Introducing Orthogonal Polynomials

Def: A polynomial approximation of a function $f(x)$ is a polynomial $p(x)$ which approximates $f(x)$ in some way. The degree of the approximation is the degree of $p(x)$.

Ex: The linear approx. of $f(x)$ at $x=a$ is

$$L(x) = f'(a)(x-a) + f(a) \longleftarrow \text{linear equation}$$

Ex: The n^{th} Taylor polynomial expansion of $f(x)$ at $x=a$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

Why polynomial approximations?

- polynomials are easier to understand
- differentiation is easier
- integration is easier
- lots more applications

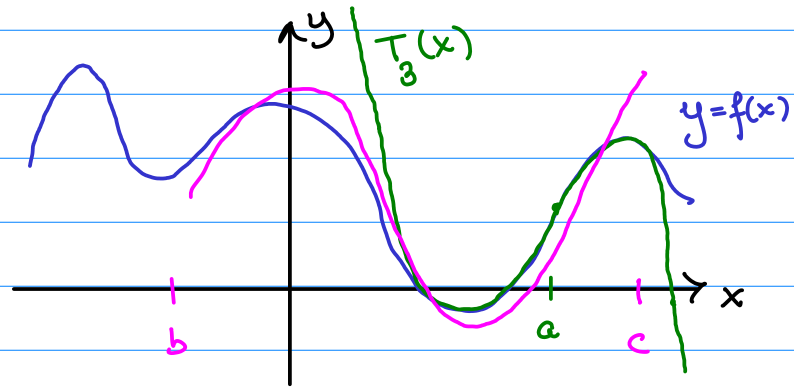
★ Big Idea ★ from harmonic analysis / approximation theory:

express something complicated as a linear combination of things which we understand.

The nth Taylor polynomial $T_n(x)$ based at $x=a$ gives the polynomial of degree $\leq n$ which most closely fits $f(x)$ near $x=a$.

That's not always what we want!

Sometimes want to understand $f(x)$ not just near $x=a$, but rather on an entire interval (b,c).



We need some metric which measures how different two functions are on an interval.

Think back to multivariate calculus:

Given vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$\|\vec{v}-\vec{w}\| = \sqrt{(v_1-w_1)^2+(v_2-w_2)^2+\ldots+(v_n-w_n)^2}$$

Measures how different $\vec{v}$ and $\vec{w}$ are!

Likewise for $f(x), g(x)$ functions on (b,c), define

$$\|f(x)-g(x)\| = \sqrt{\int_b^c (f(x)-g(x))^2\,dx}.$$

Problem: Figure out a polynomial $p(x)$ of degree n which minimizes $\|f(x)-p(x)\|$

$$\langle f(x), g(x) \rangle = \int_b^c f(x)g(x)\,dx$$

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{n} u_k v_k$$

continuous version!

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x + a_{10} \\ P_2(x) &= x^2 + a_{21}x + a_{20} \\ P_3(x) &= x^3 + a_{32}x^2 + a_{31}x + a_{30} \\ &\vdots \end{aligned}\right\} \langle P_j(x), P_k(x) \rangle = 0$$

whenever $j \neq k$.

Theorem: The polynomial $p(x)$ of degree n minimizing $\|f(x) - p(x)\|$ is given by

$$p(x) = \sum_{k=0}^{n} c_k P_k(x), \quad \text{where}$$

$$c_k = \frac{\langle f(x), P_k(x) \rangle}{\langle P_k(x), P_k(x) \rangle}.$$

The sequence $P_0(x), P_1(x), \cdots, P_n(x)$ is the key to finding our approximation and is a sequence of orthogonal polynomials.

More generally, given a function $r(x) \geq 0$ on (b,c)

$$\langle f(x), g(x) \rangle_r = \int_b^c f(x)g(x)r(x)\,dx$$

Def: A <u>sequence of orthogonal polynomials</u> for a function $r(x) > 0$ on (b, c) is a sequence of polynomials $P_0(x), P_1(x), P_2(x), \ldots$

- $\deg P_k(x) = k \quad \forall\ k \geq 0$
- $$\langle P_j(x), P_k(x) \rangle_r = \int_b^c P_j(x) P_k(x) r(x)\, dx = 0 \quad \forall\ j \neq k$$