = 0$.



a. Name two methods we could use to obtain the governing differential equations for x_1 and x_2 (2 pts)

Lagrange, Newton's method

b. Use one of the methods you listed in (a) to obtain the governing differential equations for x_1 and x_2. (8 pts)

$$T = 1/2\, m_1 \dot{x}_1^2 + 1/2\, m_2 \dot{x}_2^2 \qquad R = 1/2\, b(\dot{x}_2 - \dot{x}_1)^2$$

$$V = 1/2\, k(x_2 - x_1)^2 \qquad L = 1/2\, m_1 \dot{x}_1^2 + 1/2\, m_2 \dot{x}_2^2 - 1/2\, k(x_2 - x_1)^2$$

x_1:

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1, \quad \frac{\partial R}{\partial \dot{x}_1} = -b(\dot{x}_2 - \dot{x}_1)$$

$$\frac{\partial L}{\partial x_1} = +k(x_2 - x_1) \qquad \boxed{m_1 \ddot{x}_1 - k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) = 0}$$

$x_2 \longrightarrow$ on back

Bonus (1pt): The equations of motion can be written as a matrix-vector system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrices **M**, **B**, and **K**

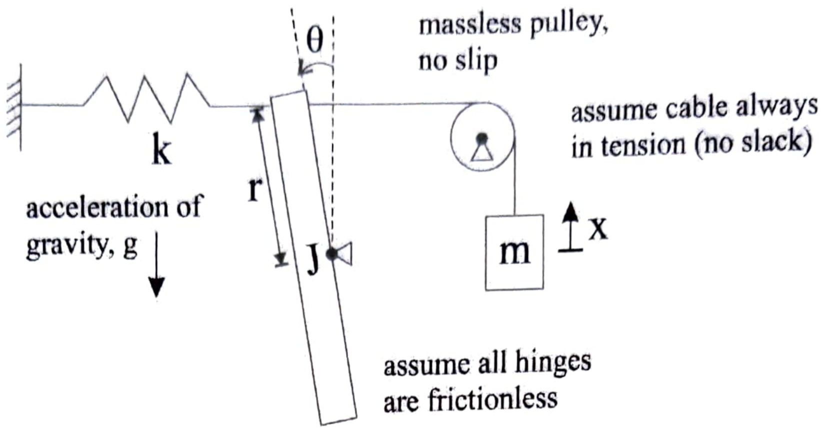
$$M = [m_1, m_2], \quad B = [b, b], \quad K = [k, k]$$

x_2: $\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2$

$\frac{\partial R}{\partial \dot{x}_2} = b(\dot{x}_2 - \dot{x}_1)$ $\qquad$ $\frac{\partial L}{\partial x_2} = -k(x_2 - x_1)$

$$m_2\ddot{x}_2 + k(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) = 0$$

4. A dynamic system is shown below. The spring is unstretched when $\theta = 0$ and $x = 0$. The lever has moment of inertia J about the hinge at its midpoint. The pulley has no inertia



a. How many degrees of freedom does this system have? (1 pt) 1 DOF

b. What is the kinematic relationship between θ and x (use the small-angle approximation $\sin\theta = \theta$)? (1 pt)

$$x = r\theta$$

a. Write down the total kinetic energy in the system, in terms of $\dot{\theta}$. (1 pt)

$$T = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2$$

b. Write down the total potential energy in the system, in terms of θ, including the gravitational potential energy of the mass due to its height. (1 pt)

$$V = \tfrac{1}{2}kr^2\theta^2 + mgr\theta$$

c. Obtain the differential equation governing θ using either (1) the principle of conservation of energy or (2) energy comparisons to find equivalent system elements. Note that since we already included the gravitational potential energy in (b), we do not include any external power input from gravity. Further note that there are no dissipative elements (dampers) in this system. (3 pts)

$$E_{tot} = T + V = \tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}kr^2\theta^2 + mgr\theta$$

$P_i = 0$

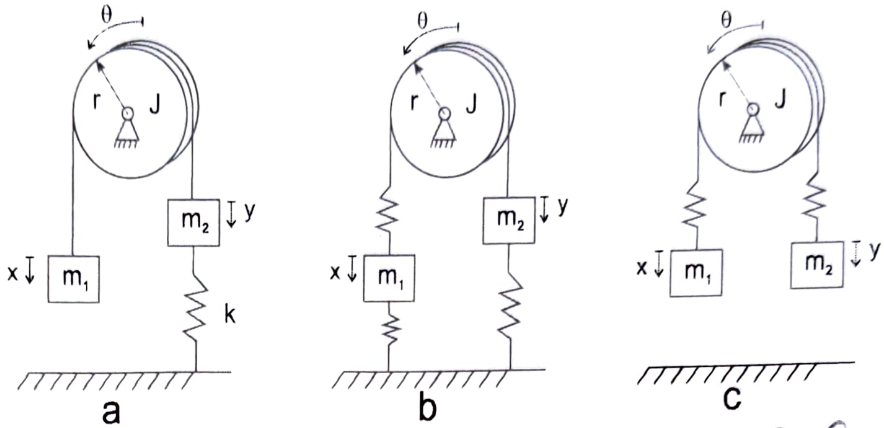
$P_d = 0$

$$\dot{E}_{tot} = P_i - P_d = 0$$

$$J\dot{\theta}\ddot{\theta} + mr^2\dot{\theta}\ddot{\theta} + kr^2\theta\dot{\theta} + mgr\dot{\theta} = 0$$

$$\boxed{(J + mr^2)\ddot{\theta} + kr^2\theta + mgr = 0}$$

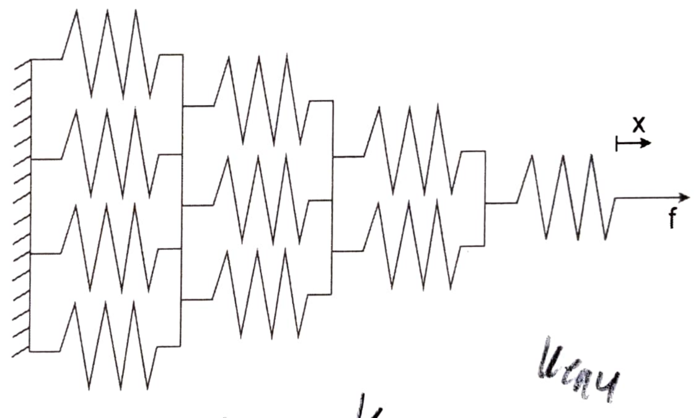
d. The figure below shows 3 similar dynamic system schematics. The cables always remain in tension with no slack, and they do not slip on the pulleys.



1 DoF 2 DoF 3 DOF

How many degrees of freedom does each system have? Write the number of degrees of freedom under the letters above. (3 pts)

Bonus (1pt): A system of springs is shown below, where each spring has the same constant $k = 1$ N/m. If a force $f = 1$ N is applied, how many meters will x displace?



$\Sigma F: 1 = \frac{1}{2}(.48)x^2$

$\boxed{X = 2.04\ m}$

k_{eq1} k_{eq2} k_{eq3} k_{eq4}

$k_{eq1} = 4k$

$\frac{1}{k_{eq2}} = \frac{1}{4k} + \frac{1}{3k}$

$\frac{1}{k_{eq3}} = \frac{1}{k_{eq2}} + \frac{1}{2k}$

$\frac{1}{k_{eq4}} = \frac{1}{k_{eq3}} + \frac{1}{k}$

$\frac{1}{k_{eq}} = \frac{1}{4k} + \frac{1}{3k} + \frac{1}{2k} + \frac{1}{k}$