$$\sum_{k=1}^{n} f\left(\tfrac{2k-1}{n}\right)\frac{1}{\sqrt{1-\left(\frac{2k-1}{n}\right)^2}}\frac{2}{n}$$

$$\int_{-1}^{1} f(x)\frac{1}{\sqrt{1-x^2}}\,dx \approx \underbrace{\frac{\pi}{n}\sum_{k=1}^{n} f\left(\cos\left(\tfrac{2k-1}{2n}\pi\right)\right)}_{\text{Riemann sum?}}$$

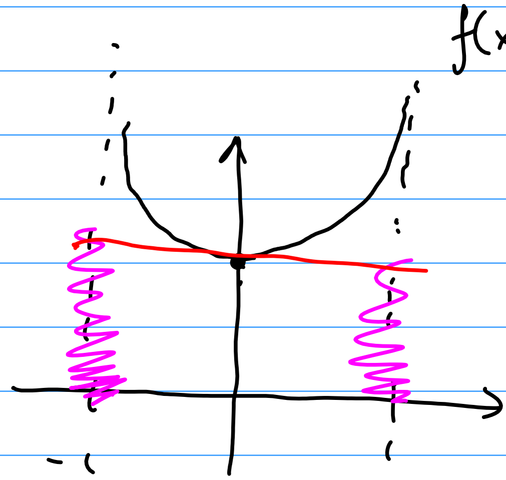
$$\int_0^{\pi} f(\cos(x))\,dx \approx \sum_{k=1}^{n} f(\cos(x_k))\Delta x, \qquad \Delta x = \frac{\pi}{n}$$

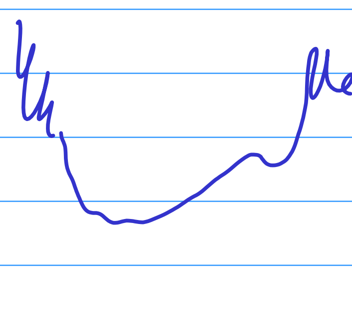
$$x_k = (k-1)\Delta x + \frac{\Delta x}{2} = \frac{\pi}{n}(k-1) + \frac{\pi}{2n} = \frac{\pi}{2n}(2k-2+1) = \frac{2k-1}{2n}\pi.$$

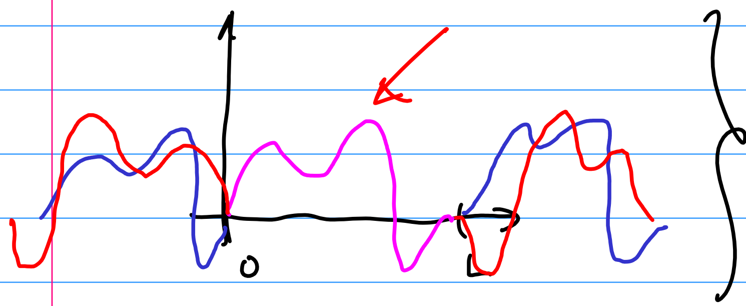
$$= \sum_{k=1}^{n} f\left(\cos\left(\tfrac{2k-1}{2n}\pi\right)\right)\frac{\pi}{n}.$$

$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} f(x)\frac{1}{\sqrt{1-x^2}}\,dx$$

$$f(x) = \sqrt{1-x^2}\,g(x)$$







Group: $\mathbb{R}$ $\longleftrightarrow$ continuous Fourier

$[0,2\pi) = \mathbb{T}$ $\longleftrightarrow$ Fourier series

$\mathbb{Z}/n\mathbb{Z}$ $\longleftrightarrow$ discrete Fourier transform

G group $\leftrightsquigarrow$ <u>dual group</u> $\hat{G}$

$\hat{G} = \{\chi \mid \chi \text{ group character}\}$

Here <u>group character</u> means a hom $\chi : G \to \{z \mid |z| = 1\}$

<u>Ex</u>: $G = \mathbb{R}$, $\xi \in \mathbb{R}$

$\chi_\xi : x \longmapsto e^{ix\xi}$, $\hat{G} = \{\chi_\xi : \xi \in \mathbb{R}\} \cong \mathbb{R}$

$\chi_\xi \cdot \chi_\omega = \chi_{\xi+\omega}$

<u>Ex</u>: $G = \mathbb{T} = [0,2\pi)$, $k \in \mathbb{Z}$

$\chi_k : x \longmapsto e^{ikx}$

$\hat{G} = \{\chi_k : k \in \mathbb{Z}\} \cong \mathbb{Z}$

$e^{ik(x+2\pi)} = e^{ikx}$

$\Rightarrow e^{ik2\pi} = 1$

$\Rightarrow k \in \mathbb{Z}$

<u>Ex</u>: $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$

$\chi_k : m \longmapsto e^{2\pi i k m/n}$

$$\hat{G} = \{ \chi_k : 0 \le k \le n-1 \} \cong \mathbb{Z}/n\mathbb{Z}$$

Def: The <u>Fourier transform</u> of $f : G \to \mathbb{C}$ is $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \int_G \overline{\chi(g)} f(g) \, dg$$

Haar measure

$G = \mathbb{R}$: $\hat{f} : \mathbb{R} \to \mathbb{C}$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx$$

$G = [0, 2\pi)$: $\hat{f} : \mathbb{Z} \to \mathbb{C}$

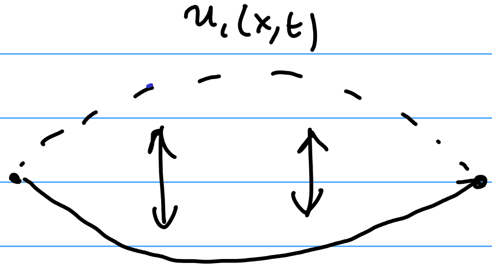
$$\hat{f}(k) = \int_0^{2\pi} e^{-ikx} f(x) \, dx$$

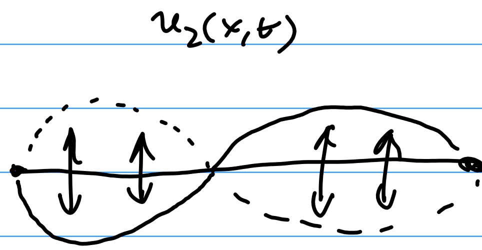
$$f(x) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \hat{f}(k) e^{ikx} \, dx$$

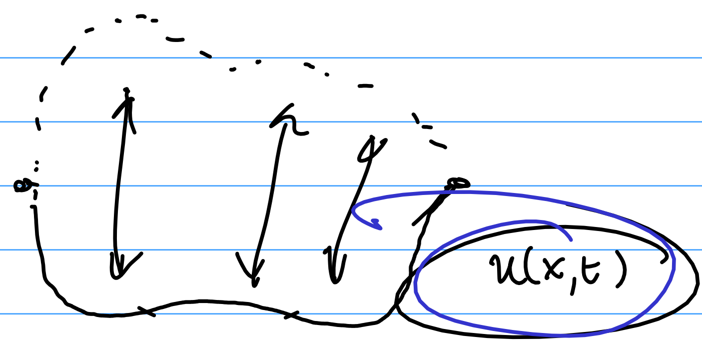
$G = \mathbb{Z}/n\mathbb{Z}$: $\hat{f} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$

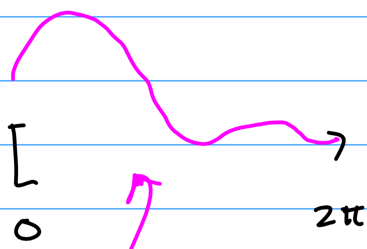
$$\hat{f}(k) = \sum_{j=0}^{n-1} e^{-2\pi i jk} f(j)$$

Fourier Series:









$$\boxed{f(x) = \frac{1}{2\pi} \sum_{k=0}^{\infty} e^{ikx} \hat{f}(k)}$$

$$f(x) \approx \frac{1}{2\pi} \sum_{k=0}^{N} e^{ikx} \hat{f}(k)$$

$$\boxed{\hat{f}(k) = \sum_{j=0}^{n} e^{2\pi i jk/n} f\left(\frac{2\pi j}{n}\right)}$$

$$\boxed{w_k = \frac{1}{P_n'(x_{n,k})} \int_b^c \frac{P_n(x)}{x - x_{nk}} r(x)\,dx}$$

$$P_n(x) = \sum_{j=0}^{n} \alpha_{nj} x^j$$

$$\sum_{k=0}^{n} \frac{P_k(x) P_k(y)}{\|P_k\|_r^2} = \frac{1}{\|P_n\|_r^2} \left(\frac{\alpha_{nn}}{\alpha_{n+1,n+1}}\right) \frac{P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)}{x - y}$$

$$\int_b^c \sum_{k=0}^{n} \frac{P_k(x) P_k(y)}{\|P_k\|_r^2} r(x)\,dx = \frac{\alpha_{nn}/\alpha_{n+1,n+1}}{\|P_n\|_r^2} \cdot \int_b^c \frac{P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)}{x - y} r(x)\,dx$$

$$\frac{\|P_n\|_r^2}{\alpha_{nn}/\alpha_{n+1,n+1}} = \int_b^c \frac{P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)}{x - y} r(x)\,dx$$

$$\frac{-1}{P_{n+1}(x_{n,k})} \frac{\|P_n\|_r^2}{\alpha_{nn}/\alpha_{n+1,n+1}} = \int_b^c \frac{P_n(x)}{x - x_{n,k}} r(x)\,dx$$

$$\boxed{w_k = \left(\frac{-\|P_n\|_r^2}{\alpha_{nn}/\alpha_{n+1,n+1}}\right) \frac{1}{P_{n+1}(x_{n,k})\, P_n'(x_{n,k})}}$$

$$\int_{-1}^{1} f(x)\,dx \qquad \int_{-1}^{1} P_n(x) P_m(x)\,dx = 0 \quad m \neq n.$$