Moments and the Stieltjes transform

Def: The *n'th moment of a function* $r(x)$ is

$$m_n = \int_{\mathbb{R}} x^n r(x)\,dx$$

Notice: if $p(x) = \sum_{k=0}^{n} \alpha_k x^k$, then

$$\int_{\mathbb{R}} p(x) r(x)\,dx = \int_{\mathbb{R}} \sum_{k=0}^{n} \alpha_k x^k\,dx = \sum_{k=0}^{n} \alpha_k m_k$$

Remark: Inner product is defined by moments

Def: Given a sequence of values $\{m_n\}_{n=0}^{\infty}$ the *moment functional* is the function

$$\mathcal{X} : \mathbb{R}[x] \longrightarrow \mathbb{R}$$
$$\sum_{k=0}^{n} \alpha_k x^k \longmapsto \sum_{k=0}^{n} \alpha_k m_k$$

In the case, the m_n's are moments of $r(x)$,

$$\mathcal{X}(p(x)) = \int_{\mathbb{R}} p(x) r(x)\,dx$$

Questions: (1) Can two different functions have the same moments?

(2) How can we recover a function from its moment data?

Ex:

The function

log-normal distribution

$$f(x) = \frac{\sin(2\pi \ln(x))}{x} e^{-\frac{1}{2}(\ln(x))^2}, \quad 0 < x < \infty$$

$$\int_0^\infty x^n f(x)\,dx = \int_0^\infty \sin(2\pi \ln(x))\, x^{n-1} e^{-\frac{1}{2}(\ln(x))^2}\,dx$$

$t = \ln x$
$dt = \frac{1}{x}dx$

$$\rightsquigarrow = \int_{-\infty}^\infty \sin(2\pi t)\, e^{nt} e^{-\frac{1}{2}t^2}\,dt$$

$$= e^{-\frac{1}{2}n^2} \int_{-\infty}^\infty \sin(2\pi t)\, e^{-\frac{1}{2}(t-n)^2}\,dt$$

$u = t - n$
$du = dt$

$$\rightsquigarrow = e^{-\frac{1}{2}n^2} \int_{-\infty}^\infty \underbrace{\sin(2\pi u)\, e^{-\frac{1}{2}u^2}}_{\text{odd!}}\,du = 0$$

★ All moments of $f(x)$ are zero!!!

Any functions $g(x)$, $g(x) + f(x)$

different functions w/ same moments

<u>Theorem</u>: (Carleman's Condition)

If $f(x)$ is a function supported on (a,b) whose moments are $m_0, m_1, m_2, \ldots$ then

Case I: interval is $[-1,1]$, unique function w/ moments

Case II: interval is $[0,\infty)$, $\sum_{n=1}^{\infty} m_n^{-1/2n} = \infty$

$\Rightarrow$ unique function w/ moments!

Case III: interval is $(-\infty,\infty)$ $\sum_{n=1}^{\infty} m_{2n}^{-1/2n} = \infty$

$\Rightarrow$ unique function w/ moments

Stieltjes Transform:

Def: Let $r(x)$ be supported on $[a,b]$. The Stieltjes Transform is

$$S_r(z) = \int_a^b \frac{r(x)}{z-x}\,dx \quad , \qquad z \in \mathbb{C} \setminus [a,b]$$

$$\frac{1}{z-x} = \frac{1}{z}\left(\frac{1}{1-x/z}\right) = \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}}$$

$$S_r(z) = \int_a^b \sum_{n=0}^{\infty} \frac{x^n}{z^{n+1}} r(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_a^b x^n r(x)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}$$

Punchline: If m_n's nice enough, then the moments give the Stieltjes transform!

$$r(x) = \lim_{\varepsilon \to 0+} \frac{S_r(x-i\varepsilon) - S_r(x+i\varepsilon)}{2\pi i}$$

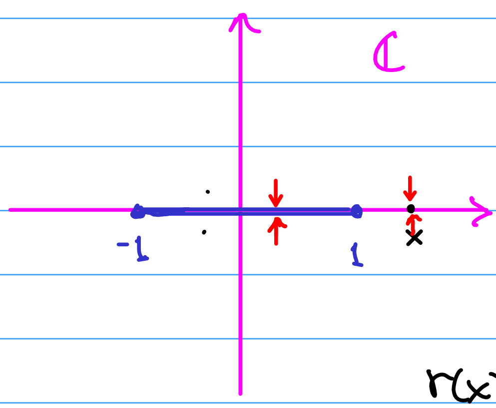
Ex: $r(x)$ function w/ moments $m_n = \begin{cases} \pi 2^{-2k}\binom{2k}{k}, & n=2k \\ 0, & n=2k+1 \end{cases}$

$$S_r(z) = \sum_{k=0}^{\infty} \frac{\pi 2^{-2k}\binom{2k}{k}}{z^{2k+1}}$$

$$\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$$

$$= \frac{\pi}{z} \sum_{k=0}^{\infty} \binom{2k}{k} (2z)^{-2k}$$

$$= \frac{\pi}{z} \frac{1}{\sqrt{1 - 1/z^2}} = \frac{\pi}{\sqrt{z^2 - 1}}$$



$r(x) = 0 \quad , \quad |x| > 1$

$$r(x) = \lim_{\varepsilon \to 0} \frac{\frac{\cancel{\pi}}{\sqrt{(x-\varepsilon i)^2 - 1}} - \frac{\cancel{\pi}}{\sqrt{(x+\varepsilon i)^2 - 1}}}{2\cancel{\pi} i}$$

$$= \lim_{\varepsilon \to 0} \frac{\frac{1}{\sqrt{(x-\varepsilon i)^2 - 1}}}{i}$$

$$= \frac{\frac{1}{\sqrt{x^2-1}}}{i} = \frac{1}{\sqrt{1-x^2}}$$

$|x| < 1$

$$\therefore \quad r(x) = \begin{cases} \frac{1}{\sqrt{1-x^2}}, & -1 < x < 1 \\ 0, & |x| > 1. \end{cases}$$