Gram-Schmidt

Weight function $r(x) \longrightarrow P_0(x), P_1(x), P_2(x), \ldots$

$$P_n(x) = \sum_{k=0}^{n} \alpha_{nk} x^k$$

$$x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$$

$$\sum_{k=0}^{n} \alpha_{n,k} x^{k+1} = \sum_{k=0}^{n+1} a_n \alpha_{n+1,k} x^k + \sum_{k=0}^{n} b_n \alpha_{n,k} x^k + \sum_{k=0}^{n-1} c_n \alpha_{n-1,k} x^k$$

- $\alpha_{n,n} = a_n \alpha_{n+1,n+1} \implies \boxed{a_n = \dfrac{\alpha_{n,n}}{\alpha_{n+1,n+1}}}$

- $\alpha_{n,n-1} = a_n \alpha_{n+1,n} + b_n \alpha_{n,n} \implies \boxed{b_n = \dfrac{\alpha_{n,n-1} - \dfrac{\alpha_{n,n}}{\alpha_{n+1,n+1}} \alpha_{n+1,n}}{\alpha_{n,n}}}$

- $\alpha_{n,n-2} = a_n \alpha_{n+1,n-1} + b_n \alpha_{n,n-1} + c_n \alpha_{n-1,n-1}$

$$\implies c_n = \left[\alpha_{n,n-2} - \frac{\alpha_{n,n}}{\alpha_{n+1,n+1}} \alpha_{n+1,n-1} - \frac{\alpha_{n,n-1} - \dfrac{\alpha_{n,n}}{\alpha_{n+1,n+1}} \alpha_{n+1,n}}{\alpha_{n,n}} \alpha_{n,n-1} \right] \Big/ \alpha_{n-1,n-1}$$

To go the other way: spectral theory.

A $n \times n$ matrix $A = A^T$. Then $\exists$ orthonormal eigenbasis

i.e., $v_1, \ldots, v_n \in \mathbb{R}^n$ w/ $\langle v_i, v_j \rangle = 0$ $i \neq j$

$A v_i = \lambda_i v_i$ $\forall i$.

If $\vec{u} \in \mathbb{R}^n$, $\vec{u} = \sum_{j=1}^{n} \langle v_j, u \rangle v_j$

$$\boxed{A\vec{u} = \sum_{j=1}^{n} \langle v_j, u\rangle \lambda_j v_j}$$ $\{\lambda_1, \dots, \lambda_n\}$

Infinite situation: spectrum of eigenvalues $[b, c] \subseteq \mathbb{R}$

A infinite linear operator on ∞-dim v.s. is

$$A\vec{u} = \int_b^c P_x(\vec{u})\, x\, \mu(x)\, dx$$

Jacobi matrix:

$$J : \begin{bmatrix} b_0 & a_1 & & 0 \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & a_3 \\ 0 & & & \ddots \end{bmatrix}$$

acts on $\ell^2(\mathbb{N})$

$$= \left\{ \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} : u_j \in \mathbb{R},\ \sum_j u_j^2 < \infty \right\}$$

Spectral Theorem:

$$J\vec{u} = \int_b^c P_x(\vec{u})\, x\, \mu(x)\, dx.$$

spectral measure of J

Can show:

The sequence of polys satisfying

$$J \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{bmatrix}$$

satisfy

$$\int_b^c p_m(x)\, p_n(x)\, \mu(x)\, dx = 0, \quad m \neq n.$$

<u>Solve this</u> : $q(x) =$ polynomial

<u>Problem</u>: Let $p_0(x), p_1(x), \ldots$ be a sequence of orthogonal polynomials for $r(x) = q(x)/\sqrt{1-x^2}$

<u>Theorem</u>: There exists a sequence of constants $\beta_{n,0}, \beta_{n,1}, \ldots$ w/

$$p_n(x) = \left(\sum_{j=0}^{d} \beta_{n,j} T_{j+n}(x) \right) \Big/ q(x)$$

$d \leftarrow$ deg of q

$$p_n(x) q(x) = \sum_{j=-n}^{d} \beta_{n,j} T_{j+n}(x)$$

$$\int_b^c T_k(x) \frac{q(x)}{\sqrt{1-x^2}} p_n(x)\, dx = \beta_{n,k-n} \frac{\pi}{2}$$

<u>Problem 0</u>: Prove this again yourself!

<u>Problem 1</u>: Show that generically the $\beta_{n,j}$'s are determined by the choice of leading coeff. and the condition $q(x) \mid \sum_{j=0}^{d} \beta_{n,j} T_{j+n}(x)$.

$\lambda_1, \ldots, \lambda_d$ $\quad \sum_{j=0}^{d} \beta_{n,j} T_{j+n}(\lambda_k) = 0 \quad \forall\ k = 1, \ldots d.$

<u>Problem 2</u>: Use problem 1 to come up w/ an equation for the coeff's of the recurrence relation

$\beta_j = \gamma_{j+n}$

$$p_n(x) q(x) = \sum_{j=0}^{n+d} \gamma_j T_j(x) = \sum_{j=n}^{n+d} \gamma_j T_j(x) = \sum_{j=0}^{n} \beta_j T_{j+n}(x)$$

$$\boxed{\int_{-1}^{1} p_n(x) q(x) \frac{T_k(x)}{\sqrt{1-x^2}}} = \sum_{j=0}^{n+d} \gamma_j \int_{-1}^{1} T_j(x) T_k(x) \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \boxed{\gamma_k \frac{\pi}{2}}\ .$$

$= 0$ when $k < n$

$\gamma_k = 0$ for $k < n$

Ex: $q(x) = x^2 - 4$, $\lambda_1 = 2$, $\lambda_2 = -2$

$$P_3(x) = \frac{\beta_{3,0} T_3(x) + \beta_{31} T_4(x) + \beta_{32} T_5(x)}{x^2-4} \leftarrow \underline{\text{poly}}\ !!$$

This means
$$\begin{cases} \beta_{30} \underbrace{\cancel{T_3(2)}}_{26} + \beta_{31} \underbrace{\cancel{T_4(2)}}_{97} + \beta_{32} \underbrace{\cancel{T_5(2)}}_{362} = 0 \\ \beta_{30} \underbrace{\cancel{T_3(-2)}}_{-26} + \beta_{31} \underbrace{\cancel{T_4(-2)}}_{97} + \beta_{32} \underbrace{\cancel{T_5(-2)}}_{-362} = 0 \end{cases}$$

$\beta_{32} = 1$

$\beta_{31} = 0$

$\beta_{30} = -\frac{362}{26}$

$\beta_{32} = 1 \leftarrow$ choice of leading coeff

$$P_3(x) = \frac{-\frac{181}{13} T_3(x) + T_5(x)}{x^2-4} =$$

Super cool:

Know: $x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$

$$P_j(x) = \frac{q_j(x)}{q(x)} \Rightarrow x q_j(x) = a_n q_{n+1}(x) + b_n q_n(x) + c_n q_{n-1}(x)$$

$\left.\begin{array}{l} q_n(\lambda_k) = 0 \\ q_{n-1}(\lambda_k) = 0 \end{array}\right\}$

$$\cancel{x q_j(\lambda_k)} = a_n q_{n+1}(\lambda_k) + \cancel{b_n q_n(\lambda_k)} + \cancel{c_n q_{n-1}(\lambda_k)}$$

automatic

$$\boxed{xP_n(x) = a_n P_{n+1} + b_n P_n + c_n P_{n-1}(x)}$$

$$P_{n+1}(x) = \frac{(x-b_n)P_n(x) - c_n P_{n-1}(x)}{a_n}$$

$$= (\tilde{a}_n - \tilde{b}_n) P_n(x) - \tilde{c}_n P_{n-1}(x)$$

$$J \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \end{bmatrix} = x \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$