Christoffel-Darboux Kernel and Root Interlacing

Definition: Let $P_0(x), P_1(x), \cdots$ be a sequence of orthogonal polynomials. The Christoffel-Darboux kernel of degree n is:

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle P_k, P_k \rangle_r}$$

Question: what's it do?

Recall: the degree n polynomial approx wrt. $r(x)$,
ie a polynomial of degree n as close as possible to $f(x)$, in terms of the norm defined by $r(x)$

$$p(x) = \sum_{k=0}^{n} \alpha_k P_k(x), \qquad \alpha_k = \frac{\langle P_k(x), f(x) \rangle_r}{\langle P_k(x), P_k(x) \rangle_r}$$

$$\int_b^c K_n(x,y) f(y) r(y)\, dy = \sum_{k=0}^{n} \frac{P_k(x)}{\langle P_k, P_k \rangle_r} \int_b^c P_k(y) f(y) r(y)\, dy$$

$$= \sum_{k=0}^{n} \frac{P_k(x)}{\langle P_k, P_k \rangle_r} \langle P_k, f \rangle_r = p(x)$$

Definition:
A function of functions defined by

$$f(x) \mapsto \int K(x,y) f(y)\, dy$$

for some function $K(x,y)$ is called an <u>integral transform</u>.
The function $K(x,y)$ is called the kernel.

<u>Ex</u>: $K(x,y) = e^{-ixy}$ gives Fourier transform
$K(x,y) = e^{-xy}$, $y > 0$ gives Laplace transform

Theorem (Christoffel-Darboux Formula):

$$K_n(x,y) = \frac{l_n}{\langle p_n,p_n\rangle l_{n+1}} \cdot \frac{p_n(y)p_{n+1}(x) - p_{n+1}(y)p_n(x)}{x-y}, \quad x \neq y$$

$$K_n(x,x) = \frac{l_n}{\langle p_n,p_n\rangle l_{n+1}} \cdot \left(p_{n+1}'(x)p_n(x) - p_{n+1}(x)p_n'(x)\right)$$

where l_j = leading coeff. of $p_j(x)$ $\forall j \geq 0$

Proof:

$$x K_n(x,y) = \sum_{k=0}^{n} \frac{x p_k(x) p_k(y)}{\langle p_k, p_k\rangle}$$

$x p_k(x) = a_k p_{k+1}(x) + b_k p_k(x) + c_k p_{k-1}(x)$

$$= \sum_{k=0}^{n} a_k \frac{p_{k+1}(x) p_k(y)}{\langle p_k,p_k\rangle} + \sum_{k=0}^{n} b_k \frac{p_k(x) p_k(y)}{\langle p_k,p_k\rangle} + \sum_{k=0}^{n} c_k \frac{p_{k-1}(x) p_k(y)}{\langle p_k,p_k\rangle}$$

$$y K_n(x,y) = \sum_{k=0}^{n} a_k \frac{p_k(x) p_{k+1}(y)}{\langle p_k,p_k\rangle} + \sum_{k=0}^{n} b_k \frac{p_k(x) p_k(y)}{\langle p_k,p_k\rangle} + \sum_{k=0}^{n} c_k \frac{p_k(x) p_{k-1}(y)}{\langle p_k,p_k\rangle}$$

$$(x-y)K_n(x,y) = \sum_{k=0}^{n-1}\left[\cancel{\frac{a_k}{\langle p_k,p_k\rangle} - \frac{c_{k+1}}{\langle p_{k+1},p_{k+1}\rangle}}\right] p_{k+1}(x)p_k(y) + \frac{a_n}{\langle p_n,p_n\rangle} p_{n+1}(x)p_n(y)$$

$a_k = \frac{\langle p_k,p_k\rangle}{\langle p_{k+1},p_{k+1}\rangle} c_{k+1}$

$\langle p_k, x p_{k-1}\rangle = \langle x p_k, p_{k-1}\rangle$

$$- \sum_{k=0}^{n-1}\left[\cancel{\frac{a_k}{\langle p_k,p_k\rangle} - \frac{c_{k+1}}{\langle p_{k+1},p_{k+1}\rangle}}\right] p_k(x)p_{k+1}(y) - \frac{a_n}{\langle p_n,p_n\rangle}$$

$$(x-y)K_n(x,y) = \frac{a_n}{\langle p_n,p_n\rangle}\left[p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\right]$$

Since $a_n = \frac{l_n}{l_{n+1}}$, this proves our theorem* $\square$

* Try proving the case when $x=y$ yourself!

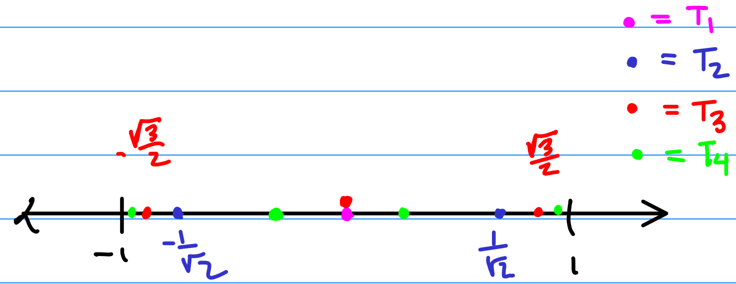
Roots of Orthogonal Polynomials

$T_n(\cos(\theta)) = \cos(n\theta)$, $T_n(x)$ Chebyshev of deg. n.

Roots of $\cos(n\theta)$: $n\theta = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$

Roots of $T_n(x)$: $x = \cos(\theta)$, $\theta = \frac{(2k+1)\pi}{2n}$, $k \in \mathbb{Z}$

$$= \left\{ \cos\left(\frac{(2k+1)\pi}{2n}\right) : 0 \le k < n \right\}$$

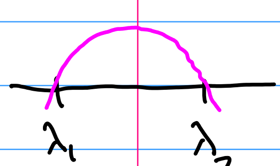


Remark: Between any two roots of $T_{n+1}(x)$, we can find a root of $T_n(x)$. This behavior is called root interlacing.

Theorem: Between any two adjacent roots of $P_{n+1}(x)$ there exists a root of $P_n(x)$.

Proof: S'pose that λ_1, λ_2 are two adjacent real, simple roots of $P_{n+1}(x)$.

$$0 < \sum_{k=0}^{n} \frac{P_k(x)^2}{\langle P_k, P_k \rangle} = \frac{a_n}{\langle P_n, P_n \rangle}\left(P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x)\right)$$



$$\frac{-a_n}{\langle P_n, P_n \rangle} P_{n+1}'(\lambda_k) P_n(\lambda_k) > 0$$

flip sign

flip sign

$\therefore$ Intermediate value theorem: P_n has a root in $[\lambda_1, \lambda_2]$

To finish this proof, use next theorem.

<u>Theorem</u>: $P_n(x)$ has exactly n real, simple roots for all $n \geq 0$.

<u>Proof</u>:

Let $\lambda_1, \ldots, \lambda_m$ be the real roots of $P_n(x)$ with odd multiplicity.

WLOG : $P_n(x)(x-\lambda_1)\ldots(x-\lambda_m) \geq 0 \Rightarrow$

$$\int_b^c P_n(x)(x-\lambda_1)\ldots(x-\lambda_m)\, r(x)\, dx > 0$$

$\Rightarrow (x-\lambda_1)\ldots(x-\lambda_m)$ has deg $\geq n$ $\therefore$ $m = n$.

Thus $P_n(x)$ has n distinct real roots

$\square$