Maximal Orders in Quaternion Algebras on Surfaces

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1 Introduction

Let X be a surface, by which we mean a nonsingular, two-dimensional integral scheme. Let η be its generic point, and K its fraction field. The short exact sequence associated to the sheaf of Cartier divisors on S induces an injection of the Brauer group $\mathrm{Br}(X)$ of X into the Brauer group of K. Thus if Δ is a central simple K-algebra, it makes sense to ask if Δ is the image of an Azumaya algebra on X, ie. if there is an Azumaya algebra on X whose restriction to η is Δ. The answer can in general be “no”. For example, if $X = \mathbb{P}^2_k$ for k an algebraically closed field, then $\mathrm{Br}(X) = 0$, even though $\mathrm{Br}(K)$ is highly nontrivial.

However, Δ can always be viewed as the restriction of a maximal $\mathcal{O}_X$-order A in Δ. Furthermore, such an A is nicely behaved – it is locally free as an $\mathcal{O}_X$-module of finite rank and is Azumaya on a dense open subset of X. Moreover, the places where A ramifies, ie. is not an Azumaya algebra, coincide with the cohomological ramification detected by the Artin-Mumford spectral sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(K) \to \bigoplus_{x\in X^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{x\in X^2} \mu^{-1} \to \mu^{-1} \to 0,$$

as well as the ramification of the Brauer-Severi scheme $\mathrm{BS}(A) \xrightarrow{\pi} A$.

In this paper, we determine the local presentation of a maximal $\mathcal{O}_X$-order A in Δ in the case that Δ is a quaternion algebra over K. Using this, we provide an exposition of Artin and Mumfords result that for suitably chosen S and Δ, the Brauer-Severi variety $\mathrm{BS}(A)$ provides an example of a nonsingular nonrational unirational variety.

2 Azumaya Algebras

2.1 Twisted Forms and Nonabelian H^1

Definition 2.1. Let X be a scheme. We define a *category of algebraic structures* on X to be a subcategory $\mathcal{C}$ of the category of sheaves on X.

We will be primarily interested in case where $\mathcal{C}/X$ is one of the following examples

- the category of $\mathcal{O}_X$-modules on X
- the category of $\mathcal{O}_X$-algebras on X
- the category of schemes over X
- the category of G-sets, for G a group scheme on X

Note that if $Y \in \underline{\mathrm{Sch}}_X$, then pullback determines from $\mathcal{C}$ a category of algebraic structures $\mathcal{C}/Y$, and a right exact functor $\mathcal{C} \to \mathcal{C}/Y$.

Definition 2.2. Let $c \in \mathcal{C}$. We define an etale (fppf) *twisted form of* c to be an object $c' \in \mathcal{C}$ such that there exists an etale (fppf) cover $\mathfrak{U} = \{f_i : U_i \to X\}$ with c', c isomorphic in $\mathcal{C}/U_i$ for all i. We denote by $\underline{\mathrm{Twist}}(c)$ the set of isomorphism classes of twisted forms of c.

The basis for understanding and classifying etale (fppf) twisted forms of $c \in C$ comes from a canonical identification of twisted forms of c as Čech cohomology classes of the sheaf of automorphisms of c. We will describe this correspondence here for etale twisted forms of c; the correspondence for fppf twisted forms is similar. For $c \in C$, let $\mathrm{Aut}_{et}(c)$ denote the sheaf on the big etale site of schemes over X, defined by

$$\mathrm{Aut}_{et}(c) : (f : Y \to X) \mapsto \mathrm{Aut}_{\mathcal{C}/Y}(f^*c).$$

Then for a twisted form c' of c, the local isomorphisms $\varphi_i : f_i^*c' \to f_i^*c$ generate a Čech cocycle $(\psi_{ij}) \in \check{C}^1(\mathfrak{U}, \mathrm{Aut}_{et}(c))$, for

$$\psi_{ij} : (p_i^*\varphi_i) \circ (p_j^*\varphi_j)^{-1} \in \mathrm{End}(f_{ij}^*c),$$

with $f_{ij} : U_i \times_X U_j \to X$ the induced map of the fiber product, and $p_i : U_i \times_X U_j \to U_i$ the canonical projection map. The natural map $\check{C}^1(\mathfrak{U}, \mathrm{Aut}_{et}(c)) \to \check{H}^1(\mathfrak{U}, \mathrm{Aut}_{et}(c)) \to \check{H}^1(X_{et}, \mathrm{Aut}_{et}(c))$ completes the construction. A descent argument then proves the following lemma

Lemma 2.1. *Let $\mathcal{C}$ be a category of algebraic structures on a scheme X, and let $c \in C$. Then the canonical association of a Čech cocycle to a twisted form c' of c defines an injection*

$$\underline{\mathrm{Twist}}(c) \hookrightarrow \check{H}^1(X_{et}, \mathrm{Aut}_{et}(c)).$$

A similar statement holds for the fppf twisted forms of c.

Proof. See Milne [6]. □

Example 2.1. Let $X = \text{Spec}(k)$, let $\mathcal{C}$ be the category of schemes over X, and let $c = \text{Spec}(L)$ for L a finite separable field extension of k. If $Z \to X$ is etale, then $Z = \amalg_{i=1}^{n} \text{Spec}(L_i)$ where each L_i is a finite, separable extension of k. Therefore $Y \times_X Z = \amalg \text{Spec}(L_i \otimes_k L)$, and since $\text{Aut}_{L_i}(L_i \otimes_k L) \cong \text{Aut}_k(L)$, and thus

$$\text{Aut}(c) : Z \mapsto \text{Aut}_{\underline{\text{Sch}}_Z}(Y \times_X Z) = \prod_{i=1}^{n} G,$$

where G is the Galois group of the extension L/k. Thus $\text{Aut}(c)$ may be identified with the constant sheaf G on the (big) etale site of X. Consequently, $\check{H}^1(X_{et}, \text{Aut}_{et}(c)) = \check{H}^1(X_{et}, G)$, and so twisted forms of c are in one-to-one correspondence with G-torsors on X; for $X = \text{Spec}(k)$, these are exactly the Galois extensions of k with Galois group G.

In the previous example, we swept under the rug the issue of whether $\underline{\text{Twist}}(c) \to \check{H}^1(X, \text{Aut}_{et}(c))$ is surjective. Since descent theory may always be applied to "glue together" the descent data coming from a representative $\check{H}^1(X, \text{Aut}_{et}(c))$ to a sheaf, the question is really one of representability: when is the sheaf that we glue together actually a member of the category $\mathcal{C}$? For example, it is clear that descent will be effective for $\mathcal{C}$ the category of G-torsor sheaves. However, whether it is effective for the category of G-torsors is equivalent to asking whether any G-torsor sheaf is representable. This question in general can be difficult to answer.

Nevertheless, in many important instances the above map is an equivalence of categories.

Example 2.2. Let $\mathcal{C}$ be the category of quasi-coherent $\mathcal{O}_X$-modules, and let $c = \mathcal{O}_X^n$. Then $\text{Aut}(c)$ is equal to the group scheme $\mathbb{GL}_n$ defined on the big fppf site of X by

$$\mathbb{GL}_n : (U \to X) \mapsto \mathbb{GL}_n(\Gamma(U, \mathcal{O}_U)).$$

Suppose $c' \in \mathcal{C}$ is a twisted form of c. Then c' is fppf locally free of rank n. By a follow-your-nose type argument, c' is therefore Zariski locally free. Thus $\underline{\text{Twist}}(c)$ is the set of isomorphism classes of locally free rank n $\mathcal{O}_X$-modules. Since fppf descent is effective for quasicoherent sheaves of $\mathcal{O}_X$-modules, we conclude that

$$\check{H}^1(X_{fppf}, \mathbb{GL}_n) \cong \underline{\text{Twist}}(c) = \{\text{iso. classes of locally free, rank } n\ \mathcal{O}_X\text{-modules}\}.$$

As a special case of this, we see that $\check{H}^1(X_{fppf}, \mathbb{G}_m) = \check{H}^1(X_{fppf}, \mathbb{GL}_1)$ classifies isomorphism classes of invertible sheaves on X, ie. $\check{H}^1(X, \mathbb{G}_m) \cong \text{Pic}(X)$.

Example 2.3. Let $\mathcal{C}$ be the category of $\mathcal{O}_X$-algebras, and let $c = M_n : (U \to X) \mapsto M_n(\Gamma(U, \mathcal{O}_U))$. The etale twisted forms of c are exactly the degree n Azumaya algebras on X. Since etale descent is effective for $\mathcal{O}_X$-algebras, it follows that $\check{H}^1(X_{et}, \text{Aut}_{et}(c))$ classifies isomorphism classes of degree n Azumaya algebras on X.

What is $\mathrm{Aut}_{et}(c)$? Conjugation provides a natural map $\mathbb{GL}_n \to \mathrm{Aut}_{et}(c)$ with kernel $\mathbb{G}_m$. A version of the Skolem-Noether Theorem says that for any local ring R, all automorphisms of the ring $M_n(R)$ are inner. Therefore, we see that the map of etale stalks $\mathbb{GL}_n \to \mathrm{Aut}_{et}(c)$ is surjective on stalks, and hence surjective. In particular, we have a short exact sequence

$$0 \to \mathbb{G}_m \to \mathbb{GL}_m \to \mathrm{Aut}(c) \to 0.$$

The cokernel of $\mathbb{G}_m \to \mathbb{GL}_n$ is exactly the group scheme $\mathbb{PGL}_n$, so we have an isomorphism $\mathrm{Aut}(c) \cong \mathbb{PGL}_n$. Hence $\check{H}^1(X_{et}, \mathbb{PGL}_n)$ classifies isomorphism classes of Azumaya algebras on X.

Example 2.4. Let $\mathcal{C}$ be the category of schemes over X, and let $c = \mathbb{P}^{n-1}_X$. Then $\mathrm{Aut}(c) = \mathbb{PGL}_n$ and we have a canonical injection $\underline{\mathrm{Twist}}(c) \hookrightarrow \check{H}^1(X_{fppf}, \mathbb{PGL}_n)$. As it so happens, this map is also surjective [6]. Therefore $\check{H}^1(X, \mathbb{PGL}_n)$ also classifies twisted forms of $\mathbb{P}^{n-1}_X$, so called Severi-Brauer varieties.

Our last example describes a sort of universal interpretation of the elements of $\check{H}^1(X, G)$ as isomorphism classes of G-torsor sheaves on X. From some perspectives, this could be considered the right definition of the first nonabelian cohomology group.

Example 2.5. Let G be a group scheme on X. By a (right) G-set on X, we mean a sheaf F on X, and a morphism of sheaves $F \times G \to F$ compatible with the canonical morphism $G \times G \to G$, ie. so that the natural map $(F \times G) \times G \to F \times (G \times G)$ factors over F. Let $\mathcal{C}$ be the category of G-sets on X with morphisms given by G-equivariant natural transformations. Let $c = G$, with the canonical G-action by multiplication on the right. We call c the *trivial* G-torsor.

If c' is an fppf-twist of G, then c' is an fppf locally isomorphic to the trivial G-torsor: this is exactly the definition of a G-torsor sheaf on X! Thus $\underline{\mathrm{Twist}}(c)$ is exactly the set of isomorphism classes of locally trivial G-torsor sheaves on X. For any $Y \to X$, the set of G-equivariant maps $G \to G$ are exactly determined by where they send the identity. Hence there is a canonical identification $\mathrm{Aut}(c) = G$. Thus $\check{H}^1(X_{fppf}, G)$ is classifies the isomorphism classes of G-torsors on X. We denote the set of isomorphism classes of G-torsors on X as $\mathrm{Torsor}(G)$.

Remark 2.1. The first cohomology group is known to agree with the standard cohomology group for G abelian on any site, so from now on we will leave off the check symbol, and simply write $H^1(X_{site}, G)$. Moreover, for many of the groups we work with, the first cohomology groups for the etale and fppf sites will agree – in such situations, we will simply write $H^1(X, G)$ to mean cohomology with respect to this site.

2.2 Gerbes and Nonabelian H^2

In this section, let $\mathcal{C}$ and $\mathcal{B}$ be categories, and $\varphi : \mathcal{C} \to \mathcal{B}$ a functor between them. The exposition is based off of the one found in [6]. The author also found [10] helpful.

Definition 2.3. Suppose that $b \in \mathcal{B}$. We define the *fiber of* b, denoted $\varphi^{-1}(b)$, to be the category whose objects objecs in $\mathcal{C}$ sent to b under φ and whose morphisms are morphisms sent to id_b under φ.

Definition 2.4. Consider two objects c, c' of $\mathcal{C}$ and a morphism $f : c' \to c$, and set $b = \varphi(c), b' = \varphi(c')$ and $g = \varphi(f) : b' \to b$. We call f *Cartesian* if for all $x \in \varphi^{-1}(b')$, and for all $h \in \mathrm{Hom}_{\mathcal{C}}(x, c)$ satisfying $\varphi(h) = g$ there exists a unique morphism $f_0 \in \mathrm{Hom}_{\varphi^{-1}(c')}(x, c')$ satisfying $f f_0 = h$.

Definition 2.5. Let $\varphi : \mathcal{C} \to \mathcal{B}$ be a functor between categories $\mathcal{C}$ and $\mathcal{B}$. We call the data $(\mathcal{C}, \mathcal{B}, \varphi)$ a *fibered category* if the following two properties hold

(i) given $b', b \in \mathcal{B}$, $c \in \varphi^{-1}(b)$ and $g : b' \to b$, there exists $c' \in \varphi^{-1}(b')$ and $f : c' \to c$ with $\varphi(f) = g$.

(ii) the composition of two Cartesian morphisms is Cartesian

In this case the object c' is unique up to canonical isomorphism and is called the *pullback of* c, denoted g^*c.

Example 2.6. The identity functor $\mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ defines $\mathcal{B}$ as a fibered category over iteself.

Example 2.7. Grothendieck Construction Let $F : \mathcal{B} \to$ 🐱 be a pseudofunctor between a category $\mathcal{B}$ and a 2-category 🐱. A construction attributed to Grothendieck allows us to use F to build a fibered category $(\int F, \mathcal{C}, \varphi)$.

Define a category $\int F$ whose objects are pairs (b, c) with $b \in \mathcal{B}$ and $c \in F(b)$, where

$$\mathrm{Hom}_{\int F}((b, c), (b', c')) = \{(g, \alpha) : b \xrightarrow{g} b',\ F(g)(c) \xrightarrow{\alpha} c'\}.$$

This category has a natural functor $\varphi : \int F \to \mathcal{B}$ defined by $(b, c) \mapsto b$ and $(g, \alpha) \mapsto g$, and one may verify that $(\int F, \mathcal{C}, \varphi)$ is a fibered category.

Definition 2.6. Let $\mathcal{C} \to \mathcal{B}$ and $\mathcal{C}' \to \mathcal{B}$ be fibered categories, and let $F : \mathcal{C}' \to \mathcal{C}$ be a 1-morphism of fibered categories over $\mathcal{B}$. Then we define the *inertia category of* $\mathcal{C}'$ *over* $\mathcal{C}$ $\mathcal{I}_{\mathcal{C}'/\mathcal{C}}$ to be the category whose objects are pairs (c', f) with $c \in \mathcal{C}'$ and $f : c' \to c'$ an automorphism satisfying $F(f) = \mathrm{id}_{F(c')}$, and whose morphisms $(c'_0, f_0) \to (c'_1, f_1)$ are morphisms $h : c'_0 \to c'_1$ such that $h f_0 = f_1 h$. We define the *inertia category of* $\mathcal{C}$ to be $\mathcal{I}_{\mathcal{C}/\mathcal{B}}$, and denote it by $\mathcal{I}_{\mathcal{C}}$

Now consider the case when the base category $\mathcal{B}$ is a site.

Definition 2.7. Let $(\mathcal{C}, \mathcal{B}, \varphi)$ be a fibered category, with $\mathcal{B}$ a site. We call this fibered category a *prestack* if for any $b \in \mathcal{B}$, and $a, a' \in \varphi^{-1}(b)$ the functor $F : \mathcal{C}/b \to \underline{\mathrm{Sets}}$ defined by

$$(f : b' \to b) \mapsto \mathrm{Hom}(f^*a', f^*a)$$

is a sheaf.

Definition 2.8. Let $(\mathcal{C}, \mathcal{B}, \varphi)$ be a fibered category, with $\mathcal{B}$ a site. Let $\mathfrak{U} = \{g_i : b_i \to b\}$ be a covering of an object $b \in \mathcal{B}$. Then there is a natural map from $\varphi^{-1}(b)$ to descent data on $\mathfrak{U}$. If all descent data on $\mathfrak{U}$ arises in this way, then we say *descent is effective on* $\mathfrak{U}$. If descent is effective for every choice of $\mathfrak{U}$ and b, we say descent is effective on φ. A prestack on which descent is effective is called a *stack*.

We will mostly be interested in the case that the base category $\mathcal{B}$ is a site on the category of schemes over X, ie. $\mathcal{B} = X_{\text{site}}$. For simplicity, we will refer to a stack $(\mathcal{C}, X_{\text{site}}, \varphi)$ as a stack on X, and denote it by $\varphi : \mathcal{C} \to X_{\text{site}}$ or just by $\varphi : \mathcal{C} \to X$, if the site structure on X is clear from context.

Example 2.8. Let F be a sheaf on X, ie. a functor $F : \underline{\text{Sch}}_X^{\text{op}} \to \underline{\text{Sets}}$. Then the Grothendieck construction gives a stack $\int F \to X$ on X.

Definition 2.9. Let $\varphi : \mathcal{C} \to X$ be a stack on X. The stack is called a *gerbe* if it satisfies the following additional properties

(i) for all $b \in \mathcal{B}$ the fiber $\varphi^{-1}(b)$ is a groupoid (ie. all morphisms are isomorphisms)

(ii) for every $b \in \mathcal{B}$, there exists a covering $\mathfrak{U} = \{b_i \to b\}$ such that each $\varphi^{-1}(b_i)$ is nonempty

(iii) for any $b \in \mathcal{B}$, and $a, a' \in \varphi^{-1}(b)$, there exists a covering $\mathfrak{U} = \{f_i : b_i \to b\}$ of b such that $f_i^* a$ and $f_i^* a'$ are isomorphic for all i

Definition 2.10. Let G be a group scheme on X, and let $\mathcal{C} \to X$ be a gerbe on X. Consider the natural projection $p : \mathcal{I}_{\mathcal{C}} \to X$. We call $\mathcal{C}$ a *G-gerbe on X* if p^{-1} is naturally isomorphic to G as sheaf of groups on X.

Example 2.9. Let G be a group scheme on X. We define the *trivial G-gerbe on X* to be $\pi : BG \to X$, where BG is the category whose objects are pairs (Y, V) with $Y \xrightarrow{f} X$ a scheme over X and V a (left) G-torsor on Y, and with morphisms $(Y', V') \to (Y, V)$ given by pairs (g, α) with $g : Y' \to Y$ a morphism of schemes over X and $\alpha : g_* V' \to V$ an isomorphism.

Example 2.10. Let G be a group scheme on X, and let V be a (left) G-torsor sheaf. We associate to V a gerbe $\mathcal{X}_V$ as follows. The objects of $\mathcal{X}_V$ are triples (Y, W, α), where $f : Y \to X$ is a scheme over X, W a (left) G-torsor on Y, and α an isomorphism of G-torsors $\alpha : f_* W \cong V$.

For reasons that will become apparent momentarily, we denote the set of isomorphism classes of G-gerbes on X by $H_g^2(X, G)$. Note that if G is abelian, then $H_g^2(X, G)$ has a natural group structure

$$H_g^2(X, G) \times H_g^2(X, G) \to H_g^2(X, G \times G) \to H_g^2(X, G),$$

induced by the product map on G and the projections $G \rightrightarrows G \times G \to G$.

Proposition 2.2. *Suppose that G is an abelian group scheme. Then there is a canonical isomorphism*

$$H_g^2(X,G) \to H^2(X_{et},G).$$

Proof. Suppose that G' a group scheme on X, which is injective as a $\mathbb{Z}$-module. Let $\mathcal{X}$ be a G'-gerbe, and choose a covering $\mathfrak{U} = \{f_i : U_i \to X\}$ on which the gerbe $\mathcal{X}$ restricts to the trivial gerbe BG'. Then let $f : \amalg_i U_i \to X$ be the induced covering map. Then since G' is injective, the natural map $G' \to f_* f^* G'$ splits. It follows that $H_g^2(X,G') \to H_g^2(X, f_* f^* G')$ splits also, and therefore since since the image of $\mathcal{X}$ is 0 in $H_g^2(X, f_* f^* G')$, we must have $\mathcal{X} = 0$.

Now let G be an arbitrary abelian group scheme G, and embed G into an injective G', with cokernel G''. Then we have a short exact sequence of abelian group schemes

$$1 \to G \to G' \to G'' \to 1,$$

which induces a long exact sequence

$$\ldots H^1(X,G') \to H^1(X,G'') \to H_g^2(X,G) \to H_g^2(X,G') \to H^2(X,G,'').$$

See Milne [6] for details. Then since $H_g^2(X,G') = 0$, this shows that $H_g^2(X,G)$ is the cokernel of $H^1(X,G') \to H^1(X,G'')$, which is $H^2(X,G)$. This completes the proof. □

Thus one interpretation of the second cohomology group $H^2(X,G)$ is as isomorphism classes of G-gerbes. Giraud proves that short exact sequences still induce long exact sequences as desired.

Lemma 2.3 (Giraud)**.** *Let X be a scheme and let*

$$1 \to G' \to G \to G'' \to 1,$$

be a short exact sequence of group schemes on X, with $G' \subseteq Z(G)$. Then there exists a long exact sequence of pointed sets

$$\begin{aligned} 1 &\to H^0(X,G') \to H^0(X,G) \to H^0(X,G'') \\ &\to H^1(X,G') \to H^1(X,G) \to H^1(X,G'') \xrightarrow{\delta} H^2(X,G') \end{aligned}$$

where the map δ in the above sends (the iso. class of) a G''-torsor sheaf V on X to the (iso. class of the) associated gerbe $\mathcal{X}_V$.

Proof. See [4]. □

2.3 Azumaya Algebras and the Brauer Group

Definition 2.11. Let X be a scheme. We define an *Azumaya algebra* A on X to be a twisted form of the $\mathcal{O}_X$-algebra M_n for some n. The corresponding value of n is called the *rank*. We denote the set of isomorphism classes of Azumaya algebras of rank n on X as $\mathrm{Az}_n(X)$. We call an Azumaya algebra A *trivial* if $A \cong \underline{\mathrm{End}}_{\mathcal{O}_X}(V)$ for some locally free $\mathcal{O}_X$-module V. We define two Azumaya algebras A and A' to be *equivalent* if $A \otimes_{\mathcal{O}_X} (A')^{op}$ is a trivial Azumaya algebra.

Example 2.11. If $X = \mathrm{Spec}(k)$ for k a field, then an Azumaya algebra on X is the same thing as a central simple k-algebra. Wedderburn's theorem tells us that each central simple algebra is isomorphic to a matrix algebra over a division ring; two central simple algebras are equivalent if and only if they have the same underlying division ring.

Remark 2.2. With the definition of equivalence provided above, two Azumaya algebras A and A' are equivalent if and only if the corresponding module categories are the equivalent. In fact, there corresponding equivalence of categories is the one induced by the A, A'-bimodule $A \otimes (A')^{op}$. Thus the definition of equivalence coincides with the notion of Morita equivalence.

Definition 2.12. We define the (geometric) *Brauer group* of X to be the set $\mathrm{Br}(X)$ of equivalence classes of Azumaya algebras on X.

Proposition 2.4. *The Brauer group* $\mathrm{Br}(X)$ *is an abelian group with binary operation* $[A] + [A'] = [A \otimes A']$.

Example 2.12. Let $X = \mathrm{Spec}(\mathbb{R})$. Then the central division rings over $\mathbb{R}$ consist of only $\mathbb{R}$ and the quaternion algebra Q. Thus $\mathrm{Br}(\mathrm{Spec}(\mathbb{R})) = \mathbb{Z}/2$.

Example 2.13. Let $X = \mathbb{P}^1_k$. Then $\mathrm{Br}(X) = \mathrm{Br}(\mathrm{Spec}(k))$.

The Brauer group $\mathrm{Br}(X)$ is intimately related to $H^2(X, \mathbb{G}_m)$, and in particular there is always an injection $\mathrm{Br}(X) \hookrightarrow H^2(X, \mathbb{G}_m)$ mapping $\mathrm{Br}(X)$ into the torsion part of $H^2(X, \mathbb{G}_m)$.

Definition 2.13. Let X be a scheme. We define the *cohomological Brauer group* $\mathrm{Br}'(X)$ of X to be the torsion subgroup of $H^2(X, \mathbb{G}_m)$.

Remark 2.3. Note that the fppf and etale cohomology of $\mathbb{G}_m$ agree. Hence we simply write $H^i(X, \mathbb{G}_m)$ to represent either cohomology group.

Given an Azumaya algebra A on X, we define a $\mathbb{G}_m$-gerbe $\mathcal{X}_A$ by taking the category fibered over X associated to the pseudo-functor

$$(Y \to X) \mapsto \{(E, \alpha) : E \text{ loc. free, coh. } \mathcal{O}_Y\text{-module}, \alpha : \mathrm{End}_{\mathcal{O}_Y}(E) \cong A_Y\}$$

Lemma 2.5. *The fibered category* $\mathcal{X}_A$ *is a* $\mathbb{G}$*-gerbe, which is trivial if and only if* A *is a trivial Azumaya algebra. The map* $A \mapsto \mathcal{X}_A$ *defines a group monomorphism of* $\mathrm{Br}(X)$ *into* $H^2_g(X, \mathbb{G}) \cong H^2(X, \mathbb{G})$

Proof. See Milne [6]. ☐

The image of $\mathrm{Br}(X)$ in $H^2(X, \mathbb{G}_m)$, is torsion. To see this, consider the commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r, "\cdot^n"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{SL}_n \arrow[r] \arrow[d] & \mathbb{GL}_n \arrow[r, "\det"] \arrow[d] & \mathbb{G}_m \arrow[r] & 1 \\
 & \mathbb{PGL}_n \arrow[r, equal] \arrow[d] & \mathbb{PGL}_n \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

where the rows and columns are all short exact sequences (in fppf topology).

Proposition 2.6. *The map $\delta : H^1(X, \mathbb{PGL}_n) \to H^2(X, \mathbb{G}_m)$ factors through the map $H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)$ induced by the Kummer sequence. In particular, the image of $H^1(X, \mathbb{PGL}_n)$ is torsion in $H^2(X, \mathbb{G}_m)$.*

Proof. Follows from mucking about with the above diagram. □

Corollary 2.7. *The geometric Brauer group embeds into the cohomological Brauer group.*

Proof. This follows from the fact that $\mathrm{Br}'(X)$ is defined to be the torsion in $H^2(X, \mathbb{G}_m)$. □

In ideal situations, one can show that the geometric Brauer group and the cohomological Brauer group are the same. What exactly constitutes an "ideal situation" is an open problem. However, agreement is know to hold for Noetherian schemes with dimension at most 1.

Proposition 2.8. *If X is a Noetherian scheme and $\dim(X) \leqslant 1$, then $\mathrm{Br}(X) = \mathrm{Br}'(X)$.*

Proof. We reduce to the case that X is reduced. To do so, we first must compare the sheaves $\mathcal{O}^*$ and $\mathcal{O}^*_{\mathrm{red}}$ on X. Define $K_i = 1 + N^i$, for N the nilradical of X. We have a short exact sequence

$$0 \to K_1 \to \mathcal{O}^* \to \mathcal{O}^*_{\mathrm{red}} \to 0.$$

Moreover, since X is noetherian, there exists an integer r such that $N^r = 0$. Therefore we have a filtration

$$1 = K_r \subseteq K_{r-1} \subseteq \cdots \subseteq K_2 \subseteq K_1.$$

Note also that $K_i/K_{i+1} \cong N^i/N^{i+1}$, and therefore the quotients K_i/K_{i+1} are coherent. Since N^i/N^{i+1} is supported on a 0-dimensional closed subscheme of K, it follows that $H^j(X, K_i/K_{i+1}) = 0$ for all $j > 0$. Hence the usual fitration trick gives us isomorphisms $H^j(X, K_i) \cong H^j(X, K_{i+1})$ for all $j > 0$. Thus $H^j(X, K_1) = H^j(X, K_r) = 0$ for all $j > 0$. It follows immediately that $H^j(X, \mathcal{O}^*) = H^j(X, \mathcal{O}^*_{\text{red}})$ for all $j > 0$, and in particular $\text{Br}'(X) = \text{Br}'(X_{\text{red}})$.

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\text{Br}(X) \arrow[r] \arrow[d] & \text{Br}'(X) \arrow[d, "\cong"] \\
\text{Br}(X_{\text{red}}) \arrow[r] & \text{Br}'(X_{\text{red}})
\end{tikzcd}$$

Where the horizontal maps are injective. Thus if $\text{Br} = \text{Br}'$ for X_{red}, then the above diagram shows that $\text{Br}(X)$ must surject onto $\text{Br}'(X)$, and therefore be an isomorphism. In this way we can reduce to the case $X = X_{\text{red}}$.

Case 1: Assume $\dim(X) = 0$. Then X is a disjoint union of points, and we can reduce to the case that $X = \text{Spec}(K)$. In this case, equality of $\text{Br}(K)$ and $H^2(\text{Spec}(K), \mathbb{G}_m)$ is classical result of Galois cohomology. See [3] for details.

Case 2: Assume $\dim(X) = 1$, and let $d \in H^2(X, \mathbb{G}_m)$. We claim that there is a dense, open subscheme U of X on which the restriction of $[d]$ to an element of $H^2(U, \mathbb{G}_m)$ lies in the image of $\text{Br}(U)$. To see this, let U_0 be the open subscheme of nonsingular points of X. Then U_0 is a disjoint union of nonsingular, integral curves $C_1 \cup \cdots \cup C_r$. Let c_i be the generic point of C_i for each i. Since C_i is nonsingular, $\text{Br}'(C_i) \hookrightarrow \text{Br}(K_i) = \text{Br}'(K_i)$, where K_i represents the fraction field of C_i. Thus the restriction d_i of d to C_i restricts to the generic point to a central simple K_i-algebra Δ_i. Let A_i be a maximal $\mathcal{O}_{C_i}$-order in Δ_i (discussed in a later section). Then $(A_i)_{c_i} = \Delta_i$, and therefore there is a Zariski open neighborhood U_i of c_i in C_i on which A_i is Azumaya: take the complement of the zero set of the descriminant. Since the pullback morphism $\text{Br}'(X) \to \text{Br}(K_i)$ factors through the injection $0 \to \text{Br}'(U_i) \to \text{Br}(K_i)$ it follows that the pullback of d to $\text{Br}'(U_i)$ lands in the image of $\text{Br}(U_i)$. Hence the pullback of d to $\text{Br}'(U)$ for $U = \bigcup_i U_i$ lands in the image of $\text{Br}(U)$. Since U is dense in X, this proves our claim.

So far, we have shown that d is represented on a dense open subscheme U of X by an Azumaya algebra A on U. If we can show that U can be taken to be all of X, then we are done! Let V be a largest dense open subscheme of X on which d is represented by a sheaf of Azumaya algebras (this makes sense since the complement of U is finitely many points). Suppose $x \in X \backslash V$. By the previous paragraph, we know that x is a closed point of X, and therefore $R = \mathcal{O}_{X,x}$ is a one-dimensional local ring. Set $V_x = \text{Spec}(R) \cap V = \text{Spec}(R) \backslash \{x\}$. The completion $\widehat{R}$ of

R is a Hensilian local ring, and therefore $\mathrm{Br}(\widehat{R}) = \mathrm{Br}'(\widehat{R})$ [6]. Hence the pullback of d to $\mathrm{Spec}(\widehat{R})$ may be represented by an Azumaya $\widehat{R}$-algebra $\widehat{B}$, which must be equivalent to A on $\widehat{V}_x = \mathrm{Spec}(\widehat{R})\backslash\{x\}$. Hence there exist integers s, t such that $M_s(A) = M_t(\widehat{B})$ on $\widehat{V}_x$. Then there exists a unique Azumaya R-algebra B such that $B \otimes \widehat{R} \cong M_t(\widehat{B})$. Gluing B and A together on $\{x\} \cup V$, extends A to an Azumaya algebra on $V \cup \{x\}$. This contradicts the maximality of V, and hence $V = X$. This completes the proof.

□

Furthermore, if X is a nonsingular, integral surface then $\mathrm{Br}(X) = \mathrm{Br}(X')$. We will prove this result in a later section.

3 Brauer-Severi Schemes

Let X be a proper integral scheme over an algebraically closed field k. We have already pointed out that the set $H^1(X, \mathbb{PGL}_n)$ classifies twisted forms of two different algebraic structures over X:

$$\begin{tikzcd} & H^1(X, \mathbb{PGL}_n) \arrow[dl, <->] \arrow[dr, <->] & \\ \mathrm{Az}_n(X) \arrow[rr, dashed] & & \underline{\mathrm{Twist}}(\mathbb{P}_X^{n-1}) \end{tikzcd}$$

Hence rank n Azumaya algebras on X and twisted forms of $\mathbb{P}_X^{n_1}$ are in bijection. One way of explicitly obtaining this bijection is by means of generating the Brauer-Severi scheme associated to an Azumaya algebra, which we now define.

Let A a sheaf of $\mathcal{O}_X$-algebras, locally free as an $\mathcal{O}_X$-module. We define a functor $\mathfrak{BS}_n(A) : \underline{\mathrm{Sch}}_X \to \underline{\mathrm{Sets}}$ by

$$\mathfrak{BS}_n(A) : (T \to X) \mapsto \{\text{quot. left } A_T\text{-modules of } A_T \text{ loc. free of rank } n^2 - n\}.$$

We will prove that this functor is representable by proving that it is a locally closed subfunctor of the Quot functor. The latter is representable, since X is proper over k [8].

Lemma 3.1. *The functor $\mathfrak{BS}_n(A)$ is representable by a quasi-projective k-scheme.*

Proof. Define a subfunctor $\mathfrak{Q}_n$ of the Quot functor $\mathfrak{Q} := \mathfrak{Quot}_{A/X/k}$ by

$$\mathfrak{Q}_n : (T \to X) \mapsto \{(\mathcal{F}, q) \in \mathfrak{Q}(T) : \mathcal{F} \text{ is locally free of rank } n\}.$$

We claim that $\mathfrak{Q}_n$ is an open subfunctor of $\mathfrak{Q}$. To see this, suppose that $T \in \underline{\mathrm{Sch}}_k$ and that we have a natural transformation $h_T \to \mathfrak{Q}$. Since $\mathfrak{Q}$ is represented by a scheme Q, there exists a natural isomorphism $\eta : \mathrm{Hom}(-, Q) \to \mathfrak{Q}$. Furthermore, Yoneda tells us $h_T \to \mathfrak{Q}$ corresponds to a unique morphism $f : T \to Q$, thereby a unique element $(\mathcal{F}, q) := \eta(T)(f) \in \mathfrak{Q}(T)$. Note that since $\mathcal{F}$ is flat over T, it is locally free over T. Let $U \subseteq T$ be the set of points $t \in T$ for which $\mathcal{F}_t$ has rank n. Then U is an open subset of T. Thus to prove that $\mathfrak{Q}_n$ is an open subfunctor of $\mathfrak{Q}$ it suffices to show that $\mathfrak{Q}_n \times_{\mathfrak{Q}} h_T = h_U$.

To see this, suppose that we have a commutative diagram

$$\begin{tikzcd} h_{T'} \arrow[r] \arrow[d] & \mathfrak{Q}_n \arrow[d] \\ h_T \arrow[r] & \mathfrak{Q} \end{tikzcd} .$$

Then we have a natural transformation $h_{T'} \to \mathfrak{Q}$, corresponding to a morphism $f' : T' \to Q$, and determining an element $(\mathcal{F}', q') := \eta(T')(f') \in \mathfrak{Q}(T')$. Since

$h_{T'} \to \mathfrak{Q}$ factors through $\mathfrak{Q}_n$, this tells us $\mathcal{F}'$ is locally free of rank n. Since $h_{T'} \to \mathfrak{Q}$ factors through h_T, this tells us that f' factors through f; meaning that there is a morphism $g : T' \to T$ with $f' = gf$. By functorality, it follows that $\eta(T)(f') = \eta(g)(f) = (g^*\mathcal{F}, g^*q)$. Thus $g^*\mathcal{F}$ is locally free of rank n. Since pulling back locally free modules doesn't affect the rank, the image of g must lie in U. Hence $T' \to T$ factors through $U \to T$, giving us a unique morphism $h_{T'} \to h_U$. Thus h_U satisfies the universal property of the pullback, and this proves our claim. In particular, this shows that $\mathfrak{Q}_n$ is representable by an open subscheme of Q.

Next, define a subfunctor $\mathfrak{Q}'$ of $\mathfrak{Q}$ by

$$\mathfrak{Q}' : (T \to X) \mapsto \{(\mathcal{F}, q) \in \mathfrak{Q}(T) : \mathcal{F} \text{ is a left } A\text{-module}\}.$$

We claim that $\mathfrak{Q}'$ is a locally closed subfunctor of $\mathfrak{Q}$. To see this, again consider a natural transformation $h_T \to \mathfrak{Q}$, let $(\mathcal{F}, q)$ be the corresponding element of $\mathfrak{Q}(T)$, and let I be the kernel of $A_T \to \mathcal{F}$. We define

$$Z = \{t \in T : aI_t \subseteq I_t,\ \forall a \in A\}.$$

Then Z is a locally closed subscheme of Y. To check this, we may immediately reduce to the case where $T = \mathrm{Spec}(R)$ for finitely generated k-algebra R and A is generated by a free R-module A. Here, it's easy to express $A_{\mathfrak{p}} I_{\mathfrak{p}} \subseteq I_p$ in terms of finitely many relations on R. By a similar argument to before, we prove that $h_Z \cong h_T \times_{\mathfrak{Q}} \mathfrak{Q}'$, and therefore that $\mathfrak{Q}'$ is locally closed subfunctor of h_Z. In particular it is representable.

Noting that $\mathfrak{B}\mathfrak{S}_n(A)$ is the intersection of $\mathfrak{Q}'$ and $\mathfrak{Q}_n$, we see that $\mathfrak{B}\mathfrak{S}_n(A)$ is representable by a locally closed subscheme $\mathrm{BS}_n(A)$ of Q. Since Q is projective over k, this also shows us that $\mathrm{BS}_n(A)$ is quasiprojective over k. $\square$

Definition 3.1. We denote the X-scheme representing the functor $\mathfrak{B}\mathfrak{S}_n(A)$ as $\mathrm{BS}_n(A)$, and call it the *Brauer-Severi scheme of* A. In the special case that $X = \mathrm{Spec}(K)$ for some finitely generated field extension K/k, and Δ is a central simple K-algebra, the Brauer-Severi scheme is also called the *Brauer-Severi variety.*

Consider the sheaf of $\mathcal{O}_X$-algebras M_n defined on the (big) etale site of X by

$$M_n : (T \to X) = M_n(\Gamma(T, \mathcal{O}).$$

Lemma 3.2. $\mathrm{BS}_n(M_n) = \mathbb{P}^{n-1}_X$

Proof. Suppose that I is a left ideal of $M_n(\mathcal{O}_T)$, locally free of rank n as an R-module. Then I is also locally free as a left $M_n(\mathcal{O}_T)$-module since $\mathcal{O}$ and M_n are Morita equivalent. Noting that a principally generated free module of I has rank n as an R-module. Hence I is principal, and this shows that

$$\begin{aligned}
\mathfrak{B}\mathfrak{S}_n(M_n)(T \to X) &= \{\text{left ideals of } M_n(\mathcal{O}_T) \text{ locally free over } T \text{ of rank } n\} \\
&= \{M_n(\mathcal{O}_T)a\text{: } a \text{ not a zero divisor}\} \\
&= \{\text{one-dimensional linear subspaces of } \mathbb{A}^n_T\} \\
&= \Gamma(\mathbb{P}^{n-1}_T, \mathcal{O}).
\end{aligned}$$

Therefore $\mathfrak{BS}_n(M_n) = \mathbb{P}_X^{n-1}$ as sheaves on the big etale site of X, and consequently $\mathrm{BS}_n(M_n) = \mathbb{P}_X^{n-1}$. □

Example 3.1. Let k be a field and $\Delta = M_n(k)$. Then $\mathrm{BS}_n(\Delta) \cong \mathbb{P}_k^{n-1}$.

Example 3.2. Let k be a field, and $a, b \in k^*$ not squares, and $\Delta = (a, b)$ the quaternion algebra

$$(a, b) = k\langle x, y\rangle/(x^2 - a, y^2 - b, xy + yx).$$

Then $\mathrm{BS}_n(\Delta) = V(X^2 - aY^2 - bZ^2) \subseteq \mathbb{P}_k^2$.

Example 3.3. Let k be a field, and $\Delta \subseteq M_2(k)$ be

$$\Delta = \left(\begin{array}{cc} k & k \\ 0 & k \end{array} \right).$$

Then $\mathrm{BS}_n(\Delta) \cong \mathbb{P}_k^1 \vee \mathbb{P}_k^1$

Lemma 3.3. *Let $f : Y \to X$ be a morphism of schemes. Then*

$$\mathrm{BS}_n(A) \times_X Y \cong \mathrm{BS}_n(f^*A).$$

Proof. Let $F : \underline{\mathrm{Sch}}_Y \to \underline{\mathrm{Sch}}_X$ be the natural map induced by X. Then for any $V \in \underline{\mathrm{Sch}}_X$,

$$h_V \circ F = h_{V \times_X Y}.$$

Thus to show that

$$\mathrm{BS}_n(A) \times_X Y \cong \mathrm{BS}_n(f^*A),$$

by Yoneda, it suffices to prove that

$$\mathfrak{BS}_n(A) \circ F = \mathfrak{BS}_n(f^*A).$$

Take $T \in \underline{\mathrm{Sch}}_Y$ with $\pi : T \to Y$. Then

$$\begin{aligned} \mathfrak{BS}_n(A)(F(T)) &= \{I \subseteq (f\pi)^*A : I \text{ loc. free rk. } n \text{ ideal}\} \\ &= \{I \subseteq \pi^* f^* A : I \text{ loc. free rk. } n \text{ ideal}\} \\ &= \mathfrak{BS}_n(f^*A)(T). \end{aligned}$$

□

Proposition 3.4. *Let A be an Azumaya algebra. Then Brauer-Severi scheme $\mathrm{BS}_n(A)$ is etale-locally isomorphic to $\mathbb{P}_X^{n-1}$, and the functor $A \mapsto \mathrm{BS}_n(A)$ induces a bijective correspondence*

$$\mathrm{Az}_n(X) \longleftrightarrow \{\textit{iso classes of twisted forms of } \mathbb{P}_K^{n-1}\}.$$

Proof. The fact that $\mathrm{BS}_n(A)$ is etale locally isomorphic to $\mathbb{P}_X^{n-1}$ follows from the previous lemma and the fact that A is etale locally isomorphic to M_n. □

4 Classical Theory of Maximal Orders

4.1 Basic Definitions and Facts

Let R be a noetherian integral domain with fraction field K, and let Δ be a central simple K-algebra, ie. a simple K-algebra with center K.

Definition 4.1. An *R-order* in Δ is an R-subalgebra A of Δ, finitely generated as an R-module, such that $A \otimes_R K = \Delta$. The set $\underline{\mathrm{Ord}}_{R,\Delta}$ forms a poset with ordering defined by inclusion. A *R-order in* Δ is *maximal* if it is not properly contained in any R-order in Δ

Note that it follows from the definition that if A is an R-order in Δ, then the center of A contains R.

Lemma 4.1. *Let $S \subseteq R\backslash\{0\}$ be a multiplicative subset of R. Then there is a morphism of posets*

$$S^{-1} : \underline{\mathrm{Ord}}_{R,\Delta} \to \underline{\mathrm{Ord}}_{S^{-1}R,\Delta}, \quad A \mapsto S^{-1}A$$

which is surjective and sends maximal orders to maximal orders.

Proof. If A is an R-order in Δ, then the fact that $S^{-1}A$ is an $S^{-1}R$-order in Δ follows from the exactness of localization.

Now suppose that A' is an $S^{-1}R$-order in A, and let $a_1, \ldots, a_m$ be a set of generators for A' as an $S^{-1}R$-module. Set $M = Ra_1 + \cdots + Ra_m$ and let

$$A = \{a \in A' : Ma \subseteq M\}.$$

It is clear from the definition that A is an R-subalgebra of A' and that $S^{-1}A = A'$; in particular this also tells us that $A \otimes_R K = \Delta$. Thus to show that A is an R-order in Δ, all that is left to show is that A is finitely generated as an R-module. To see this, note that the action of A on M on the right induces an R-module monomorphism of A into the finitely generated R-module $\mathrm{End}_R(A)$. Since R is noetherian, it follows that A is finitely generated.

Lastly, suppose that A is a maximal R-order in Δ, and let $A' = S^{-1}A$, and suppose that A' is contained in another $S^{-1}R$-order B'. Then there exists a nonzero $r \in R$ such that $rB' \subseteq A'$. Define $I = A \cap rB'$, then I is a two-sided ideal of A with $S^{-1}I = rB'$. Define $B = \{b \in B' : Ib \subseteq I\}$. By a similar argument to the one of the previous paragraph, B is an R-order in Δ and $S^{-1}B = B'$. Furthermore, B contains A and therefore by maximality $B = A$. Hence $B' = S^{-1}B = S^{-1}A = A'$. $\square$

Corollary 4.2. *Let A be an R-order in Δ. Then the following are equivalent.*

(a) A is maximal

(b) $A_\mathfrak{p}$ is maximal for every prime ideal $\mathfrak{p}$ of R

If in addition R is integrally closed in its fraction field, then (a) or (b) is also equivalent to *(c) A is a reflexive R-module and $A_{\mathfrak{p}}$ is maximal for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht}(\mathfrak{p}) = 1$.*

Proof. The fact that (a) implies (b) follows immediately from the previous lemma. Conversely, assume (b) and suppose that B is any R-order containing A. Then $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$, and therefore by maximality, $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$. It follows that $A = B$, and hence A is maximal.

Next suppose that R is also integrally closed. Note that A^{**} is an R-order containing A, and therefore if A is maximal A is reflexive. Hence by the previous lemma (a) implies (c). Conversely, assume (c). Suppose that B is any R-order containing A. Then by maximality at height one primes, we have that $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for all $\mathfrak{p}$ with $\operatorname{ht}\mathfrak{p} = 1$. Then since R is integrally closed,

$$A^{**} = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_{\mathfrak{p}} = \bigcap_{\operatorname{ht}\mathfrak{p}=1} B_{\mathfrak{p}} = B^{**}.$$

Since A is reflexive, this shows that $A = B^{**}$. However, B^{**} is an R-order containing B, and therefore A contains B. Hence $A = B$, and therefore A is maximal. Thus (c) implies (a). □

Definition 4.2. Let A be an R-order in Δ. We define the *reduced norm* $\mathrm{N}_{\mathrm{red}}(a)$, *reduced trace* $\mathrm{Tr}_{\mathrm{red}}(a)$, and *reduced characteristic polynomial* $\mathrm{Ch}_{\mathrm{red}}(x; a)$ to be the corresponding data as elements of the Azumaya algebra Δ.

Lemma 4.3. *Suppose that R is integrally closed and A is an R-order in Δ. Then for all $a \in A$, the reduced characterisic polynomial $\mathrm{Ch}_{red}(x; a)$ is in $R[x]$. Consequently, the reduced norm and reduced trace are also both in R.*

Proof. Let $a \in A$. Then since A is finitely generated over R, a is integral over R. Let $f(x) \in K[x]$ be the minimal polynomial of a; since R is integrally closed, $f(x) \in R[x]$. Note that $f(x)$ and $\mathrm{Ch}_{\mathrm{red}}(x; a)$ have the same roots over the algebraic closure $\overline{K}$ of K (not counting algebric multiplicity), since they are exactly the eigenvalues of the left action of $a \otimes 1$ on $\Delta \otimes_K \overline{K}$. This means that the roots of $\mathrm{Ch}_{\mathrm{red}}(x; a)$ are all algebraic over R. Hence the coefficients of $\mathrm{Ch}_{\mathrm{red}}(x; a)$ are algebraic over R. Since $\mathrm{Ch}_{\mathrm{red}}(x; a) \in K[x]$ and R is algebraically closed in K, this implies that $\mathrm{Ch}_{\mathrm{red}}(x; a) \in R[x]$. □

Proposition 4.4. *Let R be integrally closed. Then every R-order A in Δ is contained in a maximal R-order in Δ.*

Proof. Let B be any R-order in Δ. Then the reduced trace defines a nondegenerate bilinear form on B. Therefore the map $B \mapsto B^*$ defined by

$$b \mapsto b^* : x \mapsto \mathrm{Tr}_{\mathrm{red}}(xb)$$

is an injective R-module homomorphism.

Let $\Omega \subseteq \underline{\mathrm{Ord}}_{R,\Delta}$ be the set of all R-orders in Δ containing A. Then Ω is nonzero, since it contains A, moreover if $B_1 \subseteq B_2 \subseteq \ldots$ is a chain in Ω, then $B := \bigcup_i B_i$ is an R-subalgebra of A. Moreover the R-module monomorphisms

$B_i \to B_i^* \subseteq A^*$ induce an R-module monomorphism $B \to A^*$. Since A^* is a finitely generated R-module and R is noetherian, it follows that B is finitely generated. Hence $B \in \Omega$. Thus Zorn's lemma tells us that A is contained in a maximal R-order in Δ. □

Proposition 4.5. *If A is an Azumaya algebra on R, then A is a maximal R-order in Δ.*

Proof. We claim that the natural map $A \mapsto A^*$ given by $a \mapsto a^* : x \mapsto \mathrm{Tr}_{\mathrm{red}}(xa)$ is an isomorphism. Note that by the nondegeneracy of the reduced trace on Δ, this map is injective, so it suffices to show that it is surjective. To see this, choose a basis $a_1, \ldots, a_N$ of A as a free R-module and consider the matrix $T = \mathrm{Tr}_{\mathrm{red}}(a_i a_j) \in M_N(R)$. Since A is an Azumaya algebra, the discriminant $d = \det(T)$ of A is a unit. Therefore T is invertible in R. Hence for $\vec{r} = T^{-1}\vec{e}_1$, and $a = \sum_i a_i r_i$, we have

$$\mathrm{Tr}_{\mathrm{red}}(a_1 a) = \vec{e}_1 \cdot T\vec{r} = 1.$$

Thus A contains an element with reduced trace 1, namely y. For any $\varphi \in \mathrm{Hom}_R(A, R)$, define $\psi \in \mathrm{End}_R(A)$ by $\psi(x) = \varphi(x)y$. Then since A is Azumaya, the natural map $A \otimes_R A^{op} \to \mathrm{End}_R(A)$ is an isomorphism, and therefore there exists $u, v \in A$ such that $\psi(x) = uxv$. It follows that

$$\begin{aligned} \varphi(x) &= \varphi(x)\mathrm{Tr}_{\mathrm{red}}(y) = \mathrm{Tr}_{\mathrm{red}}(\varphi(x)y) \\ &= \mathrm{Tr}_{\mathrm{red}}(\psi(x)) = \mathrm{Tr}_{\mathrm{red}}(uxv) = \mathrm{Tr}_{\mathrm{red}}(xvu) = (vu)^*(x). \end{aligned}$$

This proves that $A \mapsto A^*$ is an isomorphism. Now if B is any R-order in Δ containing A, then $A \subseteq B \to B^* \subseteq A^*$, from which it follows that $A = B$. □

Example 4.1. Let $R = k[x]$, and let $\Delta = M_N(k(x))$. Then

$$A = \begin{pmatrix} R & R \\ xR & R \end{pmatrix}$$

is an R-order in Δ. It is not maximal, since it is contained in the R-order $M_N(R)$. Furthermore, $M_N(R)$ is an Azumaya algebra, and therefore is a maximal R-order in Δ containing A. Note that it is not the only R-order in Δ containing A, since for example A is contained in the R-order

$$B = \begin{pmatrix} R & x^{-1}R \\ xR & R \end{pmatrix}.$$

Note that A and B are isomorphic under an inner automorphism of Δ, and therefore B is also a maximal R-order in Δ.

Note that in the previous example, both of the maximal R-orders in Δ are Azumaya algebras – and in fact all of the maximal R-orders in Δ are Azumaya. This is indicative of the following more general phenomenon, which we discuss in greater detail in the section below on ramification.

4.2 Maximal Orders over a complete DVR

Let R be a complete DVR with valuation ν, uniformizer π, residue field k, and fraction field K. Let Δ be a central simple K-algebra; by Wedderburn's Theorem, we may take $\Delta = M_N(D)$, for D a division algebra with center K, with $\dim_K(D) = m$. In this section, we will show that the maximal R-order in Δ is essentially unique. More specifically, we will show that there exists a unique maximal R-order A in D, and that any maximal R-order in Δ is isomorphic to $M_N(A)$ via an inner automorphism of Δ. The exposition we provide is based on that found in Reiner [9], which in turn is based an earlier paper of Hasse.

The main ideal is that the function $w : D \to K$ defined by

$$w(x) = \frac{1}{m}\nu(\det(x)), \quad \forall x \in D$$

extends the valuation ν to a valuation on all of D. Our next goal is to prove this, and it will take a bit of doing.

Lemma 4.6. *Suppose $f \in K[x]$ is an irreducible polynomial with*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_1 x + a_0.$$

Then for all $0 \leqslant i \leqslant n$, we have that

$$\nu(a_i) \geqslant \min\{\nu(a_0), \nu(a_n)\}.$$

Proof. We will assume that there exists i such that $\nu(a_i) < \min\{\nu(a_0), \nu(a_n)\}$ and will arrive at a contradiction. Let $t = \min_i \nu(a_i)$, and set

$$m = \min\{0 \leqslant i \leqslant n : \nu(a_i) = t\}.$$

Note that $a_m \neq 0$. Then since f is irreducible in $K[x]$, so too is the polynomial

$$g(x) := a_m^{-1} f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0,$$

for $b_i = a_i/a_m$. Note that by definition $m \geqslant 1$ and for all $0 \leqslant i < m$, we have $b_i \in \pi R$. Therefore reducing g modulo the maximal ideal of R, we find

$$\overline{g}(x) = x^m(\overline{b}_n x^{n-m} + \cdots + \overline{b}_{m+1}x + \overline{b}_m).$$

By Hensel's lemma, it follows that $g(x)$ is reducible, a contradiction. This proves our lemma. □

Lemma 4.7. *Let $a \in D$ and let $f(x) \in K[x]$ be it's minimal polynomial. Then*

$$w(a) = \frac{1}{[K(a) : K]}\nu(f(0)).$$

Consequently, $w(a) \geqslant 0$ if and only if a is integral over R.

Proof. Let $g(x)$ be the characteristic polynomial of a, viewed as an endomorphism of D, and let

$$g(x) = x^m + \cdots + (-1)^m \det(a).$$

and must divide a power of $f(x)$. Hence $g(x) = f(x)^{m/r}$, from which it follows that

$$(-1)^m \det(a) = f(0)^{m/r},$$

and therefore

$$w(a) = \frac{1}{m}\nu(\det(a)) = \frac{1}{r}\nu(f(0)) = \frac{1}{[K(a) : K]}\nu(f(0)).$$

If a is integral over R, then $g(x) \in R[x]$, and therefore $\det(a) \in R$. Hence $w(a) \geqslant 0$. Conversely, suppose that $w(a) \geqslant 0$, then $\nu(f(0)) > 0$ and therefore by the previous lemma $f(x) \in R[x]$. Consequently, a is integral over R. □

We are now in a position to show that w defines a discrete valuation on D.

Theorem 4.8. *Let D be a division K-algebra with $\dim_K(D) = m < \infty$. Then the valuation ν extends to a discrete valuation w on D, defined by*

$$w(x) = \frac{1}{m}\nu(\det(x)),\ \forall x \in D$$

where $\det(x)$ is the determinant of x viewed as an endomorphism of D by left multiplication.

Proof. It's clear from the definition that $w|_K = \nu$. To prove that w is a discrete valuation on D, we must verify the following

(a) $w(x) = \infty$ if and only if $x = 0$

(b) $w(ab) = w(a) + w(b)$ for all a, b

(c) $w(D) \cong \mathbb{Z}$

(d) $w(a + b) \geqslant \min\{w(a), w(b)\}$

Note that (a), (b) follow immediately from the definition and the fact that D is a division algebra. Furthermore, $w(D)$ is an infinite subgroup of $\mathbb{Z}$, and therefore (c) holds. Thus it suffices to prove (d).

Suppose that $w(a) \geqslant 0$, and let $f(x)$ be the minimal polynomial of a. Then by the previous lemma $f(x) \in R[x]$, and since $f(x-1)$ is the minimal polynomial of $a + 1$, it follows that $a + 1$ is integral over R. Hence by the previous lemma, $\nu(a + 1) \geqslant 0$. Thus

$$\nu(a + 1) \geqslant 0 = \min\{\nu(a), \nu(1)\}.$$

Now take $b, c \in K$ and without loss of generality assume $\nu(b) \leqslant \nu(c)$. Then

$$\nu(cb^{-1} + 1) \geqslant \min\{\nu(c/b), \nu(1)\}$$

and by adding $\nu(b)$ to both sides, we obtain (d). □

As it turns out, the way that we defined w is in fact the only way that one may extend the valuation ν to D. To prove this, we establish the following lemma.

Lemma 4.9. *Suppose $f \in K[x]$ is an irreducible monic polynomial, with*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + xa_1 + a_0.$$

Then for all $0 \leqslant i \leqslant n-1$,

$$\nu(a_i) \geqslant \frac{n-i}{n}\nu(a_0).$$

Proof. Let L be the splitting field of f, and let ν' be the extension of ν to L given by $\nu'(x) = \frac{1}{n}\nu(\det(x))$. Let $b_1, \ldots, b_n \in L$ be the roots of f. Then for all i, $f(x)$ is the characteristic polynomial of b_i; hence $a_0 = f(0) = (-1)^n \det(b_i)$, and therefore $\nu'(b_i) = \frac{1}{n}\nu(a_0)$. Moreover $f(x) = \prod_i (x - b_i)$, making the coefficients of f the various elementary symmetric polynomials in b_i and therefore

$$\nu(a_i) = \nu'(a_i) \geqslant \frac{n-i}{n}\nu(a_0).$$

□

Corollary 4.10. *The valuation w is the unique extension of ν to D.*

Proof. Suppose that w' is another extension of ν to D that doesn't agree with w. Then there exists $a \in D$ with $w'(a) > w(a)$. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be the minimal polynomial of $f(x)$ (note that $a_n = 1$). Then we have that

$$a_0 = -\sum_{i=1}^{n} a_i a^i,$$

and therefore

$$\nu(a_0) = w'(a_0) \geqslant \min_i\{\nu(a_i) + iw'(a)\} > \min_i\{\nu(a_i) + iw(a)\}.$$

Furthermore, by the previous lemma

$$\nu(a_i) \geqslant \frac{n-i}{n}\nu(a_0) = (n-i)w(a),$$

and therefore the above inequality tells us

$$\nu(a_0) > nw(a),$$

which is a contradiction. Thus $w = w'$. □

Using the valuation w, it is now easy to explicitly define the maximal R-order in D. It is determined by

$$A = \{a \in D : w(a) \geqslant 0\}.$$

It's clear from the definition that A is an R-subalgebra of D, and that $A \otimes_R K = D$. Furthermore, if B is any other R-order in D, then every $b \in B$ is integral over R. This implies that $w(b) \geqslant 0$, and therefore that $b \in A$. Thus A containes all other R-orders in D. Thus to show that A is a maximal R-order in D, it suffices to prove that A is a finitely generated R-module. To prove this, we need to more finely probe the structure of A.

We fix an element $\pi_D \in A$ satisfying $\nu_D(\pi_D) = 1$.

Proposition 4.11. *Let A be the R-algebra defined above. Then*

(a) $J(A) = \pi_D A$ is the unique maximal left (or right or two-sided) ideal of A

(b) $J(A) \cap R = \pi R$

(c) $A/J(A)$ is a division k-algebra

(d) if I is a nonzero left (or right ideal of A, then $I = J(A)^m = \pi_D^m A$ for some m

Proof. Each of these statements follows from mucking about with v_D. □

Definition 4.3. Let A be a maximal R-order in D. We define the *ramification index* $e = e(D/R)$ of D over K to be the unique integer e satisfying $J(A)^e = \pi A$. We define the *inertial degree* $f = f(D/R)$ to be $f = \dim_k(A/\pi A)$.

Theorem 4.12. *The R-algebra A is a finitely generated free R-module of rank $n = \dim_K(D)$. Moreover, if e and f are the ramification index and inertial degree, respectively, then $ef = n$.*

Proof. Let e be the ramification index. Choose $a_1, \ldots, a_f \in A$ with $\overline{a}_1, \ldots \overline{a}_f \in A/\pi A$ a k-basis. We claim that the set

$$\{a_i \pi^j : 1 \leqslant i \leqslant f,\ 0 \leqslant j \leqslant e-1\},$$

is K-linearly independent. To see this, suppose otherwise. Then there exist $\alpha_{ij} \in K$ with

$$\sum_{ij} \alpha_{ij} a_i \pi^j = 0.$$

For each j set $n_j = \min_i \nu(\alpha_{ij})$, and define β_{ij} by $\alpha_{ij} = \pi^{n_j}\beta_{ij}$. Then $\beta_{ij} \in R$ for all i, j, and for all i there exists j with $\beta_{ij} \notin \pi R$. Define $b_j = \pi^{n_j}\pi_D^j$ and σ be a permutation of $\{0, \ldots, e-1\}$ such that $\nu_D(b_{\sigma(0)}) < \cdots < \nu_D(b_{\sigma(e-1)})$, set $c_j = b_{\sigma(j)}/b_{\sigma(0)}$ and $\gamma_{ij} = \beta_{i\sigma^{-1}(j)}$. Then $\sum_{ij} \gamma_{ij} a_i b_j = 0$, with $\gamma_{ij} \in R$ for all i, j, $\gamma_{i1} \notin \pi R$ for some i and $c_j \in \pi A$ for all $j \geqslant 2$. Therefore reducing modulo πA we obtain $\sum_i \overline{\gamma}_{i1}\overline{a}_i = 0$, which contradicts the assumption of k-linear independence.

Lastly, we claim that $A = R\{a_i\pi_D^j\}$. To see this, suppose that $x \in A$, and let $\nu_D(x) = ke + j$ for integer j, k with $0 \leqslant j < e$. Then we may write $x = \pi^k\pi_D^j u$ for some $u \in A^\times$. Therefore by definition of the a_i's, there exist $r_i \in R$ such that $u = \sum_i r_i a_i$ modulo πA. Setting $x_1 = \pi^k\pi_D^j(u - \sum_i r_i a_i)$ we have that $x = y_1 + x_1$ for some $y_1 \in R\{a_i\pi_D^j\}$ and $x_1 \in A$ with $\nu_D(x_1) > \nu_D(x)$. Repeating this process, we obtain a sequence $x_i = y_{i+1} + x_{i+1}$ with $\nu_D(x_{i+1}) > \nu_D(x_i)$ and for all integers $m > 0$

$$x = \sum_{\ell=1}^{m} y_\ell + x_m = \sum_{i,j} r_{ijm} a_i \pi_D^j + x_m.$$

The for each i, j the sequence of coefficients r_{ijm} is Cauchy sequences in R, and since R is complete, converge to a limit r_{ij}. It follows that $x = \sum_{i,j} r_{ij} a_i \pi_D^j$, proving our claim. □

Corollary 4.13. *The algebra A is the unique maximal R-order in D.*

Proof. Note that A is the integral closure of R in Δ. The previous theorem shows that A is finitely generated as an R-module, and it is clear from the definition that A is an R-subalgebra of Δ with $A \otimes_R K = \Delta$. Therefore A is an R-order in Δ, and since every R-order in Δ consists of elements integral over R, it is necessarily maximal. □

Theorem 4.14. *The matrix algebra $M_N(A)$ is a maximal R-order in Δ. Furthermore*

(a) $J(A) = \pi_D M_N(A)$ is the unique maximal two-sided ideal of A

(b) if I is any two-sided ideal of A, then $I = J(A)^n = \pi_D^n M_N(A)$ for some n

(c) if B is any other maximal R-order in A, then B is of the form $uM_N(A)u^{-1}$ for some invertible $u \in \Delta$

4.3 Maximal orders over DVRs

In this section, let R be an arbitrary DVR with uniformizer π and fraction field K. In the last section, we determined a great many structural theorems regarding maximal orders over complete DVRs. In this section, we will show how the theory of maximal orders over R may be related to the theory of maximal orders over their completion $\widehat{R}$. For notational convenience, in this section Δ will be a CSA$/K$, $\widehat{K}$ will be the fraction field of $\widehat{R}$, and $\widehat{\Delta}$ will be $\Delta \otimes_K \widehat{K}$. In particular, $\widehat{\Delta}$ is a CSA$/\widehat{K}$, and Δ is a division algebra if and only if $\widehat{\Delta}$ is a division algebra.

The basic idea of relating the theory of maximal orders on R to the theory of maximal orders on $\widehat{R}$ is based on fpqc descent. In particular, take $X = \mathrm{Spec}(R)$, $U_1 = \mathrm{Spec}(\widehat{R})$ and $U_2 = \mathrm{Spec}(K)$. Then the two natural maps $f_i : U_i \to X$ form an fppf covering of X. Descent data on this cover consists of a sheaves

F_i on U_i with isomorphisms $\varphi_{ij} : p_{ij1}^* F_i \to p_{ij2}^* F_j$ for all $1 \leqslant i \leqslant j \leqslant 2$, where $p_{ijk} : U_i \times_X U_j \to U_k$ is the natural projection. In our case, we have canonical identifications $U_1 \times_X U_1 = U_1$, $U_2 \times_X U_2 = U_2$ and $U_1 \times_X U_2 = \mathrm{Spec}(\widehat{K})$ with $p_{iij} = \mathrm{id}_{U_i}$ and $p_{12j} : \mathrm{Spec}(\widehat{K}) \to U_j$ the natural map. Therefore the only nontrivial descent data comes from the isomorphisms φ_{ij} for $i \neq j$. Unwinding definitions, descent theory then tells us that

$$M \mapsto (M \otimes_R \widehat{R}, M \otimes_R K)$$

defines a bijective correspondence between R-modules M and pairs $(\widehat{M}, V)$ with $\widehat{M}$ an $\widehat{R}$-module and V a K-vector space satisfying $\widehat{M} \otimes_{\widehat{R}} \widehat{K} = V \otimes_K \widehat{K}$.

Proposition 4.15. *Let $\widehat{\Delta} = \Delta \otimes_K \widehat{K}$. Then $\widehat{\Delta}$ is a central simple $\widehat{K}$-algebra and there is a bijective morphism of posets*

$$\underline{\mathrm{Ord}}_{R,\Delta} \to \underline{\mathrm{Ord}}_{\widehat{R},\widehat{\Delta}}, \quad A \mapsto A \otimes_R \widehat{R}.$$

In particular A is a maximal R-order in Δ if and only if $A \otimes_R \widehat{R}$ is a maximal $\widehat{R}$-order in $\widehat{\Delta}$.

Proof. Taking $V = \Delta$ in the above fpqc correspondence shows that $M \mapsto M \otimes_R \widehat{R}$ is a bijective correspondence between R-modules M satisfying $M \otimes_R K = \Delta$ and $\widehat{R}$-modules $\widehat{M}$ satisfying $\widehat{M} \otimes_{\widehat{R}} \widehat{K} = \widehat{\Delta}$. This restricts to the desired poset isomorphism. □

Thus the study of maximal orders on R is closely tied to the study of maximal $\widehat{R}$-orders on $\widehat{\Delta}$. Using this, we deduce some of the structure of maximal R-orders in Δ.

Corollary 4.16. *If A is a maximal R-order in Δ, then A is left and right hereditary.*

Proposition 4.17. *Let A be an R-order in Δ, and let $\overline{A} = A/\pi A$. Then*

$$A/J(A) \cong \overline{A}/J(\overline{A}) \cong (\widehat{A})/J(\widehat{A}).$$

Proof. By elementary ring theory, $A/J(A) \cong \overline{A}/J$ and $\widehat{A}/J(\widehat{A}) \cong \overline{\widehat{A}}/J(\overline{\widehat{A}})$, where $\overline{\widehat{A}} = \widehat{A}/\pi\widehat{A}$. Therefore it suffices to show that $\overline{A} \cong \overline{\widehat{A}}$. This in turn follows from the fact that $\pi\widehat{A} \cong \pi A \otimes \widehat{R}$ and the exactness of $- \otimes \widehat{R}$. □

Theorem 4.18. *Let A be a maximal R-order in Δ. The ideal $J(A)$ is the unique maximal left (or right) ideal of A, and furthermore*

$$J(\widehat{A}) = J(A) \otimes \widehat{R}, \quad J(\widehat{A}) \cap A = J(A).$$

In fact, every nontrivial two-sided ideal of A is a power of $J(A)$.

Proof. We have a natural bijective correspondence

$$\{\text{maximal left ideals of } A\} \longleftrightarrow \{\text{maximal left ideals of } A/J(A)\}$$

and

$$\{\text{maximal left ideals of } \widehat{A}\} \longleftrightarrow \{\text{maximal left ideals of } \widehat{A}/J(\widehat{A})\}$$

so by the previous proposition, there is a bijective correspondence between maximal left ideals of A and maximal left ideals of $\widehat{A}$. Since $\widehat{A}$ has exactly one maximal left ideal, it follows that A has exactly one also. Then, since $J(A)$ is the intersection of all maximal left ideals, $J(A)$ must be the maximal left ideal of A. Furthermore, tracing the above correspondence, we see that $J(A) \otimes \widehat{R} = J(\widehat{A})$ and $J(A) = J(\widehat{A}) \cap A$.

Now if I is a nontrivial two-sided ideal of A, then $I \otimes K$ is a nontrivial two-sided ideal of Δ. Hence $I \otimes K = \Delta$, and therefore there exists $a_i \in I$ and $r_i \in K$ with $\sum_i a_i r_i = 1$. Choose $r \in R$ such that $rr_i \in R$ for all i, and set $s = \nu(r)$. Then $r = \sum_i a_i(rr_i) \in I$ and therefore $rA = \pi^s A \subseteq I$ for all I. Therefore I is the preimage of a two-sided ideal of $A/\pi^s A \cong \widehat{A}/\pi^s \widehat{A}$. Therefore $I \otimes \widehat{R}$ is a preimage of a two-sided ideal of $\widehat{A}/\pi^s \widehat{A}$, and therefore $I \otimes \widehat{R} = \pi^t \widehat{A} = J(\widehat{A})^t$ for some $t \leqslant s$. This tells us that $I = (I \otimes \widehat{R}) \cap A = J(\widehat{A})^t \cap A = J(A)^t$. ☐

Lemma 4.19. *Let A be a maximal R-order in Δ, and let M be a finitely generated left A-module. Then*

(a) A is a projective A-module

(b) A is a free R-module

(c) if B is another finitely generated free R-module, then $A \cong B$ as A-modules if and only if A and B have the same rank as R-modules

Theorem 4.20. *Let A be a maximal R-order in Δ. Then*

(a) Every one-sided ideal of A is principal.

(b) If B is another maximal R-order in Δ then $B = uAu^{-1}$ for some $u \in \Delta^\times$.

(c) Let $\widehat{\Delta} = M_t(\widehat{D})$ for $\widehat{D}$ a division algebra over $\widehat{K}$, and let $\widehat{C}$ be a unique maximal R-order in $\widehat{D}$. Then $\widehat{C}/J(\widehat{C})$ is a division algebra over k and $A/J(A) = M_t(\widehat{C}/J(\widehat{C}))$.

4.4 Normal Orders

Let R be a DVR with fraction field K, maximal ideal $\mathfrak{m}$, uniformizer π, and residue field k. Also let $\Delta = M_N(D)$ with D a CDA over K.

Definition 4.4. Let A be an R-order in Δ. Then A is called *normal* if $J(A) = At$ for some $t \in A$.

Note in particular that maximal R-orders are normal.

Lemma 4.21. *If A is a normal R-order in Δ, then A is hereditary.*

Proof. See [5]. $\square$

Lemma 4.22. *Let R' be a DVR with fraction field K', let $\Delta' = \Delta \otimes_K K'$ and suppose that $R \to R'$ is an etale morphism of rings. Then if A is a normal R-order in Δ, $A \otimes_R R'$ is a normal R'-order in Δ'. Furthermore, if $J(A) = At$, then $J(A') = A'(t \otimes 1)$.*

Proof. See [5]. $\square$

In the case that R is complete and k is sufficiently nice, the normal R-orders in Δ are known up to isomorphism. In particular, if B is the unique R-order in D, then the isomorphism class of a normal R-order in Δ is determined by its ramification index and inertia degree.

Definition 4.5. Let e, f be positive integers. We define the *standard hereditary order* $A_{e,f}(B)$ by

$$A_{e,f}(B) = A_{e,1}(B) \otimes_R M_f(R),$$

with

$$A_{e,1}(B) = \begin{pmatrix} B & B & \dots & B & B \\ J(B) & B & \dots & B & B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J(B) & J(B) & \dots & B & B \\ J(B) & J(B) & \dots & J(B) & B \end{pmatrix} \subseteq M_e(B).$$

Lemma 4.23. *Let R be a complete DVR, and $ef = N$. Also let B be the unique maximal R-order in D, with $J(B) = \pi_D B = B\pi_D$. Then*

(a) $A_{e,f}(B)$ is an R-order in Δ

(b) $J(A_{e,f}) = J(A_{e,1}) \otimes_R M_f(R)$, with

$$A_{e,1}(B) = \begin{pmatrix} J(B) & B & \dots & B & B \\ J(B) & B & \dots & B & B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J(B) & J(B) & \dots & J(B) & B \\ J(B) & J(B) & \dots & J(B) & J(B) \end{pmatrix} \subseteq M_e(B).$$

(c) $A_{e,f}$ is normal, and in fact $J(A_{e,f}) = A_{e,f}t$ for

$$t = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \pi & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \otimes_R I.$$

Proof. See [5]. ☐

Theorem 4.24. *Let R be a complete DVR, and let B be the unique maximal R-order in D. Then an R-order in $\Delta = M_N(D)$ is normal if and only if it is isomorphic to $A_{e,f}(B)$ for some integers e, f.*

Proof. See [5]. ☐

4.5 Ramification of Maximal Orders over DVRs

In this section R will always be a DVR with fraction field K, uniformizer π and residue field k, and Δ will represent a central simple K-algebra of rank n.

Definition 4.6. Let A be a maximal R-order in Δ. The central simple algebra Δ is said to *ramify* over R if A is not an Azumaya algebra on R. We define the *algebraic ramification index* $e = e(\Delta/R)$ to be the unique integer satisfying $J(A)^e = \pi J(A)$, and the *cohomological ramification index* $e' = e'(\Delta/R)$ to be $e' = \dim_k(Z(A/J(A)))$. We define the *inertia degree* $f = f(\Delta/R)$ to be the dimension of $A/J(A)$ over its center.

Note that since all of the maximal R-orders in Δ are conjugate, whether or not Δ ramifies and the value of its ramification index is independent of the choice of maximal R-order in Δ.

Lemma 4.25. *Let A be a maximal R-order in Δ, with algebraic and cohomological ramification index e and e', respectively, and with inertia degree f. Then $ee'f^2 = n^2$.*

Proof. Since A is a free R-module of rank n^2, we have

$$n^2 = \dim_k(\overline{A}) = e\dim_k(\overline{A}/J(\overline{A})) = e\dim_k(A/J(A)) = ee'f^2.$$

☐

Proposition 4.26. *Let A be a maximal R-order in Δ, with algebraic and cohomological ramification e and e', respectively. Then the following are equivalent*

(a) Δ is ramified over R

(b) $e > 1$ or $e' > 1$

Proof. If Δ is unramified over R, then A is an Azumaya algebra and therefore $\overline{A} = A/\pi A$ is an Azumaya algebra on k. In particular, this means $\overline{A}$ is a CSA over k, so that $J(\overline{A}) = 0$ and $Z(\overline{A}) = k$. Since $A/J(A) \cong \overline{A}/J(\overline{A})$, this implies that $A/J(A) \cong \overline{A}$. Hence $\dim_k(Z(A/J(A))) = 1$ and $J(A) = \pi A$. Therefore (b) implies (a).

If Δ is ramified over R, then A is not Azumaya, and therefore $\overline{A}$ is not Azumaya. If $e = 1$, then $\overline{A}$ is simple with center a nontrivial cyclic extension of k, and therefore $e' > 1$. Hence (a) implies (b). ☐

Theorem 4.27. *Suppose there is a finite extension K'/K which is unramified over R and splits Δ. Then the algebraic and cohomological ramification of Δ over R are the same.*

Proof. See [5]. $\square$

5 Maximal Orders on Integral Schemes

In this section, let X be a normal integral scheme with fraction field K, and let Δ be a CSA over K. The constant sheaf on X associated to Δ defines a sheaf of $\mathcal{O}_X$-algebras, which as an abuse of notation we also denote by Δ. In writing this section, the author found useful [7], [2], and [5].

5.1 Basic Definitions and Facts

Definition 5.1. An $\mathcal{O}_X$*-order in* Δ is a coherent $\mathcal{O}_X$-module A which is also a $\mathcal{O}_X$-subalgebra of Δ. The $\mathcal{O}_X$-order A is *maximal* if it is not properly contained in any other $\mathcal{O}_X$-order in Δ.

Proposition 5.1. *Let A be a $\mathcal{O}_X$-order in Δ. Then the following are equivalent*

(a) A is maximal

(b) A_x is maximal for all $x \in X$

(c) A is reflexive and A_x is maximal for all $x \in X^1$

where here X^1 denotes the set of codimension 1 points of X.

Proposition 5.2. *Every $\mathcal{O}_X$-order in Δ is contained in a maximal $\mathcal{O}_X$-order in Δ.*

Proposition 5.3. *If A is an $\mathcal{O}_X$-order in Δ which is Azumaya, then A is a maximal $\mathcal{O}_X$-order in Δ.*

5.2 Ramification

Let X be a normal integral scheme with fraction field K, and let Δ be a CSA over K of degree n.

Definition 5.2. Let A be a maximal $\mathcal{O}_X$-order in Δ. We define the *ramification locus* of Δ over R, denoted $\mathrm{ram}(\Delta/X)$, to be the set of all $x \in X$ such that A_x is not an Azumaya algebra. The complement of the ramification locus is called the *Azumaya locus* of Δ over X.

Lemma 5.4. *The ramification locus* $\mathrm{ram}(\Delta/X)$ *is independent of the choice of maximal $\mathcal{O}_X$-order A in Δ.*

Proposition 5.5. *If Δ and Δ' are representatives of the same Brauer class over K, then* $\mathrm{ram}(\Delta/X) = \mathrm{ram}(\Delta'/X)$.

Proposition 5.6. *The ramification locus* $\mathrm{ram}(\Delta/X)$ *is a union of finitely many irreducible, codimension 1 subschemes of X.*

In particular this shows that there is a dense open subscheme of X on which Δ is represented by an Azumaya algebra.

Proposition 5.7. *Let A be a maximal $\mathcal{O}_X$-order in Δ, and consider the canonical morphism $\pi : \mathrm{BS}(A) \to X$.*

(a) $\mathrm{BS}(A)$ is nonsingular over k, and π is proper and flat

(b) for all $x \in X$, $\mathrm{BS}(A)_x$ is reducible if and only if $x \in \mathrm{ram}(\Delta/X)$

(c) for all $x \in X \backslash \mathrm{ram}(\Delta/X)$, the fiber $\mathrm{BS}(A)_x \cong \mathbb{P}^{n-1}_{\kappa(x)}$

5.3 Maximal Orders in Quaternion Algebras on Surfaces

In this section, let S be a complete, simply connected, non-singular algebraic surface over an algebraically closed field k with characteristic different from 2. We explore the presentation of maximal $\mathcal{O}_S$-orders in quaternion $K(S)$-algebras.

Proposition 5.8. *Let Δ be a quaternion algebra over $K(S)$ with nonsingular ramification curve C. Let A be a maximal $\mathcal{O}_S$-order in Δ. Then for any $s \in C$,*

$$A_s \cong \mathcal{O}_{S,s}\langle x, y\rangle/(x^2 - a, y^2 - bt, xy + yx),$$

where $a, b \in \mathcal{O}^*_{S,s}$, with a not a square (modulo t), and $t \in \mathcal{O}_{S,s}$ is a local equation for C.

Proof. Since C is nonsingular, it is a disjoint union of nonsingular algebraic curves $C_1, \dots, C_n$ in S. Let c_i be the generic point of C_i. Then $\mathcal{O}_{S,c_i}$ is a DVR (R_i, ν_i) with fraction field $K(S)$ and residue field $K(C_i)$. Furthermore, $A_i := A_{c_i}$ is a maximal R_i-order in Δ. Let t_i be a uniformizer of R_i, and let J_i be the Jacobson radical. From the classical theory of maximal orders, $L_i := A_i/J_i$ is a cyclic extension of $K(C_i)$ of degree equal to the ramification index e. Since e divides the degree of A (which is 2), and since A ramifies at c_i, we have $e = 2$ and $J_i^2 = A_i t$.

Choose $x_0 \in A_i$ such that x_0 reduces to a cyclic generator $\overline{x}_0$ of L_i modulo J_i. Then $\overline{x}_0^2 \in K(L_i)$, and since x_0 also satisfies $x_0^2 - \mathrm{Tr}_{\mathrm{red}}(x_0)x_0 + \mathrm{N}_{\mathrm{red}}(x_0) = 0$, it follows that $\mathrm{Tr}_{\mathrm{red}}(x_0) \in J_i$. Then $x := x_0 - \mathrm{Tr}_{\mathrm{red}}(x_0)$ reduces to the cyclic generator $\overline{x}_0$ modulo J_i and has reduced trace 0, so that $x^2 = -\mathrm{N}_{\mathrm{red}}(x) = a \in R_i^*$.

Next note that A_i comes equipped with an involution $z \mapsto z^* := -z + \mathrm{Tr}_{\mathrm{red}}(z)$. Since J_i is a maximal left ideal of A, J_i^* will be a maximal right ideal of A_i. However, there is only one maximal left (or right) ideal of A, namely J_i. Hence J_i is preserved by $*$. Therefore for all $z \in J_i$, $\mathrm{Tr}_{\mathrm{red}}(z) = z + z^* \in J_i \cap R_i = tR_i$. Furthermore, since $J_i^2 = tA_i$, it follows that z is equivalent to z' modulo J_i^2, where

$$z' = z - \frac{1}{2}\mathrm{Tr}_{\mathrm{red}}(z) - \frac{1}{2a}\mathrm{Tr}_{\mathrm{red}}(xz),$$

satisfies $\mathrm{Tr}_{\mathrm{red}}(z') = \mathrm{Tr}_{\mathrm{red}}(xz') = 0$. Thus by means of the L_i-linear homomorphism

$$J_i/J_i^2 \xrightarrow{\cong} J \otimes_R (L_i) \xrightarrow{\subseteq} A \otimes_R L_i \xrightarrow{\cong} M_2(L_i),$$

we may identify J_i/J_i^2 with the two-dimensional subspace of $M_2(L_i)$ consisting of matrices z satisfying $\mathrm{Tr}_{\mathrm{red}}(z) = \mathrm{Tr}_{\mathrm{red}}(xz) = 0$ (where here x is identified with its image in $M_2(L_i)$). In particular J_i/J_i^2 is two-dimensional as a vector space over L_i. Take $y \in J_i$ nonzero with $\mathrm{Tr}_{\mathrm{red}}(y) = \mathrm{Tr}_{\mathrm{red}}(xy) = 0$. Then $\{y, xy\}$ is a L_i-linearly independent set, and hence a basis for J_i/J_i^2. Nakayama's lemma then tells us that $\{y, xy\}$ generates J_i as an R_i-module. Consequently $\{1, x, y, xy\}$ generates A_i as an R_i-module. Note that since S is normal, R is integrally closed. Therefore A_i is free of rank 4, so $\{1, x, y, xy\}$ is in fact an R_i-module basis. Note also that we can calculate the discriminant of A_i using this basis to be $-16a^2b^2$.

Now since x, y, xy all have zero reduced trace,

$$-xy = (xy)^* = y^*x^* = (-y)(-x) = yx.$$

Furthermore, since $y^2 = -\mathrm{N}_{\mathrm{red}}(y) \in J_i \cap R = At$, we may write $y^2 = bt$ for some $b \in A$. Thus we may define a map

$$R\langle x, y\rangle/(x^2 - a, y^2 - bt, xy + yx) \to A_i,$$

and since the $\{1, x, y, xy\}$ form an R_i-module basis, this is an R_i-algebra isomorphism. Since x module J_i generates L_i, it's clear that x is a unit and x is not a square modulo t. Moreover, b nonzero since Δ has no nilpotent elements. If b is not a unit, then we can write $b = b_0t^m$ for some m, in which case $y \mapsto ty$ includes A_i into the larger R-order $R\langle x, y\rangle/(x^2 - a, y^2 - b_0t^{m-2}, xy + yx)$, contradicting the maximality of A_i. This proves that A_i has the presentation stated in the theorem.

Next suppose that s is a closed point of C_i, and let $A = A_s$. Then again A is locally free of rank 4 as an R-module. Let J be the kernel of the natural $\mathcal{O}_{X,s}$-module map

$$A \to A_i \to L_i.$$

This identifies the quotient A/J with a subring of L_i. Hence A/J is a finitely generated $\mathcal{O}_{C,s}$-submodule of L (hence torsion free), and since $\mathcal{O}_{C,s}$ is a PID A/J is free. Moreover, since $(A/J)\otimes_{\mathcal{O}_{C,s}} K(C_i) = L_i$, A/J has rank 2. Again as above, we may choose $x \in A$ such that $x^2 = a \in \mathcal{O}_{S,s}$ (ie. so that x has reduced trace 0) and such that x reduces to a generator for A/J, and $u, v \in J$ such that $\{1, x, u, v\}$ is a $\mathcal{O}_{S,s}$-module basis for A. Using the same strategy as before, we may choose u and v to have reduced trace 0.

Since A is a maximal $\mathcal{O}_{X,x}$-order in Δ, it should be Azumaya everywhere except on C. Therefore the bilinear form defined by the reduced trace on A degenerates exactly on C, meaning that the discriminant of A should be of the form εt^r for some integer $r > 0$ and unit $\epsilon \in A^*$. Moreover, locally at the generic point of C_i the discriminant should agree with the descriminant of A_i, so $r = 2$. However, calculating the discriminant with respect to the basis $\{1, x, u, v\}$, the fact that $\mathrm{Tr}_{\mathrm{red}}(x^2) = 2a$ and $\mathrm{Tr}_{\mathrm{red}}(ux), \mathrm{Tr}_{\mathrm{red}}(vx), \mathrm{Tr}_{\mathrm{red}}(u^2), \mathrm{Tr}_{\mathrm{red}}(v^2), \mathrm{Tr}_{\mathrm{red}}(uv)$ are all in $J \cap A_s = tA_s$, we see that the discriminant is of the form $4a\xi t^2 + \eta t^3$ for some $\xi, \eta \in \mathcal{O}_X$, from which it follows that $4a\xi = \varepsilon$ and therefore a is a unit.

Now since a is a unit $A/J = \mathcal{O}_{C,s}[x]$ is a semilocal Dedekind domain, in particular a PID. Thus J/At, being a $\mathcal{O}_{C,s}[x]$ submodule of the free module J_i/tA_i, is also free. Since $(J_i/tA_i) \otimes L_i \cong J_i/J_i^2$ is a free rank 1 module over $L_i[x]$, we see that J_i/tA_i has rank 1 over $\mathcal{O}_{C,s}[x]$. Let y be a generator; as before we may choose y such that $\mathrm{Tr}_{\mathrm{red}}(y) = \mathrm{Tr}_{\mathrm{red}}(xy) = 0$. Then again $y^2 = bt$ for some $b \in \mathcal{O}_{X,s}$, $xy = -yx$ and computing the discriminant with respect to this basis we find $-16a^2b^2t^2 = \varepsilon t^2$, and therefore b is a unit. This verifies the desired presentation in the case that s is a closed point. $\square$

Proposition 5.9. *Let $[\Delta]$ be the class of a quaternion algebra Δ over $K(S)$ with nonsingular ramification curve C, and let A be a maximal $\mathcal{O}_S$-order in Δ. Then $\pi : \mathrm{BS}(A) \to S$ satisfies the following properties*

(a) $\mathrm{BS}(A)$ is nonsingular (over k?) and π is proper and flat

(b) if $s \in S$ is a geometric point then

$$\mathrm{BS}(A)_s = \begin{cases} \mathbb{P}^1_k, & s \notin C \\ \mathbb{P}^1_k \vee \mathbb{P}^1_k, & s \in C \end{cases}$$

(c) if c_i is a generic point of an irreducible component C_i of C, then the fraction fields each of the irreducible components of $\mathrm{BS}(A)_c$ define quadratic extensions of $K(C_i)$

Proof. Let $i : \mathrm{Spec}(\kappa(s)) \to s$. By Lemma 3.3, $\mathrm{BS}(A)_s = \mathrm{BS}(i^*A)$.

(a) $\pi : \mathrm{BS}(A) \to S$ is projective, and S is nonsingular over k

(b) Since A is Azumaya at s, $i^*A \cong M_2(k)$. Therefore by Example 3.1, we have that
$$\mathrm{BS}(A)_s = \mathrm{BS}(M_2(k)) = \mathbb{P}^1_k.$$

(c) From the local presentation at $s \in C_i$, we have that
$$A_s \otimes \kappa(s) \cong \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$
which has corresponding Brauer-Severi variety $\mathbb{P}^1_k \vee \mathbb{P}^1_k$ by Example 3.3.

(d) Let $R_i = \mathcal{O}_{S,c_i}$, and let $j : \mathrm{Spec}(R_i) \to S$. Then i factors through j and therefore
$$\mathrm{BS}(A)_{c_i} = \mathrm{BS}(i^*A) = \mathrm{BS}(j^*A)_{c_i}.$$
Moreover the local presentation of A_s and Example 3.2
$$\mathrm{BS}(j^*A) = V(X^2 - aY^2 - btZ^2) \subseteq \mathbb{P}^2_{R_i},$$
where $a, b \in R_i^*$ with a not a square and t generates the the unique maximal ideal of R_i. It follows that $\mathrm{BS}(A)_{c_i} = V(X^2 - aY^2) \subseteq \mathbb{P}^2_{K(C_i)}$, which has two irreducible components whose residues define the quadratic extension $K(C_i)[\sqrt{a}]$ of $K(C_i)$.

□

Proposition 5.10. *Let Δ be a quaternion algebra over $K(S)$ with nonsingular ramification curve C, and let A be a maximal $\mathcal{O}_S$-order in Δ. If C is disconnected, then* $\mathrm{BS}(A)$ *has* 2*-torsion in* $H^4(\mathrm{BS}(A), \mathbb{Z}_2)$.

Proof. For convenience, we let $V = \mathrm{BS}(A)$ and use $\pi : V \to S$ for the canonical morphism. Let $C_1, \ldots, C_r$ be the connected components of C, and for each i let $x_i \in C_i$, and let ℓ_i be one of the irreducible components of $V_{x_i} \cong \mathbb{P}^1_{\kappa(x_i)} \vee \mathbb{P}^1_{\kappa(x_i)}$. Let j_i be the inclusion of ℓ_i into V. Then ℓ_i is an irreducible codimension 2 subvariety of V, so the cycle map cl_V sends the identity of $H^0(\ell_i, \mu_2^{\otimes -1}) \overset{\text{Gysin}}{\cong} H^4_{\ell_i}(V, \mu_2)$ to an element $\mathrm{cl}_V(\ell_i)$ of $H^4(V, \mu_2)$ of order 2. Then the composition

$$H^2(V, \mu_2) \xrightarrow{j_i^*} H^2(\ell_i, \mu_2) \overset{\text{Gysin}}{\cong} H^6_{\ell_i}(V, \mu_2^{\otimes 2}) \to H^6(V, \mu_2^{\otimes 2})$$

agrees with the cup product map $\mathrm{cl}_V(\ell_i) \cup \cdot : H^2(V, \mu_2) \to H^6(V, \mu_2^{\otimes 2})$.

The map π is proper, so by the proper base change theorem, we have that for all geometric points x of S,

$$(R^j \pi_* \mu_2)_x = H^j(V_x, \mu_2).$$

Since the Picard groups of $\mathbb{P}^1_k$ and $\mathbb{P}^1_k \vee \mathbb{P}^1_k$ are torsion-free, the Kummer sequence tells us that $H^1(V_x, \mu_2) = 0$. Therefore $R^1 \pi_2 \mu_2 = 0$. Furthermore since π is proper, $\pi_* \mu_2 = \mu_2$. Notice also that $R^2 \pi_* \mu_2$ fits into a short exact sequence

$$0 \to (\mu_2)_C \to R^2 \pi_* \mu_2 \to (\mu_2)_S \to 0.$$

Moreover, the fact that S is simply connected tells us that $H^2(S, \mu_2) = 0$. Thus the Leray spectral sequence associated to π tells us

$$0 \to H^2(S, \mu_2) \to H^2(V, \mu_2) \to H^0(S, R^2 \pi_* \mu_2) \to 0.$$

Take $\alpha \in (\mu_2)_C$ which is 1 on C_1 and 0 on C_i for $i \neq 1$, let α' be its image in $R^2 \pi_*(\mu_2)$, and let $\alpha'' \in H^2(V, \mu_2)$ be a lift of α'. Then

$$j_i^* \alpha'' = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases}$$

Therefore $(\mathrm{cl}_V(\ell_1) - \mathrm{cl}_V(\ell_2)) \cup \alpha''$ is nonzero since $j_1^* \alpha'' - j_2^* \alpha'' = 1 - 0 \neq 0$, and thus $H^4(V, \mu_2)$ contains a nontrivial element of order 2. □

6 Brauer Group of a Nonsingular Surface

6.1 Some Etale Cohomology

In this section, we turn our attention to the Brauer group of a nonsingular integral scheme X, focusing primarily on the case of a nonsingular surface. Our exposition is based on a combination of [1] and [5] In the nonsingular case, the cohomological Brauer group of X injects into the Brauer group of its function field K. Furthermore, coprime to the characteristic of K, the cokernel of this may be described by certain ramification data on the divisors of X. Specifically, we have the following main theorem

Theorem 6.1. *Let X be a nonsingular, excellent, integral scheme. If ℓ is coprime to the characteristic of K, then we have an ℓ-exact sequence*

$$0 \to \mathrm{Br}'(X) \to \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X^1} H^1(X, i_*(\mathbb{Q}/\mathbb{Z})) \to H^3(X^1, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m).$$

Here X^1 represents the set of codimension 1 points of X, and by $H^p(X^1, \mathbb{G}_m)$, we mean $\mathrm{colim} H^p(U, \mathbb{G}_m)$ where the limit is taken over all complements U of closed subsets of X with codimension at least two. In other words, we are taking cohomology viewing X^1 as a pro-object in our category. The map ρ has an interesting algebraic interpretation. In particular, given a central simple K-algebra Δ, there exists a nonempty open subset of X to which Δ extends to an Azumaya algebra. A Zorn's lemma argument then tells us that there exists a largest such open set U. In fact, if we let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-algebra in Δ, then the set of $x \in X$ for which $\mathcal{A}_x$ is Azumaya is exactly U. One may show that $X \backslash U$ is the union of the closures C_x of the $x \in X^1$ for which $\rho([\Delta])_x$ is nonzero.

The relationship extends even further than this. The cohomology group $H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$ classifies cyclic extensions of $\kappa(x)$. Since X is nonsingular, for each $x \in X^1$ the local ring $\mathcal{O}_{X,x}$ is a DVR with valuation ν_x and residue field $\kappa(x)$. Since $\mathcal{A}_x$ is a maximal order on $\mathcal{O}_{X,x}$ the theory of maximal orders on DVRs tells us that $\mathcal{A}_x/J(\mathcal{A}_x)$ is a central simple algebra, whose center is a cyclic extension of $k(x)$, and therefore represented up to isomorphism by an element of $H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$. One may show that this element is $\rho([\Delta])_x$.

To prove our main theorem, we start by recognizing that the fact that for nonsingular X the sheaf of Cartier divisors D has a simple description.

Lemma 6.2. *Let D be the sheaf of Cartier divisors on X. Then there is an isomorphism of sheaves on the small Zariski/etale site of X:*

$$D \cong \sum_{x \in X^1} i_{x*}\mathbb{Z}.$$

Proof. By definition, $D = i_{\eta*}\mathbb{G}_m/\mathbb{G}_m$. Therefore it suffices to show that the sequence

$$0 \to \mathbb{G}_m \to i_{\eta*}\mathbb{G}_m \xrightarrow{\rho} \sum_{x \in X^1} i_{x*}\mathbb{Z} \to 0$$

is exact on the small Zariski site of X, where here ρ is the map defined by $s \mapsto (\nu_x(s))_x$. This in turn is verified by checking stalks. $\square$

As a consequence of the previous lemma, we have a long exact sequence in cohomology

$$\bigoplus_{x \in X^1} H^1(X, i_{x*}\mathbb{Z}) \to H^2(X, \mathbb{G}_m) \to H^2(X, i_{\eta *}\mathbb{G}_m) \to \bigoplus_{x \in X^1} H^2(X, i_{x*}\mathbb{Z}) \to \dots \tag{1}$$

The majority of the rest of this section is devoted to showing that this exact sequence may be wrangled to give us the long exact sequence in the statement of the main theorem.

Lemma 6.3. *Let Y be an integral scheme with generic point η, and let $f : \eta \to \mathbb{Z}$ be the inclusion. Then*

(a) $H^1(Y, f_*\mathbb{Z}) = 0$

(b) $H^i(Y, f_*\mathbb{Q}) = 0$ *for all* $i > 0$

(c) $H^i(Y, f_*\mathbb{Z}) \cong H^{i-1}(Y, f_*(\mathbb{Q}/\mathbb{Z}))$ *for all* $i > 1$

Proof.

(a) The Leray spectral sequence for f shows that there is an injection $H^1(Y, f_*\mathbb{Z}) \to H^1(\eta, \mathbb{Z})$. Moreover, $H^1(\eta, \mathbb{Z}) \cong H^1(K(Y), \mathbb{Z})$ and since all finite groups map trivially into $\mathbb{Z}$, this latter cohomology group is trivial. Hence $H^1(Y, f_*\mathbb{Z})$ is trivial.

(b) For all $j > 0$ and all $y \in Y$, we have $(R^j f_*\mathbb{Q})_x \cong H^j(K(\widetilde{\mathcal{O}_{Y,y}}), \mathbb{Q})$, where here $\widetilde{\mathcal{O}_{Y,y}}$ represents the etale stalk of $\mathcal{O}_Y$ at y. Note that for *any* finite group G, the group cohomology $H^j(G, \mathbb{Q})$ is 0 for $j > 0$, because the inflation/restriction sequence shows that $|G|$ kills all the homology groups, and is also invertible. Therefore $R^j f_*\mathbb{Q} = 0$ for all $j > 0$, and the Leray spectral sequence for f tells us that $H^j(Y, f_*\mathbb{Q}) \cong H^j(K(Y), \mathbb{Q})$. By the same argument, $H^j(K(Y), \mathbb{Q}) = 0$ and therefore $H^j(Y, f_*\mathbb{Q}) = 0$.

(c) The short exact sequence of sheaves on Y

$$0 \to f_*\mathbb{Z} \to f_*\mathbb{Q} \to f_*(\mathbb{Q}/\mathbb{Z})$$

induces a long exact sequence in cohomology, which by the result of (b) gives us isomorphisms $H^i(Y, f_*\mathbb{Z}) \cong H^{i-1}(Y, f_*(\mathbb{Q}/\mathbb{Z}))$ for all $i > 1$.

$\square$

Corollary 6.4. *Let X be a nonsingular integral scheme. Then for all codimension 1 points $x \in X$*

(a) $H^1(X, (i_x)_*\mathbb{Z}) = 0$

(b) $H^i(X,(i_x)_*\mathbb{Z}) \cong H^{i-1}(X,(i_x)_*\mathbb{Q}/\mathbb{Z})$ *for all* $i > 1$

Proof. Let C be the irreducible codimension 1 subscheme of X corresponding to x. Since X is nonsingular, for any abelian group A, we have $(i_x)_*A \cong j_*f_*A$, where $j : C_x \to X$ is the inclusion of the irreducible, codimension 1 subscheme C_x of X for which x is the generic point, and $f : x \to C_x$ is the inclusion of the generic point. The corollary then follows from the previous lemma. □

As a consequence of the previous corollary, we now have that the long exact sequence obtained from the sheaf of Cartier divisors looks like

$$0 \to \mathrm{Br}'(X) \to H^2(X, i_{\eta*}\mathbb{G}_m) \to \bigoplus_{x\in X^1} H^1(X, i_{x*}(\mathbb{Q}/\mathbb{Z})) \to H^3(X,\mathbb{G}_m) \to H^3(X, i_{\eta*}\mathbb{G}_m) \tag{2}$$

To finish the proof of the main theorem, it would be nice to obtain an ℓ-exact isomophism $H^i(X,(i_\eta)_*\mathbb{G}_m) \cong H^i(K,\mathbb{G}_m)$. However, this is not actually the case! The real key to the rest of the argument lies in the fact that elements of $H^2(X,\mathbb{G}_m)$ are determined by their behavior in codimension 0 and 1. More specifically, we have the following lemma:

Lemma 6.5. *Let X be a nonsingular, excellent, integral scheme with fraction field K. Then $H^2(X,\mathbb{G}_m) \cong H^2(X^1,\mathbb{G}_m)$.*

Proof. [5] □

Therefore, by replacing X with U in our long exact sequence, and taking the limit over all complements of closed subschemes of X with codimension at least two, we obtain a long exact sequence

$$0 \to \mathrm{Br}'(X) \to H^2(X^1, i_{\eta*}\mathbb{G}_m) \to \bigoplus_{x\in X^1} H^1(X, i_{x*}(\mathbb{Q}/\mathbb{Z})) \to H^3(X^1,\mathbb{G}_m) \to H^3(X^1, i_{\eta*}\mathbb{G}_m) \tag{3}$$

where we have used the fact that $i_{x*}(\mathbb{Q}/\mathbb{Z})$ is supported in codimension 0 and 1, so that $H^1(X, i_{x*}(\mathbb{Q}/\mathbb{Z})) = H^1(X^1, i_{x*}(\mathbb{Q}/\mathbb{Z}))$. Thus to finish the proof of our big theorem, it remains to prove that $H^p(X^1, i_{\eta*}\mathbb{G}_m) \cong H^j(K,\mathbb{G}_m)$ for $j > 1$.

Lemma 6.6. *Let X be a nonsingular integral scheme, with generic point η and fraction field K. Let ℓ be a prime different from the characteristic of K. Then for $j > 0$ the sheaf $R^j(i_\eta)_*\mathbb{G}_m$ has support in codimension $\geqslant 2$.*

Proof. It suffices to show that the (etale) stalks at each codimension 1 point of the sheaf $R^j(i_\eta)_*\mathbb{G}_m$ are trivial. Suppose that $x \in X$. Then there is an isomorphism

$$(R^j(i_\eta)_*\mathbb{G}_m)_x \cong H^j(K(\widetilde{\mathcal{O}_{X,x}}),\mathbb{G}_m).$$

If x is a point of codimension 1, then $K(\widetilde{\mathcal{O}_{X,x}})$ is the fraction field of an irreducible curve over a separably closed field, and hence has ℓ-cohomological dimension 1. Hence the ℓ-principal part of $(R^j(i_\eta)_*\mathbb{G}_m)_x$ is 0 for $j > 1$. Moreover, this also shows that $R^1(i_\eta)_*\mathbb{G}_m = 0$ by Hilbert's Theorem 90. □

Lemma 6.7. *Let X be a nonsingular, integral scheme with fraction field K. Let ℓ be a prime number different from the characteristic of K. Then*

$$H^j(X^1, (i_\eta)_*\mathbb{G}_m)(\ell) \cong H^j(K, \mathbb{G}_m)(\ell).$$

Proof. Let η be the generic point of X. Then since $R^1(i_\eta)_*\mathbb{G}_m = 0$ and $R^j(i_\eta)_*\mathbb{G}_m(\ell)$ is supported in codimension $\geqslant 2$ for all $j > 1$, the Leray spectral sequence for $i_\eta : \eta \to X^1$ tells us that $H^j(X^1, \mathbb{G}_m)(\ell) \cong H^j(K, \mathbb{G}_m)(\ell)$ for all $j > 1$. $\square$

Putting this all together, the main theorem follows immediately.

Proof of the Main Theorem. Clear from the above discussion. $\square$

As a consequence of the main theorem, we see that there are canonical inclusions

$$\mathrm{Br}(X) \subseteq \mathrm{Br}'(X) \subseteq H^2(X, (i_\eta)_*\mathbb{G}_m) \subseteq \mathrm{Br}(K).$$

In the case that X is a nonsingular, 2-dimensional integral scheme, the first inclusion is actually an isomorphism

Proposition 6.8. *Let X be a nonsingular, 2-dimensional integral scheme. Then the canonical inclusion $\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$ is an isomorphism.*

Proof. Suppose that $[d] \in \mathrm{Br}'(X)$, and let $[\Delta]$ be its image in $\mathrm{Br}(K)$. Let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in Δ. Then $\mathcal{A}_x$ is a maximal $\mathcal{O}_{X,x}$-order in Δ. Since the latter factors through $\mathrm{Br}(X) \to \mathrm{Br}(\mathcal{O}_{X,x})$, it follows that $\mathcal{A}_x$ is Azumaya. Furthermore, since $\mathcal{A}$ is maximal, it must be reflexive. Hence $\mathcal{A}$ is Azumaya, and $[\mathcal{A}] \mapsto [\Delta] \in \mathrm{Br}(K)$. Hence $[\mathcal{A}] \mapsto [d] \in \mathrm{Br}'(X)$. $\square$

6.2 Artin-Mumford Spectral Sequence

In this section, we prove the following main theorem of Artin and Mumfords paper.

Theorem 6.9 (Artin-Mumford)**.** *Let X be a simply connected smooth projective surface over an algebraically closed field k. Then for any prime integer ℓ invertible in K, there is an ℓ-exact sequence*

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{a} \bigoplus_{x \in X^2} \mu^{-1} \xrightarrow{\Sigma} \mu^{-1} \to 0,$$

where X^i represents the set of codimension i points of X.

Lemma 6.10. *Let $f : X^1 \to X$. Then there is an exact sequence*

$$0 \to H^3(X, \mathbb{G}_m) \to H^3(X^1, \mathbb{G}_m) \to \bigoplus_{x \in X_0} (j_x)_*\mu^{-1} \to H^4(X, \mathbb{G}_m) \to 0.$$

Proof. Since $R^0 f_* \mathbb{G}_m = \mathbb{G}_m$, $R^3 f_* \mathbb{G}_m = \bigoplus_{x \in X_0} (j_x)_* \mu^{-1}$ and $R^q f_* \mathbb{G}_m = 0$ otherwise, this just falls out of the Leray spectral sequence for f. In the above, we used the fact that $H^4(X^1, \mathbb{G}_m) = 0$, which follows from Poincare duality. $\square$

Proof of Theorem 6.9. First note that since X is simply connected, $H^3(X, \mathbb{G}_m) = 0$. Also, since K is the fraction field of a nonsingular surface over an algebraically closed field k, the ℓ-cohomological dimension of K is two. In particular $H^3(K, \mathbb{G}_m) = 0$. Taking the ℓ-exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X_0^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \to H^3(X^1, \mathbb{G}_m) \to \overbrace{H^3(K, \mathbb{G}_m)}^{=0}$$

and combining it with the exact sequence of the previous lemma, we have the ℓ-exact sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X_0^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{x \in X_0} (j_x)_* \mu^{-1} \to H^4(X, \mathbb{G}_m) \to 0.$$

The last part then follows from $H^4(X, \mathbb{G}_m) = \mu^{-1}$. $\square$

7 A Nonrational Unirational Variety

In this section, we construct a surface S with quotient field $K = k(x, y)$ and quaternion algebra Δ over K whose ramification locus is a union of two nonsingular, disjoint curves C_1, C_2 on S such that the Brauer-Severi scheme of a maximal order $\mathcal{A}$ of Δ over S is unirational. It will then follow from cohomological considerations that $\mathrm{BS}(\mathcal{A})$ is a nonrational unirational variety.

Fix a nonsingular conic in $Z = V(\alpha) \subseteq \mathbb{P}^2_k$, and choose six distinct points $P_i^{(j)} \in Z$, $i = 1, 2$, $j = 1, 2, 3$.

Lemma 7.1. *There exist nonsingular cubic curves $E_i = V(\beta_i) \subseteq \mathbb{P}^2_k$ such that*

(a) E_i has double intersection with Z at $P_i^{(j)}$ for $j = 1, 2, 3$

(b) E_i and E_j transversally at nine distinct points $O_1, \ldots, O_9$

Proof. See [1]. $\square$

Let q be the rational function on $\mathbb{P}^2_k$ whose divisor is $Z - 2L$, with L the line at infinity.

Lemma 7.2. *The restriction $\overline{q}$ of q to E_i is not a square in $K(E_i)$ for $i = 1, 2$.*

Proof. Let $\mathcal{L}$ be the line bundle on $\mathbb{P}^2_k$ corresponding to the divisor L. We claim that if $f \in \Gamma(E_i, \mathcal{L}|_{E_i})$, then f has at most two zeros on $Z \cap E_i$. To see this, let $f \in \Gamma(E_i, \mathcal{L}|_{E_i})$. The natural map

$$\Gamma(Z, \mathcal{L}) \to \Gamma(E_i, \mathcal{L}|_{E_i})$$

is surjective, and therefore f may be extended to a global section $\widetilde{f}$ of the sheaf $\mathcal{L}$. Therefore $(\widetilde{f}) + L$ is a degree 1 effective divisor, so we may choose a line L' such that $L' = (\widetilde{f}) + L$. It follows that the zero set of $f = \widetilde{f}|_{E_i}$ is exactly $L' \cap E_i$, and therefore the zeros of f on $Z \cap E_i$ are contained in $Z \cap L'$, this latter set having only two elements. This proves our claim.

Now suppose that $\overline{q} = f^2$ for some $f \in K(E_i)$. Then f has the same number of zeros on $Z \cap E_i$ as $\overline{q}$, namely 3. Furthermore, since $(q) = Z - 2L$ we know that $(\overline{q}) + 2E_i \cdot L$ is effective, and therefore $(f) + E_i \cdot L$ is effective. Hence $f \in \Gamma(E_i, \mathcal{L}|_{E_i})$. By the argument of the previous paragraph, this is a contradiction. $\square$

Lemma 7.3. *Let $L_i := K(E_i)[\overline{q}^{1/2}]$ and let E_i' be the normalization of E_i in L_i. Then $E_i' \to E_i$ is unramified.*

Proof. Since $E_i' \to E_i$ is locally of finite type, it suffices to show that for any point $P' \in E_i'$ with image $P \in E_i$, the induced map of residue fields $\kappa(P) \to \kappa(P')$ is a finite, separable field extension. Then the local ring $R = \mathcal{O}_{E_i,P}$ is a DVR (R, ν) with residue field k and fraction field $K(L_i)$. Let t be a uniformizer for R. Depending on whether or not P is one of the intersection points $P_i^{(j)}$ of E_i and Z, the value of $\nu(\overline{q})$ is either 0 or 2, meaning we may write $\overline{q} = ut^{2j}$ for some $u \in R^*$ and integer j. Note that $R[u^{1/2}]$ is a semilocal Dedekind domain, hence a PID, and therefore integrally closed in its fraction field L_i. Therefore since $u^{1/2} \in L_i$ is integral over R, we have $\mathcal{O}_{E_i',P'} = R[u^{1/2}]$. Furthermore $R[u^{1/2}] = R[x]/(x^2 - u)$ is unramified since u is a unit. $\square$

Define S to be the blow up of $\mathbb{P}^2_k$ at the intersection points of $O_1, \ldots, O_9$ of E_1 and E_2, and let C_i be the proper transform of E_i for $i = 1, 2$. Then S is a simply connected rational surface, on which the nonsingular curves C_1 and C_2 are disjoint. Note that since S is rational, $\mathrm{Br}(S) = 0$. Therefore the Artin-Mumford sequence tells us there exists a unique $\Delta \in \mathrm{Br}_2(K)$ whose ramification locus is $C = C_1 \cup C_2$ and whose ramification data on C_i is the nowhere-ramified extension L_i for $i = 1, 2$.

Choose a cubic curve $E_0 = V(\beta) \subseteq \mathbb{P}^2_k$ such that the divisor E_0 pulls back on Z to the divisor $\sum_{ij} P_i^{(j)}$. In particular, this means that $\beta_1\beta_2 - \beta$ is identically zero on α, so that there exists a homogeneous polynomial γ of degree 4 satisfying $4\alpha\gamma = (\beta_1\beta_2 - \beta)$. Consider the field $L := K[q^{1/2}]$, and let Y_0 be the quartic in $\mathbb{P}^3_k$ defined by

$$Y_0 = V(\alpha(X_0, X_1, X_2)X_3^2 + \beta(X_0, X_1, X_2)X_3 + \gamma(X_0, X_1, X_2)) \subseteq \mathbb{P}^3_k.$$

Note that $Q_0 = [0 : 0 : 0 : 1]$ is a singular point of Y_0. Moreover, The projection $Y_0 \backslash \{Q_0\} \to \mathbb{P}^2_k$ is a double cover of $\mathbb{P}^2_k$ ramified at $V(\beta^2 - 4\alpha\gamma) = V(\beta_1\beta_2) = E_1 \cup E_2$. The remaining singular points of Y_0 must occur at ramification points, and therefore correspond to the singular points of $E = E_1 \cup E_2$, ie. the points Q_i mapping to the intersection points O_i of the curves E_1 and E_2.

Let Y be the blow up of Y at $Q_0, \ldots, Q_9$. Then Y_0 is the normalization of $\mathbb{P}^2_k$ in L, so Y is the desingularization of the normalization of S in L. Furthermore, Y is rational since it is birational to Y_0, which itself is rational because it is a quadric in $\mathbb{P}^3_k$. Thus if we consider the commutative diagram induced by the morphism $f : Y \to S$ and the Artin-Mumford exact sequences for S and Y

$$\begin{tikzcd}
\mathrm{Br}(S) \arrow[r] \arrow[d] & \mathrm{Br}(K) \arrow[r] \arrow[d] & \bigoplus_{C \subseteq S} K(C)^*/K(C)^{*2} \arrow[d] \\
\mathrm{Br}(Y) \arrow[r] & \mathrm{Br}(L) \arrow[r] & \bigoplus_{C \subseteq Y} K(C)^*/K(C)^{*2}
\end{tikzcd}$$

The image of $[\Delta]$ in $\mathrm{Br}(L)$ is $[\Delta_L]$, and the ramification data becomes trivial. Furthermore, the fact that Y is rational implies that $\mathrm{Br}(Y) = 0$. Thus the class $[\Delta_L]$ is trivial. Since the Brauer class of Δ splits over a quadratic extension, we conclude that Δ may be taken to be a quaternion algebra.

Lemma 7.4. *Let Δ be the quaternion algebra defined above, and let $\mathcal{A}$ be a maximal $\mathcal{O}_S$-order in Δ. Then the Brauer-Severi variety* $\mathrm{BS}(\mathcal{A})$ *of $\mathcal{A}$ is unirational.*

Proof. Using the same $f : Y \to S$ as above, we have that

$$\mathrm{BS}(\mathcal{A}) \times_S Y \cong \mathrm{BS}(f^*\mathcal{A}) \cong \mathbb{P}^1_Y.$$

Projection onto the first factor then gives us a surjection $\mathbb{P}^1_Y \to \mathrm{BS}(\mathcal{A})$, and since Y is rational this proves that $\mathrm{BS}(\mathcal{A})$ is unirational. $\square$

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