$$\tilde{P}_n(x) = \boxed{\det \begin{bmatrix} P_n(\lambda_1) & P_{n+1}(\lambda_1) & \cdots & P_{n+d}(\lambda_1) \\ P_n(\lambda_1) & P_{n+1}(\lambda_1) & \cdots & \\ \vdots & & & \vdots \\ P_n(\lambda_d) & P_{n+1}(\lambda_d) & \cdots & P_{n+d}(\lambda_d) \\ P_n(x) & P_{n+1}(x) & \cdots & P_{n+d}(x) \end{bmatrix}} \qquad \lambda_i \neq \lambda_j \quad i \neq j$$

<u>Vandermonde Determinant</u> :

$$\begin{bmatrix} 1 & \lambda_0 & \lambda_0^2 & \lambda_0^3 & \cdots & \lambda_0^n \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \cdots & \lambda_1^n \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \cdots & \lambda_2^n \\ 1 & \vdots & & & & \\ 1 & \lambda_n & \lambda_n^2 & \lambda_n^3 & \cdots & \lambda_n^n \end{bmatrix} = \prod_{0 \le i < j \le n} (\lambda_i - \lambda_j)$$

$$\begin{bmatrix} P_0(\lambda_0) & P_1(\lambda_0) & \cdots & P_n(\lambda_0) \\ P_0(\lambda_1) & P_1(\lambda_1) & \cdots & P_n(\lambda_1) \\ \vdots & \vdots & \ddots & \vdots \\ P_0(\lambda_n) & P_1(\lambda_n) & \cdots & P_n(\lambda_n) \end{bmatrix} = \boxed{\prod_{0 \le i < j \le n} (\lambda_i - \lambda_j)}$$

Slater Determinants of Orthogonal Polynomials

$$S_n^{m_1,\dots,m_r}(t_1,\dots,t_r) = \det \begin{bmatrix} P_n(t_1) & P_{n+1}(t_1) & \dots & P_{n+m+1}(t_1) \\ P_n'(t_1) & P_{n+1}'(t_1) & \dots & P_{n+m+1}'(t_1) \\ \vdots & & & \\ P_n^{(m-1)}(t_1) & P_{n+1}^{(m-1)}(t_1) & \dots & P_{n+m+1}^{(m-1)}(t_1) \\ P_n(t_2) & P_{n+1}(t_2) & \dots & \\ & \vdots & & \end{bmatrix}$$

$$S^{1,1,1\dots,1}(t_1,\dots,t_r)$$

Theorem 2.4 $\qquad d\mu(s_j) = \dfrac{1}{\sqrt{1-s_j^2}}\, ds_i$

$$T_n = C_n^0$$

$$\frac{\det \begin{bmatrix} C_n^\lambda(x) & C_{n+1}^\lambda(x) & & C_{n+m-1}^\lambda(x) \\ C_n^{\lambda\prime}(x) & C_{n+1}^{\lambda\prime}(x) & & C_{n+m-1}^{\lambda\prime}(x) \\ C_n^{\lambda\prime\prime}(x) & C_{n+1}^{\lambda\prime\prime}(x) & \cdot & C_{n+m-1}^{\lambda\prime\prime}(x) \\ \vdots & \vdots & & \\ C_n^{\lambda(m-1)}(t) & C_{n+1}^{\lambda,(m-1)}(t) & & C_{n+m-1}^{\lambda(m-1)}(t) \end{bmatrix}}{(t-x)^m}$$

$$\frac{1}{\sqrt{1-t^2}}(t-x)^m$$

$$H = \begin{bmatrix} h_0 & h_1 & h_2 & \\ h_1 & h_2 & & \\ h_2 & & \ddots & \\ & & & \end{bmatrix} \qquad H_{i,j} = h_{i+j-2}$$

$$M = \begin{bmatrix} m_0 & m_1 & m_2 & m_3 \\ m_1 & m_2 & m_3 & \\ m_2 & m_3 & & \\ m_n & & & \end{bmatrix} \quad \cdots \leftarrow$$

Moment matrix of $r(x)$

$$p(x) = \sum_{k=0}^{n} a_k x^k \qquad q(x) = \sum_{k=0}^{n} b_k x^k.$$

$$\int p(x) q(x) r(x) dx = [a_0 \; a_1 \ldots a_n] \, M \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Moment matrix $\longleftrightarrow$ seq of orth polys

$$r(x) = \frac{1}{q(x)\sqrt{1-x^2}}, \quad q(x) \text{ polynomial}$$

$q(x) \neq 0$ m

Sequence of polys for $r(x)$?

Szegö: for $n > 2\deg(q)+1$,

$$p_n(x) = \sum_{k=0}^{\deg(q)} \alpha_k T_{n-k}(x)$$

CONSTANT !! indep of n.

three-term recursion relation:

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n + c_n p_{n-1}(x)$$

for $n > 2\deg(q)+1$, $\quad a_n = c_n = \frac{1}{2}, \quad b_n = 0.$

$$\oint_C f(z)\,dz = 0 \quad \text{if } \underline{f(z)} \text{ is holomorphic inside } C$$

- $q(x) = h(e^{i\theta})h(e^{-i\theta}) \qquad x = \cos\theta$

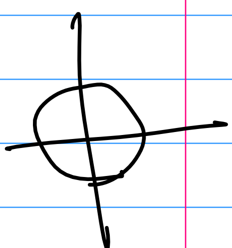
$$p_n(x) = \operatorname{Re}\left\{ e^{in\theta} h(e^{-i\theta}) \right\}, \quad x = \cos\theta$$

$$h(z) = \sum_{k=0}^{d} \alpha_k z^k \rightsquigarrow e^{in\theta}h(e^{-i\theta}) = \sum_{k=0}^{d} \alpha_k e^{i(n-k)\theta}$$

$$P_n(x) = \sum_{k=0}^{d} \alpha_k \cos((n-k)\theta) = \sum_{k=0}^{d} \alpha_k T_{n-k}(x)$$

$$\int_{-1}^{1} \mathrm{Re}\{ e^{in\theta} h(e^{-i\theta}) \} \frac{x^m}{q(x)\sqrt{1-x^2}} \, dx$$

$$= \frac{1}{2} \mathrm{Re}\left\{ \int_{-\pi}^{\pi} e^{in\theta} \cancel{h(e^{-i\theta})} \frac{(e^{i\theta}+e^{-i\theta})^m}{h(e^{i\theta})\cancel{h(e^{-i\theta})}} \, d\theta \right\}$$



$$\frac{1}{2} \mathrm{Re} \oint_{|z|=1} z^n \frac{(z+\frac{1}{z})^m}{h(z)} \, d\theta \qquad n \geq m$$

no roots in $|z| \leq 1$

$$= 0$$

$$\int_{-1}^{1} \underline{P_n(x)} \, x^m \frac{1}{q(x)\sqrt{1-x^2}} \, dx = 0 \qquad m < n$$

∴ P_n = seq of orth polys!

true for n large enough!!