Classical Orthogonal Polynomials

Three special families ~

- Hermite polynomials
- Laguerre polynomials
- Jacobi polynomials
  - Gegenbauer polynomials
    - Legendre polynomials
    - Chebyshev polynomials

Hermite Polynomials

- orthogonal wrt. Gaussian-type weight
- arise naturally for energies of a quantum harmonic oscillator

$$h_0(x) = 1$$
$$h_1(x) = 2x$$
$$h_2(x) = 4x^2 - 2$$
$$h_3(x) = 8x^3 - 12x$$
$$\vdots$$

$$h_{n+1}(x) = 2x h_n(x) - 2n h_{n-1}(x) \longleftarrow \text{ recurrence relation}$$

Orthogonality : $$\int_{-\infty}^{\infty} h_m(x) h_n(x) e^{-x^2}\, dx = \begin{cases} 0, & m \neq n. \\ \sqrt{\pi}\, 2^n n!, & m = n \end{cases}$$

Quantum Harmonic Oscillator:

Planck constant

wave function

energy

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

mass

angular frequency

Solutions:

Hermite poly!

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \underbrace{h_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)}, \quad n \geq 0.$$

<u>Laguerre Polynomials</u>

- orthogonal wrt. exponential function
- arise from energy levels of hydrogen atom

$$L_0(x) = 1$$

$$L_1(x) = -x + 1$$

$$L_2(x) = \frac{1}{2}x^2 - 2x + 1$$

$$L_3(x) = -\frac{1}{6}x^3 + \frac{3}{2}x^2 - 3x + 1$$

$$\vdots$$

$$L_{n+1}(x) = \frac{(2n+1-x)L_n(x) - nL_{n-1}(x)}{n+1} \quad \leftarrow$$

recurrence relation

Orthogonality: $$\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

<u>Generalized Laguerre Polynomials</u>:

α = constant parameter

$$\begin{cases} L_0^{(\alpha)}(x) = 1 \\ L_1^{(\alpha)}(x) = -x + \alpha + 1 \\ \vdots \\ L_{k+1}^{(\alpha)}(x) = \dfrac{(2k+1+\alpha-x)L_k^{(\alpha)}(x) - (k+\alpha)L_{k-1}^{(\alpha)}(x)}{k+1} \end{cases}$$

Orthogonality: $$\int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \begin{cases} 0, & m \neq n \\ \Gamma(n+\alpha+1)/n!, & m = n \end{cases}$$

Hydrogen Atom:

Planck constant

electron charge

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]-\frac{e^2}{4\pi\varepsilon_0 r}\psi=E\psi$$

reduced mass

vacuum permittivity

r, θ, ϕ spherical coordinates

Solutions:

Spherical harmonics

$$\psi_{n,l,m}(r,\theta,\phi)=\sqrt{\left(\frac{\mu e^2}{2\pi\varepsilon_0\hbar^2}\right)^3\frac{(n-l-1)!}{2n(n+l)!}}\;e^{-\rho/2}\rho^{l}\,\underbrace{L_{n-l-1}^{(2l+1)}(\rho)}_{\text{Laguerre poly!}}\,Y_l^m(\theta,\psi), \quad \rho=\frac{\mu e^2 r}{2\pi\varepsilon_0\hbar^2}$$

Jacobi Polynomials

- orthogonal with respect to $(1-x)^a(1+x)^b$
- related to hypergeometric function ${}_2F_1$
- arise from group representation theory

$$P_0^{(a,b)}(x) = 1$$

$$P_1^{(a,b)}(x) = (a+1) + \frac{a+b+2}{2}(x-1)$$

$$\vdots$$

$$P_n^{(a,b)}(x) = \Big[(2n+a+b+1)\left((2n+a+b)(2n+a+b-2)x+a^2-b^2\right)P_{n-1}^{(a,b)}(x)$$

$$-2(n+a+1)(n+b+1)(2n+a+b)P_{n-2}^{(a,b)}(x)\Big]\Big/\big[2n(n+a+b)(2n+a+b-2)\big]$$

Orthogonality: $$\int_{-1}^{1}(1-x)^a(1+x)^b P_m^{(a,b)}(x)P_n^{(a,b)}(x)\,dx = \begin{cases} 0, & m\neq n \\ *, & m=n. \end{cases}$$

Special cases: $P_m^{(0,0)}(x)$ = Legendre polynomial!

Take $a = b = -\frac{1}{2}$. $T_n(x) = P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x)\, dx$$

The polynomials $T_0(x), T_1(x), \ldots$ are Chebyshev polynomials

$$\cos(x)^2 = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\cos(x)^3 = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$$

General Formula: $\cos(nx) = T_n(\cos(x))$

Quick Recap:

- three special families: Hermite, Laguerre, Jacobi
- polynomials have recurrence relations
- orthogonal polynomials arise naturally