UCLA QCBio Collaboratory

Introduction to Modern Statistics

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2025 Winter

UCLA Bioinformatics

Notation of the slides

- `Code` or `Pseudo-Code` chunk starts with " ➢ ", e.g.
  - `print("Hello world!")`

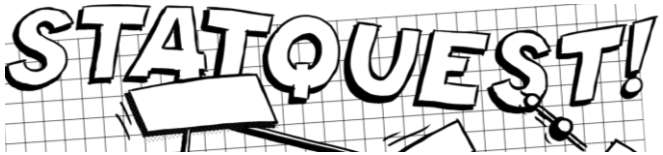
- Link is underlined

- Important terminology is in **bold** font

- Practice comes with

Workshop goals

- Recognize the uncertainty in life and put chance to work



- Understand the nuts and bolts of statistics

- Use R to perform exploratory data analysis and solve practical problems

Agenda

- Day 1: Probability and Statistics basics
  - Uncertainty; Probability; Distribution
  - Descriptive statistics
- Day 2: Inference
  - Hypothesis testing and *p*-values
  - Permutation test and bootstrap
  - False discovery rate control
- Day 3: Modeling
  - Regression techniques
  - Model selection

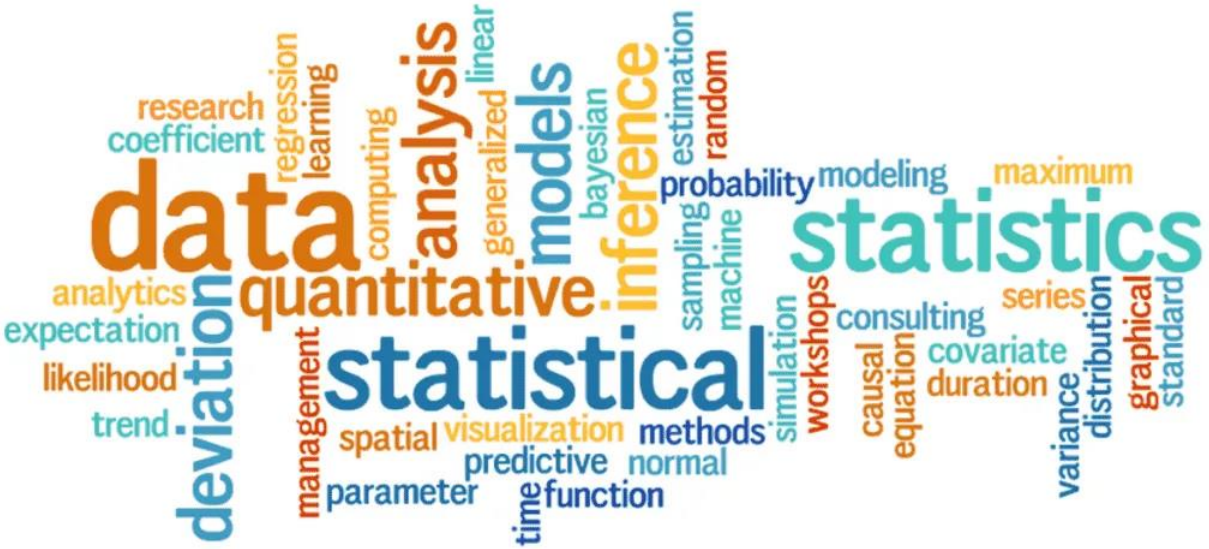
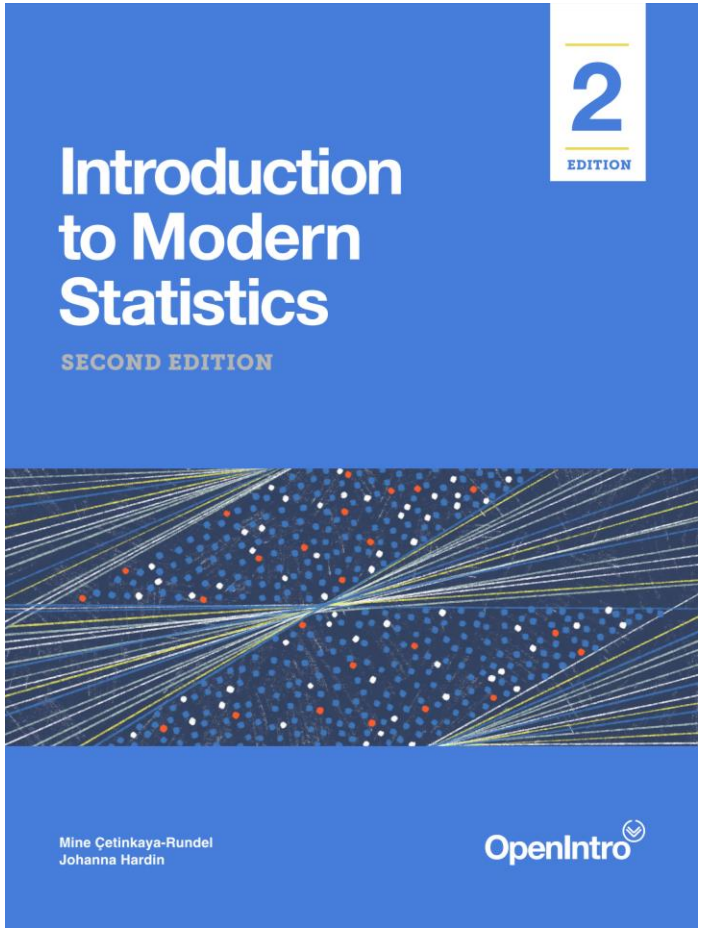


Image from Internet

Reference



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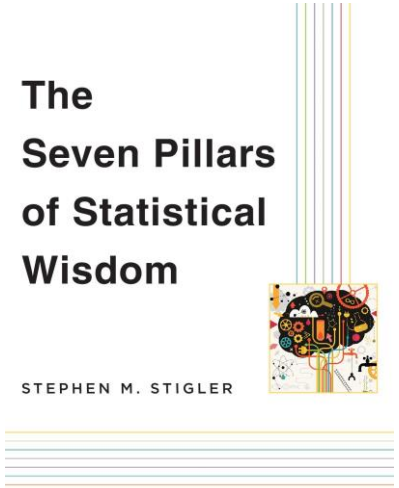
- Other useful references



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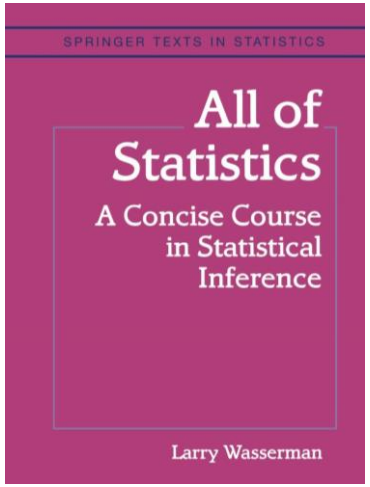

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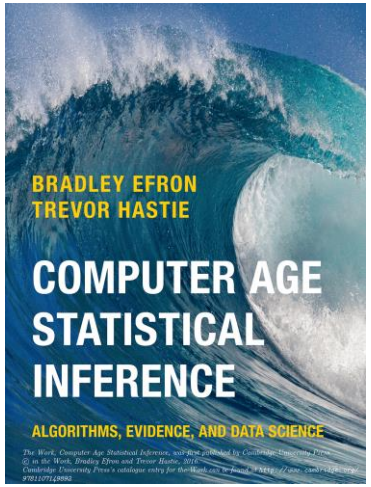


link

Working environment

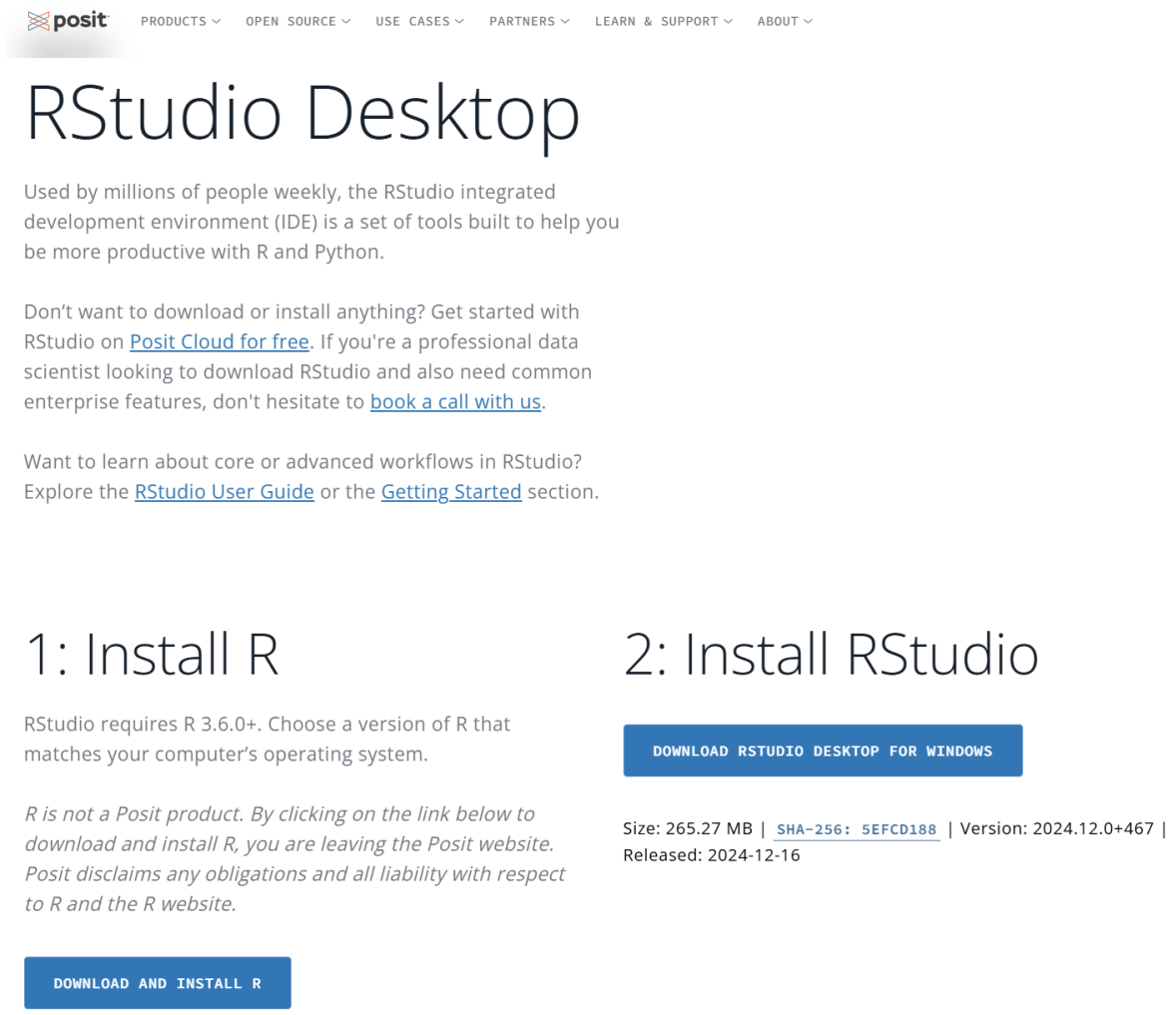
Language

Software

Image from Internet

Environment setup

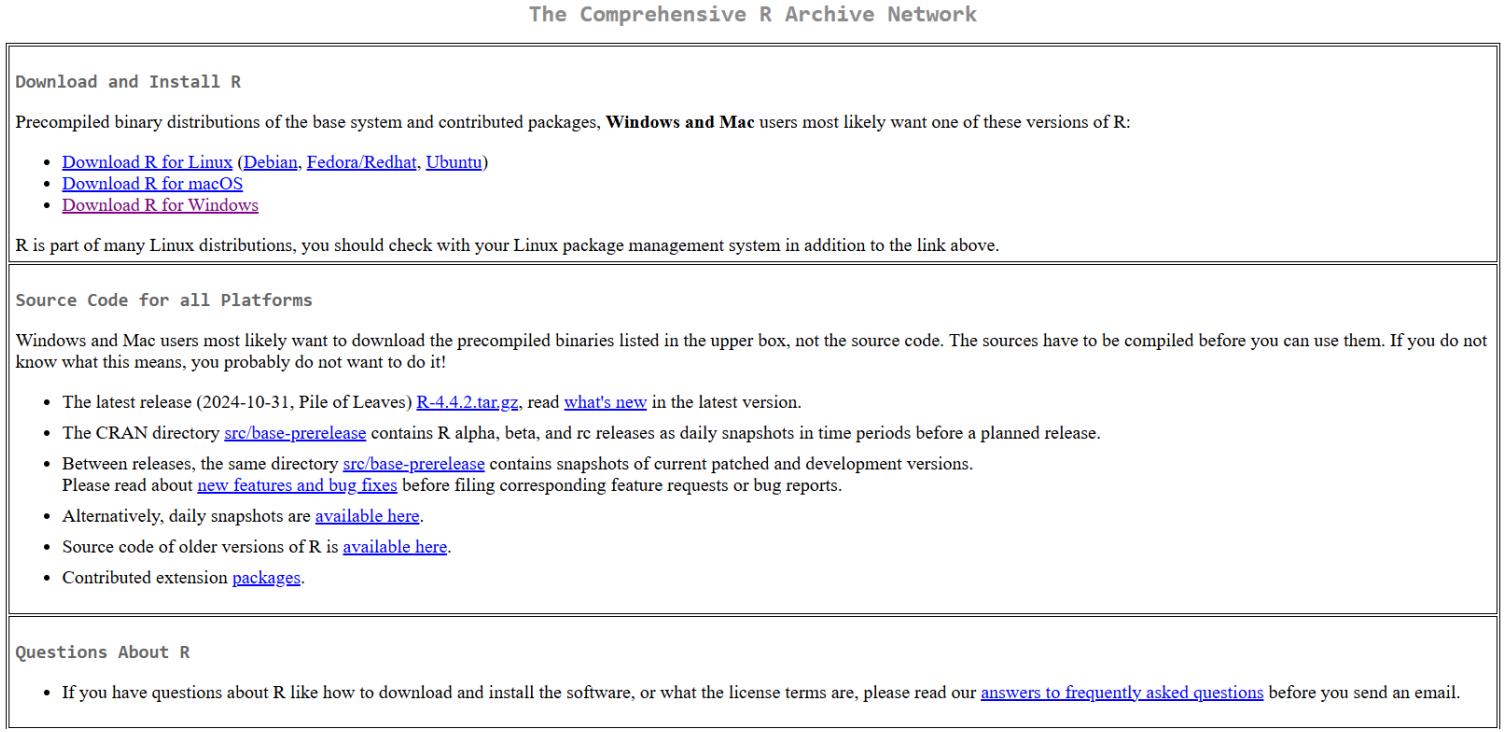
- Go to the official download website



Environment setup

- Go to the official download website

- Install R and RStudio desktop based on your operating system



The Comprehensive R Archive Network

Download and Install R

Precompiled binary distributions of the base system and contributed packages, **Windows and Mac** users most likely want one of these versions of R:

- Download R for Linux (Debian, Fedora/Redhat, Ubuntu)
- Download R for macOS
- Download R for Windows

R is part of many Linux distributions, you should check with your Linux package management system in addition to the link above.

Source Code for all Platforms

Windows and Mac users most likely want to download the precompiled binaries listed in the upper box, not the source code. The sources have to be compiled before you can use them. If you do not know what this means, you probably do not want to do it!

- The latest release (2024-10-31, Pile of Leaves) R-4.4.2.tar.gz, read what's new in the latest version.
- The CRAN directory src/base-prerelease contains R alpha, beta, and rc releases as daily snapshots in time periods before a planned release.
- Between releases, the same directory src/base-prerelease contains snapshots of current patched and development versions.
  Please read about new features and bug fixes before filing corresponding feature requests or bug reports.
- Alternatively, daily snapshots are available here.
- Source code of older versions of R is available here.
- Contributed extension packages.

Questions About R

- If you have questions about R like how to download and install the software, or what the license terms are, please read our answers to frequently asked questions before you send an email.

Environment setup

- Go to the official download website

- Install R and RStudio desktop based on your operating system

- Install the necessary package(s) in RStudio Console
  - `install.packages("tidyverse")`

Workshop materials

- git clone https://github.com/wbvguo/qcbio-Intro2ModernStats.git

UCLA QCBio Collaboratory

Day 1: Probability and Statistics basics

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2025 Winter

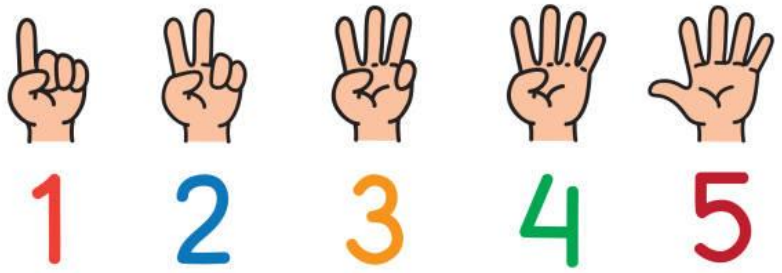
Overview

Time

- 2 hours workshop (45min + 45min + practice/Q&A)

Topics

- ❑ Introduction to Statistics
- ❑ Uncertainty
- ❑ Probability
- ❑ Distribution
- ❑ Descriptive statistics



Counting as a fundamental part of statistics

Image from Internet

What is statistics?

Statistics is the discipline that concerns the collection, organization, analysis, interpretation, and presentation of **data**

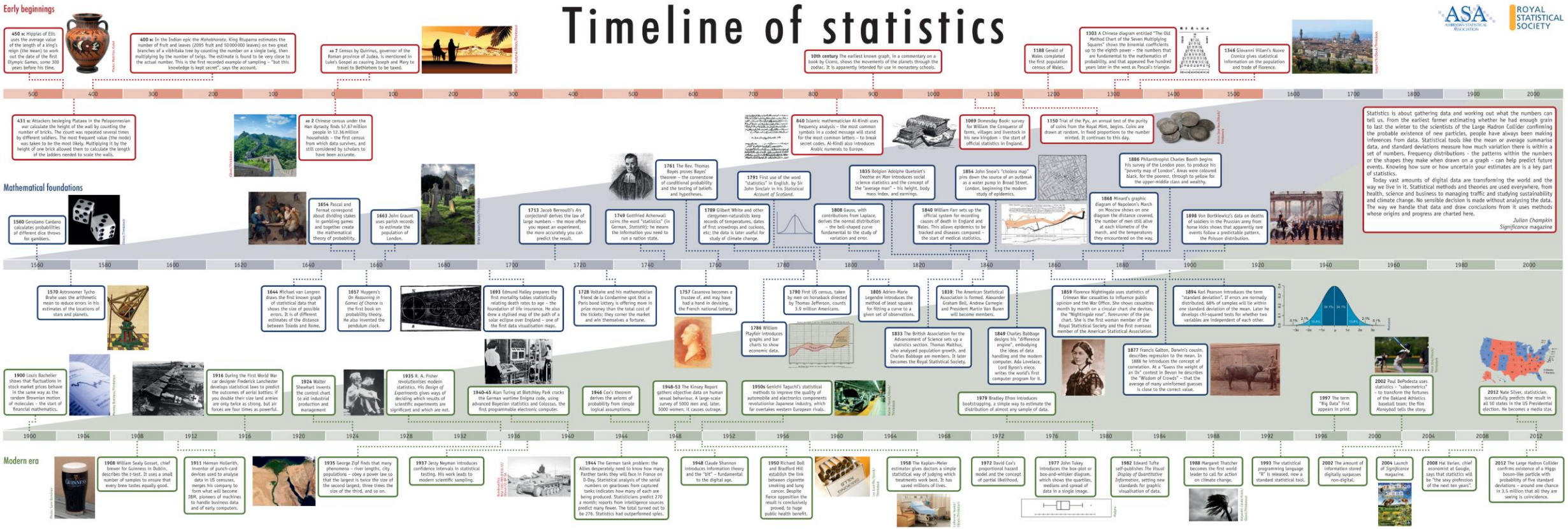
- ❑ experiment design, sampling
- ❑ descriptive summary
- ❑ hypothesis testing, significance
- ❑ visualization
- ❑ …

Data: samples drawn from a population

Two main statistical methods:

- ❑ **Descriptive** statistics: summarize the data (describe properties of sample distribution, such as central tendency, dispersion, etc.)
- ❑ **Inferential** statistics: draw conclusions from the data that are subject to random variation (use probabilistic framework to analyze random phenomena)

History of Statistics



Early beginnings

Photo: Matthias Kabel

450 BC Hippias of Elis uses the average value of the length of a king's reign (the mean) to work out the date of the first Olympic Games, some 300 years before his time.

400 BC In the Indian epic the *Mahabharata*, King Rtuparna estimates the number of fruit and leaves (2095 fruit and 50 000 000 leaves) on two great branches of a vibhitaka tree by counting the number on a single twig, then multiplying by the number of twigs. The estimate is found to be very close to the actual number. This is the first recorded example of sampling – "but this knowledge is kept secret", says the account.

AD 7 Census by Quirinus, governor of the Roman province of Judea, is mentioned in Luke's Gospel as causing Joseph and Mary to travel to Bethlehem to be taxed.

500 400 300 200 100 0 100

431 BC Attackers besieging Plataea in the Peloponnesian war calculate the height of the wall by counting the number of bricks. The count was repeated several times by different soldiers. The most frequent value (the mode) was taken to be the most likely. Multiplying it by the height of one brick allowed them to calculate the length of the ladders needed to scale the walls.

iStock/Thinkstock

AD 2 Chinese census under the Han dynasty finds 57.67 million people in 12.36 million households – the first census from which data survives, and still considered by scholars to have been accurate.

Mathematical foundations

1654 Pascal and Fermat correspond

Ancient times

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iStock/Thinkstock

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Mathematical foundations

1560 Gerolamo Cardano calculates probabilities of different dice throws for gamblers.

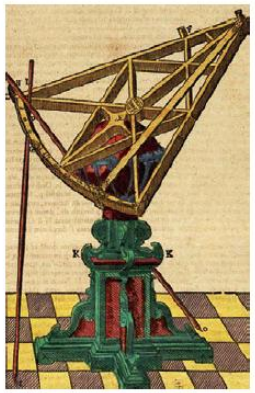
iStock/Thinkstock

1654 Pascal and Fermat correspond about dividing stakes in gambling games and together create the mathematical theory of probability.

1663 John Graunt uses parish records to estimate the population of London.

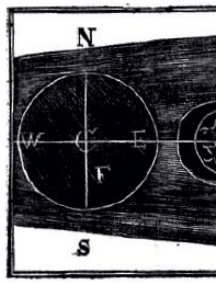
1560 1580 1600 1620 1640 1660

1570 Astronomer Tycho Brahe uses the arithmetic mean to reduce errors in his estimates of the locations of stars and planets.



1644 Michael van Langren draws the first known graph of statistical data that shows the size of possible errors. It is of different estimates of the distance between Toledo and Rome.

1657 Huygens's *On Reasoning in Games of Chance* is the first book on probability theory. He also invented the pendulum clock.



16-17th century

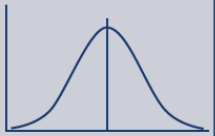
1761 The Rev. Thomas Bayes proves Bayes' theorem – the cornerstone of conditional probability and the testing of beliefs and hypotheses.

1713 Jacob Bernoulli's *Ars conjectandi* derives the law of large numbers – the more often you repeat an experiment, the more accurately you can predict the result.

1749 Gottfried Achenwall coins the word "statistics" (in German, *Statistik*); he means the information you need to run a nation state.

1789 Gilbert White and other clergymen-naturalists keep records of temperatures, dates of first snowdrops and cuckoos, etc; the data is later useful for study of climate change.

1791 First use of the word "statistics" in English, by Sir John Sinclair in his *Statistical Account of Scotland*.



1808 Gauss, with contributions from Laplace, derives the normal distribution – the bell-shaped curve fundamental to the study of variation and error.

1835 Belgian Adolphe Quetelet's *Treatise on Man* introduces social science statistics and the concept of the "average man" – his height, body mass index, and earnings.

1840 William Far official system fo causes of death in Wales. This allows e tracked and disease the start of medic

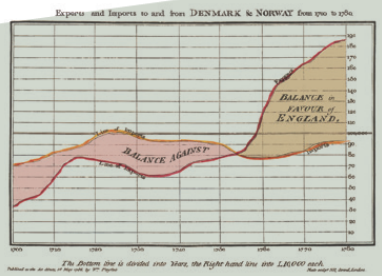
1700 1720 1740 1760 1780 1800 1820

dmund Halley prepares the rtality tables statistically death rates to age – the n of life insurance. He also ylised map of the path of a ipse over England – one of t data visualisation maps.

1728 Voltaire and his mathematician friend de la Condamine spot that a Paris bond lottery is offering more in prize money than the total cost of the tickets; they corner the market and win themselves a fortune.

1757 Casanova becomes a trustee of, and may have had a hand in devising, the French national lottery.

1786 William Playfair introduces graphs and bar charts to show economic data.



1790 First US census, taken by men on horseback directed by Thomas Jefferson, counts 3.9 million Americans.

1805 Adrien-Marie Legendre introduces the method of least squares for fitting a curve to a given set of observations.

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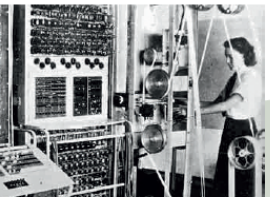
1833 The British Association for Advancement of Science sets u statistics section. Thomas Malth who analysed population growth, Charles Babbage are members. It becomes the Royal Statistical Soc



18th century

19th century

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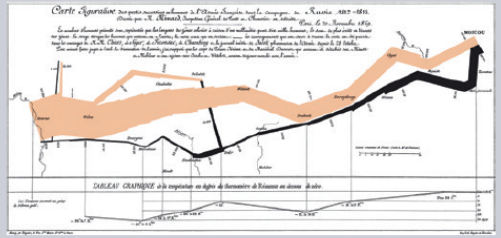
1840 William Farr sets up the official system for recording causes of death in England and Wales. This allows epidemics to be tracked and diseases compared – the start of medical statistics.

1854 John Snow's "cholera map" pins down the source of an outbreak as a water pump in Broad Street, London, beginning the modern study of epidemics.



1868 Minard's graphic diagram of Napoleon's March on Moscow shows on one diagram the distance covered, the number of men still alive at each kilometre of the march, and the temperatures they encountered on the way.

1886 Philanthropist Charles Booth begins his survey of the London poor, to produce his "poverty map of London". Areas were coloured black, for the poorest, through to yellow for the upper-middle class and wealthy.

1898 Von Bortkiewicz's data on deaths of soldiers in the Prussian army from horse kicks shows that apparently rare events follow a predictable pattern, the Poisson distribution.

1800 | 1820 | 1840 | 1860 | 1880 | 1900 | 1920

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1833 The British Association for the Advancement of Science sets up a statistics section. Thomas Malthus, who analysed population growth, and Charles Babbage are members. It later becomes the Royal Statistical Society.

1839: The American Statistical Association is formed. Alexander Graham Bell, Andrew Carnegie and President Martin Van Buren will become members.

1849 Charles Babbage designs his "difference engine", embodying the ideas of data handling and the modern computer. Ada Lovelace, Lord Byron's niece, writes the world's first

1859 Florence Nightingale uses statistics of Crimean War casualties to influence public opinion and the War Office. She shows casualties month by month on a circular chart she devises, the "Nightingale rose", forerunner of the pie chart. She is the first woman member of the Royal Statistical Society and the first overseas member of the American Statistical Association.

1877 Francis Galton, Darwin's cousin, describes regression to the mean. In 1888 he introduces the concept of correlation. At a "Guess the weight of an Ox" contest in Devon he describes

1894 Karl Pearson introduces the term "standard deviation". If errors are normally distributed, 68% of samples will lie within one standard deviation of the mean. Later he develops chi-squared tests for whether two variables are independent of each other.

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drew a stylised map of the path of a solar eclipse over England – one of the first data visualisation maps.

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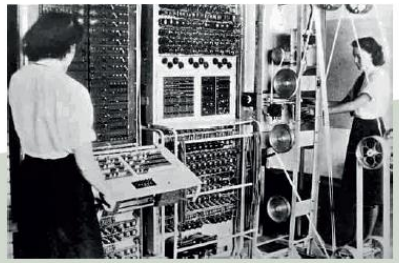

1916 During the First World War car designer Frederick Lanchester develops statistical laws to predict the outcomes of aerial battles: if you double their size land armies are only twice as strong, but air forces are four times as powerful.

1924 Walter Shewhart invents the control chart to aid industrial production and management

1935 R. A. Fisher revolutionises modern statistics. His *Design of Experiments* gives ways of deciding which results of scientific experiments are significant and which are not.

1940-45 Alan Turing at Bletchley Park cracks the German wartime Enigma code, using advanced Bayesian statistics and Colossus, the first programmable electronic computer.

1946 Cox's theorem derives the axioms of probability from simple logical assumptions.

1916 1920 1924 1928 1932 1936 1940 1944

rman Hollerith, of punch-card sed to analyse US censuses, is company to at will become eers of machines e business data rly computers.

1935 George Zipf finds that many phenomena – river lengths, city populations – obey a power law so that the largest is twice the size of the second largest, three times the size of the third, and so on.

1937 Jerzy Neyman introduces confidence intervals in statistical testing. His work leads to modern scientific sampling.

Bundesarchiv, Bild 1011-783-0110-12/ Dörner/CC-BY-SA

1944 The German tank problem: the Allies desperately need to know how ma Panther tanks they will face in France o D-Day. Statistical analysis of the serial numbers on gearboxes from captured tanks indicates how many of each are being produced. Statisticians predict 27 a month; reports from intelligence sourc predict many fewer. The total turned out be 276. Statistics had outperformed spie

Modern era

Modern era

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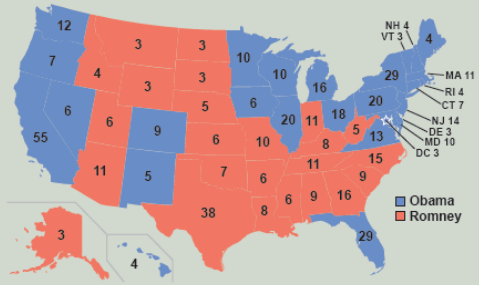
877 Francis Galton, Darwin's cousin, describes regression to the mean. In 1888 he introduces the concept of correlation. At a "Guess the weight of an Ox" contest in Devon he describes the "Wisdom of Crowds" – that the average of many uninformed guesses is close to the correct value.

Peter Kim/iStock/Thinkstock



1997 The term "Big Data" first appears in print.

2002 Paul DePodesta uses statistics – "sabermetrics" – to transform the fortunes of the Oakland Athletics baseball team; the film *Moneyball* tells the story.

2012 Nate Silver, statistician, successfully predicts the result in all 50 states in the US Presidential election. He becomes a media star.

1984 | 1988 | 1992 | 1996 | 2000 | 2004 | 2008 | 2012

2 Edward Tufte
blishes *The Visual*
of Quantitative
tion, setting new
ards for graphic
isation of data.

1988 Margaret Thatcher becomes the first world leader to call for action on climate change.

Alejandro Catalá Rubio/
iStock/Thinkstock

1993 The statistical programming language "R" is released, now a standard statistical tool.

2002 The amount of information stored digitally surpasses non-digital.

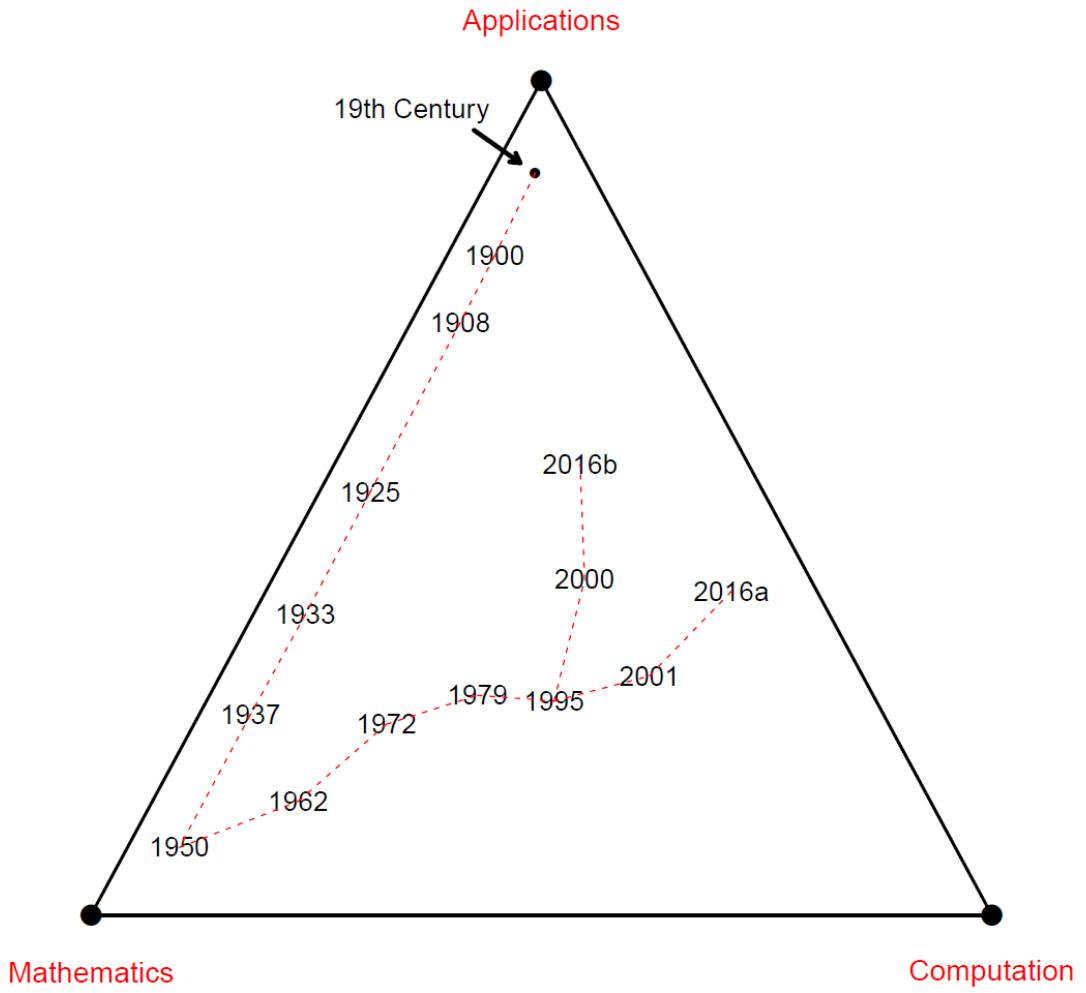
2004 Launch of *Significance* magazine.

2008 Hal Varian, chief economist at Google, says that statistics will be "the sexy profession of the next ten years".

2012 The Large Hadron Collider confirms existence of a Higgs boson-like particle with probability of five standard deviations – around one chance in 3.5 million that all they are seeing is coincidence.

Modern Statistics

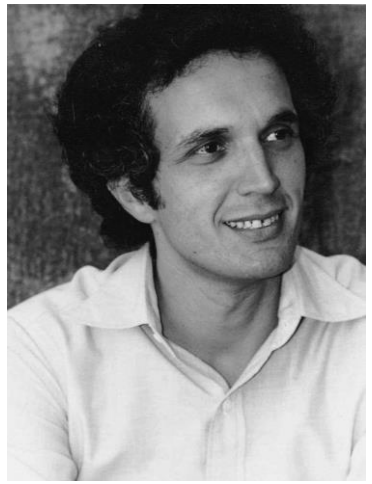
- 1900: Chi-squared test
- 1908: *t* statistic
- 1925: Maximum likelihood theory, etc.
- 1933: Neyman–Pearson lemma
- 1937: Confidence interval
- 1950: Decision theory
- 1962: Data analysis
- 1972: Cox's proportional hazards ratio
- 1979: Bootstrap; Monte-Carlo Markov Chain (MCMC)
- 1995: False Discovery Rate; Lasso
- 2000: Large scale inference (microarray data)
- 2001: Random Forest, boosting
- 2016a: Data science (even without parametric models or formal inference)
- 2016b: Big data application (Genome-wide association studies)



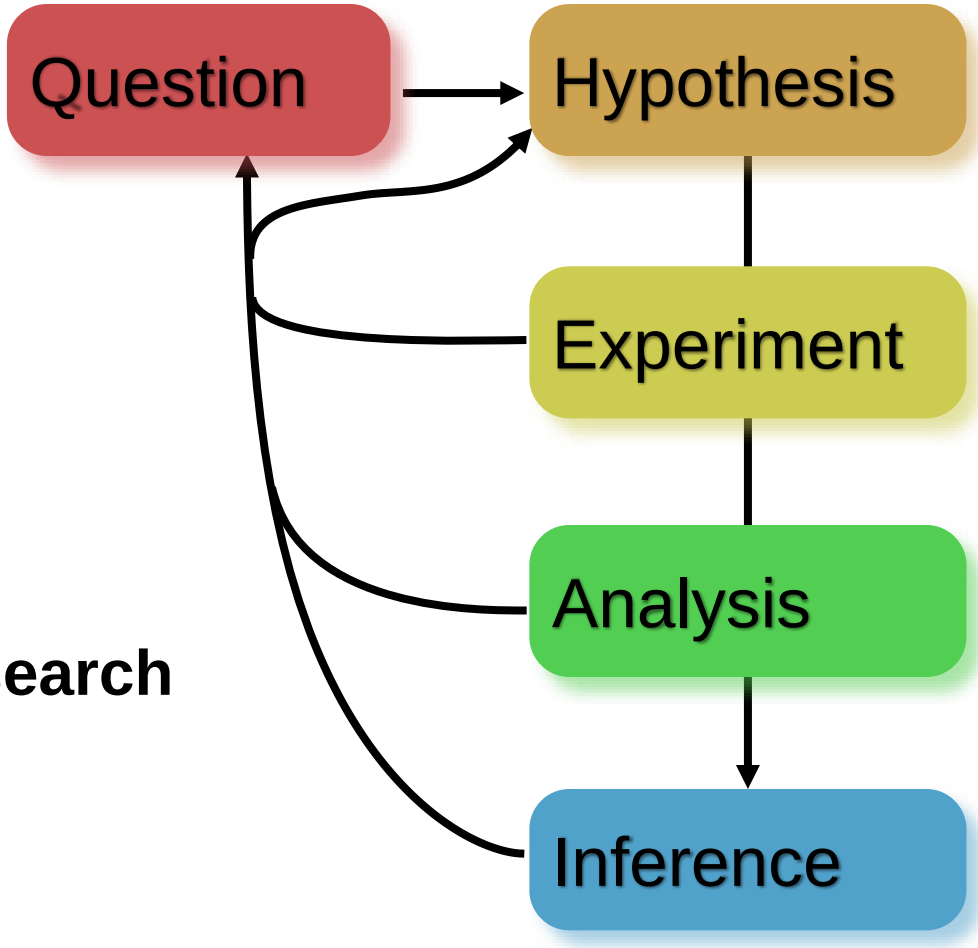
Development of the statistics discipline since the end of 19-th century

Why statistics?

“Those who ignore statistics are condemned to reinvent it”

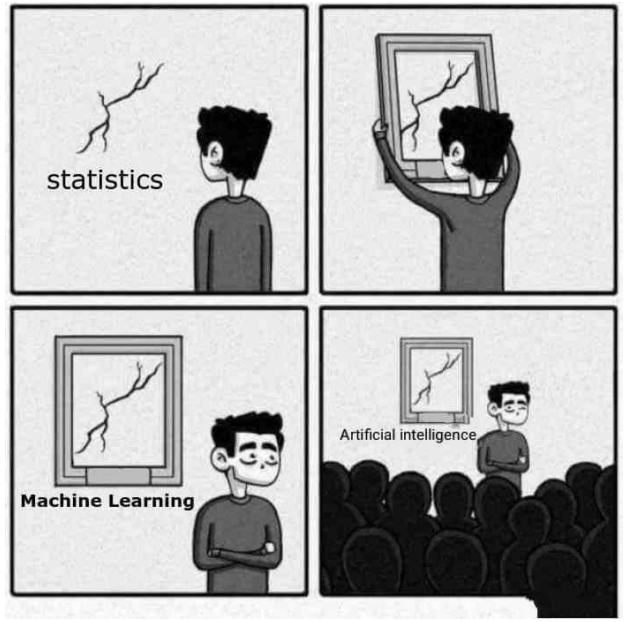


Bradley Efron
(1938-)



Statistics in scientific research

Statistics and learning



- Training data come from a probability distribution

$$(x_i, y_i, i = 1, ..., n) \sim p(x, y)$$

- Learn about this probability distribution

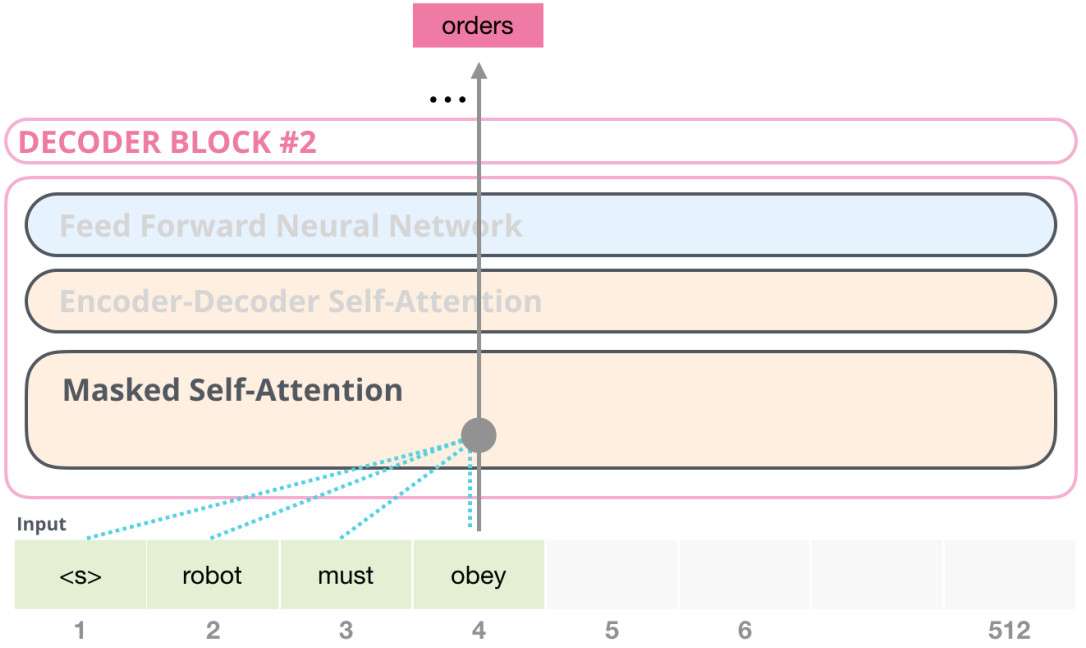
$$\max_{\theta} \frac{1}{n} \sum_{i=1}^{n} \log p_\theta(y_i|x_i) \rightarrow p_{\hat{\theta}}(y|x)$$

- Generalize to testing data (Memorization and generalization)

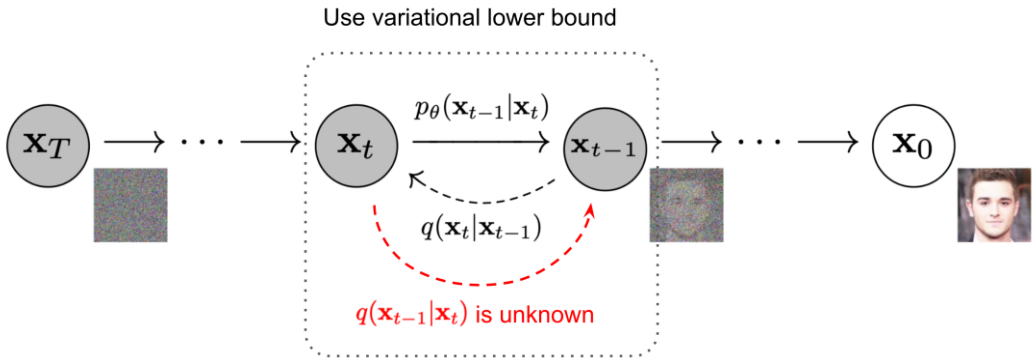
Statistics and learning

- Natural Language Processing:

next word prediction $\prod_t p_\theta(y_t|y_{<t}, x)$



- Diffusion model:



Diffusion denoising probability model $\prod_t p_\theta(y_{t-1}|y_t, x)$

Statistics and decision

Knowledge is what we know
Also, what we know we do not know.
We discover what we do not know
Essentially by what we know.

Thus knowledge expands.

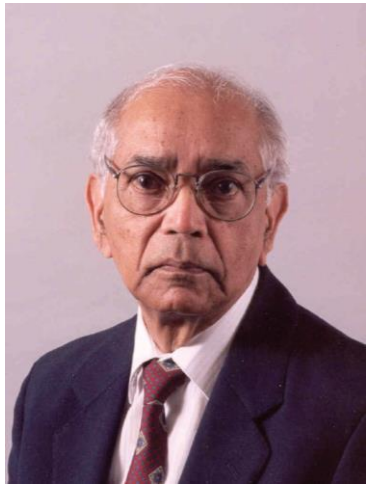
With more knowledge we come to know
More of what we do not know.

Thus knowledge expands endlessly.

* * *

All knowledge is, in final analysis, history.
All sciences are, in the abstract, mathematics.
All judgements are, in their rationale, statistics.

— *Statistics And Truth: Putting Chance To Work*, 1997

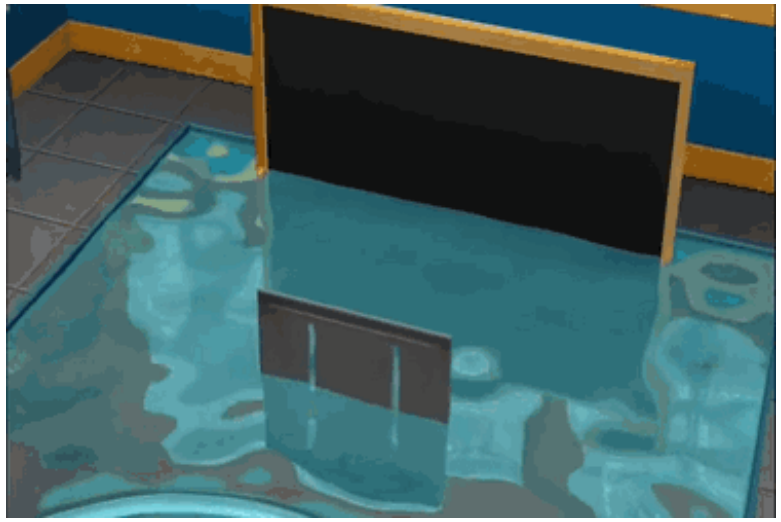


C.R. Rao
(1920-2023)

One of the greatest statisticians

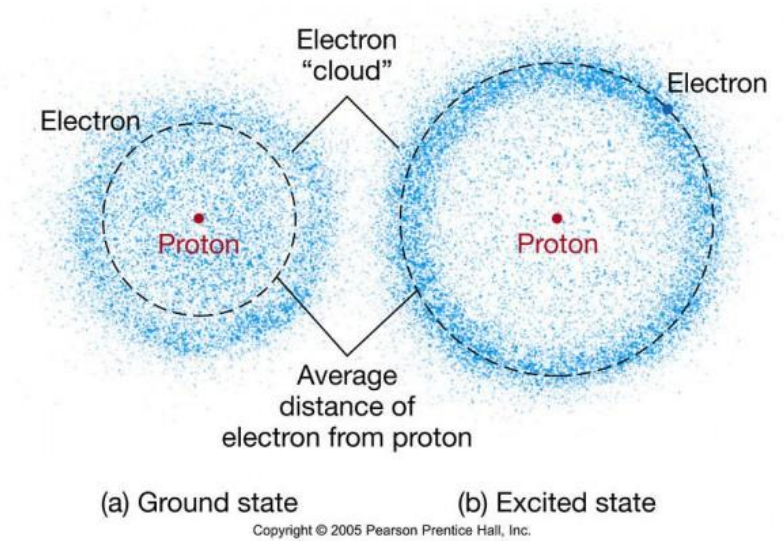
Why statistics?

- The real world is full of randomness
  - At the most fundamental level, the physical laws are probabilistic

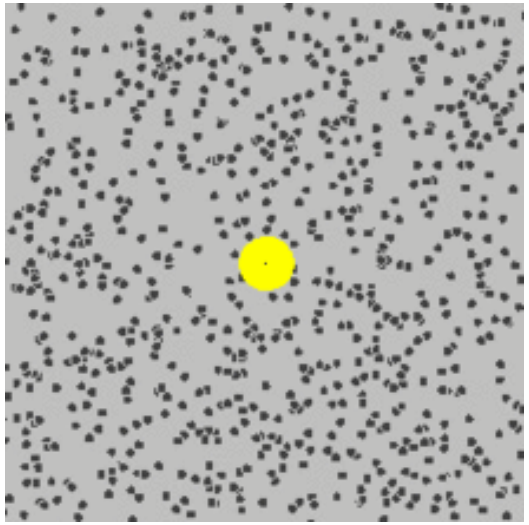


Double-slit experiment

Wave function → probability density

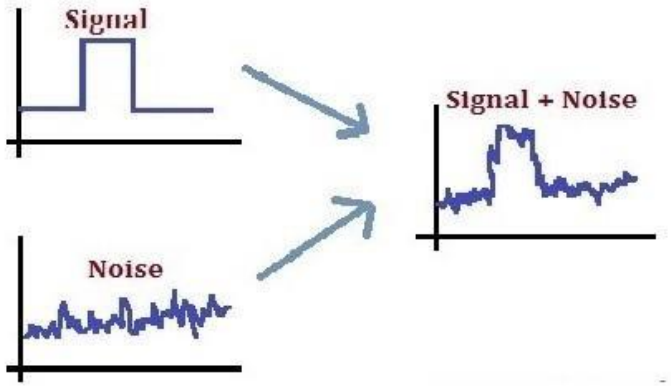


Electron cloud



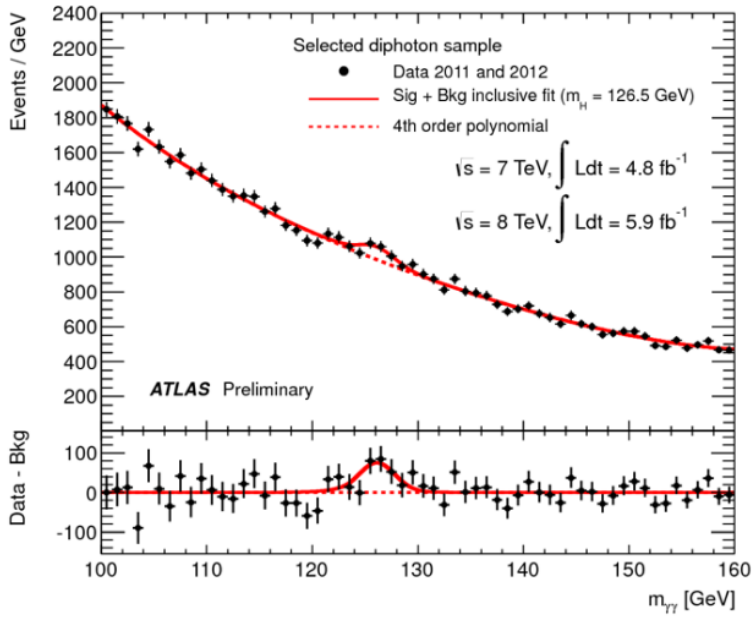
Brownian motion

Why statistics?



- The real world is full of randomness

2013 Nobel prize in physics (Higgs-boson particle)



- Our observations are not perfect
  - Limited observations due to resource constrains
  - Measurement errors (noise, bias)
  - …

We need to understand the world, and make decisions under uncertainties

Uncertainty

The only certainty in life is uncertainty itself.

Recognizing uncertainty in life

Uncertainty: A state of limited knowledge where the current state or future outcome cannot be precisely determined.

UNCERTAINTY

Objective UNCERTAINTY

Flip a coin, and the outcome is uncertain

Subjective UNCERTAINTY

Without seeing the data, someone is unsure if the coin is fair

Source of uncertainty

- lack of knowledge/Information

  e.g. go to the bus station, and see the bus is there

Question:

If we have all information, there will be no uncertainty?

No…The Heisenberg uncertainty principle

Measuring the position of a particle with high accuracy inevitably introduces uncertainty in its momentum, and vice versa, meaning that **knowing both with perfect precision at the same time is impossible.**

$$\Delta x \, \Delta p \geq \frac{h}{4\pi}$$ (h is plank constant)

Uncertainty in position ← $\Delta x \, \Delta p$ → Uncertainty in momentum

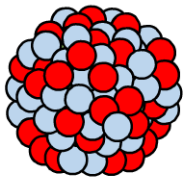
Source of uncertainty

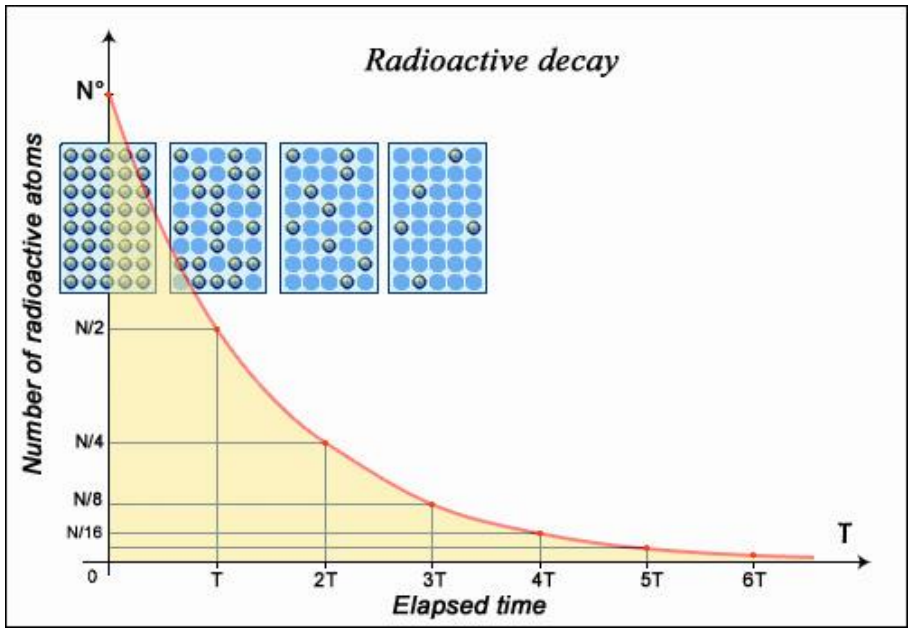
- lack of knowledge/Information

  e.g. go to the bus station, and see the bus is there

- fundamental property of nature (true randomness)

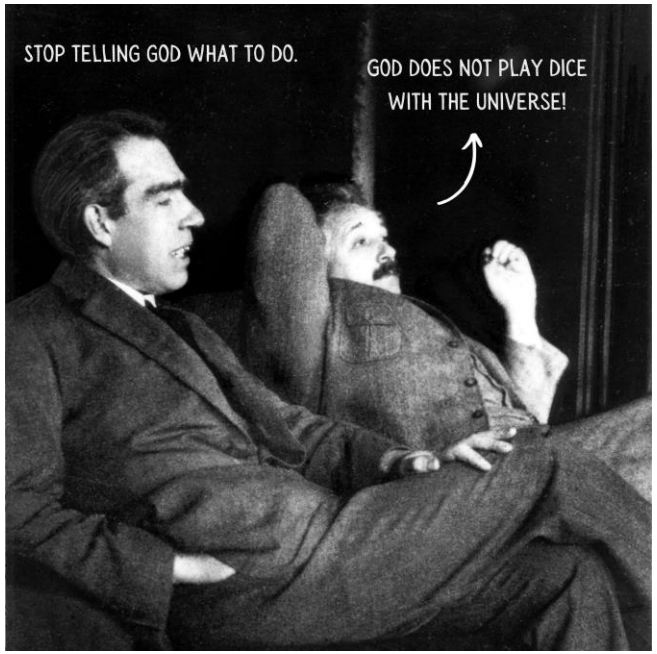
  e.g. Atom's radioactive decay

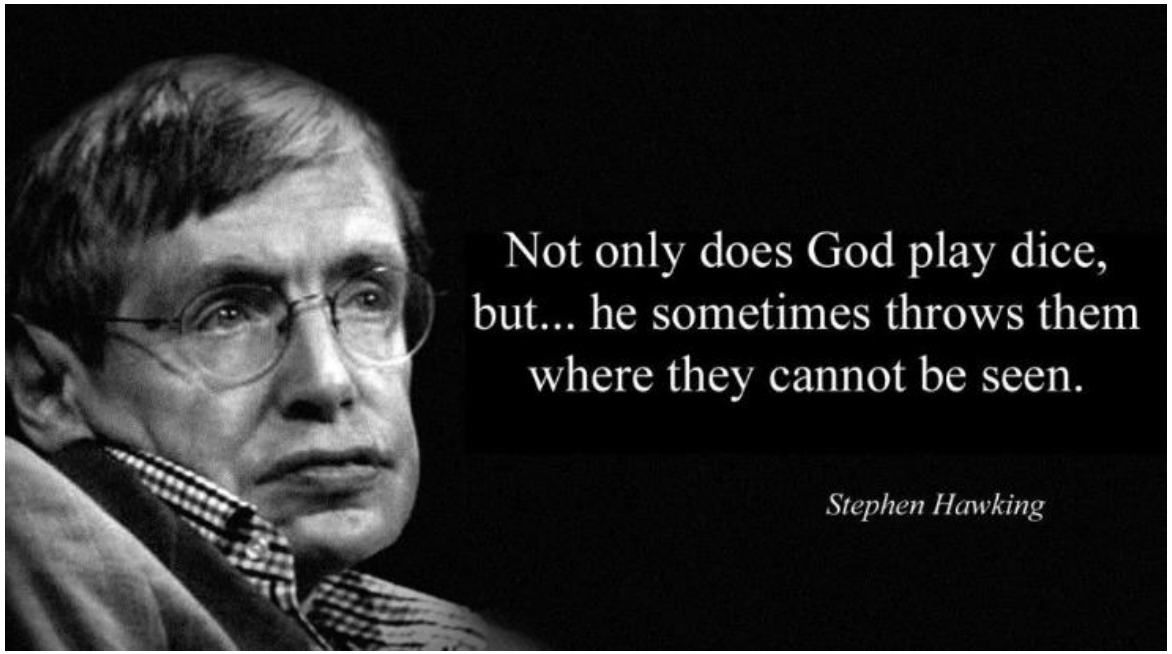




Deterministic vs Stochastic view of the world

Einstein famously declared, **'He does not play dice**', in response to Bohr's probabilistic interpretation of quantum mechanics, expressing his belief in a deterministic universe.



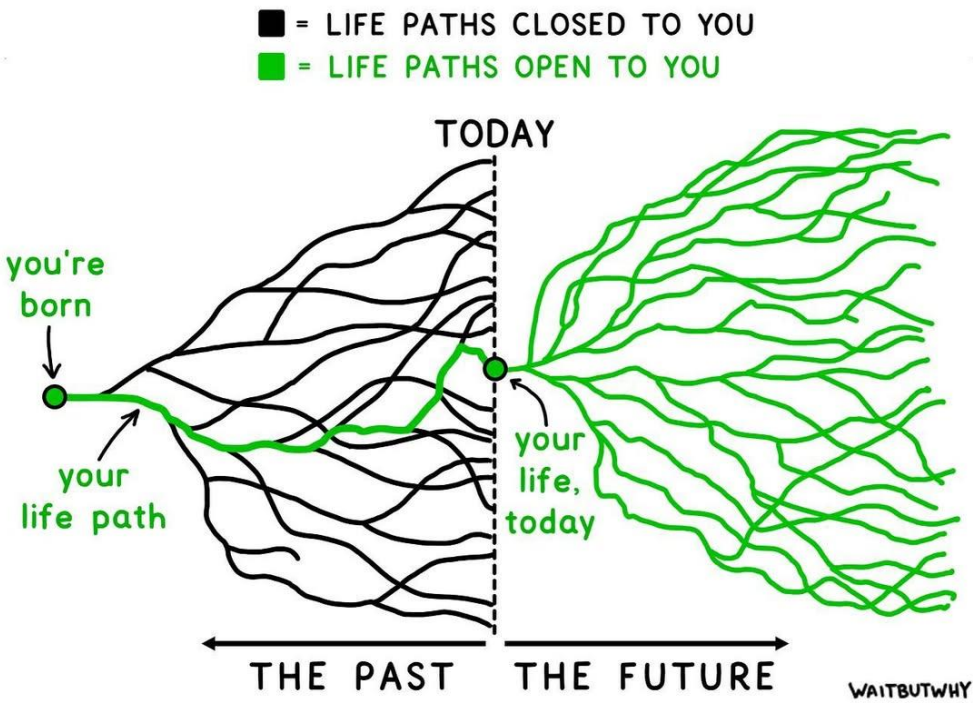


More examples of uncertainty in life

Lottery ball

Weather forecasting

Earthquake



Your life

How do we fell about uncertainty?

- We hate it, we try to avoid it
  - Drive slowly when the vision is not good
  - Invest less if a stock fluctuates too much
  - …

- We like it, we play with it

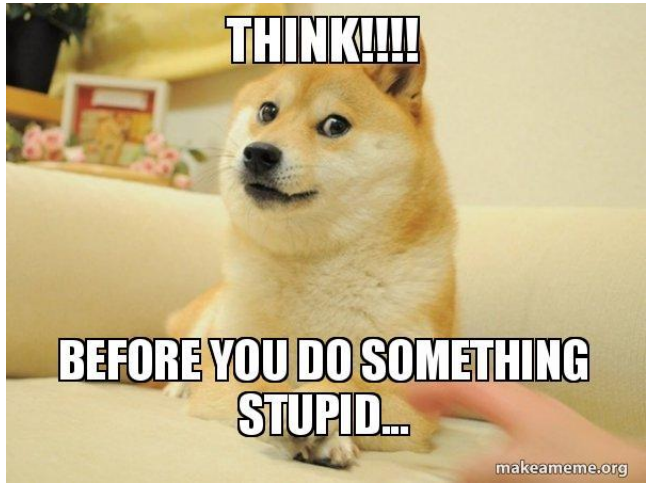
Texas poker

Monopoly

Quant trading

How do we deal with uncertainty?

We try to make a guess on the consequence before we act



Did you notice that, we seems to do some implicit calculations in our mind when making decisions with uncertainty?

Probability

"*Nothing but common sense reduced to calculation*" – Laplace

Not all events are created equal

Consider two events

a) I flip a coin, and get a head
   - Odds: 1/2

b) I buy a lottery ticket, and win $1,000,000
   - Odds: 1/1,219,304

Both of them are uncertain, which one is more likely?

We need a quantitative measure for the uncertainty!

P.S. Possible doesn't imply probable

Probability (take coin toss as an example)

Probability is a mathematical language for quantifying uncertainty

- **Sample space Ω**: the set of possible outcomes of an experiment

- **Event A**: a subset of sample space

If each outcome is equally likely, then **probability** P(**A**) = $\frac{|\mathbf{A}|}{|\mathbf{\Omega}|}$

Properties of probability

Probability P(**A**) = $\frac{|\mathbf{A}|}{|\mathbf{\Omega}|}$

Axiom:

Axiom 1: $\mathbb{P}(A) \geq 0$ *for every* A

Axiom 2: $\mathbb{P}(\Omega) = 1$

Axiom 3: *If* $A_1, A_2, \ldots$ *are disjoint then*

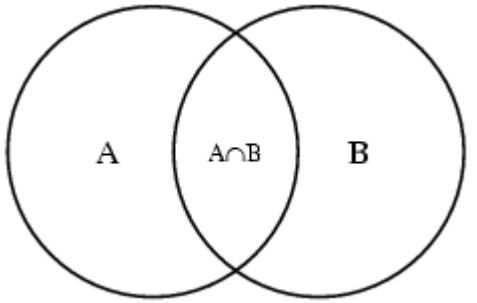
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$$\begin{aligned} \mathbb{P}(\emptyset) &= 0 \\ A \subset B &\implies \mathbb{P}(A) \leq \mathbb{P}(B) \\ 0 \leq &\ \mathbb{P}(A) \leq 1 \\ \mathbb{P}(A^c) &= 1 - \mathbb{P}(A) \end{aligned}$$

Lemma:

For any events A and B

$$P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A}) + P(\mathbf{B}) - P(\mathbf{AB})$$



Let’s do a small exercise

- Consider tossing a die twice and summing the values. What is the probability that the sum equals 10?

- Given the number of people in the classroom, assuming birthdays are equally likely, calculate the probability that at least two students share the same birthday (same month and day)

- Code practice: estimate π

Independent events

Two events are **independent** if

$$P(\mathbf{AB}) = P(\mathbf{A})\,P(\mathbf{B})$$

e.g. Flip the coin twice, what's the probability of getting both heads?

Solution 1:

- $\boldsymbol{\Omega}$ = {HH, HT, TH, TT}; **A** = {HH};
- $P(\mathbf{A}) = \frac{|\mathbf{A}|}{|\boldsymbol{\Omega}|} = ¼$

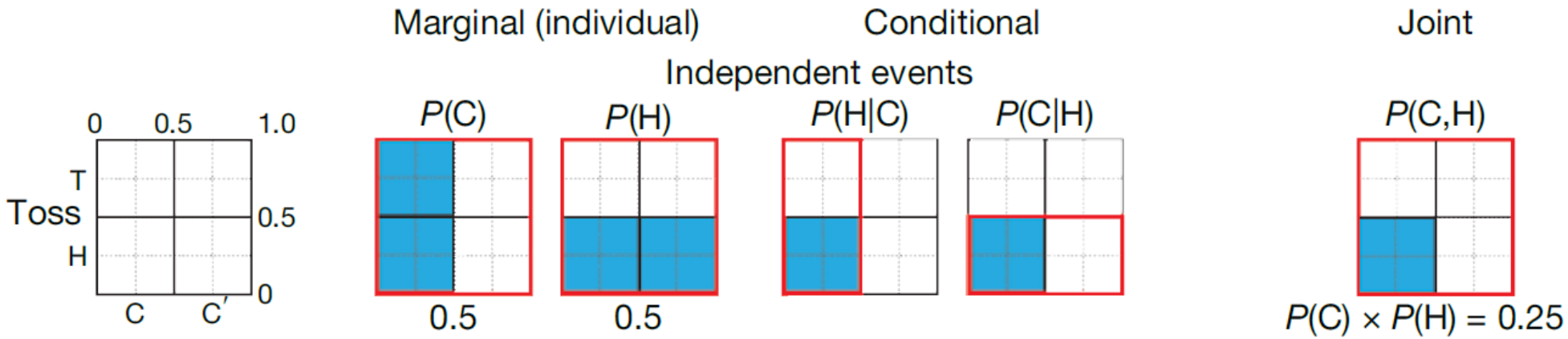
Solution 2:

$P(A_1=H) = ½$; $P(A_2=H) = ½$; $P(\mathbf{A}= A_1\ A_2 = HH) = ¼$

Conditional Probability

If P(**B**) > 0, the conditional probability of **A** given **B** is given by

$$\mathrm{P}(\mathbf{A} \mid \mathbf{B}) = \frac{\mathrm{P}(\mathbf{AB})}{\mathrm{P}(\mathbf{B})}$$



Apparently, **A** and **B** are independent events, if and only if

$$\mathrm{P}(\mathbf{A} \mid \mathbf{B}) = \mathrm{P}(\mathbf{A})$$

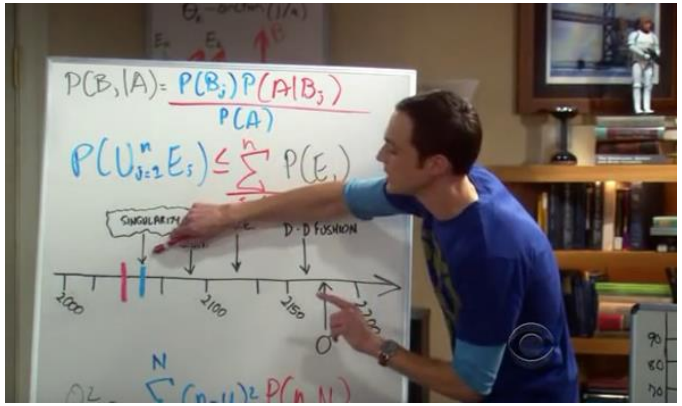
Bayes theorem

Given that P(**AB**) = P(**A | B**) * P(**B**) = P(**B | A**) * P(**A**), we derive

Prior

$$P(\mathbf{A}|\mathbf{B}) = \frac{P(\mathbf{B}|\mathbf{A})\,P(\mathbf{A})}{P(\mathbf{B})}$$

Posterior



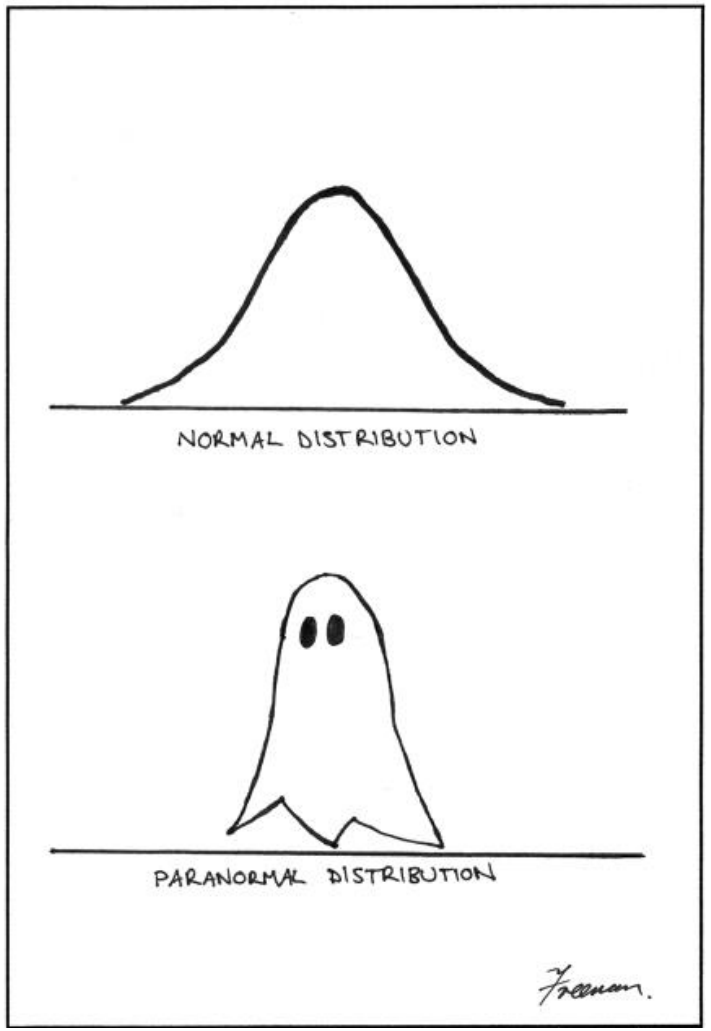
Let’s do a small exercise!

- A particular heart disease has a prevalence of 1/1000 people.
- The test used to detect this disease has a 5% false positive rate, meaning that the probability of a positive result given that a person does not have the disease (P(B | not A)) is 0.05.
- Assume the test correctly identifies every individual who does have the disease.

- What is the chance that a randomly selected person found to have a positive result actually has the disease?

Distribution

A map of possibilities.



Probability function

For a **random variable** X, the **cumulative distribution function** (CDF) $F_X: R \rightarrow [0,1]$ is defined as

$$F_X(x) = P(X \leq x)$$

- If X is discrete, define **probability mass function** (PMF) as

$$f_X = P(X = x)$$

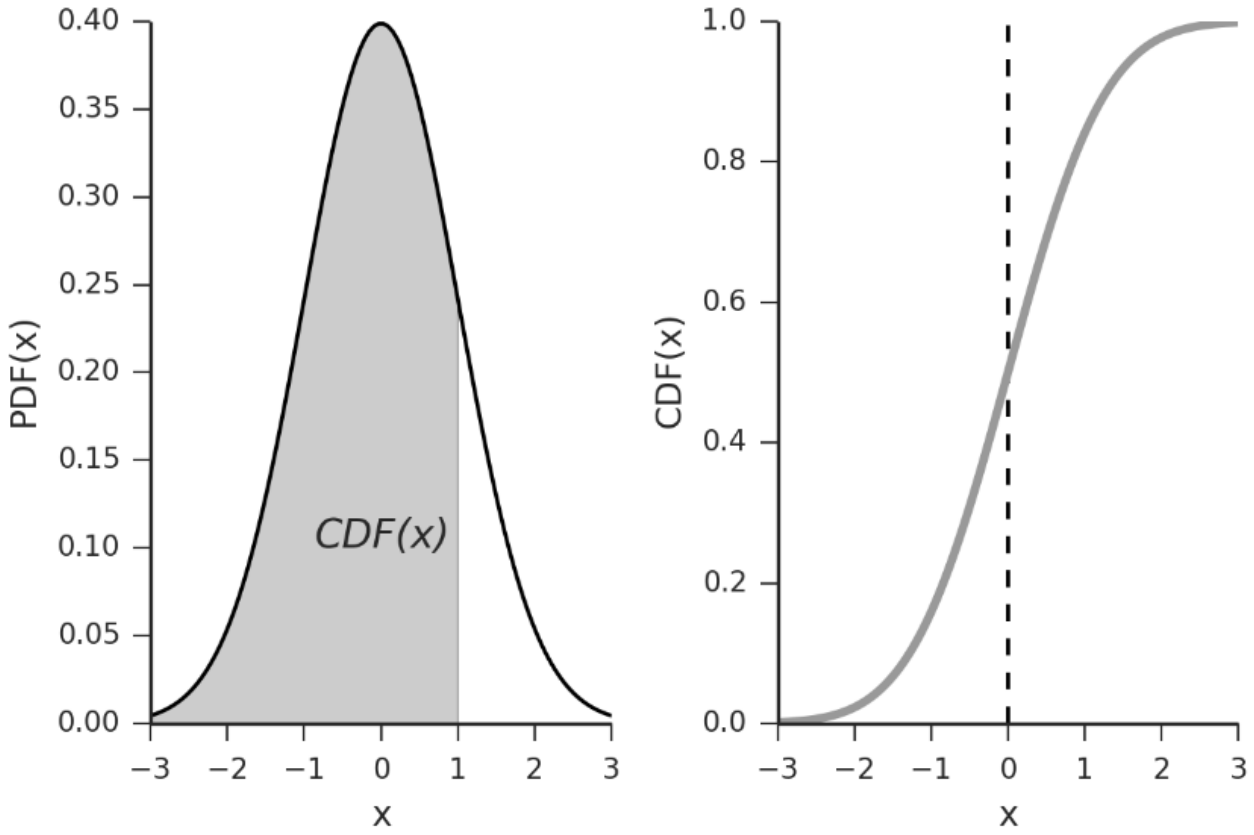
- If X is continues, define **probability density function** (PDF) as

$$f_X = F'_X(x)$$

That is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Comparison between PDF and CDF



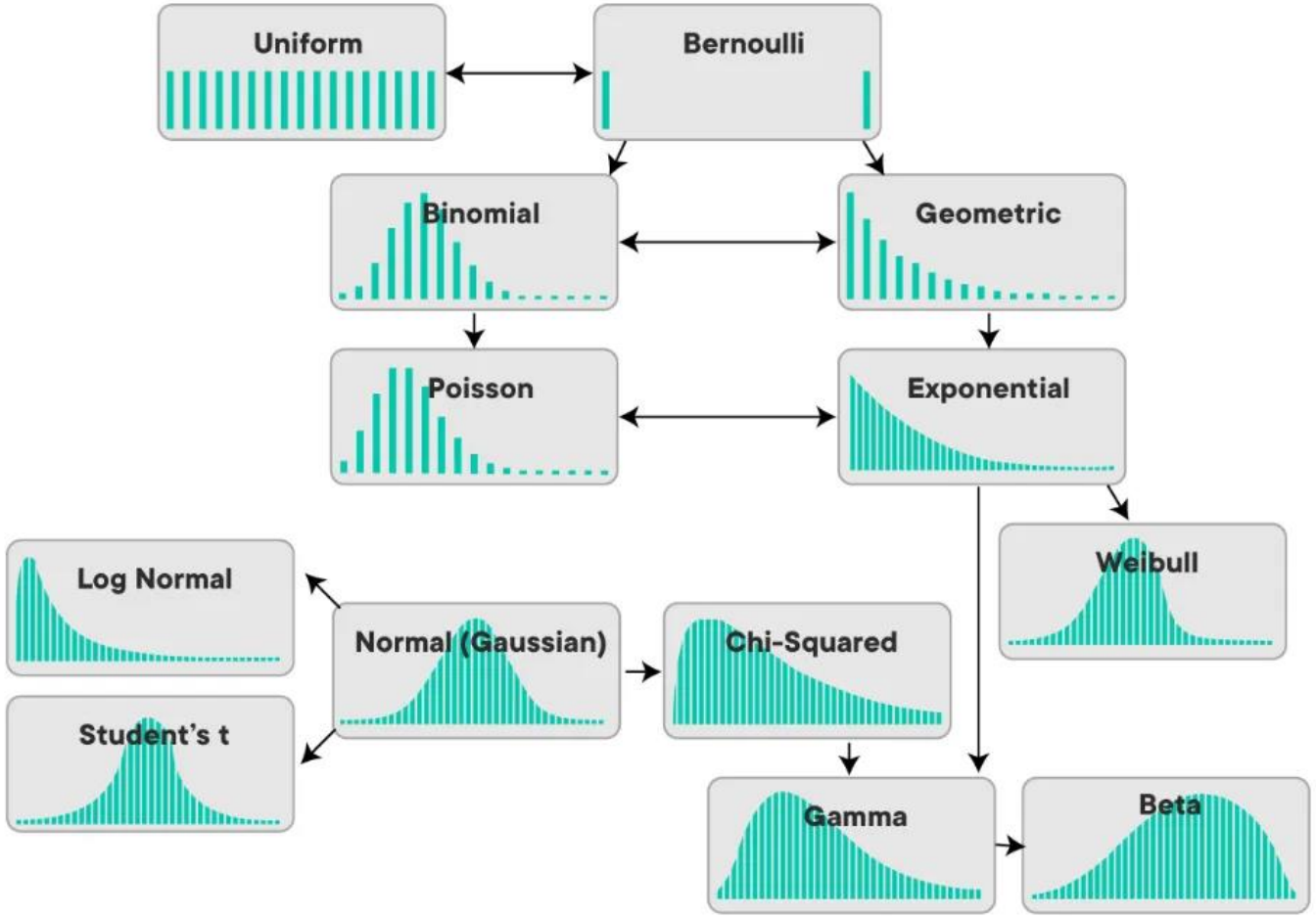
Distributions

Discrete

- ❑ Uniform
- ❑ Bernoulli
- ❑ Binomial
- ❑ Negative Binomial
- ❑ Geometric
- ❑ Hypergeometric
- ❑ Poisson

Continuous

- ❑ Uniform
- ❑ Normal
- ❑ *t*
- ❑ Chi-square
- ❑ F
- ❑ Exponential
- ❑ Beta

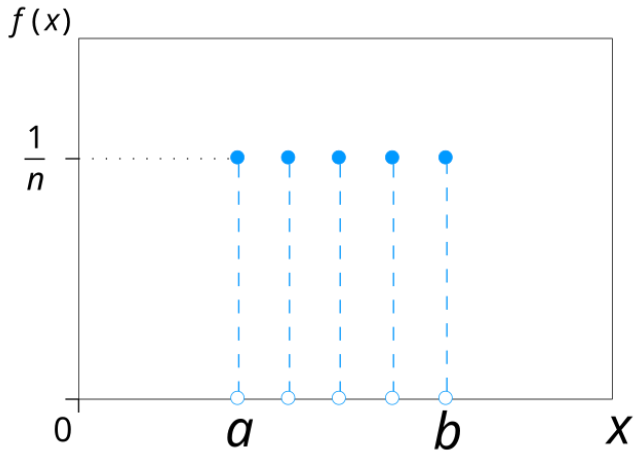


Uniform (discrete)

Intuition: A discrete uniform distribution assigns equal probability to every integer between a and b

$$P(X = x) = \frac{1}{b - a + 1}, \quad x = a, a + 1, \ldots, b$$

Example: Rolling a fair 6-sided die is a uniform distribution with possible outcomes 1 to 6, each with equal probability 1/6

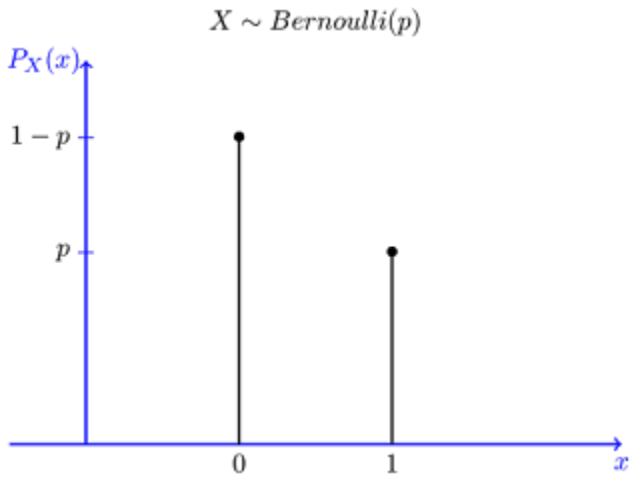


Bernoulli

Intuition: A Bernoulli distribution represents a **single trial** with two possible outcomes: success (1) with probability p, and failure (0) with probability $1 - p$.

$$P(X = x) = p^x(1-p)^{1-x}, \quad x = 0,1$$

Example: Flipping a coin once — with heads as success and tails as failure — is a Bernoulli trial with $p = 0.5$.

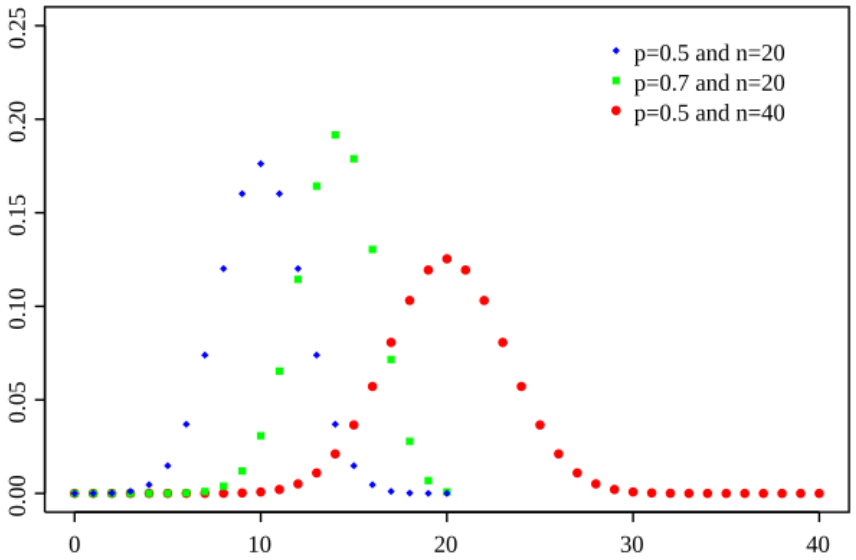


Binomial

Intuition: The binomial distribution represents the number of successes in n independent Bernoulli trials, each with success probability p.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0,1,\dots,n$$

Example: Flipping a coin n times and counting the number of heads is a binomial experiment.

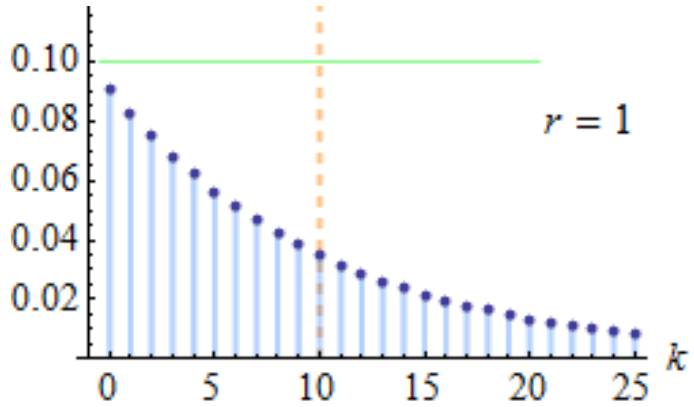


Negative binomial

Intuition: The negative binomial distribution represents the number of failures before achieving r successes in a series of independent Bernoulli trials with success probability p.

$$P(X = k) = \binom{k + r - 1}{k} p^r (1 - p)^k, \quad k = 0,1,2,$$

Example: The number of times you need to roll a die before getting a 6 three times is a negative binomial experiment.

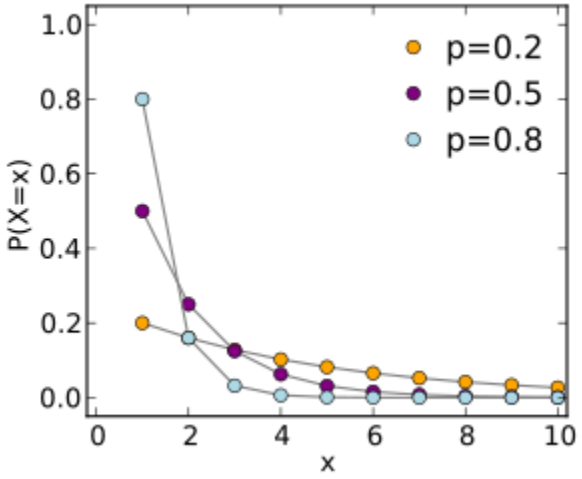


Geometric

Intuition: The geometric distribution represents the number of trials needed to get the **first success** in a series of Bernoulli trials with success probability p.

$$P(X = k) = (1 - p)^{k-1}p, \quad k = 1,2,3,$$

Example: The number of coin flips needed to get the first heads is a geometric distribution.



Hypergeometric

Intuition: The hypergeometric distribution models the number of successes in a sample of size n drawn **without replacement** from a population of size N containing K successes. This test is usually used in enrichment analysis.

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad \max(0, n - N + K) \leq k \leq \min(n, K)$$

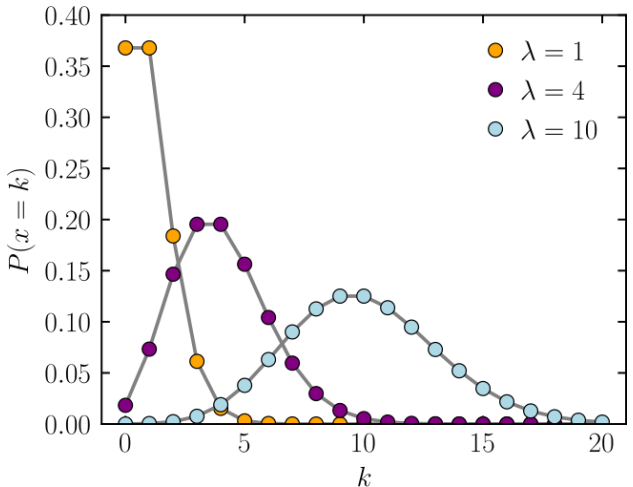
Example: Drawing cards from a deck — the number of red cards in a hand of 5 cards is a hypergeometric distribution.

Poisson

Intuition: The Poisson distribution models the number of events occurring in a fixed interval of time or space, assuming the events occur randomly and independently at a constant average rate λ.

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0,1,2,$$

Example: The number of phone calls received at a call center in an hour follows a Poisson distribution.

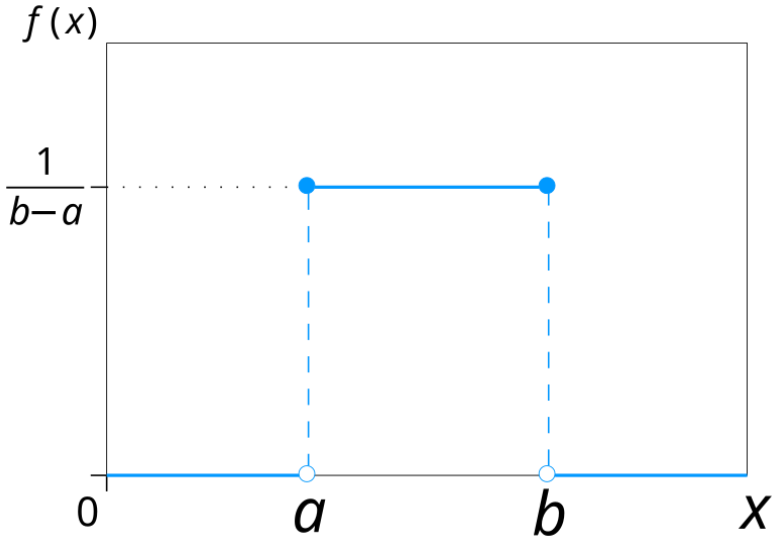


Uniform (Continuous)

Intuition: Every value in the interval $[a,b]$ is equally likely.

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

Example: The waiting time for a bus that arrives at a random time within a known 10-minute window is uniformly distributed.

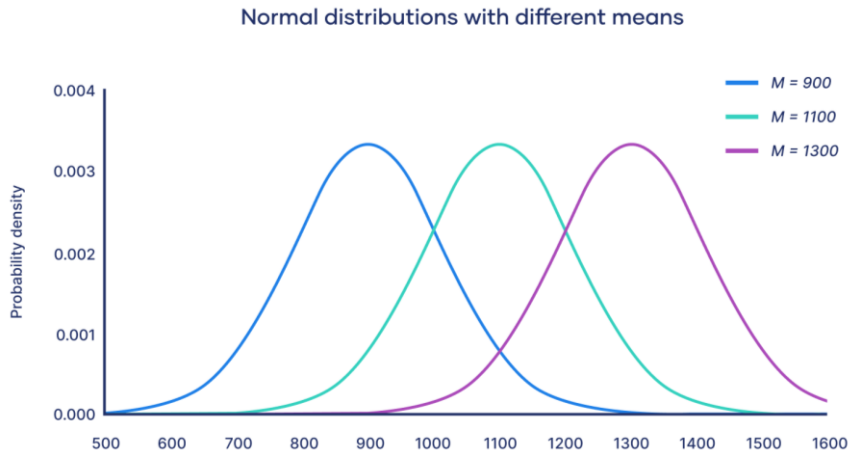


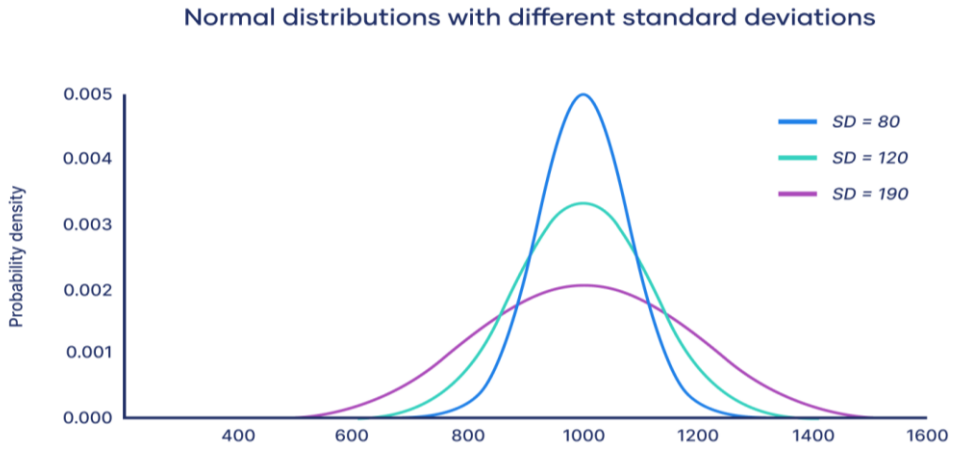
Normal

Intuition: The normal distribution (bell curve) models many natural phenomena like heights, weights, and measurement errors. It is symmetric around the mean μ, with standard deviation σ.

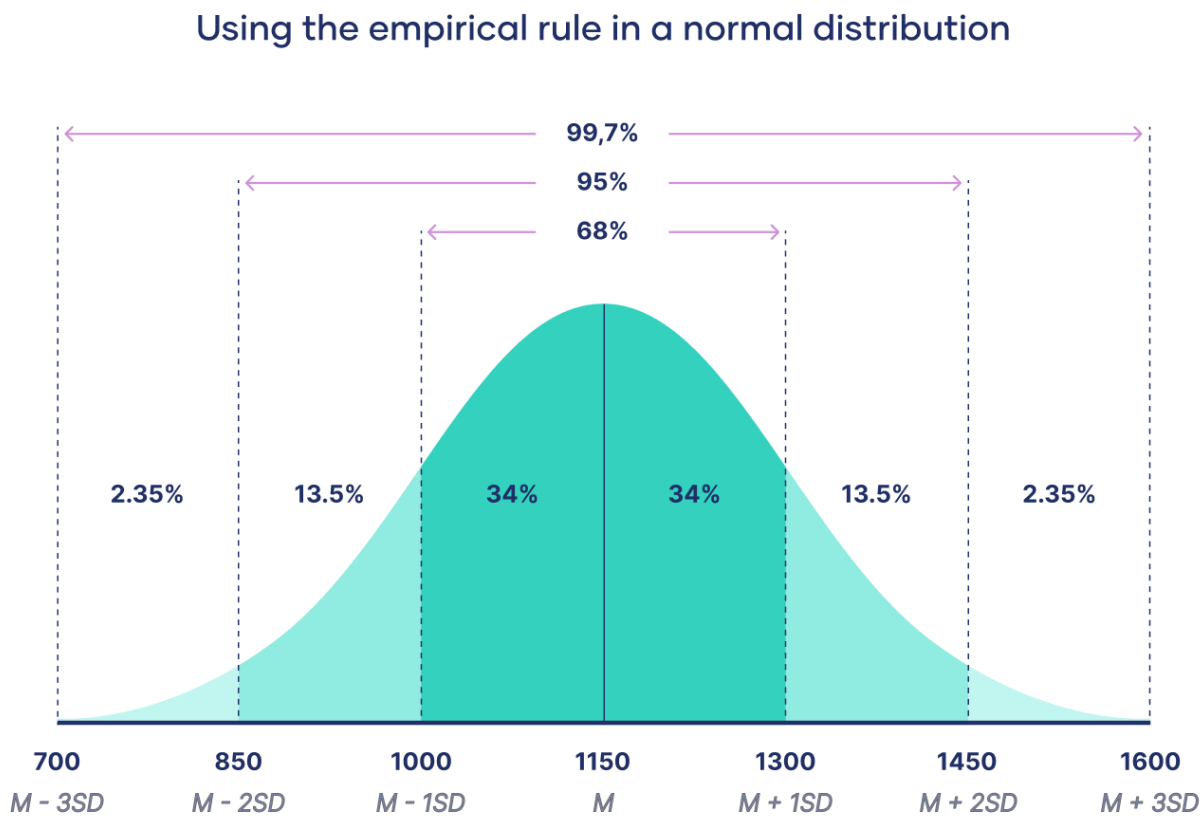
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Example: The distribution of heights in a large population





Normal distribution and three-sigma rule of thumb



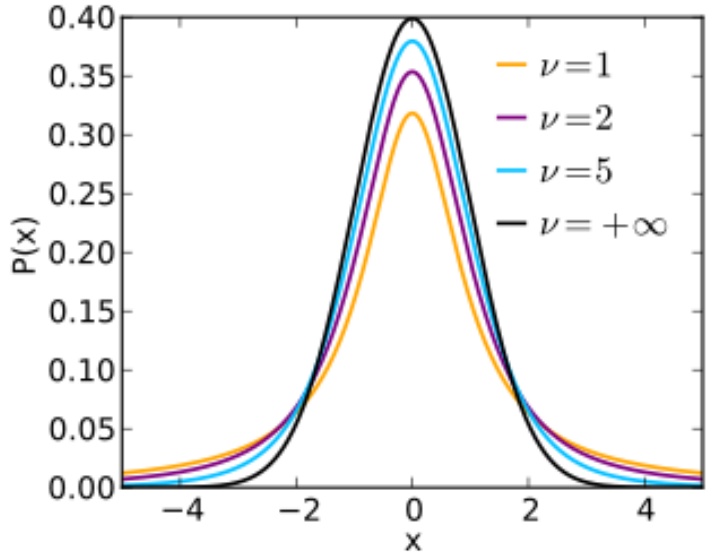
nearly all values are taken to lie within three standard deviations of the mean

t-distribution

Intuition: The *t*-distribution is similar to the normal distribution but has heavier tails. It is used in hypothesis testing when the sample size is small.

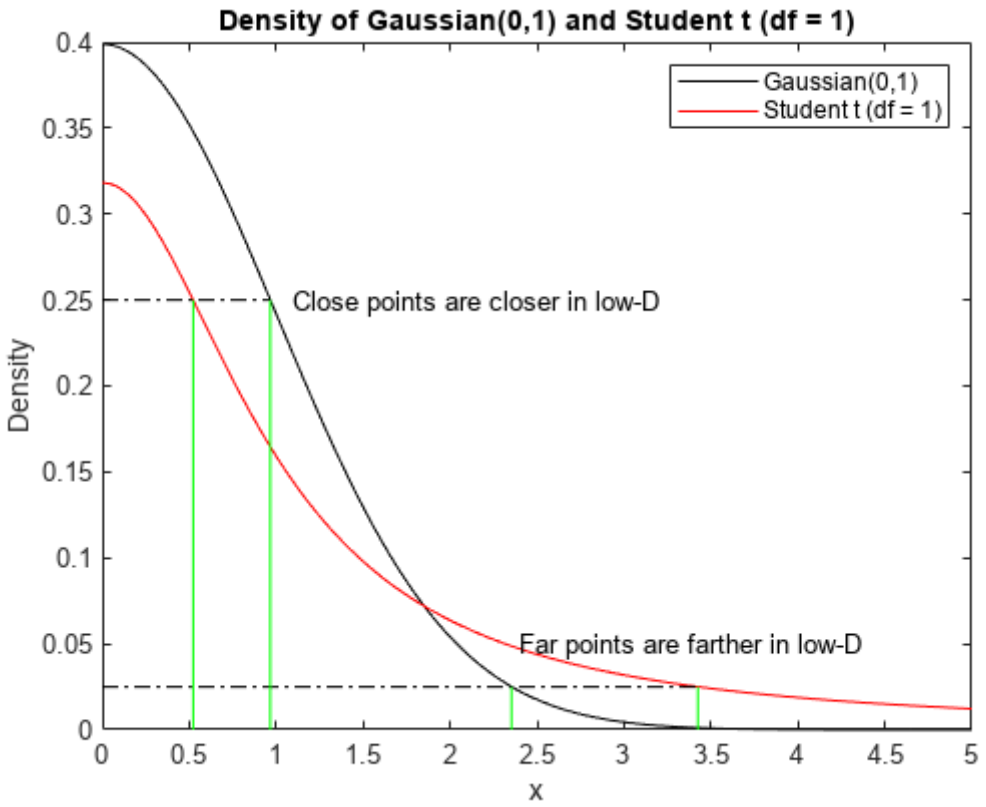
$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Where ν is the degrees of freedom.



t-Distributed Stochastic Neighbor Embedding (*t*-SNE)

- Nonlinear dimensionality reduction
  - PCA uses the global covariance matrix
  - t-SNE focus more on the local structure



https://www.mathworks.com/help/stats/t-sne.html

- Core algorithm
  - In high dimensional space

$$p_{ij} = \frac{\exp(-\|x_i - x_j\|^2 / 2\sigma_i^2)}{\sum_{k \neq l} \exp(-\|x_k - x_l\|^2 / 2\sigma_i^2)}$$

  - In lower dimensional space

$$q_{ij} = \frac{(1 + \|y_i - y_j\|^2)^{-1}}{\sum_{k \neq l} (1 + \|y_k - y_l\|^2)^{-1}}$$

  - Try to minimize the difference of 2 distributions

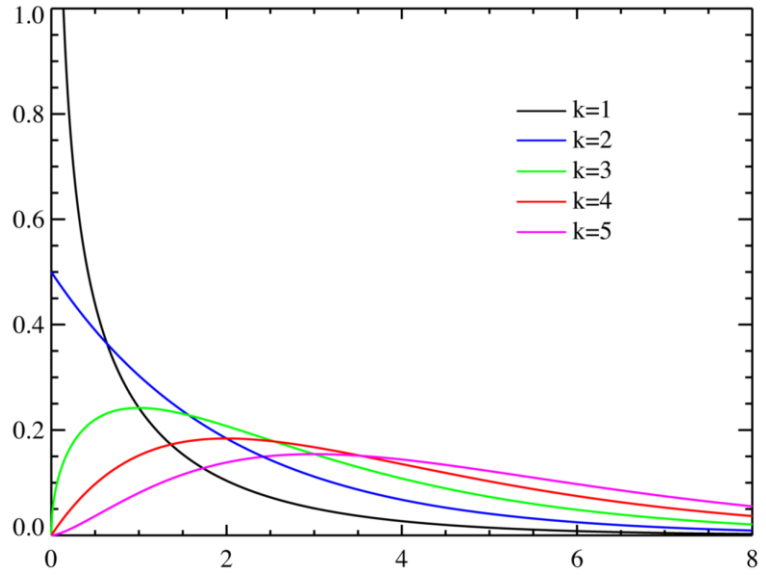
$$C = D_{\mathrm{KL}}(P \parallel Q) = \sum_{x \in \mathcal{X}} P(x) \log \left(\frac{P(x)}{Q(x)} \right)$$

Chi-square

Intuition: The chi-square distribution models the sum of the squares of k independent standard normal variables. It's commonly used in tests of independence and goodness-of-fit.

$$f(x) = \frac{1}{2^{\mathrm{k}/2}\Gamma(\mathrm{k}/2)} x^{\mathrm{k}/2-1} e^{-x/2}, \quad x \geq 0$$

where k is the degrees of freedom.

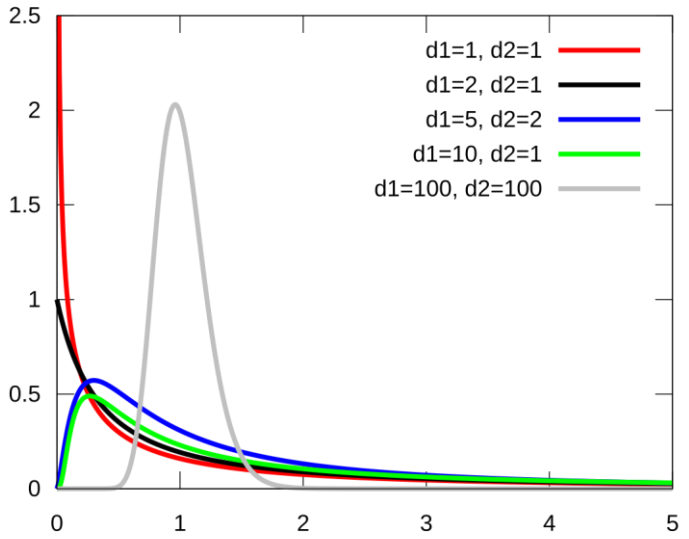


F

Intuition: The F-distribution is used to compare variances between two groups. It arises in ANOVA and regression analysis.

$$f(x) = \frac{\left(\frac{d_1}{d_2}\right)^{d_1/2} x^{d_1/2-1}}{B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)\left(1 + \frac{d_1}{d_2}x\right)^{(d_1+d_2)/2}}, \quad x \geq 0$$

where d_1 and d_2 are degrees of freedom.

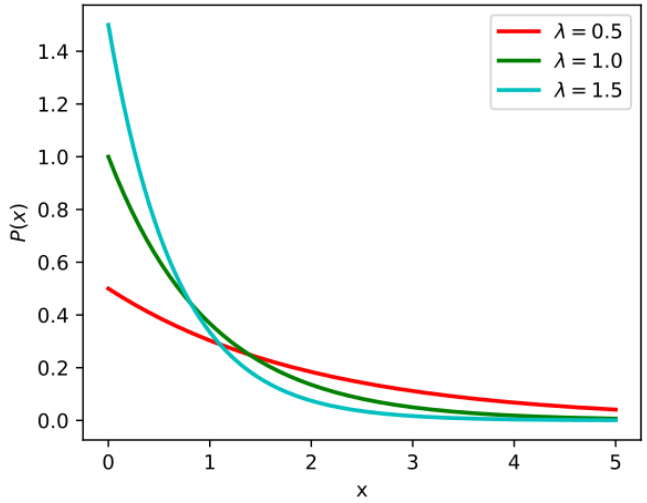


Exponential

Intuition: The exponential distribution models the time between events in a Poisson process.

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

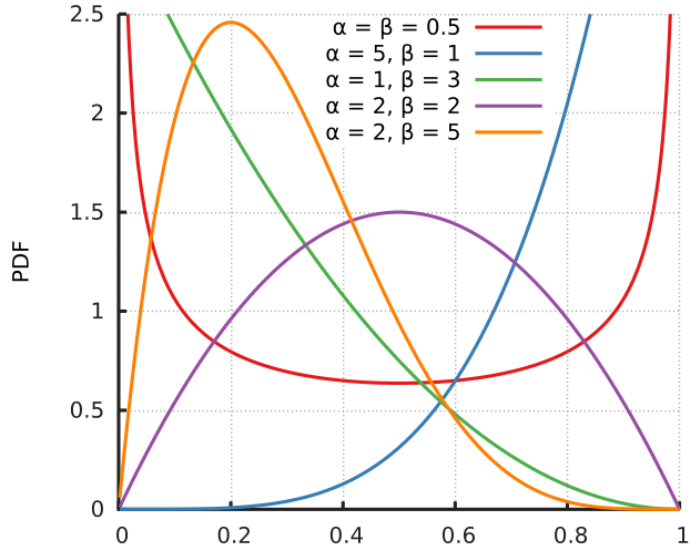
Example: The time between arrivals of customers at a service desk follows an exponential distribution.



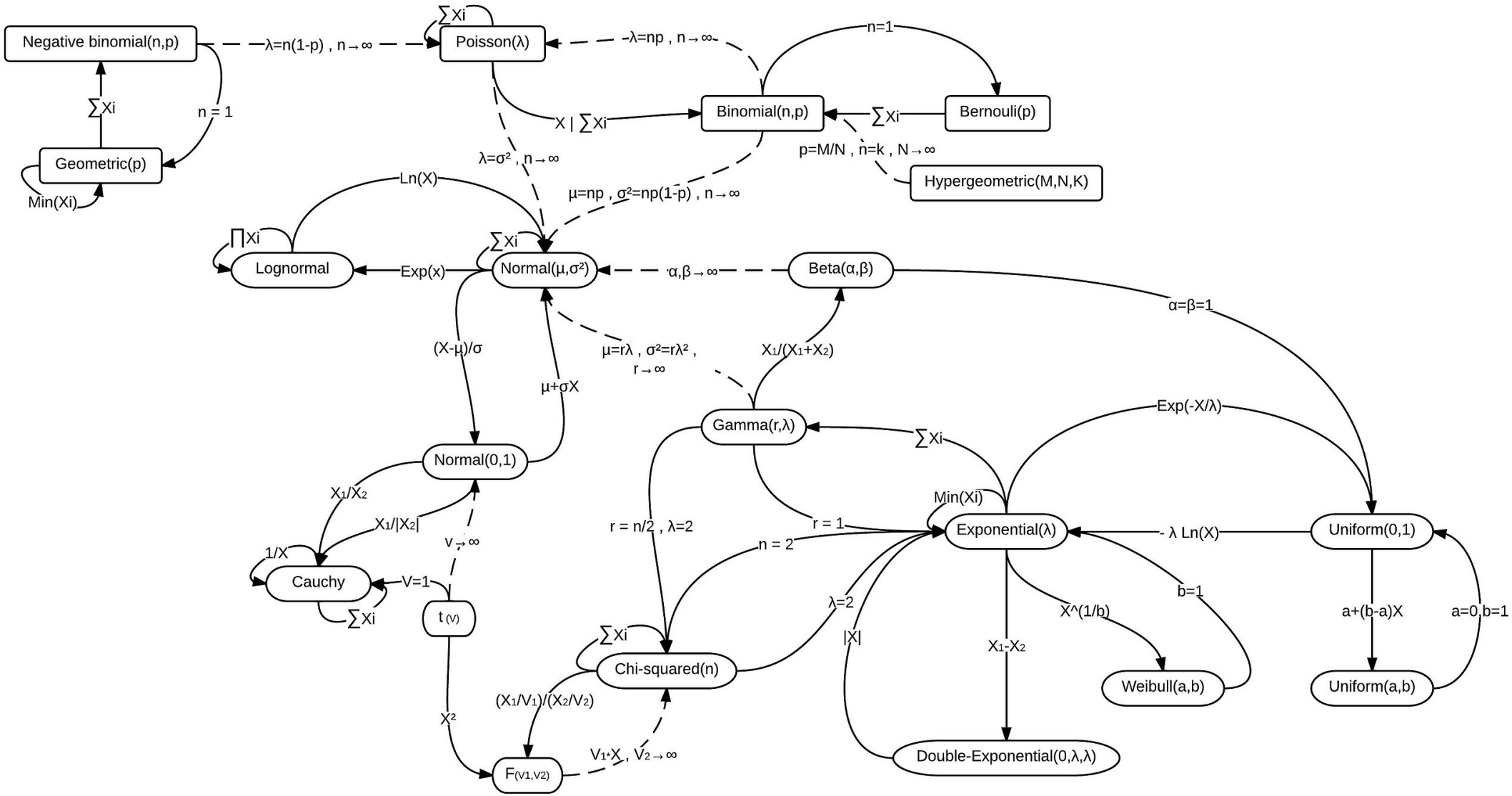
Beta

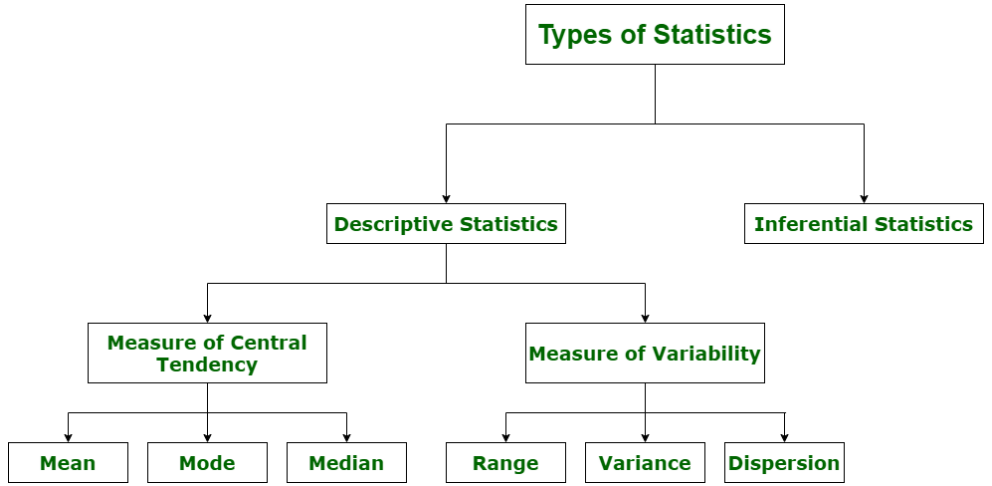
Intuition: The beta distribution is a continuous distribution on [0,1]. It's often used to model proportions and probabilities.

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 \leq x \leq 1$$



Relationship between distributions





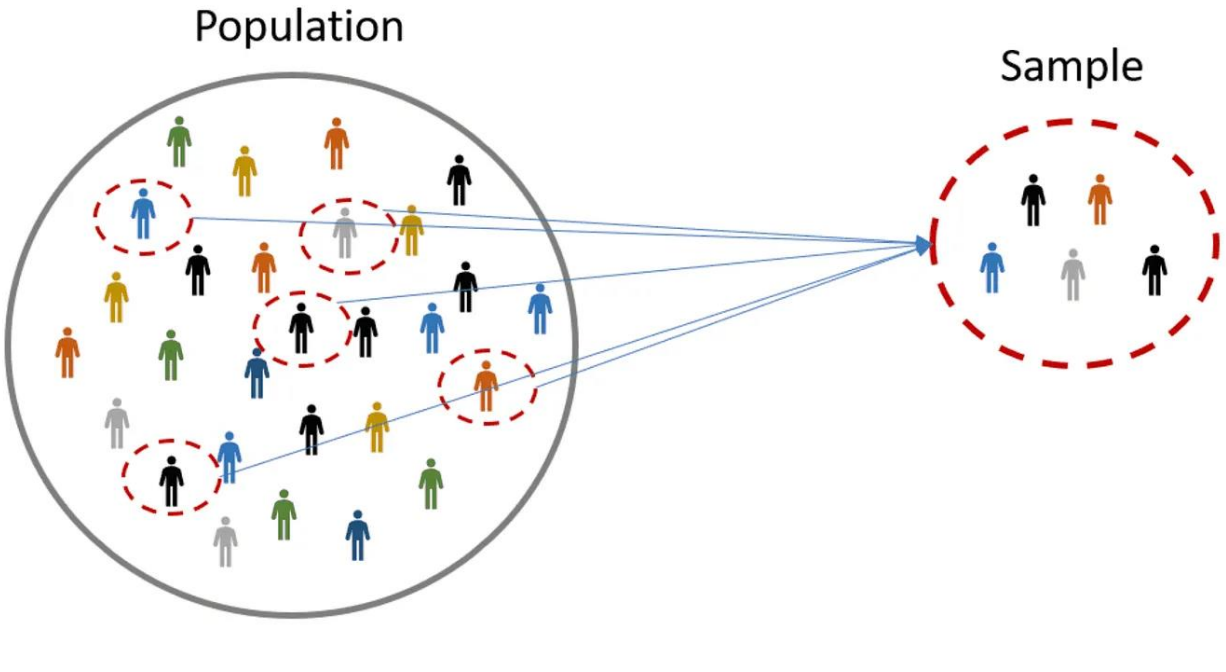
Descriptive statistics

The first step in understanding is to describe.

Population and samples

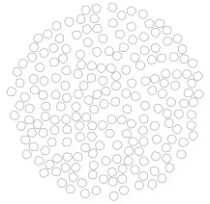
We study a finite set of samples from the population

- It’s too costly to measure the entire population
- It’s impossible to iterate through the entire population
- …

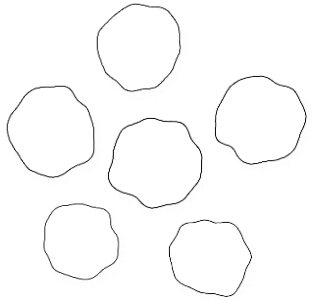


Common Type of Experiment

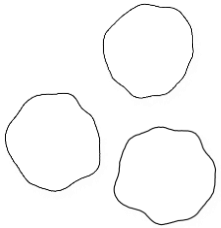
The whole population



Random sample

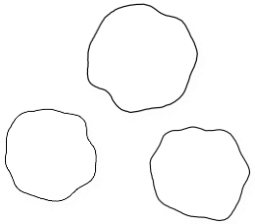


Random assign

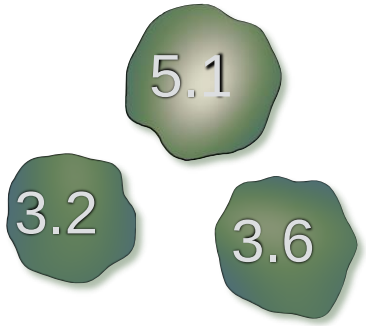


Control

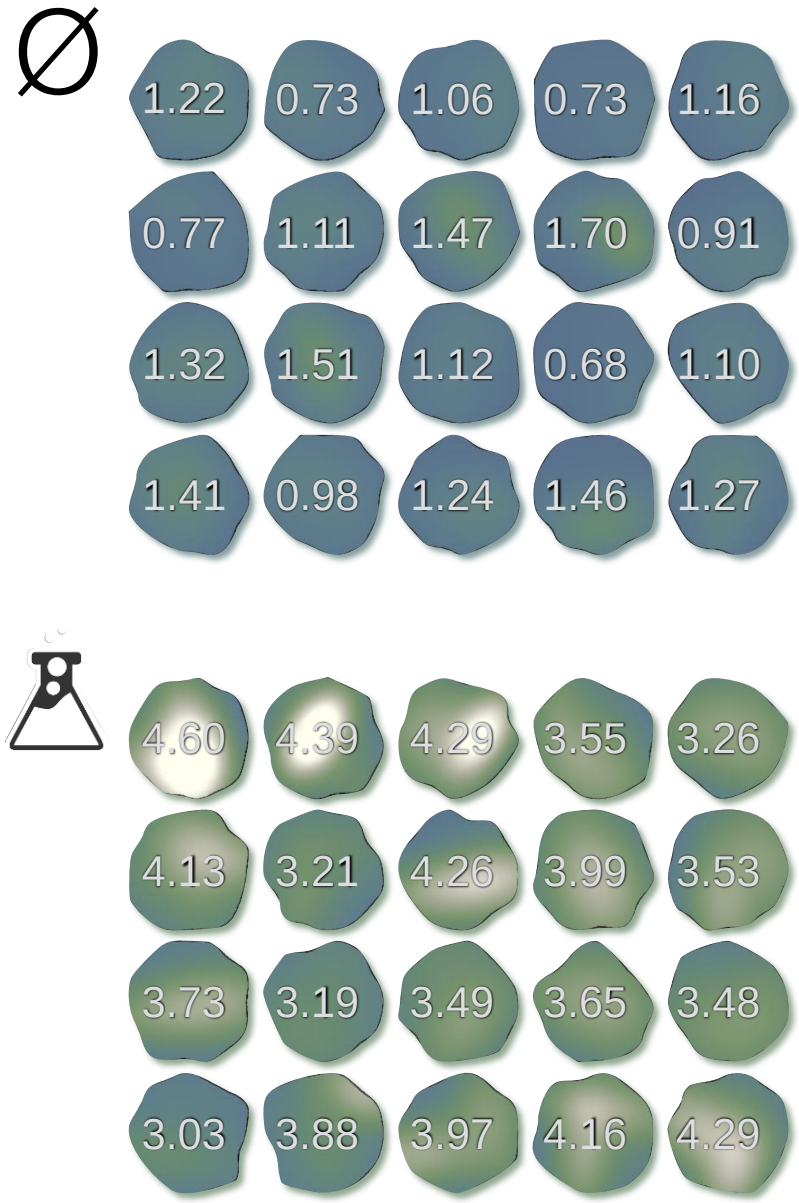
Observed Outcomes

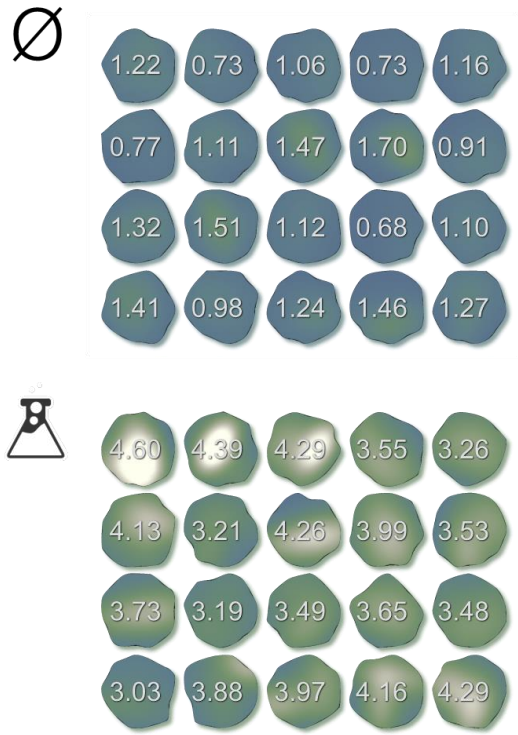


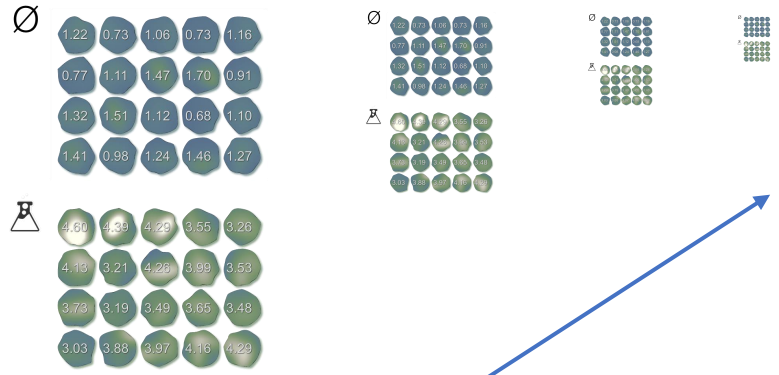
Treated



Summarizing data







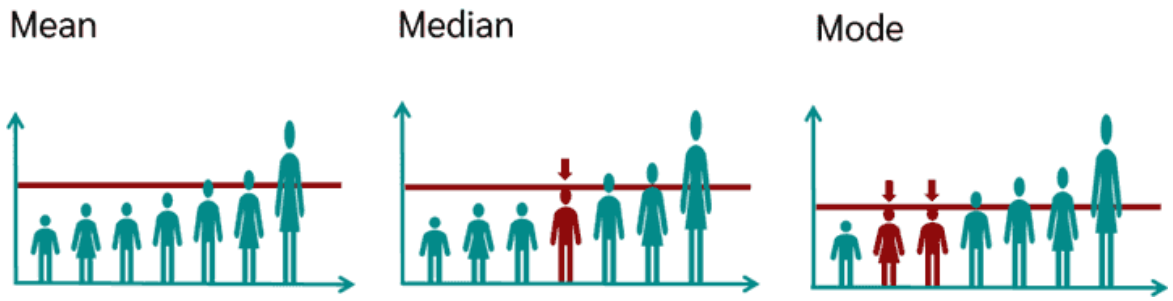
Consider a **data compression** process

Data → Information

Some summary statistics

Tendency: characterize the distribution's central or typical value

- **Mean**: $\frac{\sum_{i=1}^{n} x_i}{n}$
- **Median**:
  - For odd number observations: $x_{(\frac{n+1}{2})}$
  - For even number observations: $\frac{x_{(n/2)}+x_{(n/2+1)}}{2}$
- **Mode**: value or values that appear most frequently in a dataset.



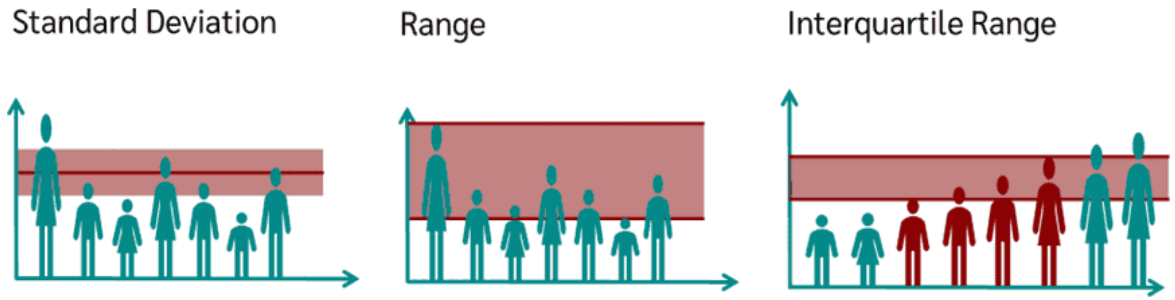
Some summary statistics

Dispersion: measures how far distribution members deviate from the center and each other.

- ❑ Variance/Standard deviation
  - Population variance: $\sigma^2 = \frac{\sum_{i=1}^{N}(x_i-\mu)^2}{N}$
  - Sample variance: $s^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$

- ❑ Quantiles: divide the data into **4 parts**
  - Q1 (25th percentile); Q2 (50th percentile or median), Q3 (75th percentile).

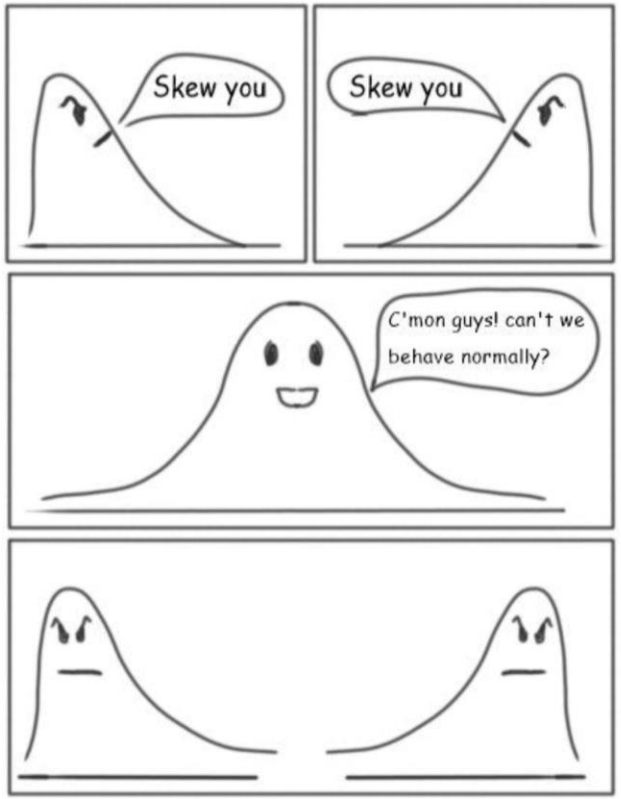
- ❑ Range/Inter Quantile Range (IQR)
  - Range: $x_{(n)} - x_{(1)}$
  - IQR: Q3 – Q1



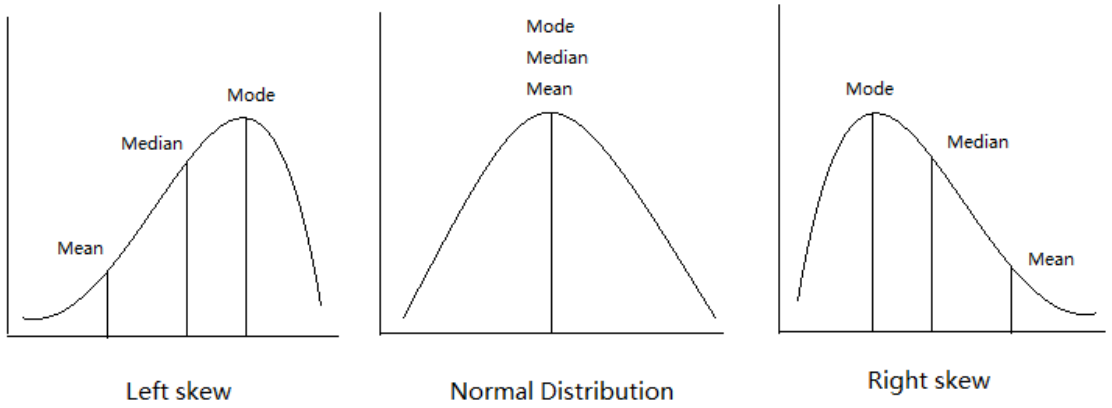
Some summary statistics

Skewness: measures the asymmetry of the distribution of data.

$$\text{Skewness} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)^{\frac{3}{2}}}$$

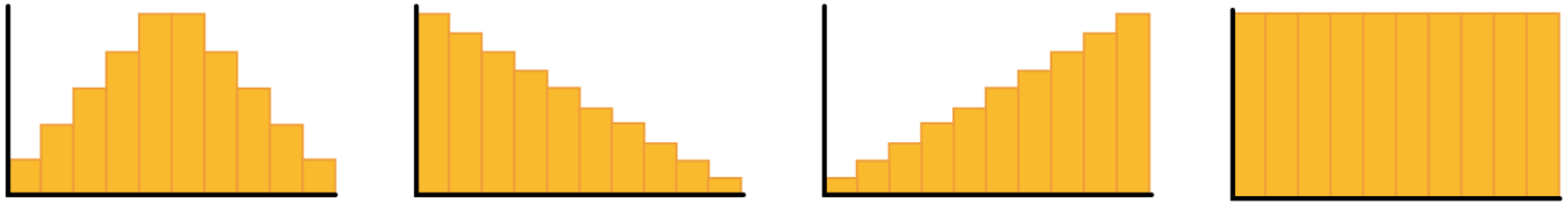


- Skewness = 0: Symmetric Distribution
- Skewness < 0: Negative Skewness (Left-Skewed)
- Skewness > 0: Positive Skewness (Right-Skewed)



Visualize distribution -- histogram

Symmetric (normal) vs skewed and uniform distriutions



Normal distribution
(unimodal, symmetric,
the “bell curve”)

Right-skewed distribution
(Positively-skewed)

Left-skewed distribution
(Negatively-skewed)

Uniform distribution
(equal spread,
no peaks)

Unimodal vs bimodal distributions

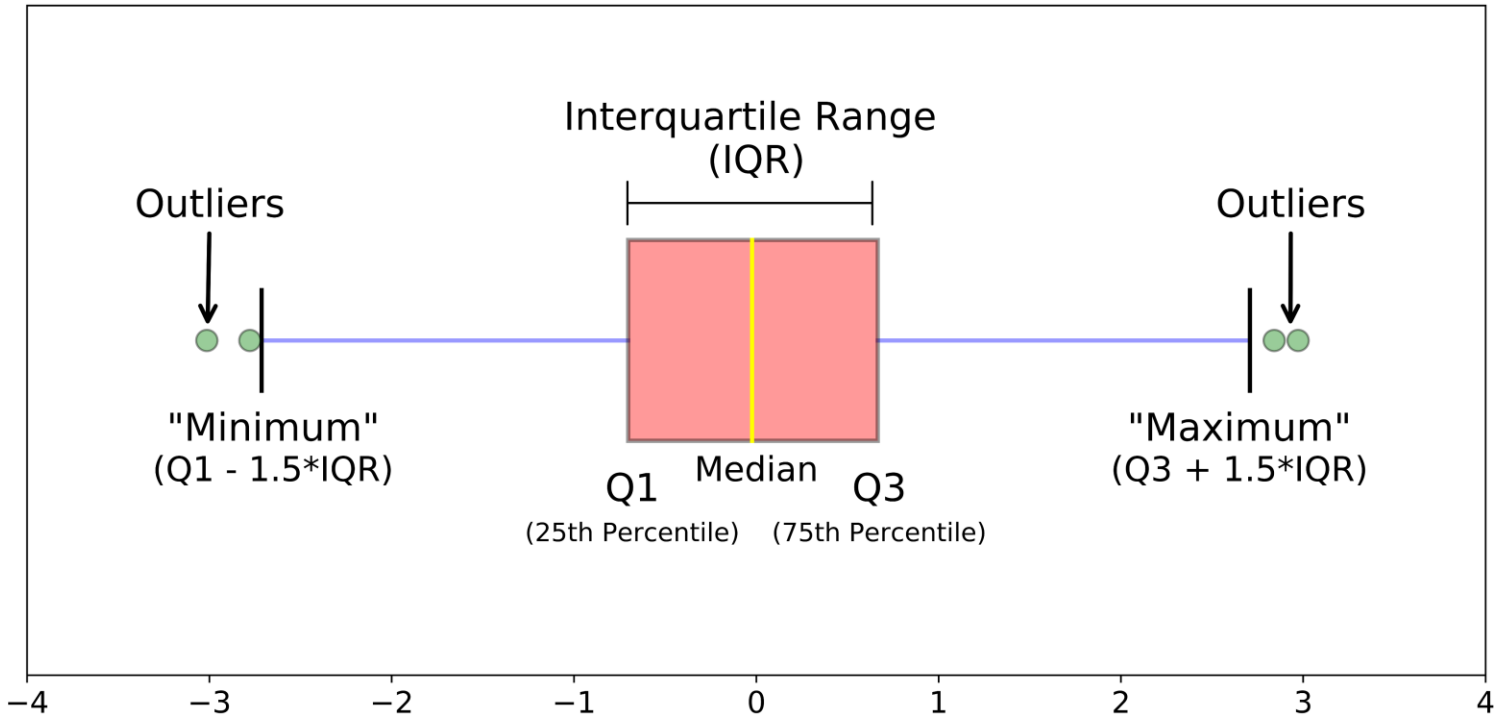
Normal distribution
(unimodal, symmetric,
the “bell curve”)

Symmetric bimodal distribution
(two modes)

Non-symmetric bimodal distribution
(two modes)

Always check the distribution

Visualize distribution -- boxplot



Thanks

Q & A