Three-term Recurrence Relations

Theorem: Every sequence of orthogonal polynomials satisfies a three-term recurrence relation:

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$$

Here a_n, b_n, c_n are some sequences of constants

Monic case: $x p_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$

Normalized: $x p_n(x) = a_n p_{n+1} + b_n p_n(x) + a_{n-1} p_{n-1}(x)$

Ex: Chebyshev polynomials $T_0(x), T_1(x), T_2(x), \ldots$

$$T_n(\cos\theta) = \cos(n\theta)$$

$$\cos(\theta)\cos(n\theta) = \tfrac{1}{2}\cos((n+1)\theta) + \tfrac{1}{2}\cos((n-1)\theta)$$
$$\cos(\theta)\, T_n(\cos\theta) = \tfrac{1}{2} T_{n+1}(\cos\theta) + \tfrac{1}{2} T_{n-1}(\cos\theta)$$

$$\boxed{x\, T_n(x) = \tfrac{1}{2} T_{n+1}(x) + 0\, T_n(x) + \tfrac{1}{2} T_{n-1}(x)}$$

$a_n = \frac{1}{2}$ $\qquad b_n = 0$ $\qquad c_n = \frac{1}{2}$

Ex: Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}$$

$$\begin{aligned} H_{n+1}(x) &= (-1)^{n+1} e^{x^2} \left(e^{-x^2}\right)^{(n+1)} \\ &= (-1)^{n+1} e^{x^2} \left(-2x e^{-x^2}\right)^{(n)} \\ &= (-1)^{n+1} e^{x^2} \left[-2x\left(e^{-x^2}\right)^{(n)} - 2n\left(e^{-x^2}\right)^{(n-1)}\right] \\ &= (2x)(-1)^n e^{x^2}\left(e^{-x^2}\right)^{(n)} - 2n(-1)^{n-1} e^{x^2}\left(e^{-x^2}\right)^{(n-1)} \\ &= 2x\, H_n(x) - 2n H_{n-1}(x) \end{aligned}$$

$$\boxed{x H_n(x) = \tfrac{1}{2} H_{n+1}(x) + 0 H_n(x) + 2n H_{n-1}(x)}$$

$a_n = \frac{1}{2}$ $\quad$ $b_n = 0$ $\quad$ $c_n = 2n$

Proof of the Theorem:

Consider $\mathcal{P}_n = \{ p(x) \mid \deg p(x) \leq n \}$.

This is a vector space of dimension $(n+1)$.

Now suppose that $p_0(x), p_1(x), p_2(x)$ are orthogonal polynomials for some weight $r(x)$.

Remember $\deg(p_n(x)) = n \quad \forall n$, so
$p_0(x), p_1(x), p_2(x), \cdots, p_n(x) \in \mathcal{P}_n$

They have different degrees, so they are lin. indep
Thus $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is a basis for $\mathcal{P}_n$.

Now suppose $q(x) \in \mathcal{P}_n$. What is $\langle P_{n+1}(x), q(x) \rangle_r$?

$$q(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \dots + \alpha_n P_n(x)$$

$$\langle P_{n+1}(x), q(x) \rangle_r = \alpha_0 \overset{0}{\cancel{\langle P_{n+1}(x), P_0(x) \rangle_r}} + \dots + \alpha_n \overset{0}{\cancel{\langle P_{n+1}(x), P_n(x) \rangle_r}}$$

Thus $\langle P_{n+1}(x), q(x) \rangle_r = 0 \quad \forall \; q(x) \in \mathcal{P}_n$.

$$x P_n(x) = \beta_{n,0} P_0(x) + \beta_{n,1} P_1(x) + \dots + \beta_{n,n+1} P_{n+1}(x)$$

$$\langle P_{n+1}(x), x P_n(x) \rangle_r = \beta_{n,n+1} \langle P_{n+1}(x), P_{n+1}(x) \rangle_r$$

$$\langle P_n(x), x P_n(x) \rangle_r = \beta_{n,n} \langle P_n(x), P_n(x) \rangle_r$$

$$\langle P_k(x), x P_n(x) \rangle_r = \beta_{n,k} \langle P_k(x), P_k(x) \rangle_r \quad \forall \; 0 \le k \le n-1$$

$$\langle P_k(x), x P_n(x) \rangle_r = \int_a^b P_k(x)\, x\, P_n(x)\, r(x)\, dx = \int_a^b x P_k(x)\, P_n(x)\, r(x)\, dx$$

$$= \langle \underbrace{x P_k(x)}_{\deg k+1}, \underbrace{P_n(x)}_{\deg n} \rangle$$

Thus $\langle P_k(x), x P_n(x) \rangle_r = 0$ if $k+1 < n$

must be zero!

$$\beta_{n,k} \underbrace{\langle P_k(x), P_k(x) \rangle}_{\text{non-zero}} = 0 \quad \text{if} \quad k+1 < n$$

$$x P_n(x) = \cancel{\beta_{n,0}} P_0(x) + \cancel{\beta_{n,1}} P_1(x) + \dots + \beta_{n,n+1} P_{n+1}(x)$$

$$x P_n(x) = \beta_{n,n-1} P_{n-1}(x) + \beta_{n,n} P_n(x) + \beta_{n,n+1} P_{n+1}(x)$$

$$a_n = \beta_{n,n+1} \qquad b_n = \beta_{n,n} \qquad c_n = \beta_{n,n-1}$$

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$$

$\square$

<u>Favard's Theorem</u>: A sequence of polynomials $P_0(x), P_1(x), P_2(x), \ldots$ satisfying a three-term recursion relation

$$xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$$

then there exists a function $r(x)$ on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) r(x)\, dx = \begin{cases} 0, & m \neq n. \\ h_n > 0, & m = n. \end{cases}$$

Interesting question:

$$r(x) \rightleftarrows (a_n, b_n, c_n)$$

<u>Jacobi Matrices</u>

$$J = \begin{bmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ c_1 & b_1 & a_1 & 0 & \cdots \\ 0 & c_2 & b_2 & a_2 & \cdots \\ 0 & 0 & c_3 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \leftrightarrow (a_n, b_n, c_n)$$

The orthogonal polynomials are connected to J via eigenvalues!

$$\begin{bmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ c_1 & b_1 & a_1 & 0 & \cdots \\ 0 & c_2 & b_2 & a_2 & \cdots \\ 0 & 0 & c_3 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ \vdots \end{bmatrix}$$

$$J_n = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 \\ c_1 & b_1 & a_1 & \cdots & 0 & 0 \\ 0 & c_2 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-2} & a_{n-2} \\ 0 & 0 & 0 & \cdots & c_{n-1} & b_{n-1} \end{bmatrix}$$

eigenvalues

$= \text{roots of } P_n(x)$