$r(x)$ weight function

n^{th} moment $\quad m_n = \int_b^c r(x)\, x^n \, dx$

<u>Def</u>: The sequence of values $m_0, m_1, m_2, \ldots$ is called the <u>moment sequence</u> of $r(x)$

$$p(x) = \sum_{k=0}^{n} a_k x^k \quad , \quad q(x) = \sum_{j=0}^{m} b_j x^j$$

$$\langle p, q \rangle_r = \int_b^c p(x) q(x) r(x)\, dx = \sum_{k=0}^{n} \sum_{j=0}^{m} a_k b_j \int_b^c x^{j+k} r(x)\, dx$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{m} a_k b_j \, m_{j+k}$$

<u>Def</u>: Let V be a real vector space. A <u>linear functional</u> $\mathcal{L}$ on V is a linear map $\mathcal{L} : V \to \mathbb{R}$.

<u>Prototypical example</u>: $V = \mathbb{R}[x]$

$$\mathcal{L} : p(x) \mapsto \int_{14}^{27} p(x)\, dx$$

<u>Def</u>: The <u>moment functional</u> of a sequence $m_0, m_1, \ldots$ is

$$\mathcal{L} : \mathbb{R}[x] \longrightarrow \mathbb{R}$$
$$\sum_{k=0}^{n} a_k x^k \longmapsto \sum_{k=0}^{n} a_k m_k$$

<u>Remark</u>: If $m_0, m_1, \ldots$ is the sequence of moments for $r(x)$ then

$$\mathcal{L}(p(x)) = \int_b^c p(x) r(x)\, dx$$

Moment Problem: Give necessary + sufficient conditions for a sequence $\{m_k\}_{k=0}^{\infty}$ to define a unique function $r(x)$ w/ $\int x^k r(x) dx = m_k \;\; \forall k \geq 0$

Connection to Laplace Transform:

$$\mathcal{L}\{f\}(s) = \int_s^\infty e^{-sx} f(x)\, dx$$

$$r(x) = \begin{cases} 0, & x \notin [b,c] \\ *, & \text{otherwise} \end{cases}$$

$$\mathcal{L}\{r(x)\} = \int_s^\infty e^{-sx} r(x)\, dx = \int_s^\infty \sum_{n=0}^\infty \frac{(-1)^n s^n x^n}{n!} r(x)\, dx$$

$$= \sum_{n=0}^\infty \frac{(-1)^n s^n}{n!} \int_s^\infty x^n r(x)\, dx$$

$$= \sum_{n=0}^\infty \frac{(-1)^n s^n}{n!} m_n \quad , \text{ assuming } s < b$$

Assuming $r(x)$ nice enough, having $\{m_n\}_{n=0}^\infty \Rightarrow$ having $\mathcal{L}\{r\}$

Then to get $r(x)$, use inverse Laplace transform.

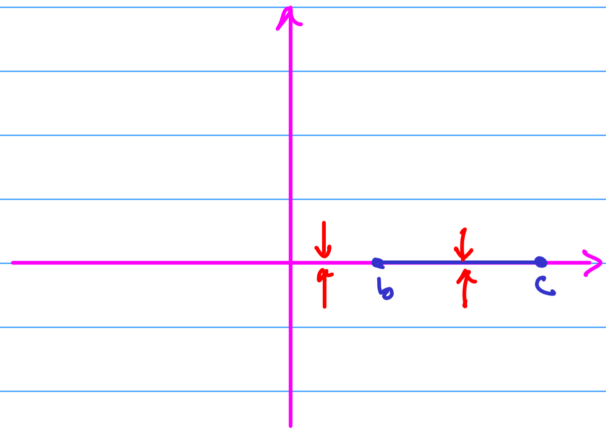
Instead, more robust to use Stieltjes transform

$$S\{f\}(z) = \int_b^c \frac{f(x)}{z-x} dx \quad , \text{ defined for } z \in \mathbb{C} \setminus [b,c].$$

$$S\{r\}(z) = \frac{1}{z}\int_b^c \frac{r(x)}{1-(x/z)} dx = \frac{1}{z}\sum_{k=0}^\infty \int_b^c \frac{x^n}{z^n} r(x)\, dx$$

$$= \sum_{n=0}^\infty \frac{m_n}{z^{n+1}}$$

$$r(x) = \lim_{\varepsilon \to 0^+} \frac{S\{r\}(x - i\varepsilon) - S\{r\}(x + i\varepsilon)}{2\pi i}$$



$$T_n(x) = \cos(n\theta) \quad (\text{for} \quad x = \cos\theta)$$

$$\underbrace{T_n(\cos\theta) = \cos(n\theta)} \longleftarrow$$

$$r(x) = \frac{1}{\sqrt{1-x^2}} = (1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}$$

Nice form! $\quad r(x) = (1-x)^{a-\frac{1}{2}}(1+x)^{a-\frac{1}{2}} \qquad a > 0$

$$= \frac{(1-x^2)^a}{\sqrt{1-x^2}} \leftarrow$$ Gegenbauer poly's representation theory!

$$r(x) = \frac{1}{\sqrt{1-x^2}\, q(x)} \rightsquigarrow$$

$$P_n(x) = \sum_{k=0}^{d} \alpha_k T_{n-k}(x) \Rightarrow$$ 3-term recursion relation

$$x P_n(x) = \underline{a_n} P_{n+1}(x) + \underline{b_n} P_n(x) + \underline{c_n} P_{n-1}(x)$$

messy for low n, but for $n \gg 0$

$a_n = \frac{1}{2}$, $b_n = 0$, $c_n = \frac{1}{2}$

<u>Ex</u> : $r(x) = \dfrac{x-\lambda}{\sqrt{1-x^2}}$, $|\lambda| > 1$.

<u>Theorem</u>: $P_n(x) = \dfrac{T_{n+1}(x) + \beta_{n,0} T_n(x)}{x-\lambda}$

$T_{n+1}(\lambda) + \beta_{n,0} T_n(\lambda) = 0 \Rightarrow \beta_{n,0} = -\dfrac{T_{n+1}(\lambda)}{T_n(\lambda)}$

$$P_n(x) = \left(\frac{T_{n+1}(x)}{T_n(x)} - \frac{T_{n+1}(\lambda)}{T_n(\lambda)} \right) \frac{T_n(x)}{x-\lambda} \quad \star$$

$$= \frac{T_{n+1}(x) - \frac{T_{n+1}(\lambda)}{T_n(\lambda)} T_n(x)}{x-\lambda} \quad \star$$

$(x-\lambda) P_{n-1}(x) = T_n(x) + \beta_{n-1,0} T_{n-1}(x)$

$(x-\lambda) P_n(x) = T_{n+1}(x) + \beta_{n,0} T_n(x)$

$(x-\lambda) P_{n+1}(x) = T_{n+2}(x) + \beta_{n+1,0} T_{n+1}(x)$

$x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$

$x \underline{(x-\lambda) P_n(x)} = a_n \underline{(x-\lambda) P_{n+1}(x)} + b_n \underline{(x-\lambda) P_n(x)} + c_n \underline{(x-\lambda) P_{n-1}(x)}$

$$\underbrace{x T_{n+1}(x)}_{\frac{1}{2}T_{n+2}(x) + \frac{1}{2}T_n(x)} + \beta_n \underbrace{x T_n(x)}_{\frac{1}{2}T_{n+1} + \frac{1}{2}T_{n-1}} = a_n T_{n+2}(x) + a_n \beta_{n+1} T_{n+1}$$
$$+ b_n T_{n+1}(x) + b_n \beta_n T_n$$
$$+ c_n T_n(x) + c_n \beta_{n-1} T_{n-1}$$

- $\frac{1}{2} = a_n$

- $\frac{1}{2} = a_n \beta_{n+1} + b_n \Rightarrow b_n = \frac{1}{2}(1 - \beta_{n+1})$
- $\frac{1}{2} = c_n \beta_{n-1} \Rightarrow c_n = \dfrac{1}{2\beta_{n-1}}$

$$\boxed{x P_n(x) = \frac{1}{2} P_{n+1}(x) + \frac{1}{2}\left(1 + \frac{T_{n+2}(\lambda)}{T_{n+1}(\lambda)}\right) P_n(x) - \frac{1}{2} \frac{T_{n-1}(\lambda)}{T_n(\lambda)} P_{n-1}(x)}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cos i\theta = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh(\theta)$$

$$T_n(\cosh(\theta)) = T_n(\cos(i\theta)) = \cos(in\theta) = \cosh(n\theta)$$

$$x P_n(x) = \frac{1}{2} P_{n+1}(x) + \frac{1}{2}\left(1 + \frac{\cosh((n+2)\theta)}{\cosh((n+1)\theta)}\right) P_n(x) - \frac{1}{2} \frac{\cosh((n-1)\theta)}{\cosh(n\theta)} P_{n-1}(x)$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\cosh((n+2)\theta) = \cosh((n+1)\theta)\cosh(\theta) + \sinh((n+1)\theta)\sinh(\theta)$$

Special case $\theta = 0$: $\quad (x-1)/\sqrt{1-x^2}$

$$x P_n(x) = \frac{1}{2} P_{n+1}(x) + P_n(x) - \frac{1}{2} P_{n-1}(x)$$

$$\theta = \log t \qquad \cosh\theta = \frac{t^n + t^{-n}}{2}$$