$$w(x) = (x-\lambda)/\sqrt{1-x^2}$$

$$\boxed{p_n(x) = \frac{T_{n+1}(x) - \frac{T_{n+1}(\lambda)}{T_n(\lambda)} T_n(x)}{x-\lambda}}$$

pp. 35 eqn. 7.3

$$K_n(x,y) = \sum_{j=0}^{n} \frac{p_j(x)\, p_j(y)}{\langle p_j(x), p_j(y) \rangle}$$

Christoffel-Darboux Kernel

$$p_n(x) = \text{CONST} \cdot K_n(x,\lambda)$$

$$r(x) = (x-\lambda_1)\cdots(x-\lambda_d)/\sqrt{1-x^2}$$

$$p_n(x) = \frac{\sum_{k=0}^{d} \beta_{n,k} T_{k+n}(x)}{(x-\lambda_1)\cdots(x-\lambda_d)}$$

$p_n(x)$ poly $\Rightarrow$ $\sum_{k=0}^{d} \beta_{n,k} T_{n+k}(\lambda_j) = 0 \quad \forall\ j = 1, \dots, d$

$$\begin{bmatrix} T_n(\lambda_1) & T_{n+1}(\lambda_1) & \cdots & T_{n+d}(\lambda_1) \\ T_n(\lambda_2) & T_{n+1}(\lambda_2) & \cdots & T_{n+d}(\lambda_2) \\ \vdots & & & \\ T_n(\lambda_d) & T_{n+1}(\lambda_d) & \cdots & T_{n+d}(\lambda_d) \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \beta_{n,0} \\ \beta_{n,1} \\ \beta_{n,2} \\ \vdots \\ \beta_{n,d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Cramers Rule: $B_{n,j} = \dfrac{j^{th} \text{ minor of matrix}}{d^{th} \text{ minor of matrix}}$

$$P_n(x) = \sum_{j=0}^{n} \frac{j^{th} \text{ minor of matrix}}{d^{th} \text{ minor of matrix}} T_{n+j}(x) \Big/ g(x)$$

$$= \frac{\det \begin{bmatrix} T_n(\lambda_1) & T_{n+1}(\lambda_1) & \cdots & T_{n+d}(\lambda_1) \\ T_n(\lambda_2) & T_{n+1}(\lambda_2) & \cdots & T_{n+d}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_n(\lambda_d) & T_{n+1}(\lambda_d) & & T_{n+d}(\lambda_d) \\ T_n(x) & T_{n+1}(x) & \cdots & T_{n+d}(x) \end{bmatrix} \dfrac{1}{\left(d^{th} \text{ minor}\right)}}{(x-\lambda_1)\cdots(x-\lambda_d)}$$

Christoffel's Formula according to Szegö

Paper in 2016: relates value of this kind of determinant to Selberg integral:

$$\det \begin{bmatrix} T_n(\lambda_1) & T_{n+1}(\lambda_1) & \cdots & T_{n+d}(\lambda_1) \\ T_n(\lambda_2) & T_{n+1}(\lambda_2) & \cdots & T_{n+d}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_n(\lambda_d) & T_{n+1}(\lambda_d) & & T_{n+d}(\lambda_d) \\ T_n(x) & T_{n+1}(x) & \cdots & T_{n+d}(x) \end{bmatrix}$$

$$= \text{const} \cdot \prod_{i=1}^{d}(x-\lambda_i) \prod_{i\neq j}(\lambda_i-\lambda_j) \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} \prod_{i=1}^{n}(y_i-x) \prod_{j=1}^{d}\prod_{k=1}^{n}(y_k-\lambda_j) \prod_{\substack{i,k=1 \\ i\neq k}}^{n}(y_i-y_k)^2 \frac{dy_1\cdots dy_n}{\sqrt{(1-y_1^2)\cdots(1-y_n^2)}}$$

exciting that they show up!

Related to some sort of Gamma functions or hypergeo. functions or whatever.

Slater determinant

$$S(x_0,\dots,x_d;n_0,\dots,n_d) = \det \begin{bmatrix} T_{n_1}(x_1) & T_{n_2}(x_1) & \cdots & T_{n_d}(x_1) \\ T_{n_1}(x_2) & T_{n_2}(x_2) & \cdots & T_{n_d}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n_1}(x_d) & T_{n_2}(x_d) & \cdots & T_{n_d}(x_d) \end{bmatrix}$$

Cool and fun to look at...

$$p_n(x) = S(\lambda_1,\dots,\lambda_d,x;n,n+1,\dots,n+d)\Big/(x-\lambda_1)\cdots(x-\lambda_d)$$

Does it satisfy any differential equations?

$$(1-x^2)T_n''(x) - xT_n'(x) = -n^2T_n(x)$$

Consequently:

$$\sum_{j=0}^{d}\left[(1-x_j^2)\frac{\partial^2 S}{\partial x_j^2} - x_j\frac{\partial S}{\partial x_j}\right] = \left(-\sum_{j=0}^{d} n_j^2\right)S$$

$$\left(\prod_{j=0}^{d}\left[(1-x_j^2)\frac{\partial^2}{\partial x_j^2} - x_j\frac{\partial}{\partial x_j}\right]\right)\cdot S = \prod_{j=0}^{d}(-n_j^2)\cdot S$$

$$\boxed{\sum_{j=0}^{n} \tfrac{1}{2}S(\vec{x};\vec{n}+\vec{e}_j) + \tfrac{1}{2}S(\vec{x};\vec{n}-\vec{e}_j) = \left(\sum_{j=0} x_j\right)S(\vec{x};\vec{n}) \quad \forall j.}$$

$$S(\vec{x}, n, n+1, n+2, \dots, n+d)$$

$$\det \begin{bmatrix} T_{n_1}(x_1) & \frac{1}{2}T_{n_2+1}(x_1) & \cdots & T_{n_d}(x_1) \\ T_{n_1}(x_2) & \frac{1}{2}T_{n_2+1}(x_2) & \cdots & T_{n_d}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_{k_1}(x_d) & \cdots \frac{1}{2}T_{n_2+1}(x_d) & \cdots & T_{n_d}(x_d) \end{bmatrix}$$

$$+ \det \begin{bmatrix} T_{n_1}(x_1) & \frac{1}{2}T_{n_2-1}(x_1) & \cdots & T_{n_d}(x_1) \\ T_{n_1}(x_2) & \frac{1}{2}T_{n_2-1}(x_2) & \cdots & T_{n_d}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_{k_1}(x_d) & \cdots \frac{1}{2}T_{n_2-1}(x_d) & \cdots & T_{n_d}(x_d) \end{bmatrix}.$$

$$A = (a_{jk})_{j,k=1}^{d}$$

$$\det(A) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{j=1}^{d} a_{\sigma(j),j}$$

$S_d = \{\text{bijections } \{1,\dots,d\} \to \{1,\dots,d\}\}$

$$= \det \begin{bmatrix} T_{n_1}(x_1) & x_1 T_{n_2}(x_1) & \cdots & T_{n_d}(x_1) \\ T_{n_1}(x_2) & x_2 T_{n_2}(x_2) & \cdots & T_{n_d}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ T_{k_1}(x_d) & \cdots x_d T_{n_d}(x_d) & \cdots & T_{n_d}(x_d) \end{bmatrix} = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)\, x_{\sigma(2)} \prod_{j=1}^{d} T_{n_j}(x_{\sigma(j)})$$

$$\sum_{k=0}^{n} \frac{1}{2} S(\vec{x}; \vec{n}+\vec{e}_j) + \frac{1}{2} S(\vec{x}; \vec{n}-\vec{e}_j)$$

$$= \sum_{k=0}^{n} \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma)\, x_{\sigma(k)} \prod_{j=1}^{d} T_{n_j}(x_{\sigma(j)})$$

$$= \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \underbrace{\sum_{k=0}^{d} x_{\sigma(k)}}_{\sum_{k=0}^{d} x_k} \prod_{j=1}^{d} T_{n_j}(x_{\sigma(j)})$$

$$= \left(\sum_{k=0}^{d} x_k\right) \cdot \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{j=1}^{d} T_{n_j}(x_{\sigma(j)}) = \left(\sum_{k=0}^{d} x_k\right) S(\vec{x}, \vec{n})$$