Gauss Quadrature Formula

- one of the main tools for using orthogonal polynomials in numerical methods

Gives a nice way to evaluate integrals of polynomials times our weight function

$r(x)$ weight function

$P_0(x), P_1(x), P_2(x), \dots$ sequence of orthogonal polys

$\{x_{n1}, \dots, x_{nn}\}$ = roots of $P_n(x)$ in increasing order

Theorem: There exist constants $w_1, \dots, w_n$ with

$$\int_b^c q(x) r(x) dx = \sum_{k=1}^{n} w_k q(x_{nk})$$

for all polynomials $q(x)$ of deg $< 2n$.

Proof:

$$\tilde{q}(x) = \sum_{k=1}^{n} q(x_{nk}) \frac{P_n(x)}{(x - x_{nk}) P_n'(x_{nk})}$$

$$\tilde{q}(x_{nj}) = \sum_{k=1}^{n} q(x_{nk}) \frac{P_n(x_{nj})}{(x_{nj} - x_{nk}) P_n'(x_{nk})}$$

$$= q(x_{nj})$$

$$\underbrace{q(x) - \tilde{q}(x)}_{\deg < 2n} = P_n(x) \underbrace{f(x)}_{\deg < n}$$

$\Longrightarrow$

$$\int_b^c [q(x) - \tilde{q}(x)]\, r(x)\,dx = \int_b^c P_n(x) f(x)\, r(x)\,dx = 0$$

$$\therefore \quad \int_b^c q(x)\, r(x)\,dx = \int_b^c \tilde{q}(x)\, r(x)\,dx$$

$$= \sum_{k=1}^{n} \left(\frac{1}{P_n'(x_{nk})} \int_b^c \frac{P_n(x) r(x)}{(x - x_{nk})} dx \right) q(x_{nk})$$

$$= \sum_{k=1}^{n} w_k\, q(x_{nk})$$

$\square$

<u>Def</u>: The constants

$$w_k = \frac{1}{P_n'(x_{nk})} \int_b^c \frac{P_n(x) r(x)}{x - x_{nk}}\, dx$$

are called the <u>quadrature weights of degree n.</u>

What about for $f(x)$ not nec. a polynomial?

$$f(x) \approx q(x) = \sum c_k P_k(x)$$

$$\int_b^c f(x)\, r(x)\,dx \approx \int_b^c q(x)\, r(x)\,dx = \sum_{k=1}^{n} w_k\, q(x_{nk})$$

$$\approx \sum_{k=1}^{n} w_k\, f(x_{nk})$$

$$\boxed{\int_b^c f(x)\, r(x)\,dx \approx \sum_{k=1}^{n} w_k\, f(x_{nk})}$$

$$f(x) \approx \sum_{k=0}^{n-1} \alpha_k P_k(x) \quad , \qquad \alpha_k = \frac{\int_b^c f(x) P_k(x) r(x) dx}{\int_b^c P_k(x)^2 r(x)\, dx}$$

$$\boxed{\alpha_j \approx \frac{\sum_{k=1}^{n} w_k f(x_{nk}) p_j(x_{nk})}{\sum_{k=1}^{n} w_k\, p_j(x_{nk})^2}}$$

Special Case: Chebyshev polynomials

$$w_k = \frac{1}{T_n'(x_{nk})} \int_{-1}^{1} \frac{T_n(x)}{x - x_{nk}} \frac{dx}{\sqrt{1-x^2}}$$

$$T_n(x) = \cos(n\theta) \quad , \qquad \cos\theta = x$$

Roots: $x_{nk} = \cos\left(\frac{2k-1}{2n}\pi\right)$

$$\frac{dx}{d\theta} T_n'(x) = \frac{d}{d\theta} T_n(x) = \frac{d}{d\theta}\cos(n\theta) = -n \sin n\theta$$

$$T_n'(x) = \frac{n \sin(n\theta)}{\sin\theta} \Rightarrow T_n'(x_{nk}) = \frac{(-1)^{k+1} n}{\sin(x_{nk})}$$

$$\frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{n} T_k(x) T_k(y) = \frac{1}{\pi} \frac{T_n(y) T_{n+1}(x) - T_{n+1}(y) T_n(x)}{x - y}$$

$y = x_{nk} \rightsquigarrow$

$$\frac{T_n(x)}{x - x_{nk}} = -\frac{\pi}{T_{n+1}(x_{nk})}\left(\frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{n} T_k(x) T_k(x_{nk})\right)$$

$$\int_{-1}^{1} \frac{T_n(x)}{x - x_{nk}} \frac{1}{\sqrt{1-x^2}} dx = - \frac{\pi}{T_{n+1}(x_{nk})}$$

$$w_k = \left(\frac{(-1)^k n}{\sin(x_{nk})} \right)^{-1} \frac{\pi}{T_{n+1}(x_{nk})} , \quad T_{n+1}(x_{nk}) = (-1)^k \sin(x_{nk})$$

$$\boxed{w_k = \frac{\pi}{n}}$$

$$\int_{-1}^{1} f(x) \frac{1}{\sqrt{1-x^2}} dx \approx \sum_{k=1}^{n} f\left(\cos\left(\frac{2k-1}{2n}\right)\right) \frac{\pi}{n}$$

Chebyshev expansion of $f(x)$ of deg n

$\approx$ <u>discrete cosine transform.</u>

$$\{f_k\}_{k=1}^{n} \qquad \hat{f}_k = \sum_{j=1}^{n} f_k \cos\left(\frac{2j-1}{2n}\right) \frac{\pi}{n}$$