EE 4302 ADVANCED CONTROL SYSTEM

State-Feedback Controller Design

CA1 Report

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1. Open-Loop System Analysis

Consider the following plant described in the state-space notation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.20 & -3.71 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = x_1 \tag{1.1}$$

and where both state-variables x_1 and x_2 are measurable.

Use MATLAB to obtain plots of frequency response from u to x_1 and x_2 separately. We can obtain the open-loop system performance.

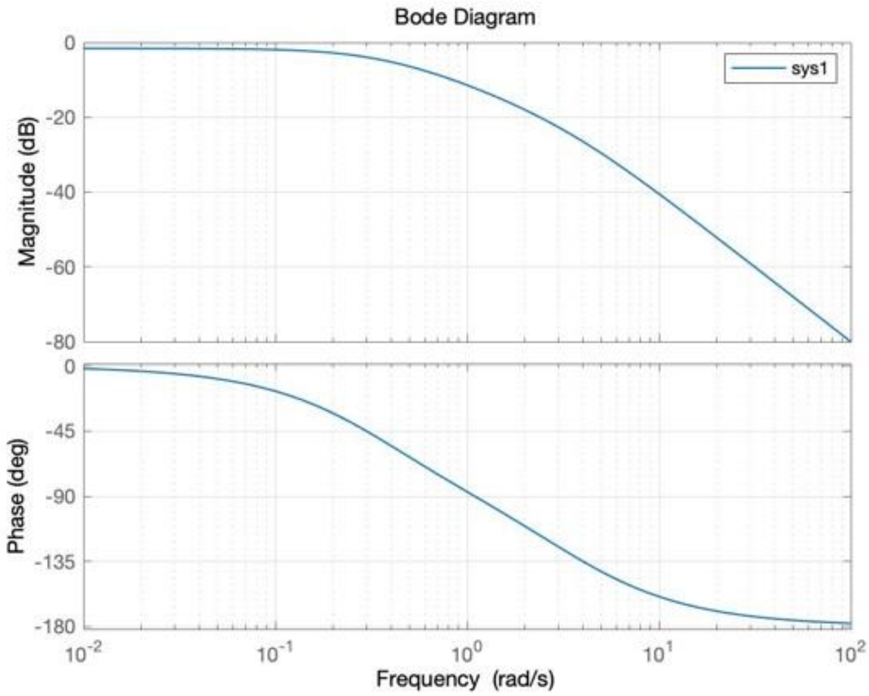

Figure 1 Frequency response from u to x1 of open-loop system

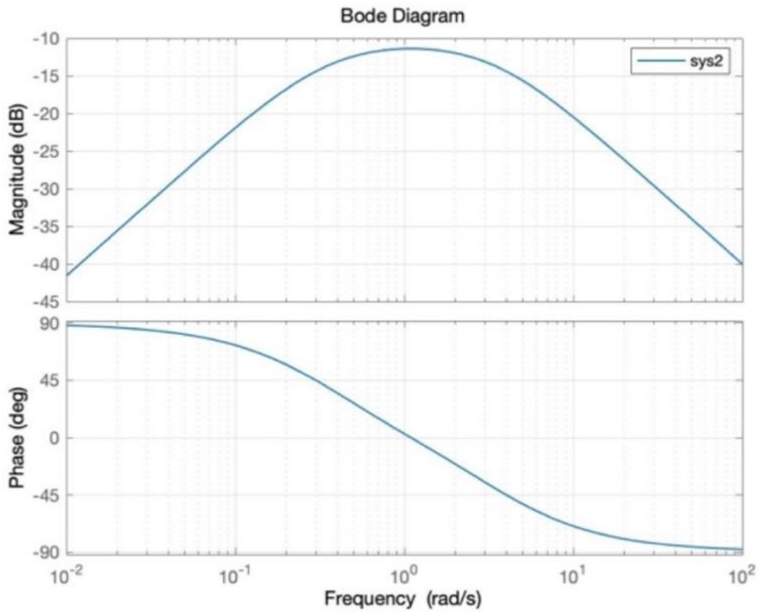

Figure 2 Frequency response from u to x2 of open-loop system

And calculate by MATLAB, the open-loop transfer function poles are located at:

$$p_1 = -0.3580 \quad \mathrm{p}_2 = -3.3520$$

2. Design Using Ackermann's Formula

To apply Ackermann's Formula, we need to get the controllability matrix of the plant firstly.

The controllability matrix of the plant is:

$$\zeta = [G \quad FG] \tag{2.1}$$

i.e.

$$\zeta = \begin{bmatrix} 0 & 1 \\ 1 & -3.71 \end{bmatrix} \tag{2.2}$$

Calculate $\det(\zeta)$ and check that the controllability matrix is nonsingular.

So, we can apply Ackermann's Formula to design the feedback controller.

Assume that the following control signal is applied to the plant

$$u = -Kx + K_S r \tag{2.3}$$

Where K is the controller gain; r is the reference signal; K_S is the scaling gain of r.

In order to apply the algorithm, we choose $K = [0.1 \quad 0.1]$ and $K_S = 1$ in the controller and apply it to the system. Then, use function FEEDBACK to obtain the closed-loop system.

It can be calculated that the closed-loop transfer function poles from r to y becomes:

$$p_1 = -0.3789, \qquad p_2 = -3.4311$$

Plot the frequency response of the closed-loop system.

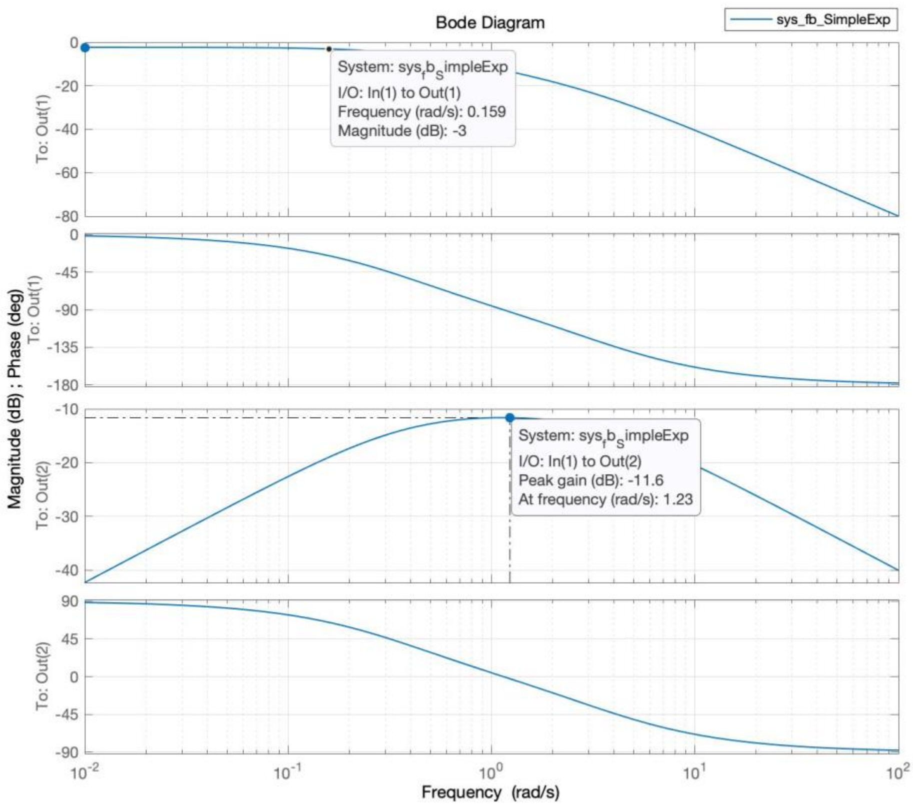

Figure 3 The frequency response from r to x1 and x2

The first two lines are frequency response from r to x_1. Since $y = x_1$, we can get the bandwidth of the closed-loop system is 0.159.

So, by changing the value of K and Ks, we can change the properties of the open-loop system and design the system as we desired.

(In the Bode diagram of closed-loop system, the intersection of the gain curve with the -3dB line is called the closed-loop cut-off frequency, also known as the closed-loop bandwidth frequency, denoted by ω_b.)

2.1 Apply ITEA criterion

2.1.1 System Design

The specified requirements of the designs are：

Closed-loop bandwidth	not lower than 3.5 rad/s
Resonant peak Mr	not larger than 2dB
Steady-state gain between r and y	0

Table 1 Design requirements

To meet the desired requirements, we choose the closed-loop poles based on ITEA table.

The bandwidth criterion is

$$\text{bandwidth} \geq 3.5\ dB$$

Refer to prototype response poles table for ITAE[1], the system is second order and ω_0 is 4 rad/s, so the closed-loop poles are chosen as follows.

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 4 \begin{bmatrix} -0.7071 + 0.7071i \\ -0.7071 - 0.7071i \end{bmatrix} \quad (2.4)$$

With determined pole position, the desired closed-loop pole polynomial can be calculated as

$$\alpha_c(s) = s^2 + \alpha_1 s + \alpha_2 \quad (2.5)$$

and the controller gain K can be obtained using

$$K = [0 \quad 1]\alpha_c \zeta^{-1}(F) \quad (2.6)$$

However, using MATLAB function ‘acker’ and ‘dcgain’, we can easily get the controller gain K_{ITAE} and the scaling gain Ks_{ITEA}.

$$K_{ITAE} = 14.7997 \quad 1.9468 \qquad Ks_{ITEA} = 15.9997 \quad (2.7)$$

i.e. the feedback controller is:

$$u = -[14.7997 \quad 1.9468]x + 15.9997r \tag{2.8}$$

MATLAB function acker and dcgain:

1. ACKER :'Pole placement gain selection using Ackermann's formula'.
   $K = acker(A, B, P)$ calculates the feedback gain matrix K such that the single input system $\dot{x} = Ax + Bu$ with a feedback law of $u = -Kx$ has closed loop poles at the values specified in vector P, i.e. $P = eig(A - B * K)$.
2. DCGAIN： $k = dcgain(sys)$ computes the DC gain k of the LTI model sys. The continuous-time DC gain is the transfer function value at the frequency s=0. For state-space models with matrices (A,B,C,D), the value is $K = D - CA^{-1}B$.

2.1.2 Closed-loop System Analysis

Then we can plot the frequency response and step response for closed loop system.

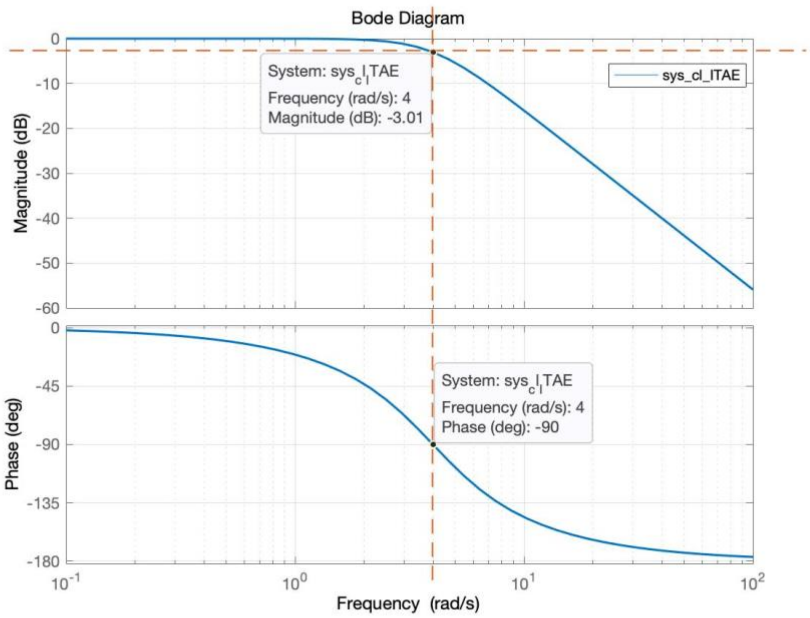

Figure 4 Frequency response for closed loop system using ITEA and Ackermann's formula

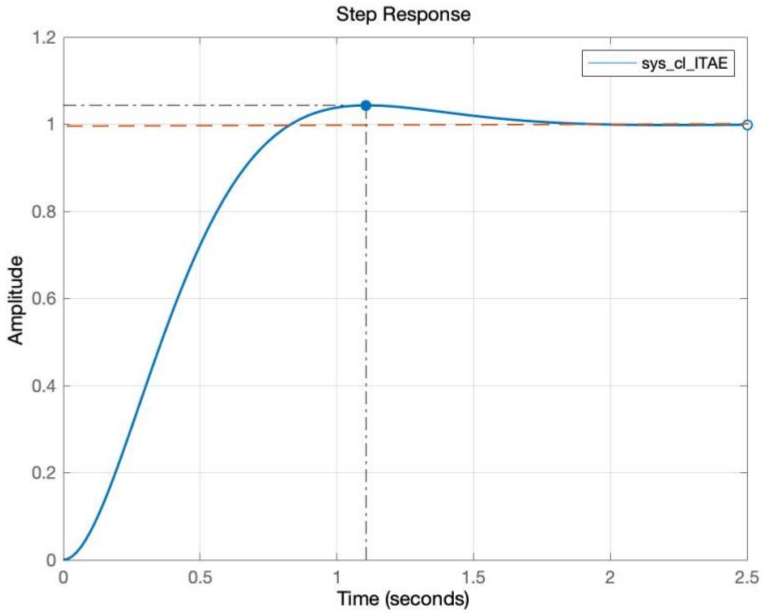

Figure 5 Step response for closed loop system using ITEA and Ackermann's formula

From figure 4 above we can draw the -3dB line and get the bandwidth frequency is 4 rads/s and also find the resonant peak Mr is equal to 0.

From figure 5, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets all the design requirements.

2.2 Apply Bessel criterion

2.2.1 System Design

Similar to 2.1, we choose the closed-loop poles based on Bessel table.

The bandwidth criterion is

$$\text{bandwidth} \geq 3.5\ dB$$

Refer to prototype response poles table for Bessel[2], the system is second order and ω_0 is 4 rad/s, so the closed-loop poles are chosen as follows.

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 4 \begin{bmatrix} -0.8660 + 0.5000i \\ -0.8660 - 0.5000i \end{bmatrix} \tag{2.9}$$

Use the same method as ITEA criterion, we can get the

$$K_{Bessel} = [14.7993 \quad 3.2180] \qquad \mathrm{Ks}_{\mathrm{Bessel}} = 15.9993 \tag{2.10}$$

Then the feedback controller can be obtained

$$u = -[14.7993 \quad 3.2180]x + 15.9993r \tag{2.11}$$

2.2.2 Closed-loop System Analysis

Plot the frequency response and step response for closed loop system.

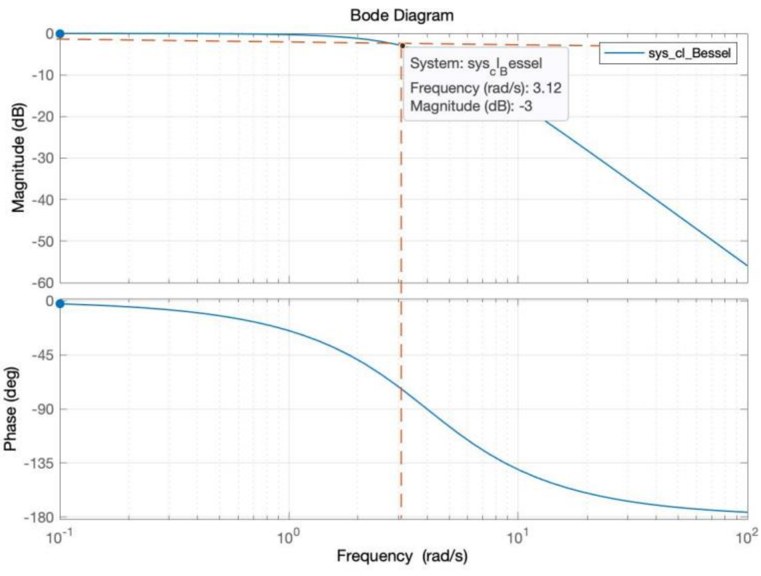

Figure 6 Frequency response for closed loop system using Bessel

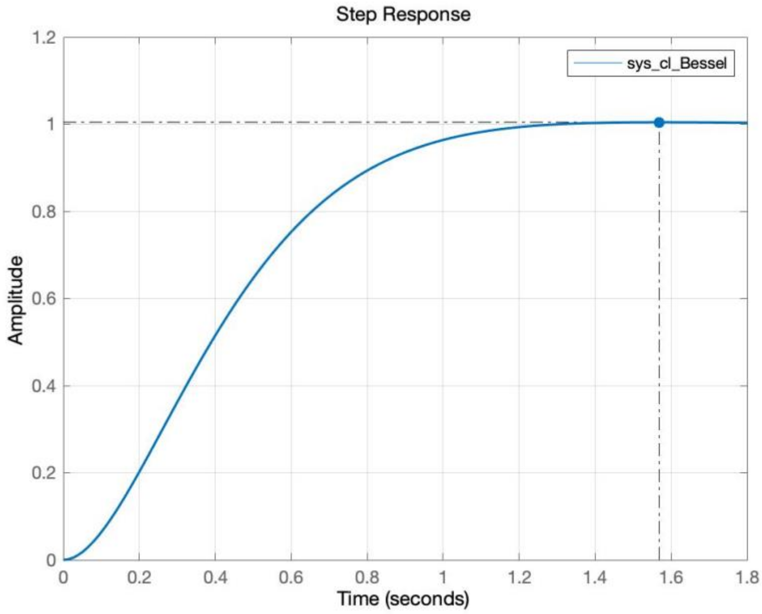

Figure 7 Step response for closed loop system using Bessel

From figure 6 above we can draw the -3dB line and get the bandwidth frequency is 3.12 rads/s, which is less than the requirement that the $\text{bandwidth} \geq 3.5\ dB$

And we can also get the resonant peak Mr is equal to 0.

From figure 7, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system seems do not meet the design requirements.

2.3 Apply second-order dominant poles methodology

2.3.1 System Design

First of all, we need to design a second-order reference model for the closed-loop system.

$$\mathrm{H(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2.12}$$

As it often assumed that $\zeta = 0.707$.
Then choose wn = 3.6 and substitute them to property expressions.

$$\omega_b = \omega_n \left((1 - 2\zeta^2) + \sqrt{(1 - 2\zeta^2)^2 + 1} \right)^{\frac{1}{2}} \tag{2.13}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}\,, \qquad M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2.14}$$

Then we can calculate $\omega_b = 3.6005, M_r = 1.0000$, which meets the design requirements.

We can also calculate the poles of system

$$P_1 = -2.5452 + 2.5460i, \qquad p_2 = -2.5452 - 2.5460i \tag{2.15}$$

Still use Ackermann's Formula and MATLAB we can easily get the controller gain K_{SOD} and the scaling gain Ks_{SOD}.

$$K_{SOD} = [11.7600 \quad 1.3804], \qquad \mathrm{Ks}_{\mathrm{SOD}} = 12.9600 \tag{2.16}$$

Then the feedback controller can be obtained

$$u = -[11.7600 \quad 1.3804]x + 12.9600r \tag{2.11}$$

2.3.2 Closed-loop System Analysis

Plot the frequency response and step response for closed loop system.

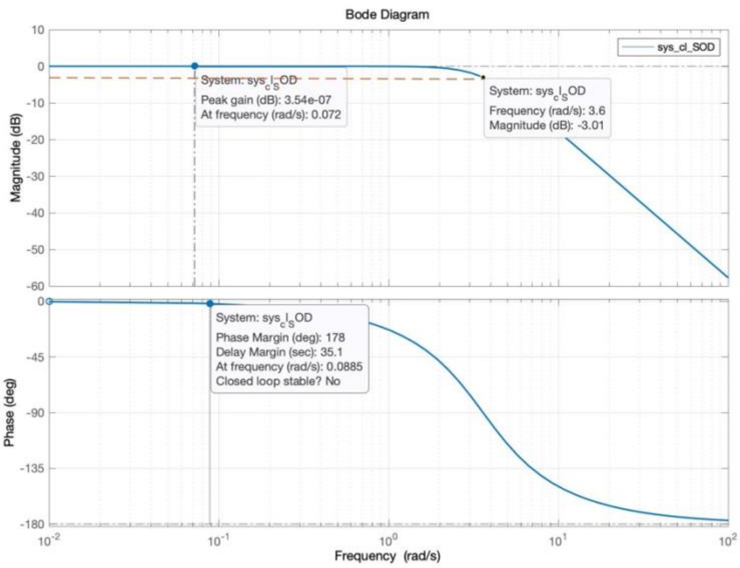

Figure 8 Frequency response for closed loop system

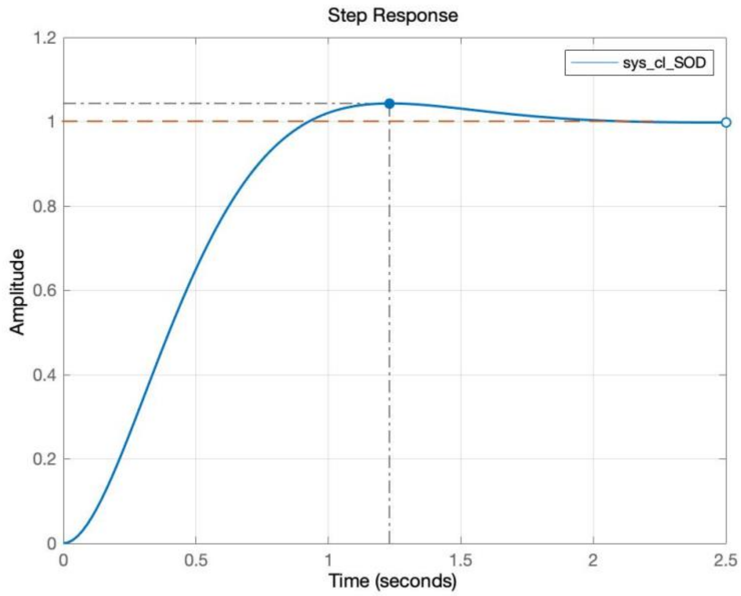

Figure 9 Step response for closed loop system

From figure 8 above we can draw the -3dB line and get the bandwidth frequency is about 3.6 rads/s and we can also get the resonant peak Mr is equal to 0.

From figure 9, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets the design requirements.

2.4 Part Conclusion

In a short, I think the second-order dominant poles methodology and ITEA criterion are much better methods among those three. And I prefer the second-order dominant poles methodology more, because we can design a system by designing its natral ratio and damping ratio and also meet the requorements precisely.

Besides,Bessel prototype methodology will have a lag in thebandwidth w_0 of the real system to the designed one. For example, we assume w_0 as 4 rad/s atthe beginning but the real bandwidth is less, that 3.6 rad/s.

3. Design Using Linear Quadratic Regulator (LQR)

3.1 System Design

The LQR approach minimizes the following cost function (or performance index) during the regulation.

$$J = \int_0^\infty (x^T Q x + u^T r u) \tag{3.1}$$

We fixed the value of r, and optimized the function by changing the value of Q. When there is only one variable, r becomes a scalar.
So, in the experiment Eq (3.1) can be stated as

$$J = \int_0^\infty (x^T Q x + r u^2) \tag{3.2}$$

To meet the design requirements, finally we choose

$$Q_1 = \begin{bmatrix} 500 & 0 \\ 0 & 0.01 \end{bmatrix}, r = 1 \tag{3.3}$$

Then the feedback controller gain K_{LQR} and the scaling gain Ks_{LQR} can be calculated.

$$K_{LQR} = [\ 21.1929 \quad 3.7840], \quad Ks_{LQR} = 22.3929 \tag{3.4}$$

Thus, the feedback controller can be obtained as

$$u = -[21.1929 \quad 3.7840]x + 22.3929r \tag{3.5}$$

3.2 Closed-loop System Analysis

Plot the frequency response and step response for closed loop system.

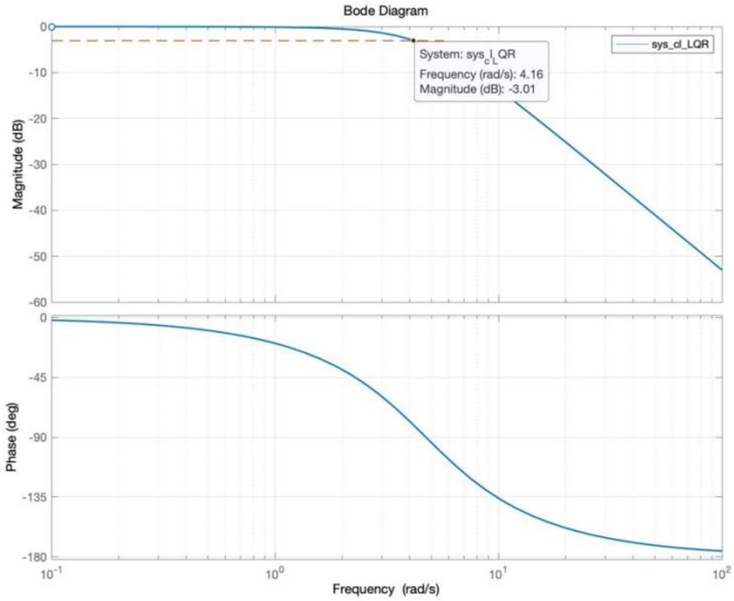

Figure 10 Frequency response for closed loop system

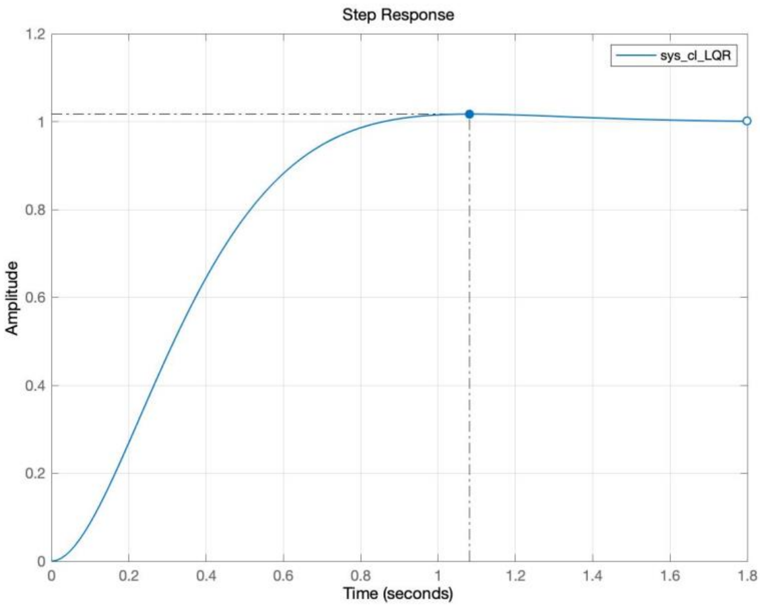

Figure 11 Step response for closed loop system

From figure 10 above we can draw the -3dB line and get the bandwidth frequency is about 4.16 rads/s and we can also get the resonant peak Mr is equal to 0.

From figure 11, we can easily read that the steady-state gain between r and y is 1, which means the input and output are same, so the steady-state gain is 0dB.

The system meets the design requirements.

3.3 The effect of Matrix Q

Since Q = diag(q_1, q_2) with q_1, q_2, r > 0. We adjust these parameters to verify how the penalties setting would influence the closed-loop system performance.

Still let r = 1 and we changed the coefficient related to x1, and obverse the effect on x1. i.e.

$$Q_1 = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \tag{3.6}$$

Change the value of i from 1 to 5.

We can get controller gain K_{LQR} and the scaling gain Ks_{LQR} of each controller.

controller gain K_LQR	scaling gain Ks_LQR
[0.3620 0.2255]	1.5620
[0.6547 0.2992]	1.8547
[0.9071 0.3617]	2.1071
[1.1324 0.4166]	2.3324
[1.3377 0.4661]	2.5377

Table 2

Plot the Bode Diagram of each controller, we can get the figure below.

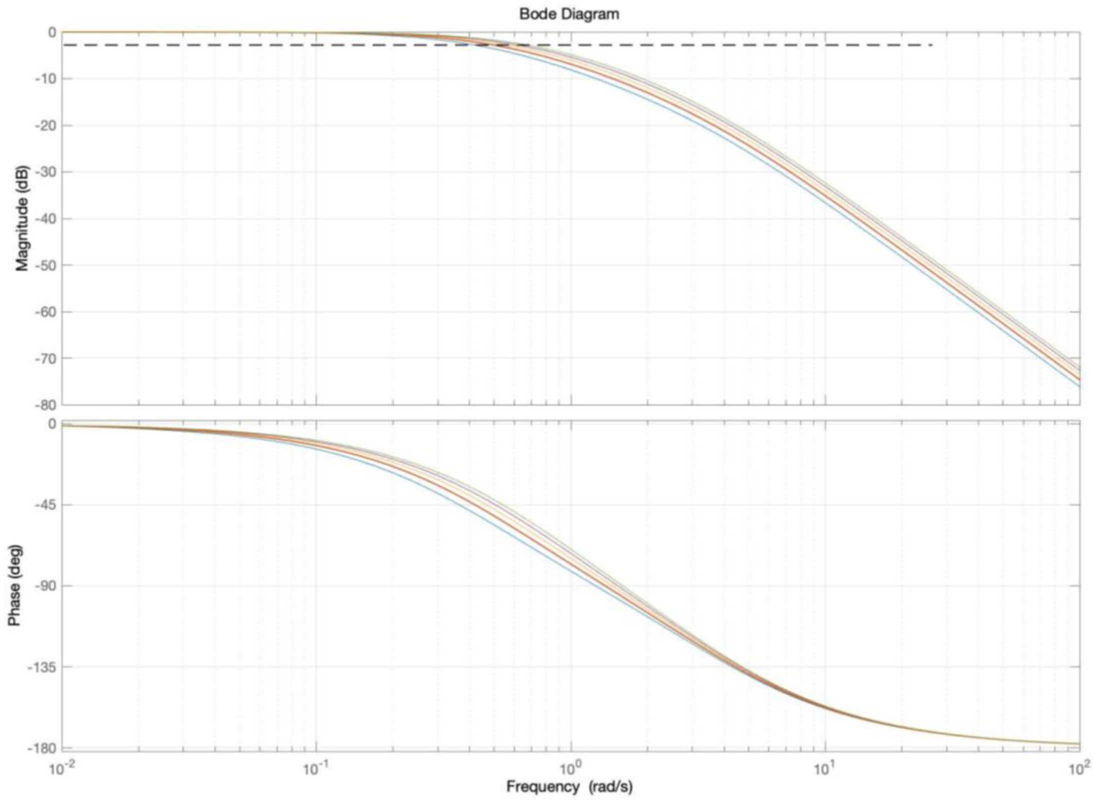

Figure 12 Frequency response for closed loop system

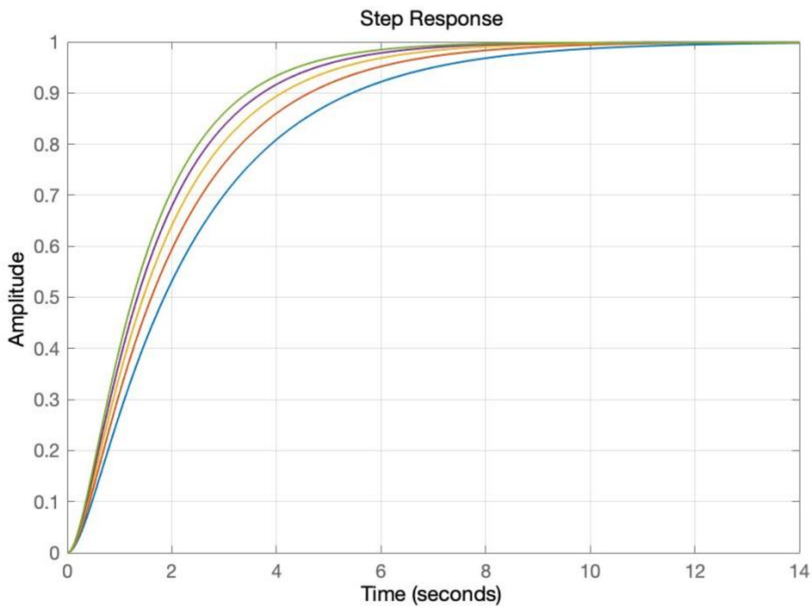

Figure 13 Step response for closed loop system

By increasing the related coefficient to x1, we can see in figure10 that the bandwidths of the closed loop system become larger, and the step response figures in figure11 move up with i increasing, which means the control force on x1 is more and more strengthened.

Then, we changed the coefficient related to x2, and obverse the effect on x1 and x2. i.e.

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{3.7}$$

Change the value of i from 1 to 5. We can get controller gain K_{LQR} and the scaling gain Ks_{LQR} of each controller.

controller gain K_LQR	scaling gain Ks_LQR
[0.3620 0.2255]	1.5620
[0.3620 0.3506]	1.5620
[0.3620 0.4719]	1.5620
[0.3620 0.5898]	1.5620
[0.3620 0.7045]	1.5620

Table 3

Plot the Bode Diagram of each controller, we can get the figure below.

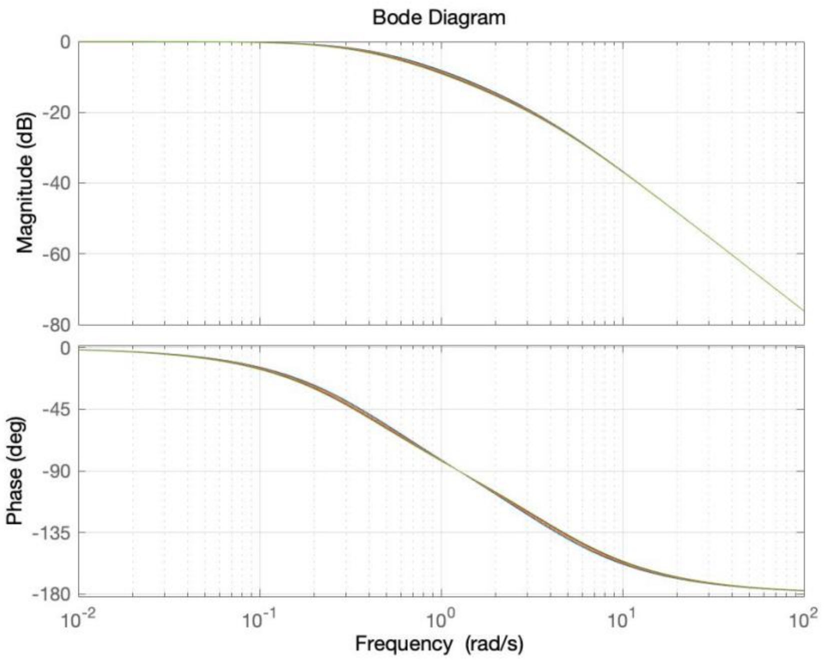

Figure 14 Frequency response for closed loop system

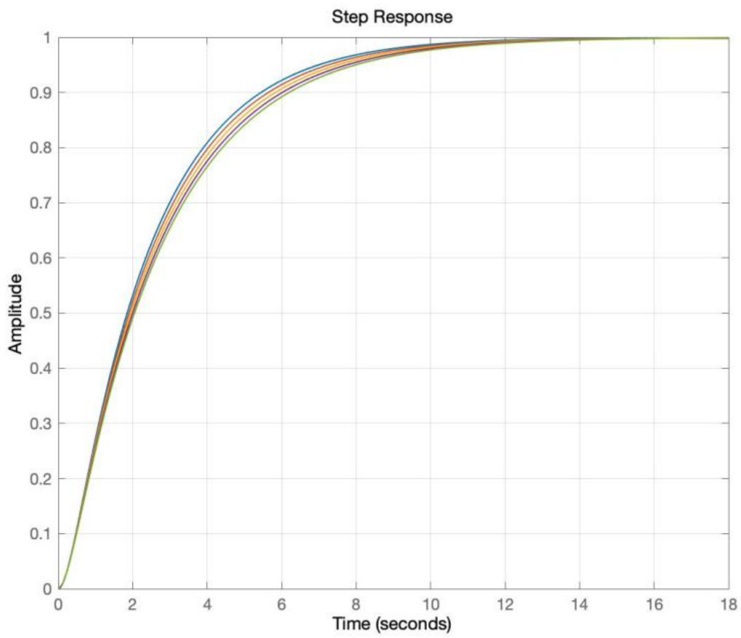

Figure 15   Step response for closed loop system

By increasing related coefficient to x2, the frequency respond figures in Figure12 stay basically the same, and the step response figures in Figure 13 move down with i increasing, which means the control force on x1 is more and more weakened.

Appendix 1

[1] ITEA criterion

[2] Bessel criterion

TABLE 6.1
Prototype Response Poles

(a) ITAE transfer functions	k	Pole locations for $\omega_0 = 1$ rad / s†
	1	$s + 1$
	2	$s + 0.7071 \pm j0.7071$‡
	3	$(s + 0.7081)(s + 0.5210 \pm j1.068)$
	4	$(s + 0.4240 \pm j1.2630)(s + 0.6260 \pm j0.4141)$
	5	$(s + 0.8955)(s + 0.3764 \pm j1.2920)(s + 0.5758 \pm j0.5339)$
	6	$(s + 0.3099 \pm j1.2634)(s + 0.5805 \pm j0.7828)(s + 0.7346 \pm j0.2873)$

(b) Bessel transfer functions	k	Pole locations for $\omega_0 = 1$ rad / s†
	1	$s + 1$
	2	$s + 0.8660 \pm j0.5000)$‡
	3	$(s + 0.9420)(s + 0.7455 \pm j0.7112)$
	4	$(s + 0.6573 \pm j0.8302)(s + 0.9047 \pm j0.2711)$
	5	$(s + 0.9264)(s + 0.5906 \pm j0.9072)(s + 0.8516 \pm j0.4427)$
	6	$(s + 0.5385 \pm j0.9617)(s + 0.7998 \pm j0.5622)(s + 0.9093 \pm j0.1856)$

† Pole locations for other values of ω_0 can be obtained by substituting s/ω_0 for s everywhere.
‡ The factors $(s + a + jb)(s + a - jb)$ are written as $(s + a \pm jb)$ to conserve space.

Appendix 2 MATLAB Part Code

1. Mr, wr and bandwidth(-3dB) frequency calculate.

```
function [ Mr,Wr,Wr_3db] = mr(S_close)
%  The closed-loop resonant amplitude Mr And resonant frequency Wr of the system
   [mag,phase,w] = bode(S_close);
   c = size(mag,3);
   mag1 = zeros(c,1);
   for i=1:c
      mag1(i) = 20*log10(mag(1,1,i));%Get the amplitude of each point
   end
   [M,i] = max(mag1);% Get the maximum amplitude and corresponding frequency
   %show resonant amplitude Mr And resonant frequency Wr
   Mr = M;
   Wr = w(i);
   %show the -3dB cutoff frequency
   Wr_3db = interp1(mag1,w,-3,'Newton');%Interpolation
   c = size(phase,3);
   pha1 = zeros(c,1);
   for i=1:c
      pha1(i) = phase(1,1,i);%Get the frequency of each point
   end
end
```

2. Open-loop system analysis

```
F_Plant = [0,1;-1.20,-3.71]; % Process Matrix
G_Plant = [0;1]; % Input Matrix
H_Plant_State1 = [1,0]; % Output Matrix
H_Plant_State2 = [0,1]; % Output Matrix Assuming x2 is the Output
H_Plant_FullState = [1,0;0,1]; % Output Matrix Assuming Both States are Outputs
sys1 = ss(F_Plant,G_Plant,H_Plant_State1,0);%function 'ss' to establish state-space
model
sys2 = ss(F_Plant,G_Plant,H_Plant_State2,0);
figure(1) % Open a new figure window named 'figure 1'
bode(sys1) % Bode plot from u to x1; Use function 'bode' to plot bode graph
grid on
figure(2)
bode(sys2) % Bode plot from u to x2
grid on
Poles = eig(F_Plant); % The open-loop poles are the eigenvalue of the original plant
process matrix
% Use function 'eig' to calculate the eigenvalue of a square matrix
disp('The open-loop system transfer function poles are located at:')
display(Poles)
```

```
ControllabilityMatrix = [G_Plant,F_Plant*G_Plant];
if (det(ControllabilityMatrix)==0)
   error('The original plant is not controllable.')
% An 'error' command will stop the MATLAB program and pop-up an error message
% If the controllability matrix has determinant of zero, the plant is uncontrollable.
else
   disp('The original plant is controllable.')
end
```

3. Sample system

```
sys_fullstateplant = ss(F_Plant,G_Plant,H_Plant_FullState,[0;0]);
% The original plant with full state vector as output
K_SimpleExp = [0.1,0.1];
Signal_KX_SimpleExp = ss(0,[0,0],0,K_SimpleExp); % Dummy system representing the value
of K*x
sys_fb_SimpleExp = feedback(sys_fullstateplant,Signal_KX_SimpleExp,-1); % Form a
feedback system.
% Use fcn 'feedback' to establish a closed-loop system with a feedback controller
% Note that this system is 1-input-2-output system. The input is r (with
% Ks=1 by default) and the outputs are x1 and x2.
figure(3)
bode(sys_fb_SimpleExp) % Bode plot from r to x1 and x2
F_ClosedLoop = F_Plant-G_Plant*K_SimpleExp;
Poles = eig(F_ClosedLoop); % The closed-loop poles are the eigenvalue of the closed-loop
process matrix
display(Poles)
```

4. ITEA

```
ITAEPoles = 4*[-0.7071+0.7071*1i;-0.7071-0.7071*1i]; % Poles as a column vector
K_ITAE = acker(F_Plant,G_Plant,ITAEPoles)
% Use function 'acker' to calculate the controller gain using Ackermann's Formula
Signal_KX_ITAE = ss(0,[0,0],0,K_ITAE)
sys_fb_ITAE = feedback(sys_fullstateplant,Signal_KX_ITAE,-1)
Ks_ITAE = 1/dcgain(sys_fb_ITAE) % Calculate the steady state gain of the feedback system
% Use function 'dcgain' to calculate the steady state gain of a system
sys_cl_ITAE = Ks_ITAE*sys_fb_ITAE;
% The feedback system together with the scaling gain Ks forms the final design of closed-
loop system
% Note that sys_cl_ITAE is SINGLE output system with output y=x1
```

6. Bessel

```
% Similarly for Bessel Prototype
BesselPoles = 4*[-0.8660+0.5000*1i;-0.8660-0.5000*1i];
K_Bessel = acker(F_Plant,G_Plant,BesselPoles)
Signal_KX_Bessel = ss(0,[0,0],0,K_Bessel);
sys_fb_Bessel = feedback(sys_fullstateplant,Signal_KX_Bessel,-1);
Ks_Bessel = 1/dcgain(sys_fb_Bessel)
sys_cl_Bessel = Ks_Bessel*sys_fb_Bessel;
```

7. Second order

```
DampRatio = 0.707; % Define Damping Ratio for Reference Model
NatrualFreq = 3.6; % Define Natural Frequency for Reference Model
BandWidthDesire = NatrualFreq*sqrt((1-2*DampRatio^2)+sqrt((1-2*DampRatio^2)^2+1))
% Calculate the Bandwidth of the reference model
SODPoles = roots([1,2*DampRatio*NatrualFreq,NatrualFreq^2])
% Calculate the pole positions of the reference model, the root of pole polynomial
% Use function 'roots' to calculate the roots of a polynomial
K_SOD = acker(F_Plant,G_Plant,SODPoles);
Signal_KX_SOD = ss(0,[0,0],0,K_SOD);
sys_fb_SOD = feedback(sys_fullstateplant,Signal_KX_SOD,-1);
Ks_SOD = 1/dcgain(sys_fb_SOD);
sys_cl_SOD = Ks_SOD*sys_fb_SOD;
```

8. LQR(Loop part)

```
% Design Using LQR
for i = 1:5
    Q_LQR = [1,0;0,i];
    R_LQR = 1; % Define penalty matrix R (in this case, a scalar)
    [K_LQR,~,~] = lqr(F_Plant,G_Plant,Q_LQR,R_LQR)
    Signal_KX_LQR = ss(0,[0,0],0,K_LQR);
    sys_fb_LQR = feedback(sys_fullstateplant,Signal_KX_LQR,-1);
    Ks_LQR = 1/dcgain(sys_fb_LQR)
    sys_cl_LQR = Ks_LQR*sys_fb_LQR;
    figure(1)
    bode(sys_cl_LQR)
    hold on
    grid on
    figure(2)
    step(sys_cl_LQR)
    grid on
    hold on
end
```