Orthogonal Matrix Polynomials

Def: A *weight matrix* is a function $W:(a,b) \longrightarrow M_n(\mathbb{R})$ with

- $\int_a^b W(x)|x|^n \, dx < \infty \quad \forall \; n \geq 0$
- $W(x)^T = W(x) \quad \forall \; x \in (a,b)$ (symmetric)
- eigenvalues of $W(x) > 0 \quad \forall \; x \in (a,b)$ (pos-definite)

Remark: An *inner product* is a function
$$V \times V \longrightarrow \mathbb{R}, \quad u,v \longmapsto \langle u,v \rangle$$
satisfying

- $\langle u,v \rangle = \langle v,u \rangle$
- $\langle u,u \rangle = 0 \iff u=0$ & $\langle u,u \rangle \geq 0$
- $\langle \lambda u, v \rangle = \lambda \langle u,v \rangle$ and $\langle u_1+u_2, v \rangle = \langle u_1,v \rangle + \langle u_2,v \rangle$

Thm: Every inner product on $V=\mathbb{R}^n$ is of the form
$$\langle u,v \rangle = u \cdot Av = u^T A v$$
for some positive def + symm. matrix A.

Ex:
$$W(x) = \begin{bmatrix} e^{-x^2} & xe^{-x^2} \\ xe^{-x^2} & e^{-x^2}(1+x^2) \end{bmatrix}, \quad -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} |x|^n W(x) \, dx = \begin{bmatrix} \int_{-\infty}^{\infty} |x|^n e^{-x^2} dx & \int_{-\infty}^{\infty} |x^n| x e^{-x^2} \\ \int_{-\infty}^{\infty} |x^n| x e^{-x^2} & \int_{-\infty}^{\infty} |x^n| (1+x^2) e^{-x^2} \end{bmatrix}$$

$$W(x) = e^{-x^2}\begin{bmatrix} 1 & x \\ x & 1+x^2 \end{bmatrix}$$

$$tr(W(x)) = e^{-x^2}(2+x^2) > 0 \qquad \lambda_1+\lambda_2 > 0$$

$$det(W(x)) = e^{-2x^2} > 0 \qquad \lambda_1\lambda_2 > 0$$

Ex: $W(x) = \dfrac{1}{\sqrt{1-x^2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Def: A sequence of orthogonal polynomials for a weight matrix $W(x)$ is a sequence of matrix-valued polys $P_0(x), P_1(x), \ldots$ satisfying

- $P_n(x) = \sum_{j=0}^{n} P_{nj}x^j$, $P_{nj} \in M_n(\mathbb{R})$, $det(P_{nn}) \neq 0$
- $\underbrace{\int_b^c P_n(x)W(x)P_m(x)^T dx}_{\langle P_n, P_m \rangle_W} = 0$, $m \neq n$.

Theorem: The sequence $P_0, P_1, \ldots$ satisfies a three-term recursion relation

$$xP_n(x) = A_nP_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x)$$

Quest: $W(x) = \dfrac{1}{\sqrt{1-x^2}}Q(x)$,

$Q(x)$ is pos-def, symm poly on $[-1,1]$

What is the three-term recursion relation for a sequence of orthogonal polys for $W(x)$?

Where to begin? CODE!

- $P_n(x) = P_n(\cos\theta) = \sum_{j=0}^{n} F_{nj} \cos(j\theta)$

- $\int_{-1}^{1} P_n(x) W(x) P_m(x)^* = \int_0^{\pi} P_n(\cos\theta) \underbrace{Q(\cos\theta) P_m(\cos\theta)^*}\, d\theta$

$$\int_0^{\pi} \cos(m\theta)\cos(n\theta) = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n > 0 \\ \pi, & m = n = 0 \end{cases}$$

$$Q(\cos\theta) = \sum_{k=0}^{d} Q_k \cos(k\theta)$$

Code input: $Q_0, \ldots, Q_d$, n_{max}

Code output: A_n, B_n, C_n values in a recursion relation for $P_n(x)$

$A_0, A_1, \ldots, A_{n_{max}}, \ldots$

- A_n = function of n, $B_n =$, $C_n =$

Ex: $Q(x) = \begin{bmatrix} 1 & ax \\ ax & a^2x^2+1 \end{bmatrix}$

$$Q(\cos\theta) = \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & \frac{3}{2} + \frac{1}{2}\cos(2\theta) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}}_{Q_0} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{Q_1} \cos\theta + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}_{Q_2} \cos 2\theta$$

<u>Conjecture</u> : If $\det Q(x) = \text{constant}$
then

$$A_n = \frac{1}{2} I \qquad B_n = 0 \qquad C_n = \frac{1}{2} I, \quad n \gg 1$$

$\therefore$

$$P_n(x) = \sum_{j=0}^{d} \Omega_j T_{n-j}(x) \qquad n >> 0$$

<u>Wonder</u> : $P_n(x) = \text{qdet} \begin{bmatrix} & \end{bmatrix} \cdot Q(x)^{-1}$