- $r(x) = q(x)/\sqrt{1-x^2}$ poly

- $P_0(x), P_1(x), P_2(x), \dots$ seq. of orth. polys.

$$\boxed{P_n(\cos\theta) = \sum_{j=0}^{n} a_{nj}\cos(j\theta)} \qquad a_{nn} = 1$$

Consider the vector space

$$V = \left\{ \sum_{j=0}^{n} c_j \cos(j\theta) \;\middle|\; n \geq 0 \text{ integer}, \ c_0, \dots, c_n \in \mathbb{R} \right\}.$$

- $\{P_0(\cos\theta), P_1(\cos\theta), \dots\}$ is a basis for V
- $\{1, \cos\theta, \cos(2\theta), \dots\}$ is a basis for V.

Change of basis:

$$L^{-1} : \cos(n\theta) \to P_n(\cos\theta).$$

Matrix form

$$\begin{bmatrix} a_{00} & 0 & 0 & 0 & 0 & \cdots \\ a_{10} & a_{11} & 0 & 0 & 0 & \cdots \\ a_{20} & a_{21} & a_{22} & 0 & 0 & \cdots \\ a_{30} & a_{31} & a_{32} & a_{33} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ \cos\theta \\ \cos 2\theta \\ \cos 3\theta \\ \vdots \end{bmatrix} = \begin{bmatrix} P_0(\cos\theta) \\ P_1(\cos\theta) \\ P_2(\cos\theta) \\ P_3(\cos\theta) \\ \vdots \end{bmatrix}$$

$$L = \begin{bmatrix} a_{00} & a_{10} & a_{20} & a_{30} & a_{40} & \cdots \\ 0 & a_{11} & a_{21} & a_{31} & a_{41} & \cdots \\ 0 & 0 & a_{22} & a_{32} & a_{42} & \cdots \\ 0 & 0 & 0 & a_{33} & a_{43} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{10} \\ a_{11} \\ 0 \\ \vdots \end{bmatrix}$$

Orthogonality: $\displaystyle\int_0^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, q(\cos\theta) = 0, \quad m \neq n.$

$$q(\cos\theta) = \sum_{j=0}^{r} q_j \cos(j\theta)$$

$$q(\cos\theta)\cos(k\theta) = \sum_{j=0}^{r} \tfrac{1}{2} q_j \left[\cos((j+k)\theta) + \cos((j-k)\theta)\right]$$

$$q_0\cos 2\theta + \tfrac{1}{2}q_1\cos(3\theta) + \tfrac{1}{2}q_{-1}\cos\theta + \tfrac{1}{2}q_2\cos(4\theta) + \tfrac{1}{2}q_2 + \tfrac{1}{2}q_3\cos(5\theta) + \tfrac{1}{2}q_2\cos\theta + \dots$$

$$Q = \begin{bmatrix} q_0 & q_1/2 & q_2/2 & & 0 \\ q_1 & q_0 + q_2/2 & (q_1+q_3)/2 & & \\ q_2 & (q_1+q_3)/2 & q_0 + q_4/2 & & \\ q_3 & (q_2+q_4)/2 & (q_1+q_5)/2 & & \\ \vdots & \vdots & \vdots & & \\ q_r & \times & 0 & & \\ 0 & 0 & \vdots & & \end{bmatrix}$$

$$\left(L \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \right)^T \begin{bmatrix} \pi & & \\ & \frac{\pi}{2} & 0 \\ 0 & & \frac{\pi}{2} \\ & & & \ddots \end{bmatrix} Q L \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = 0 \quad \text{if } j \neq k$$

$$L^T \begin{bmatrix} 1 & & \\ & 1/2 & 0 \\ 0 & & 1/2 \\ & & & \ddots \end{bmatrix} Q L = \text{diagonal!}$$

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1/2 & 0 \\ 0 & & 1/2 \\ & & & \ddots \end{bmatrix} Q}_{\text{Known}} = \underbrace{(L^{-1})^T}_{\text{lower triang}} \overbrace{D}^{\text{diag}} \underbrace{L^{-1}}_{\text{upper triang.}}$$

$$\begin{bmatrix} * & & & \\ * & * & & 0 \\ * & * & * & \\ & & \ddots & \ddots \end{bmatrix}\begin{bmatrix} d_0 & & & 0 \\ & d_1 & & \\ & & d_2 & \\ 0 & & & \ddots \end{bmatrix}\begin{bmatrix} * & * & * & * \\ & * & * & * \\ 0 & & *_1 & \\ & & & *_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & 0 \\ & 1/2 & & \\ & & 1/2 & \\ 0 & & & 1/2 \end{bmatrix} Q$$

STEPS : (1) Calculate $\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & & & 0 \\ 0 & & 1/2 & & \\ 0 & & & \ddots & \\ 0 & & 0 & & \end{bmatrix} Q$

(2) Perform LDU decomp

(3) inverse of the L = coeff's of our expansion of $P_n(\cos\theta)$

Compare : $P_n(x)$ w/ $q(x)T_m(x)$

$$\int P_m(x)\,q(x)\,P_n(x)\,dx = 0 \quad \forall\ m \neq n \Rightarrow \int p(x)\,q(x)\,P_n(x) = 0$$

$\forall\ p(x)$ w/ $\deg(p(x)) < n$

$p(x) = \underline{P_0 + P_1 + P_2}$

$$\int p(x)\,q(x)\,P_1(x) = \int P_1(x)^2\,q(x)\,dx$$

*

$$q(x)P_n(x) = \sum_{j=0}^{n+d} \tilde{a}_{nj} T_j(x) = \sum_{j=n}^{n+d} \tilde{a}_{nj} T_j(x)$$

$$\int_{-1}^{1} P_n(x) T_j(x) \frac{q(x)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \tilde{a}_{nj}.$$

$$\boxed{P_n(x) = \frac{1}{q(x)} \sum_{j=n}^{n+d} \tilde{a}_{nj} T_j(x)} \leftarrow$$

Better weight : $\dfrac{1}{q(x)\sqrt{1-x^2}}$??