Ex: $r(x) = (x+1)/\sqrt{1-x^2}$

Cosine version: find $P_0(\theta)$, $P_1(\theta)$, $P_2(\theta)$, ...

where

$$P_j(\theta) = \sum_{k=0}^{j} a_{jk} \cos(k\theta) \quad = \text{Poly of degree } j \text{ evaluated @ } \cos(\theta)$$

WLOG, $a_{jj} = 1 \quad \forall j$.

$$P_0(\theta) = 1$$
$$P_1(\theta) = \cos\theta - \tfrac{1}{2}$$
$$P_2(\theta) = \cos 2\theta - 1 \cos\theta + \tfrac{1}{2}$$
$$\vdots$$

$$\int_0^{\pi} P_0(\theta) P_1(\theta) (\cos\theta + 1)\, d\theta$$

$$= \int_0^{\pi} \cos^2\theta + a_{10}\cos\theta + \cos\theta + a_{10}\, d\theta = \pi a_{10} + \frac{\pi}{2} = 0 \Rightarrow a_{10} = -\frac{1}{2}$$

$$\int_0^{\pi} P_2(\theta) P_0(\theta)(\cos\theta + 1)\, d\theta = \int_0^{\pi} \cancel{\cos(2\theta)\cos\theta} + a_{21}\cos^2\theta + a_{20}\cancel{\cos\theta}$$
$$+ \cancel{\cos(2\theta)} + a_{21}\cancel{\cos\theta} + a_{20} \quad d\theta$$

$$= \left(\tfrac{1}{2} a_{21} + a_{20}\right)\pi = 0$$

$$a_{21} = -2a_{20} \qquad a_{20} = \frac{1}{2}$$

$$\int_0^{\pi} P_2(\theta) \underbrace{P_1(\theta)(\cos\theta + 1)}\, d\theta = \int_0^{\pi} \left[\cos(2\theta) + a_{21}\cos\theta + a_{20}\right]\left[\cos(2\theta) + \cos\theta\right] \tfrac{1}{2}\, d\theta$$

$$\cos^2\theta + \tfrac{1}{2}\cos\theta - \tfrac{1}{2}$$

$$\tfrac{1}{2}\cos(2\theta) + \tfrac{1}{2}\cos\theta$$

$$= \int_0^{\pi} \left[\cos^2(2\theta) + a_{21}\cos^2\theta\right] \tfrac{1}{2}\, d\theta$$

$$= \frac{\pi}{4}\left[1 + a_{21}\right] = 0$$

$a_{21} = -1$

$$\cos(a\theta)\cos(b\theta) = \frac{1}{2}\cos((a+b)\theta) + \frac{1}{2}\cos((a-b)\theta) \quad \checkmark$$

$$P_m(\theta) = \sum_{j=0}^{m} a_{mj}\cos(j\theta)$$

$$\cos\theta\, P_m(\theta) = \sum_{j=0}^{m} a_{mj}\cos(j\theta)\cos\theta$$

$$= \frac{1}{2}\sum_{j=0}^{m} a_{mj}\cos((j+1)\theta) + a_{mj}\cos((j-1)\theta)$$

$a_{m,m+1} = 0$

$$= \sum_{j=0}^{m} \left(\frac{a_{m,|j-1|} + a_{m,j+1}}{2}\right)\cos(j\theta)$$

$$(\cos\theta + 1)P_m(\theta) = \sum_{j=0}^{m}\left(\frac{a_{m,|j-1|} + a_{m,j+1}}{2} + a_{mj}\right)\cos(j\theta)$$

$$\int_0^{\pi} P_n(\theta)(\cos\theta+1)P_m(\theta)\,d\theta = \int_0^{\pi}\left(\sum_{k=0}^{n} a_{n,k}\cos(k\theta)\right)\left(\sum_{j=0}^{m}\left(\frac{a_{m,|j-1|}+a_{m,j+1}}{2} + a_{mj}\right)\cos(j\theta)\right)d\theta$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{m} a_{nk}\left(\frac{a_{m,|j-1|}+a_{m,j+1}}{2} + a_{mj}\right)\int_0^{\pi}\cos(k\theta)\cos(j\theta)\,d\theta$$

$$= \pi\left[a_{n0}(a_{m1}+a_{m0}) + \frac{1}{2}\sum_{k=1}^{\min(n,m)} a_{nk}\left(\frac{a_{m,k-1}+a_{m,k+1}}{2} + a_{m,k}\right)\right]$$

automated version ...

Think about $P_n(\theta)$ as a vector!

$$P_n(\theta) \longleftrightarrow \begin{bmatrix} a_{n0} \\ a_{n1} \\ a_{n2} \\ \vdots \\ a_{nn} \end{bmatrix}$$

$$r(x) = \frac{x+1}{\sqrt{1-x^2}}$$

$$\begin{bmatrix} a_{n0} & a_{n1} & \cdots & a_{nn} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} \end{bmatrix}}_{n \times m \text{ matrix}} \begin{bmatrix} a_{m0} \\ a_{m1} \\ a_{m2} \\ \vdots \\ a_{mm} \end{bmatrix} = 0$$

$\Phi_{m,n}$

$$P_0 \longleftrightarrow [1]$$

$$P_1 \longleftrightarrow \begin{bmatrix} a_{10} \\ 1 \end{bmatrix}$$

$$[a_{10} \; 1] \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} [1] = 0$$

$$a_{10} = -\frac{1}{2}$$

$$P_2 \longleftrightarrow \begin{bmatrix} a_{20} \\ a_{21} \\ 1 \end{bmatrix}$$

$$\begin{cases} [a_{20} \; a_{21} \; 1] \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} [1] = 0 \\ [a_{20} \; a_{21} \; 1] \begin{bmatrix} 1 & 0 \\ 1/2 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a_{10} \\ 1 \end{bmatrix} \end{cases}$$

$$\begin{bmatrix} -1/2 \\ 0 \\ 1/4 \end{bmatrix}$$

$$\frac{1}{4} - \frac{1}{2} a_{20} = 0$$

$$a_{20} = \frac{1}{2}$$

$$r(x) = (r_0 + r_1 x + r_2 x^2 + \cdots + r_d x^d) / \sqrt{1-x^2}$$

$\Phi_{m,n}$

$$r_0/\sqrt{1-x^2} \longleftrightarrow \Phi_{m,n} = \begin{bmatrix} r_0 & & & 0 \\ & r_0/2 & & \\ & & r_0/2 & \\ 0 & & & \ddots \\ & & & & r_0/2 \end{bmatrix}$$

$$(r_0 + r_1 x)/\sqrt{1-x^2} \longleftrightarrow \Phi_{mn} = \begin{bmatrix} r_0 & r_1/2 & & & 0 \\ r_1/4 & r_0/2 & r_1/4 & & \\ & r_1/4 & r_0/2 & r_1/4 & \\ & & & \ddots & \\ 0 & & & r_1/4 & r_0/2 \end{bmatrix}$$

$$\updownarrow$$

$$r_0 + r_1\cos\theta$$

$$\tilde{r}_0 + \tilde{r}_1\cos\theta + \tilde{r}_2\cos(2\theta)$$

$$\frac{r_0 + r_1 x + r_2 x^2}{\sqrt{1-x^2}} \longleftrightarrow \Phi_{m,n} = \begin{bmatrix} r_0 & r_1/2 & \tilde{r}_2/4 & & 0 \\ r_1/4 & r_0/2 & r_1/4 & \tilde{r}_2/4 & \\ \tilde{r}_2/4 & r_1/4 & r_0/2 & r_1/4 & \tilde{r}_2/4 \\ & & & \ddots & \\ 0 & \tilde{r}_2/4 & & r_1/4 & r_0/2 \end{bmatrix}$$