EE 4302 ADVANCED CONTROL SYSTEM

CA3- State-Feedback Controller Design Including State-Augmentation

Report

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1. Simple Feedback controller System

1.1 shortcomings

Consider the following plant described in the state-space notation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.20 & -3.71 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = x_1 \tag{1.1}$$

Assume that both state-variables x_1 and x_2 are measurable. In CA1 we designed the feedback controller using scaling gain K.

However, if there is a constant disturbance v with unknown magnitude enters at the plant input to system shown in (1.1). Then the equation could be modified as

$$\dot{x} = \mathrm{Fx} + \mathrm{Gu} + \mathrm{G_v v}$$

$$y = Hx \tag{1.2}$$

The simple feedback controller cannot track disturbances and against it.

Also, if there are differences between what the plant actually is and what we assume it to be, then our calculation of the scaling gain might not be accurate enough.

1.2 Disturbance Rejection

Therefore, we need to design a system so that the output y(t) will track asymptotically any step reference input even with the presence of a disturbance v(t) and with plant parameter variations.

Firstly, let's consider the tracking error $e = y - r$. And define it a state variable x_I such that

$$x_I = y - r = Hx - r \tag{1.3}$$

And in order to achieve this design, in addition to introducing state feedback, we introduce an integrator and a feedback from the output as shown in figure 1.

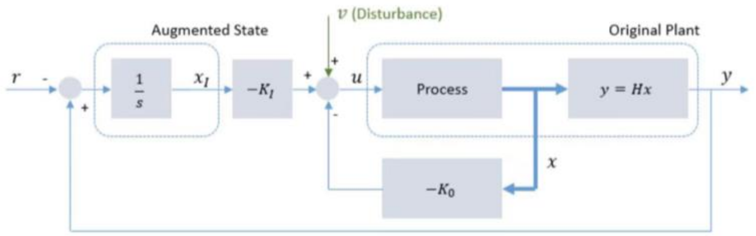

Figure 1 Designed system

Define that the joint state vector $\bar{x} = [x^T \quad x_I]^T$. The augmented state-space model of the system becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1.20 & -3.71 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_I \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0]\bar{x} \tag{1.4}$$

Name the matrix above to $\bar{F}, G_u, G_r$ and $\bar{H}$ respectively.

To stabilize the system, we design $u = -[k_1 \quad k_2 \quad k_I]$. Then at the steady state, $\dot{x} = 0$, i.e. $\dot{x_I} = y - r = 0,\ y = r$.

Substitute the equation $u = -[k_1 \quad k_2 \quad k_I]$ to (1.2), we can get the closed-loop state-space model.

$$\dot{x} = (\bar{F} - G_u\bar{K})x + G_r r$$

$$y = \bar{H}x \tag{1.5}$$

Choose appropriate K so that $\bar{F} - G_u\bar{K}$ has desired eigenvalues, which would be the poles for the transfer function from y to r.

The why the closed-loop system meeting the specs for the two items of steady-state requirement can be proved by a theorem below, which expressed the system in another way.

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{bk} & \mathbf{b}k_a \\ -\mathbf{c} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} w$$

$$y = [\mathbf{c} \ \ 0] \begin{bmatrix} \mathbf{x} \\ x_a \end{bmatrix}$$

Theorem 8.5[1]: If (A,b) is controllable and if $\hat{g}(s) = c(sI - A)^{-1}b$ has no zero at s = 0, then all eigenvalues of the new A matrix above can be assigned arbitrarily by selecting a feedback gain $[K\ K_I]$.)

For further proof, you can reach to the book.

[1] Chi Tsong Chen, Linear System Theory and Design, 3th ed., Oxford University Press, 1998.

2. State-Augmentation Controller Design

The system expression is in the standard state-space form, we can apply all the methods we talked out in CA1.

And as the hint note saying, since the system is linear, it suffices to consider disturbance as zero to meet the remaining frequency response requirements.

2.1 ITEA Criterion Methodology

2.1.1 Controller Design

The specified requirements of the designs are：

Closed-loop bandwidth	not lower than 1.5 rad/s
Resonant peak Mr	not larger than 2dB
Steady-state gain between r and y	0 dB

Table 1 Design requirements

To meet the desired requirements, we choose the closed-loop poles based on ITEA table.

The bandwidth criterion is

$$\text{bandwidth} \geq 1.5\ dB$$

The system is 3rd order, and the bandwidth needs to be not lower than 1.5 rad/s, so we choose $w_0\ as\ 2\ rad/s$. Refer to prototype response poles table for ITAE[1], the poles are as follows,

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 2 * \begin{bmatrix} -0.7081 \\ -0.5210 + 1.0680i \\ -0.5210 - 1.0680i \end{bmatrix} \tag{2.1}$$

Using Ackermann's formula to calculate the controller gain $\bar{K}_{ITAE}$.

$$\bar{K}_{ITAE} = [\ 7.3996 \quad -0.2098 \quad 7.9991] \tag{2.2}$$

2.1.2 System Analysis

The frequent domain response and unit step response of the system are as follow.

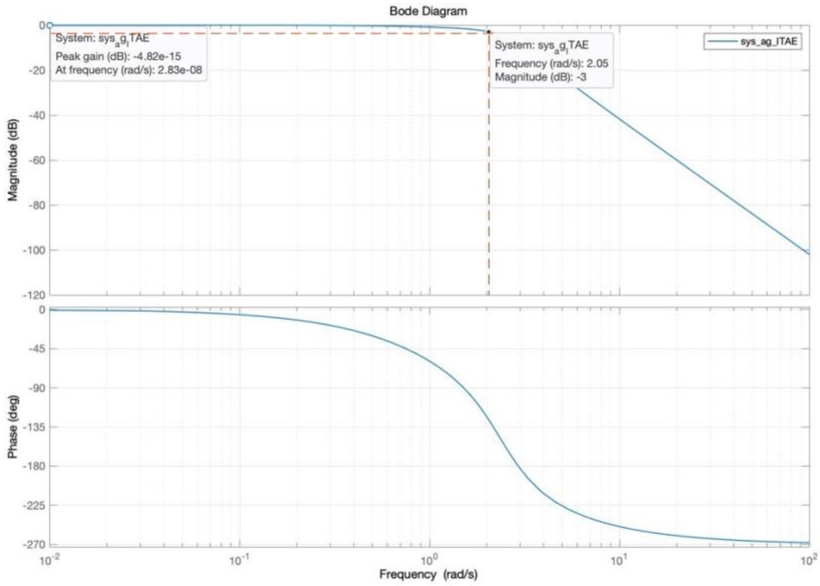

Figure 1 Frequent Response of System_ITEA

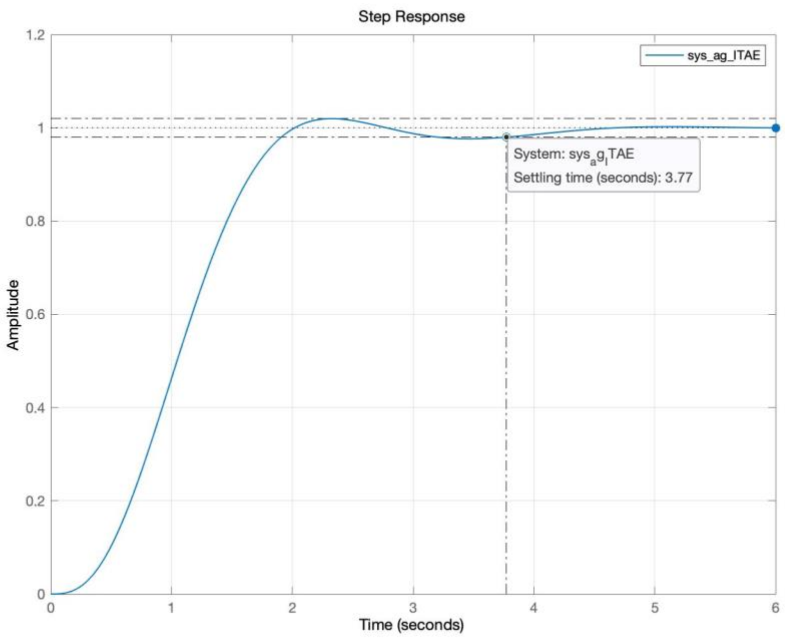

Figure 2 Step Response of System_ITEA

From figure 1 above we can draw the -3dB line and get the bandwidth frequency is about 2.05 rads/s and we can also get the resonant peak Mr is equal to 0.

From figure 2, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets the design requirements.

2.1.3 Disturbance analysis

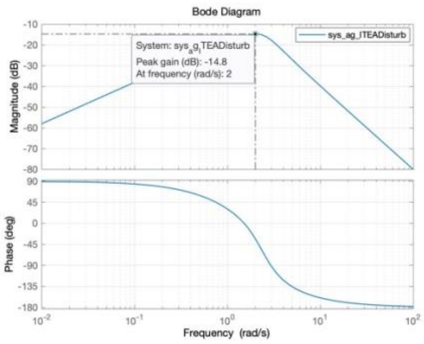

Figure 3 Frequency Response of the Output to Disturbance

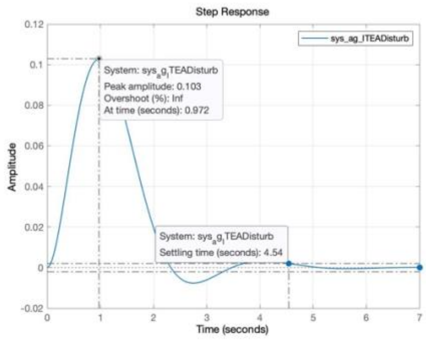

Figure 4 Step Response of the Output to Disturbance

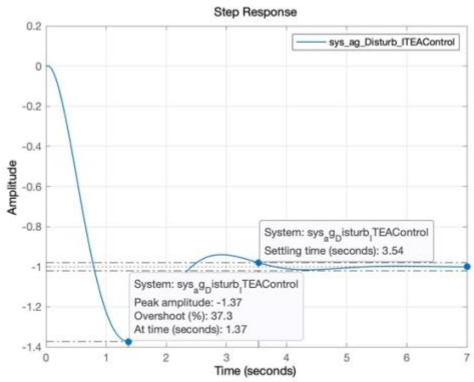

Figure 5 The Control Signal Step Response

Form the figures above, we can get the peak gain Mr of the output y to the disturbance v is -14.8dB. And there is a damped oscillation during the settling process and finally get steady to 0dB, which means the disturbance is vanished. Besides the settling time is 4.54s.

And for the control signal u, there is an effect form disturbance v during the settling process but becomes steady finally.

2.2 LQR Methodology

2.2.1 Controller Design

Similar to CA1, the LQR approach minimizes the following cost function (or performance index) during the regulation.

$$J = \int_0^\infty (x^T Q x + r u^2) \tag{2.3}$$

We fixed the value of r, and optimized the function by changing the value of Q.

Since Q = diag(q_1, q_2, q_3) with q_1, q_2, q_3, r > 0. We adjust these parameters to verify how the penalties setting would influence the system performance.

(1) Firstly, we change the weight value of x_1 and choose i = 100, 200, 300, 400, 500 and try several times.

i.e.

$$Q_{LQR} = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, r = 1 \tag{2.4}$$

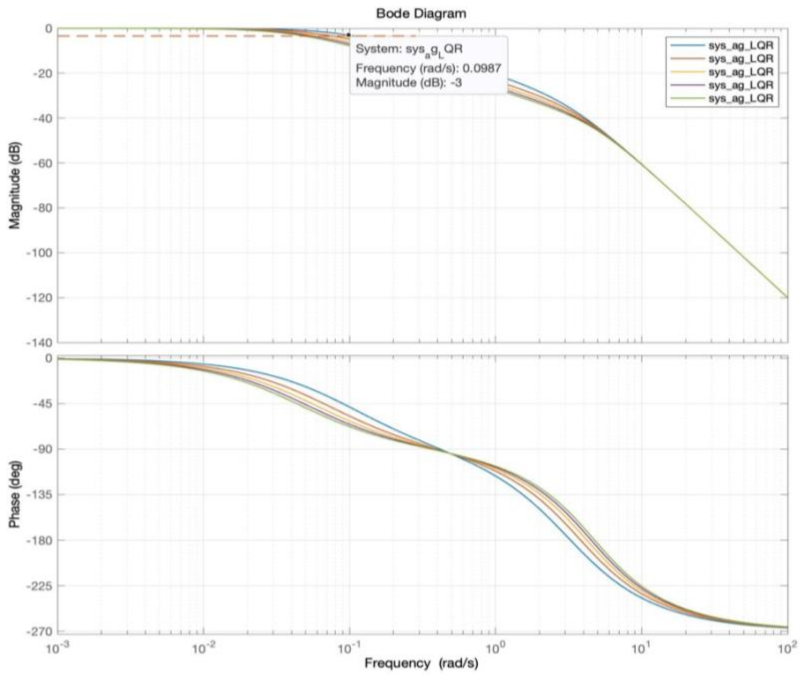

Figure 6 Frequency Response of Systems

The bandwidth is becoming smaller with i increasing but all of them didn’t meet the design requirement.

(2) Then we set that i = 1, 2, 3, 4, 5.

The figure is as follows.

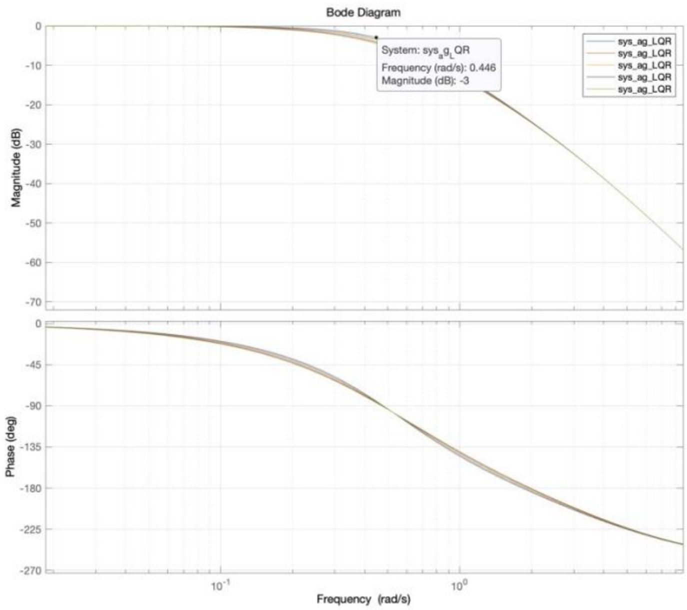

Figure 7 Frequency Response of Systems

It seems that the bandwidth is becoming larger with i increasing but the change is not that obvious. Besides, all of them didn’t meet the design requirement.

(3) Thirdly, we changed the weight value of x_2, i.e.

Let

$$Q_{LQR} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}, r = 1 \tag{2.5}$$

We still set i = 1, 2, 3, 4, 5, but they still don’t meet the requirement of bandwidth.

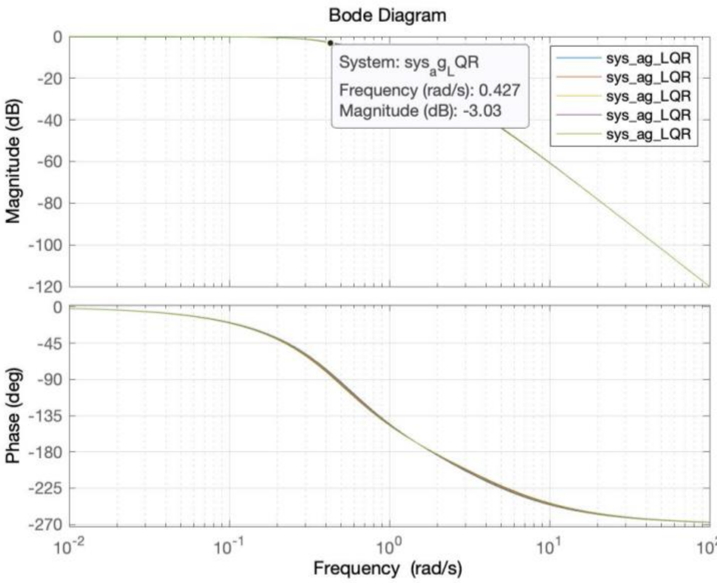

Figure 8 Frequency Response of Systems

(4) Finally, we change the value of x_I.

i.e.

$$Q_{LQR} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}, r = 1 \tag{2.6}$$

Set i = 100, 200, 300, 400, 500, they all meet the bandwidth requirement. The figures are as follows.

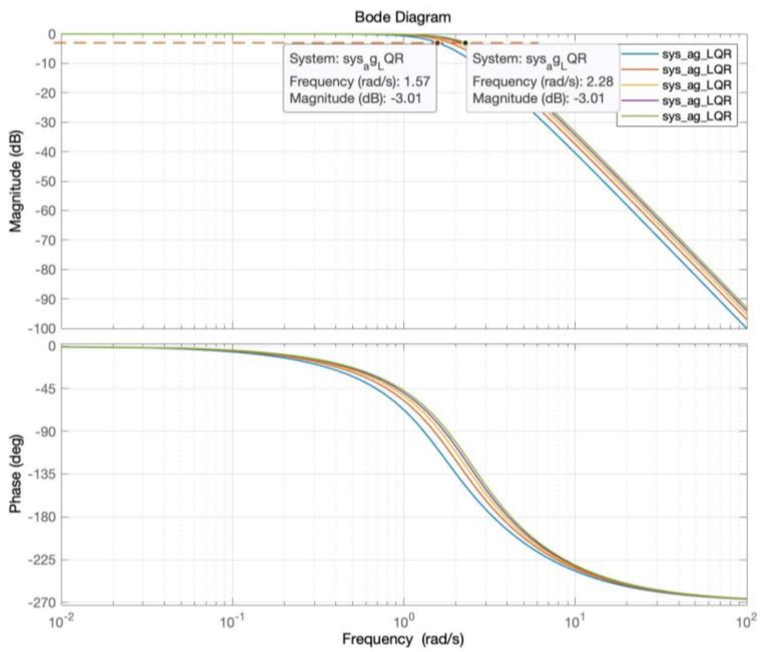

Figure 9 Frequency Response of Systems

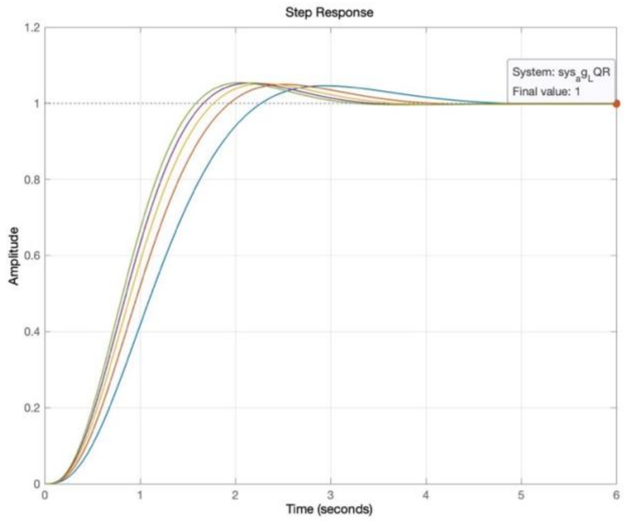

Figure 10 Step Response of Systems

From figure 9 above, we draw a -3dB cut-off line and get the bandwidth of those systems, we can also get the resonant peak Mr is equal to 0. And by increasing the control weight of x_I, the bandwidth is also increasing.

From figure 10, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets the design requirements.

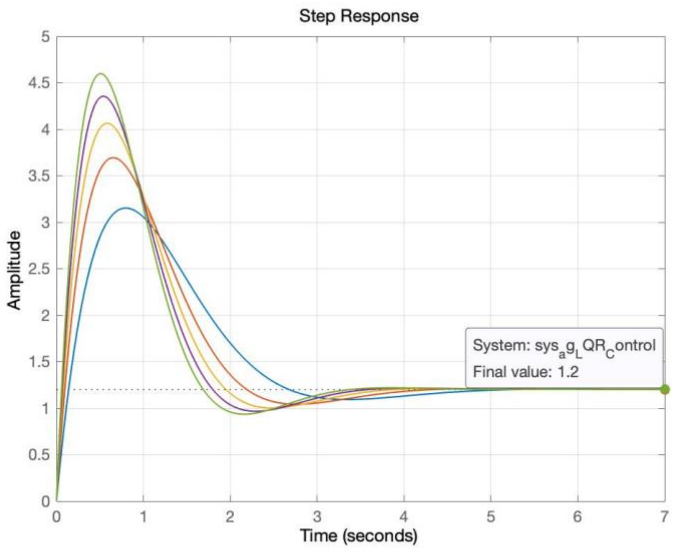

Figure 11 The Control Signal Step Response

And the controller gains K_{LQR} can be also obtained.

i	controller gain K_LQR
100	[2.1420 0.6544 1.0000]
200	[2.6903 0.7783 1.4142]
300	[3.0758 0.8634 1.7321]
400	[3.3826 0.9300 2.0000]
500	[3.6413 0.9854 2.2361]

Table 2

2.2.2 Disturbance Analysis

Set the Q matrix as

$$Q_{LQR} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 200 \end{bmatrix}, r = 1 \qquad (2.7)$$

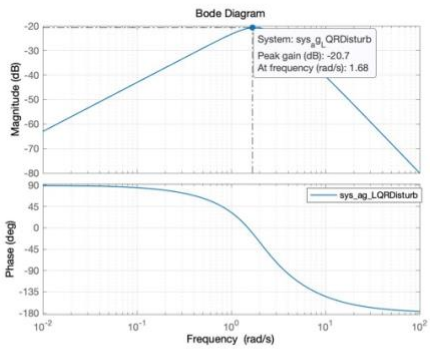

Figure 12 Frequency Response of Disturbance to the output

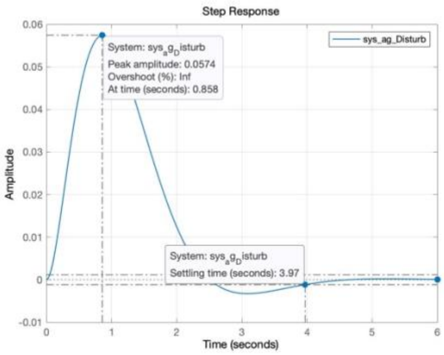

Figure 13 Step Response of Disturbance to the output

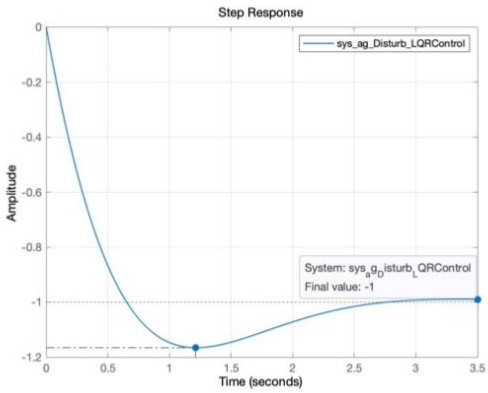

Figure 14 The Control Signal Step Response

Form the figures above, we can get the peak gain Mr of the output y to the disturbance v is -20.7dB. There is a fluctuation during the disturbance settling process and finally get steady to 0dB, which means the disturbance is vanished. The settling time is 3.97s.

And for the control signal u, there is an effect form disturbance v during the settling process but becomes steady finally.

2.3 Bessel Criterion Methodology

2.3.1 Controller Design

Similar to ITEA methodology, we choose the poles based on Bessel table.

The bandwidth criterion is

$$\text{bandwidth} \geq 1.5\ dB$$

We choose $w_0\ as\ 2\ rad/s$ at first, and the frequent domain response can be obtained as follows.

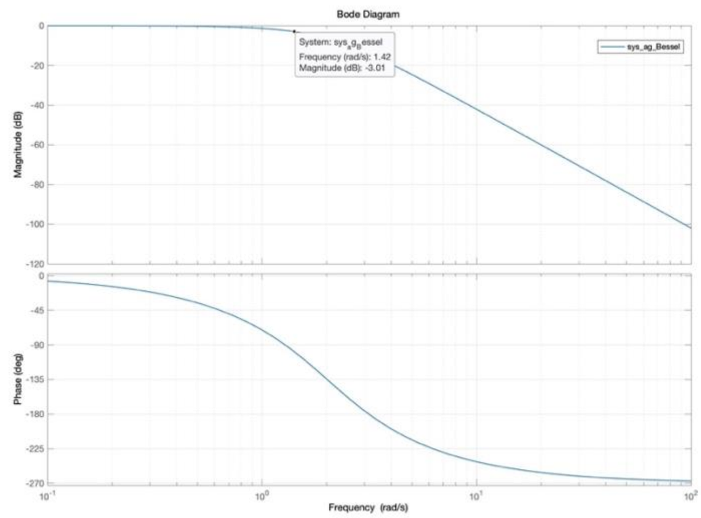

Figure 15 Frequent Response of System_BESSEL

From figure 14, we draw the -3dB line and read the corresponding bandwidth is 1.42 rad/s, which didn't meet the design requirement.

Therefore, we change to choose $w_0\ as\ 3\ rad/s$, the poles can be calculated referring to prototype response poles table for Bessel[2], which are

$$p_1 = -2.8260\ , p_{2,3} = -2.2365 \pm 2.1336i \tag{2.8}$$

Then, using Ackermann's formula to calculate the controller gain.

$$\overline{K}_{Bessel} = [20.9949 \quad 3.589 \quad 27.0001] \tag{2.9}$$

2.3.2 System Analysis

The frequent domain response and unit step response ($w_0 = 3\ rad/s$) of the system are as follow.

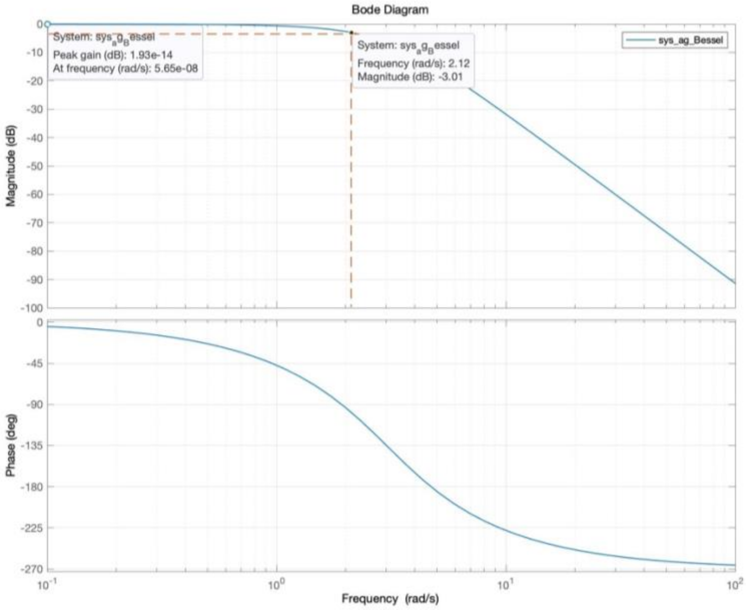

Figure 16 Frequent Response of System_BESSEL

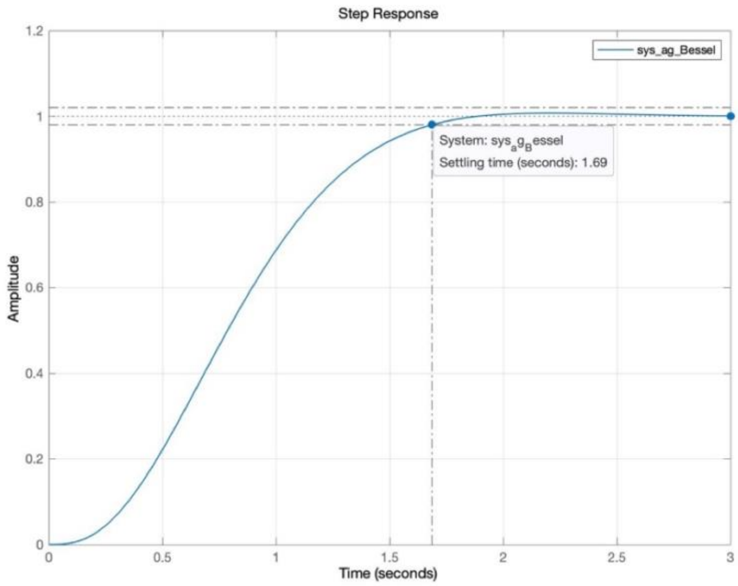

Figure 3 Step Response of System_BESSEL

From figure 15 above we can draw the -3dB line and get the bandwidth frequency is 2.12 rads/s, we can also get the resonant peak Mr is equal to 0.

From figure 16, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets the design requirements.

2.3.3 Disturbance Analysis

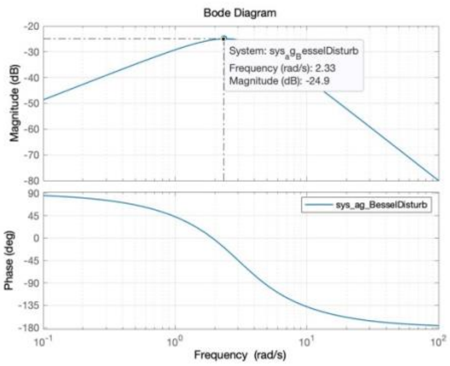

Figure 4 Frequency Response of Disturbance to the output

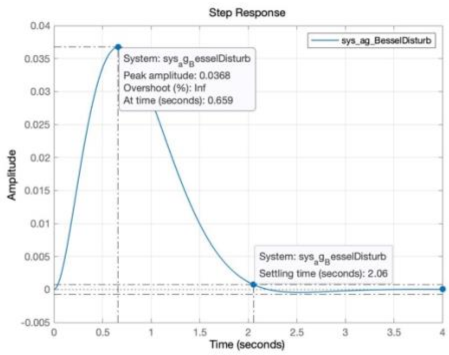

Figure 19 Step Response of Disturbance to the output

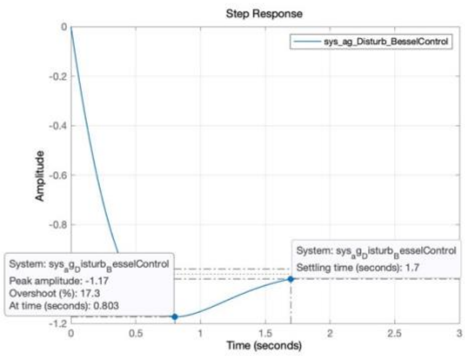

Figure 20 The Control Signal Step Response

Form the figures above, we can get the peak gain Mr of the output y to the disturbance v is -24.9dB. There is a fluctuation during the disturbance settling process and finally get steady to 0dB, which means the disturbance is vanished. The settling time is 2.06s.

And for the control signal u, there is an effect form disturbance v during the settling process but becomes steady finally.

2.4 Second-order Dominant Poles Methodology

2.4.1 Controller Design

First of all, we design a second-order reference model.

$$\mathrm{H(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2.10}$$

Assume $\zeta = 0.707$(which is often used when system designing) and $\omega_n = 2.0$, substitute them to property expressions.

$$\omega_b = \omega_n\left((1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2 + 1}\right)^{\frac{1}{2}} \tag{2.11}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}, \qquad M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2.12}$$

Then we can calculate out that $\omega_b = 2.003, M_r = 1.0000,$ which meets the design requirements.

We can also calculate the poles of system use command ‘roots’.

$$P_1 = -1.4140 + 1.4144i, \qquad p_2 = -1.4140 - 1.4144i \tag{2.13}$$

Since the system is third order, we need to choose another pole P_3 that meet the condition that the pair of roots P_1 and P_2 are the dominant poles in order to use the Second-order Dominant Poles Methodology.

P_3 shall be much far away from the zero, that is much larger in absolute value than the dominant poles, 3 ~ 5 times be appropriate in practice. So we choose $P_3 = -7.0$.

Still use Ackermann’s Formula in MATLAB we can easily get the controller gain K_{SOD}.

$$K_{SOD} = [22.5959 \quad 6.1180 \quad 27.9995] \tag{2.14}$$

2.4.2 System Analysis

The frequent domain response and unit step response of the system are as follow.

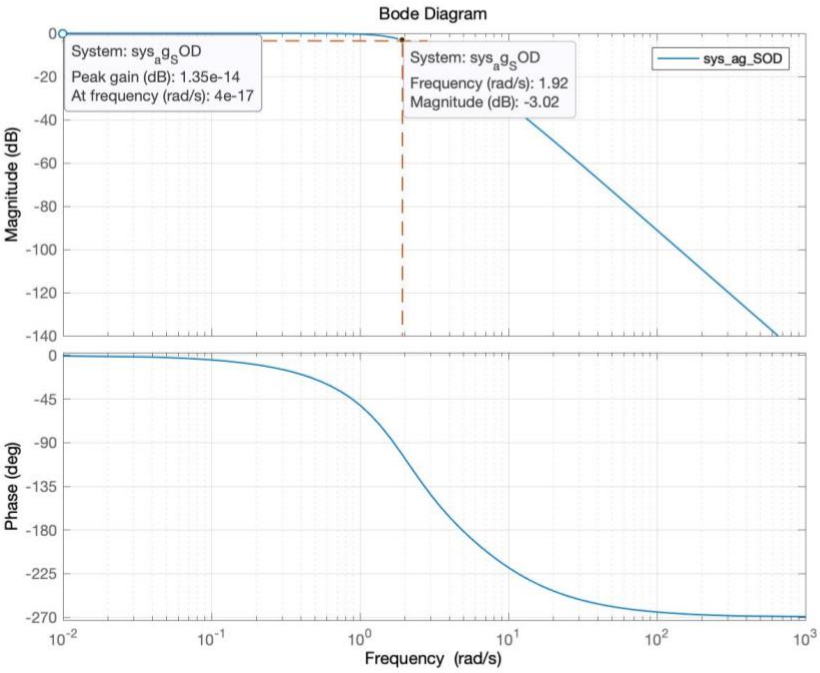

Figure 25 Frequency Response of the system

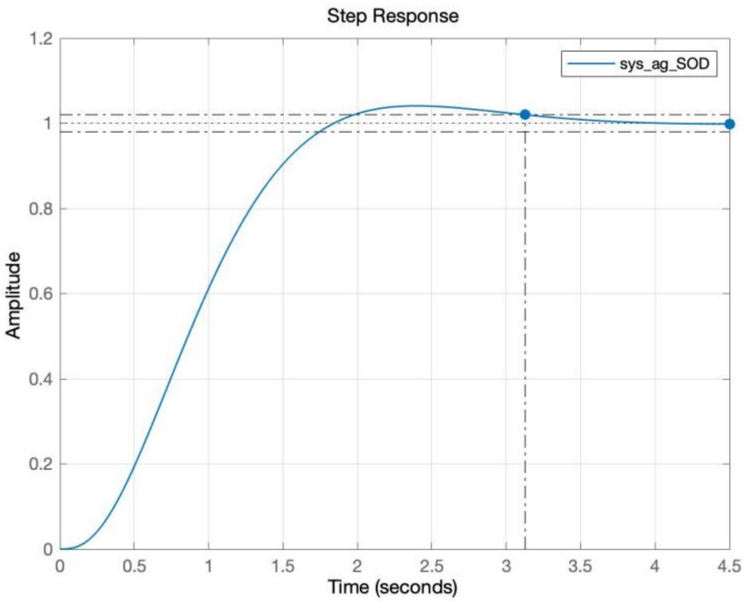

Figure 22 Step Response of the system

From figure 21 we can draw the -3dB line and get the bandwidth frequency is 1.92 rads/s, we can also get the resonant peak Mr is equal to 0.

From figure 22, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets the design requirements.

2.4.3 Disturbance Analysis

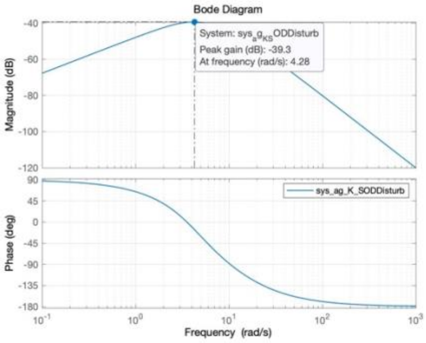

Figure 6 Frequency Response of Disturbance to the output

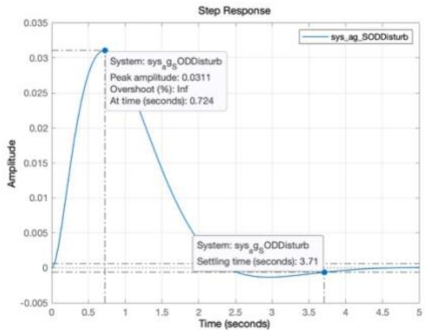

Figure 7 Step Response of Disturbance to the output

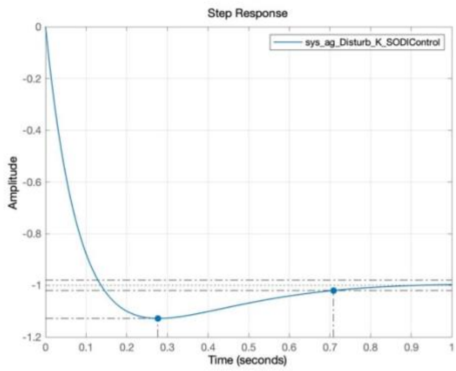

Figure 8 The Control Signal Step Response

Form the figures above, we can get the peak gain Mr of the output y to the disturbance v is -39.3dB. There is a fluctuation during the disturbance settling process and finally get steady to 0dB, which means the disturbance is vanished. The settling time is 3.71s.

And for the control signal u, there is an effect form disturbance v during the settling process but becomes steady finally.

2.5 Conclusion

Applying State-Augmentation Feedback controller to the system, y will track asymptotically and robustly any step reference input r(t) = unit and reject any step disturbance with unknown magnitude.

3. Additional Thoughts

3.1 Closed-Loop Transfer Functions

Use function 'tf' to calculate the transfer function of each suitable control system.

(1) In ITEA Criterion Methodology, we used w_0 = 2 rad/s for design, and the transfer function can be obtained.

$$\mathrm{sys}_{\mathrm{ITEA}} = \frac{7.999}{\mathrm{s}^3 + 3.5\,\mathrm{s}^2 + 8.6\,\mathrm{s} + 7.999} \tag{3.1}$$

(2) In Bessel Criterion Methodology, we used w_0 = 3 rad/s for design, and the transfer function can be obtained.

$$\mathrm{sys}_{\mathrm{Bessel}} = \frac{27}{\mathrm{s}^3 + 7.299\,\mathrm{s}^2 + 22.19\,\mathrm{s} + 27} \tag{3.2}$$

(3) In Second Order Criterion Dominant Poles methodology, we used $\zeta = 0.707$ and $\omega_n = 2.0$ for design, and the transfer function can be obtained.

$$\mathrm{sys}_{\mathrm{SOD}} = \frac{28}{s^3 + 9.828\,s^2 + 23.8\,s + 28} \tag{3.3}$$

(4) In LQR methodology, we used diagonal Q matrix $Q_{LQR} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 200 \end{bmatrix}, r = 1$ for design, and the transfer function can be obtained.

$$\mathrm{sys}_{\mathrm{LQR}} = \frac{14.14}{\mathrm{s}^3 + 6.257\,\mathrm{s}^2 + 13.39\,\mathrm{s} + 14.14} \tag{3.4}$$

3.2 Unknown change in system.

In our real life, the state equation and transfer function developed to describe a plant may change due to change of load, environment, or aging. Thus, plant parameter variations often occur in practice.

For example,

The system changes from

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.20 & -3.71 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.5}$$

to

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3.51 & -0.77 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.6}$$

But we don't know the change state variables

So, design a full-state estimator and realize it in simulink.

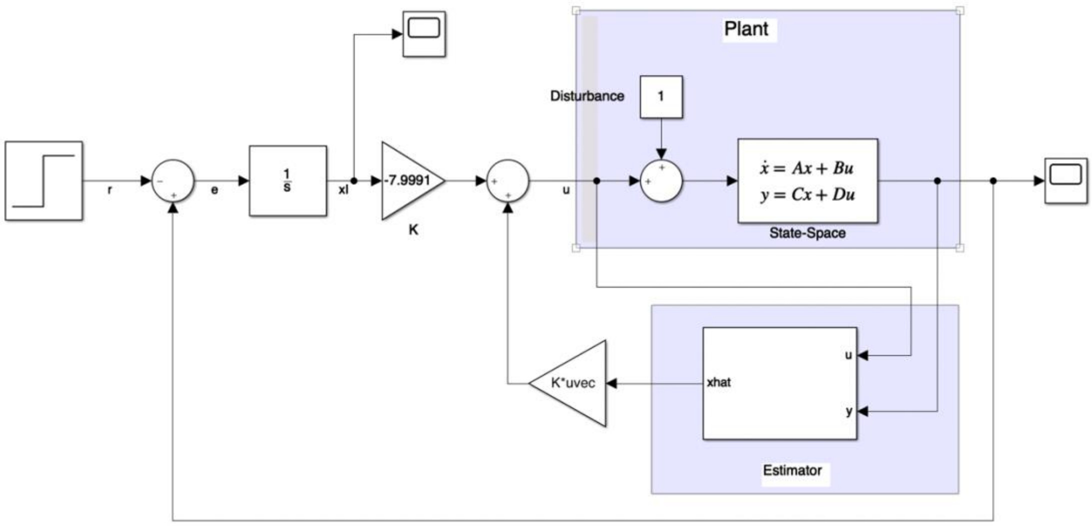

Figure 24 Simulink of System

The system used ITEA criterion State-Augmentation feedback controller and the estimator $L = [0 \;\; 1]\zeta^{-1}(F^T, H^T)\alpha_0(F^T)$

1. Using a “Scaling Gain” to detect the changes in plant parameter.

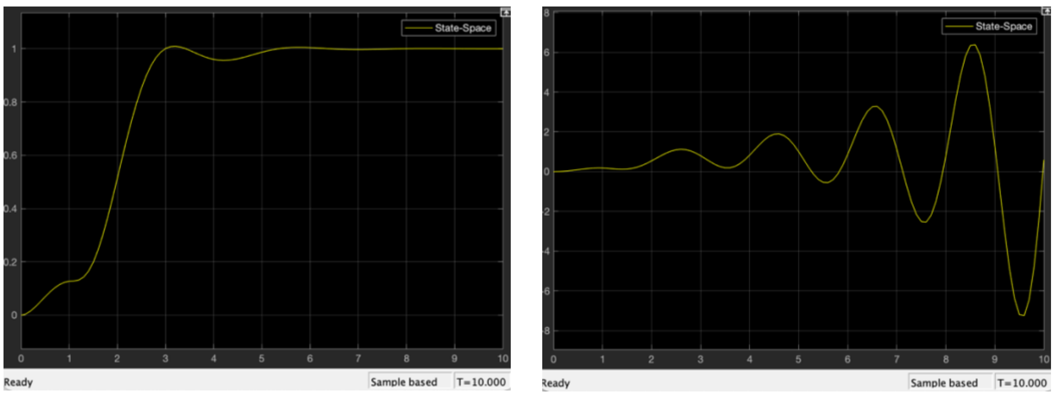

Figure 35 Output Change of the System

Just change the plant parameter of state-space, the change of scaling gain is very obvious.

And we can see in the first output, the controller can track and reject the disturbance.

2. Using the augmented state variable $x_I = y - r$ to detect the changes in plant parameter.

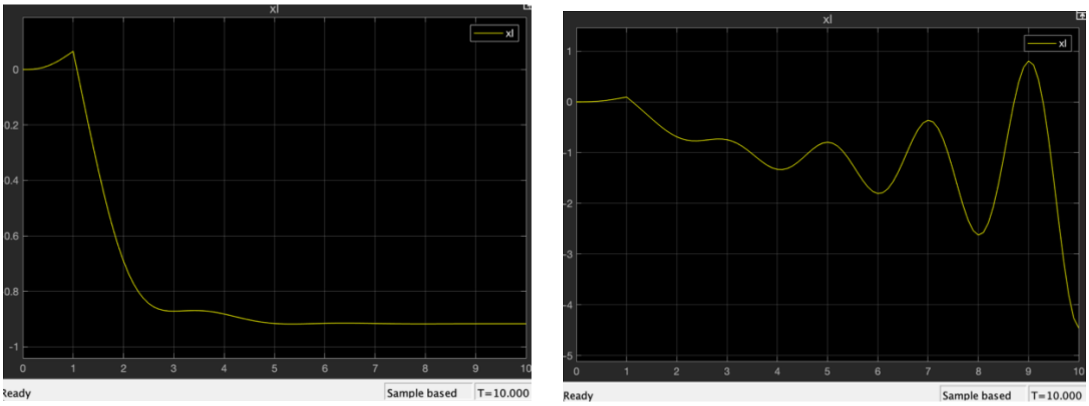

Figure 26 Output Change of the System

The change of augmented state variable is also obvious.

So we will know when the plant parameter changes through scaling gain detecting and augmented state variable detecting.

Appendix 1

Part Code

1. LQR and loop

```
clc
close all
% Formulate the Augmented State-Space Model
Fbar = [0,1,0;-1.20,-3.71,0;1,0,0]; % The process matrix for the augmented system
Gu = [0;1;0]; % The input matrix for u
Gr = [0;0;-1]; % The input matrix for r
Gv = [0;1;0]; % Used for Disturbance Analysis; The input matrix for disturbance v
Hbar = [1,0,0]; % The output matrix for y
% Design Using LQR
for i = 1:5
  Q_LQR = [1,0,0;0,1,0;0,0,i*100];
  R_LQR = 1;
  [K_LQR,~,~] = lqr(Fbar,Gu,Q_LQR,R_LQR)
  sys_ag_LQR = ss(Fbar-Gu*K_LQR,Gr,Hbar,0);
  figure(1)
  bode(sys_ag_LQR)
  grid on
  hold on
  figure(2)
  step(sys_ag_LQR)
  grid on
  hold on
  sys_ag_LQR_Control = ss(Fbar-Gu*K_LQR,Gr,-K_LQR,0);
% In sys_ag_LQR_Control, Hbar is changed to -K_LQR
% In this case, MATLAB would treat -K_LQR*x as the output of the system,
% which is nothing else but the control signal
% Therefore, by doing a unit step response analysis, the control signal
% response can be displayed
  figure(3)
  step(sys_ag_LQR_Control)
  grid on
  hold on
end
```

2. Other ways design and corresponding disturbance figures

```
ITAEPoles = 2*[-0.7081; -0.5210+1.0680*1i;-0.5210-1.0680*1i];
K_ITAE = acker(Fbar,Gu,ITAEPoles)
sys_ag_ITAE = ss(Fbar-Gu*K_ITAE,Gr,Hbar,0);

BesselPoles = [-2.8260; -2.2365+2.1336*1i; -2.2365-2.1336*1i];
K_Bessel = acker(Fbar,Gu,BesselPoles)
sys_ag_Bessel = ss(Fbar-Gu*K_Bessel,Gr,Hbar,0);

SODPoles = [-7.0;-1.4140+1.4144*1i;-1.4140-1.4144*1i];
K_SOD = acker(Fbar,Gu,SODPoles)
sys_ag_SOD = ss(Fbar-Gu*K_SOD,Gr,Hbar,0)

% Disturbance Study
sys_ag_SODDisturb = ss(Fbar-Gu*K_SOD,Gv,Hbar,0);
% Gr is changed to Gv
% For sys_ag_Disturb, the input matrix is set Gv in order to study the
% system response when disturbance v comes in
% In this case study, the reference signal r is zero.
figure(21)
bode(sys_ag_SODDisturb)
grid on
figure(22)
step(sys_ag_SODDisturb)
grid on
sys_ag_SODDisturbControl = ss(Fbar-Gu*K_SOD,Gv,-K_SOD,0);
figure(23)
step(sys_ag_SODDisturbControl)
grid on
```

3. Get the transfer functions

```
sys1 = tf(sys_ag_ITAE)
K1 = dcgain(sys1)
sys2 = tf(sys_ag_Bessel)
K2 = dcgain(sys2)
sys3 = tf(sys_ag_SOD)
K3 = dcgain(sys3)
sys4 = tf(sys_ag_LQR)
k4 = dcgain(sys4)
```