

# Maximal Orders in Quaternion Algebras on Surfaces

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## 1 Introduction

## 2 Azumaya Algebras

### 2.1 Twisted Forms and Nonabelian $H^1$

**Definition 2.1.** Let  $X$  be a scheme. We define a *category of algebraic structures* on  $X$  to be a subcategory  $\mathcal{C}$  of the category of sheaves on  $X$ .

We will be primarily interested in case where  $\mathcal{C}/X$  is one of the following examples

- the category of  $\mathcal{O}_X$ -modules on  $X$
- the category of  $\mathcal{O}_X$ -algebras on  $X$
- the category of schemes over  $X$
- the category of  $G$ -sets, for  $G$  a group scheme on  $X$

Note that if  $Y \in \underline{\text{Sch}}_X$ , then pullback determines from  $\mathcal{C}$  a category of algebraic structures  $\mathcal{C}/Y$ , and a right exact functor  $\mathcal{C} \rightarrow \mathcal{C}/Y$ .

**Definition 2.2.** Let  $c \in \mathcal{C}$ . We define an etale (fppf) *twisted form* of  $c$  to be an object  $c' \in \mathcal{C}$  such that there exists an etale (fppf) cover  $\mathfrak{U} = \{f_i : U_i \rightarrow X\}$  with  $c', c$  isomorphic in  $\mathcal{C}/U_i$  for all  $i$ . We denote by  $\underline{\text{Twist}}(c)$  the set of isomorphism classes of twisted forms of  $c$ .

The basis for understanding and classifying etale (fppf) twisted forms of  $c \in \mathcal{C}$  comes from a canonical identification of twisted forms of  $c$  as Čech cohomology classes of the sheaf of automorphisms of  $c$ . We will describe this correspondence here for etale twisted forms of  $c$ ; the correspondence for fppf twisted forms is similar. For  $c \in \mathcal{C}$ , let  $\text{Aut}_{et}(c)$  denote the sheaf on the big etale site of schemes over  $X$ , defined by

$$\text{Aut}_{et}(c) : (f : Y \rightarrow X) \mapsto \text{Aut}_{\mathcal{C}/Y}(f^*c).$$

Then for a twisted form  $c'$  of  $c$ , the local isomorphisms  $\varphi_i : f_i^*c' \rightarrow f_i^*c$  generate a Čech cocycle  $(\psi_{ij}) \in \check{C}^1(\mathfrak{U}, \text{Aut}_{et}(c))$ , for

$$\psi_{ij} : (p_i^*\varphi_i) \circ (p_j^*\varphi_j)^{-1} \in \text{End}(f_{ij}^*c),$$

with  $f_{ij} : U_i \times_X U_j \rightarrow X$  the induced map of the fibre product, and  $p_i : U_i \times_X U_j \rightarrow U_i$  the canonical projection map. The natural map  $\check{C}^1(\mathfrak{U}, \text{Aut}_{et}(c)) \rightarrow \check{H}^1(\mathfrak{U}, \text{Aut}_{et}(c)) \rightarrow \check{H}^1(X_{et}, \text{Aut}_{et}(c))$  completes the construction. A descent argument then proves the following lemma

**Lemma 2.1.** *Let  $\mathcal{C}$  be a category of algebraic structures on a scheme  $X$ , and let  $c \in \mathcal{C}$ . Then the canonical association of a Čech cocycle to a twisted form  $c'$  of  $c$  defines an injection*

$$\underline{\text{Twist}}(c) \hookrightarrow \check{H}^1(X_{et}, \text{Aut}_{et}(c)).$$

*A similar statement holds for the fppf twisted forms of  $c$ .*

*Example 1.* Let  $X = \operatorname{spec}(k)$ , let  $\mathcal{C}$  be the category of schemes over  $X$ , and let  $c = \operatorname{spec}(L)$  for  $L$  a finite separable field extension of  $k$ . If  $Z \rightarrow X$  is étale, then  $Z = \coprod_{i=1}^n \operatorname{spec}(L_i)$  where each  $L_i$  is a finite, separable extension of  $k$ . Therefore  $Y \times_X Z = \coprod \operatorname{spec}(L_i \otimes_k L)$ , and since  $\operatorname{Aut}_{L_i}(L_i \otimes_k L) \cong \operatorname{Aut}_k(L)$ , and thus

$$\operatorname{Aut}(c) : Z \mapsto \operatorname{Aut}_{\underline{\operatorname{sch}}_Z}(Y \times_X Z) = \prod_{i=1}^n G,$$

where  $G$  is the Galois group of the extension  $L/k$ . Thus  $\operatorname{Aut}(c)$  may be identified with the constant sheaf  $G$  on the (big) étale site of  $X$ . Consequently,  $\check{H}^1(X_{\text{ét}}, \operatorname{Aut}_{\text{ét}}(c)) = \check{H}^1(X_{\text{ét}}, G)$ , and so twisted forms of  $c$  are in one-to-one correspondence with  $G$ -torsors on  $X$ ; for  $X = \operatorname{spec}(k)$ , these are exactly the Galois extensions of  $k$  with Galois group  $G$ .

In the previous example, we swept under the rug the issue of whether  $\underline{\operatorname{Twist}}(c) \rightarrow \check{H}^1(X, \operatorname{Aut}_{\text{ét}}(c))$  is surjective. Since descent theory may always be applied to “glue together” the descent data coming from a representative  $\check{H}^1(X, \operatorname{Aut}_{\text{ét}}(c))$  to a sheaf, the question is really one of representability: when is the sheaf that we glue together actually a member of the category  $\mathcal{C}$ ? For example, it is clear that descent will be effective for  $\mathcal{C}$  the category of  $G$ -torsor sheaves. However, whether it is effective for the category of  $G$ -torsors is equivalent to asking whether any  $G$ -torsor sheaf is representable. This question in general can be difficult to answer.

Nevertheless, in many important instances the above map is an isomorphism.

*Example 2.* Let  $\mathcal{C}$  be the category of quasi-coherent  $\mathcal{O}_X$ -modules, and let  $c = \mathcal{O}_X^n$ . Then  $\operatorname{Aut}(c)$  is equal to the group scheme  $\operatorname{GL}_n$  defined on the big fppf site of  $X$  by

$$\operatorname{GL}_n : (U \rightarrow X) \mapsto \operatorname{GL}_n(\Gamma(U, \mathcal{O}_U)).$$

Suppose  $c' \in \mathcal{C}$  is a twisted form of  $c$ . Then  $c'$  is fppf locally free of rank  $n$ . By a follow-your-nose type argument,  $c'$  is therefore Zariski locally free. Thus  $\underline{\operatorname{Twist}}(c)$  is the set of isomorphism classes of locally free rank  $n$   $\mathcal{O}_X$ -modules. Since fppf descent is effective for quasicoherent sheaves of  $\mathcal{O}_X$ -modules, we conclude that

$$\check{H}^1(X_{\text{fppf}}, \operatorname{GL}_n) \cong \underline{\operatorname{Twist}}(c) = \{\text{iso. classes of locally free, rank } n \text{ } \mathcal{O}_X\text{-modules}\}.$$

As a special case of this, we see that  $\check{H}^1(X_{\text{fppf}}, \mathbb{G}_m) = \check{H}^1(X_{\text{fppf}}, \operatorname{GL}_1)$  classifies isomorphism classes of invertible sheaves on  $X$ , ie.  $\check{H}^1(X, \mathbb{G}_m) \cong \operatorname{Pic}(X)$ .

*Example 3.* Let  $\mathcal{C}$  be the category of  $\mathcal{O}_X$ -algebras, and let  $c = M_n : (U \rightarrow X) \mapsto M_n(\Gamma(U, \mathcal{O}_U))$ . The étale twisted forms of  $c$  are exactly the rank  $n$  Azumaya algebras on  $X$ . Since étale descent is effective for  $\mathcal{O}_X$ -algebras, it follows that  $\check{H}^1(X_{\text{ét}}, \operatorname{Aut}_{\text{ét}}(c))$  classifies isomorphism classes of rank  $n$  Azumaya algebras on  $X$ .

What is  $\operatorname{Aut}_{\text{ét}}(c)$ ? Conjugation provides a natural map  $\operatorname{GL}_n \rightarrow \operatorname{Aut}_{\text{ét}}(c)$  with kernel  $\mathbb{G}_m$ . A version of the Skolem-Noether Theorem says that for any local ring  $R$ , all automorphisms of the ring  $M_n(R)$  are inner. Therefore, we see

that the map of etale stalks  $\mathbb{G}\mathbb{L}_n \rightarrow \text{Aut}_{et}(c)$  is surjective on stalks, and hence surjective. In particular, we have a short exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}\mathbb{L}_m \rightarrow \text{Aut}(c) \rightarrow 0.$$

The cokernel of  $\mathbb{G}_m \rightarrow \mathbb{G}\mathbb{L}_n$  is exactly the group scheme  $\mathbb{P}\mathbb{G}\mathbb{L}_n$ , so we have an isomorphism  $\text{Aut}(c) \cong \mathbb{P}\mathbb{G}\mathbb{L}_n$ . Hence  $\check{H}^1(X_{et}, \mathbb{P}\mathbb{G}\mathbb{L}_n)$  classifies isomorphism classes of Azumaya algebras on  $X$ .

*Example 4.* Let  $\mathcal{C}$  be the category of schemes over  $X$ , and let  $c = \mathbb{P}_X^{n-1}$ . Then  $\text{Aut}(c) = \mathbb{P}\mathbb{G}\mathbb{L}_n$  and we have a canonical injection  $\text{Twist}(c) \hookrightarrow \check{H}^1(X_{fppf}, \mathbb{P}\mathbb{G}\mathbb{L}_n)$ . As it so happens, this map is also surjective, though it's a bit harder to show. Therefore  $\check{H}^1(X, \mathbb{P}\mathbb{G}\mathbb{L}_n)$  also classifies twisted forms of  $\mathbb{P}_X^{n-1}$ , so called Severi-Brauer varieties.

Our last example describes a sort of universal interpretation of the elements of  $\check{H}^1(X, G)$  as isomorphism classes of  $G$ -torsor sheaves on  $X$ . From some perspectives, this could be considered the right definition of the first nonabelian cohomology group.

*Example 5.* Let  $G$  be a group scheme on  $X$ . By a (right)  $G$ -set on  $X$ , we mean a sheaf  $F$  on  $X$ , and a morphism of sheaves  $F \times G \rightarrow F$  compatible with the canonical morphism  $G \times G \rightarrow G$ , ie. so that the natural map  $(F \times G) \times G \rightarrow F \times (G \times G)$  factors over  $F$ . Let  $\mathcal{C}$  be the category of  $G$ -sets on  $X$  with morphisms given by  $G$ -equivariant natural transformations. Let  $c = G$ , with the canonical  $G$ -action by multiplication on the right. We call  $c$  the *trivial*  $G$ -torsor.

If  $c'$  is an fppf-twist of  $G$ , then  $c'$  is an fppf locally isomorphic to the trivial  $G$ -torsor: this is exactly the definition of a  $G$ -torsor sheaf on  $X$ ! Thus  $\text{Twist}(c)$  is exactly the set of isomorphism classes of locally trivial  $G$ -torsor sheaves on  $X$ . For any  $Y \rightarrow X$ , the set of  $G$ -equivariant maps  $G \rightarrow G$  are exactly determined by where they send the identity. Hence there is a canonical identification  $\text{Aut}(c) = G$ . Thus  $\check{H}^1(X_{fppf}, G)$  classifies the isomorphism classes of  $G$ -torsors on  $X$ . We denote the set of isomorphism classes of  $G$ -torsors on  $X$  as  $\text{Torsor}(G)$ .

*Remark 2.1.* The first cohomology group is known to agree with the standard cohomology group for  $G$  abelian on any site, so from now on we will leave off the check symbol, and simply write  $H^1(X_{site}, G)$ . Moreover, for many of the groups we work with, the first cohomology groups for the etale and fppf sites will agree – in such situations, we will simply write  $H^1(X, G)$  to mean cohomology with respect to this site.

## 2.2 Gerbes and Nonabelian $H^2$

UNDER CONSTRUCTION

## 2.3 Azumaya Algebras and the Brauer Group

**Definition 2.3.** Let  $X$  be a scheme. We define an *Azumaya algebra*  $A$  on  $X$  to be a twisted form of the  $\mathcal{O}_X$ -algebra  $M_n$  for some  $n$ . The corresponding value

of  $n$  is called the *rank*. We denote the set of isomorphism classes of Azumaya algebras of rank  $n$  on  $X$  as  $\text{Az}_n(X)$ . We call an Azumaya algebra  $A$  *trivial* if there exist integers  $s, t$  satisfying  $A \otimes_{\mathcal{O}_X} M_s \cong M_t$ . We define two Azumaya algebras  $A$  and  $A'$  to be *equivalent* if  $A \otimes_{\mathcal{O}_X} (A')^{op}$  is a trivial Azumaya algebra.

*Example 6.* If  $X = \text{spec}(k)$  for  $k$  a field, then an Azumaya algebra on  $X$  is the same thing as a central simple  $k$ -algebra. Wedderburn's theorem tells us that each central simple algebra is isomorphic to a matrix algebra over a division ring; two central simple algebras are equivalent if and only if they have the same underlying division ring.

*Remark 2.2.* With the definition of equivalence provided above, two Azumaya algebras  $A$  and  $A'$  are equivalent if and only if the corresponding module categories are the equivalent. In fact, there corresponding equivalence of categories is the one induced by the  $A, A'$ -bimodule  $A \otimes (A')^{op}$ . Thus the definition of equivalence coincides with the notion of Morita equivalence.

**Definition 2.4.** We define the (geometric) *Brauer group* of  $X$  to be the set  $\text{Br}(X)$  of equivalence classes of Azumaya algebras on  $X$ .

**Proposition 2.2.** *The Brauer group  $\text{Br}(X)$  is an abelian group with binary operation  $[A] + [A'] = [A \otimes A']$ .*

*Example 7.* Let  $X = \text{spec}(\mathbb{R})$ . Then the central division rings over  $\mathbb{R}$  consist of only  $\mathbb{R}$  and the quaternion algebra  $Q$ . Thus  $\text{Br}(\text{spec}(\mathbb{R})) = \mathbb{Z}/2$ .

*Example 8.* Let  $X = \mathbb{P}_k^1$ . Then  $\text{Br}(X) = \text{Br}(\text{spec}(k))$ .

The Brauer group  $\text{Br}(X)$  is intimately related to  $H^2(X, \mathbb{G}_m)$ , and in particular there is always an injection  $\text{Br}(X) \hookrightarrow H^2(X, \mathbb{G}_m)$  mapping  $\text{Br}(X)$  into the torsion part of  $H^2(X, \mathbb{G}_m)$ .

**Definition 2.5.** Let  $X$  be a scheme. We define the *cohomological Brauer group*  $\text{Br}'(X)$  of  $X$  to be the torsion component of  $H^2(X, \mathbb{G}_m)$ .

*Remark 2.3.* Note that the fppf and etale cohomology of  $\mathbb{G}_m$  agree. Hence we simply write  $H^i(X, \mathbb{G}_m)$  to represent either cohomology group.

One way to think about the canonical map  $\text{Br}(X) \rightarrow \text{Br}'(X)$  is through the short exact sequence of group schemes

$$0 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_n \rightarrow \text{PGL}_n \rightarrow 0.$$

As we mentioned in the last section, for any group scheme  $G$  on  $X$ , we can think about the cohomology class  $H^1(X, G)$  as isomorphism classes of  $G$ -torsor sheaves on  $X$ . As it turns out, we also have an interpretation of  $H^2(X, G)$  as isomorphism classes of  $G$ -gerbs on  $X$ .

**Lemma 2.3** (Giraud). *Let  $X$  be a scheme and let*

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1,$$

be a short exact sequence of group schemes on  $X$ . Then there exists a long exact sequence of pointed sets

$$\begin{aligned} 1 \rightarrow H^0(X, G') \rightarrow H^0(X, G) \rightarrow H^0(X, G'') \\ \rightarrow H^1(X, G') \rightarrow H^1(X, G) \rightarrow H^1(X, G'') \xrightarrow{\delta} H^2(X, G') \end{aligned}$$

All of the maps, save for  $\delta$ , may be thought of in terms of Čech cohomology, and agree with the usual manipulation involving cocycles and restriction. The map  $\delta$  is perhaps a bit mysterious, so we take a little more time to explain it.

Associated to any  $G''$ -torsor sheaf  $F$  on  $X$ , we may define a pseudo-functor  $\underline{\text{Sch}}_X \rightarrow \underline{\text{Sets}}$  by

$$\mathcal{X}_F : (Y \rightarrow X) \mapsto \{(E, \varphi) : E \text{ a } G''\text{-torsor sheaf on } X, \varphi : E \xrightarrow{\cong} F|_Y\}.$$

This pseudo-functor defines a  $G'$ -gerb on  $X$ . The map  $\delta$  is then given by  $F \mapsto \mathcal{X}_F$ . Exactness then follows from the fact that  $\mathcal{X}_F$  and  $BG'$  are isomorphic as stacks if and only if  $\mathcal{X}_F(X) \neq \emptyset$ , which is true if and only if  $F$  is the trivial torsor sheaf.

As a consequence of the above, there exists an exact sequence

$$\text{Pic}(X) \rightarrow H^1(X, \mathbb{G}_m) \rightarrow H^1(X, \mathbb{PGL}_n) \xrightarrow{\delta} H^2(X, \mathbb{G}_m).$$

**Proposition 2.4.** *The fibres of  $\delta$  are equivalence classes of Azumaya algebras. Consequently  $\delta$  induces a group monomorphism  $\text{Br}(X) \hookrightarrow H^2(X, \mathbb{G}_m)$ .*

The image of  $\text{Br}(X)$  in  $H^2(X, \mathbb{G}_m)$ , is torsion. To see this, consider the commutative diagram

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{G}_m & \xrightarrow{\cdot^n} & \mathbb{G}_m \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \text{SL}_n & \longrightarrow & \text{GL}_n & \xrightarrow{\det} & \mathbb{G}_m \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \\ & & \text{PGL}_n & = & \text{PGL}_n & & \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

where the rows and columns are all short exact sequences (in fppf topology).

**Proposition 2.5.** *The map  $\delta : \check{H}^1(X, \mathbb{PGL}_n) \rightarrow H^2(X, \mathbb{G}_m)$  factors through the map  $H^2(X, \mu_n) \rightarrow H^2(X, \mathbb{G}_m)$  induced by the Kummer sequence. In particular, the image of  $\check{H}^1(X, \mathbb{PGL}_n)$  is torsion in  $H^2(X, \mathbb{G}_m)$ .*

**Corollary 2.6.** *The geometric Brauer group embeds into the cohomological Brauer group.*

In ideal situations, one can show that the geometric Brauer group and the cohomological Brauer group are the same. What exactly constitutes an “ideal situation” is an open problem. However, agreement is known to hold for Noetherian schemes with dimension at most 1.

**Proposition 2.7.** *If  $X$  is a Noetherian scheme and  $\dim(X) \leq 1$ , then  $\mathrm{Br}(X) = \mathrm{Br}'(X)$ .*

*Proof.* Since the small étale sites of  $X$  and  $X_{\mathrm{red}}$  are equivalent, it suffices to consider the case when  $X$  is reduced.

Case 1: Assume  $\dim(X) = 0$ . Then  $X$  is a disjoint union of points, and we can reduce to the case that  $X = \mathrm{spec}(K)$ . In this case, equality of  $\mathrm{Br}(K)$  and  $H^2(\mathrm{spec}(K), \mathbb{G}_m)$  is a classical result of Galois cohomology.

Case 2: Assume  $\dim(X) = 1$ , and let  $d \in H^2(X, \mathbb{G}_m)$ . We claim that there is a dense, open subscheme  $U$  of  $X$  on which the restriction of  $[d]$  to an element of  $H^2(U, \mathbb{G}_m)$  lies in the image of  $\mathrm{Br}(U)$ . To see this, let  $U_0$  be the open subscheme of nonsingular points of  $X$ . Then  $U_0$  is a disjoint union of nonsingular, integral curves  $C_1 \cup \dots \cup C_r$ . Let  $c_i$  be the generic point of  $C_i$  for each  $i$ . Since  $C_i$  is nonsingular,  $\mathrm{Br}'(C_i) \hookrightarrow \mathrm{Br}(K_i) = \mathrm{Br}'(K_i)$ , where  $K_i$  represents the fraction field of  $C_i$ . Thus the restriction  $d_i$  of  $d$  to  $C_i$  restricts to the generic point to a central simple  $K_i$ -algebra  $\Delta_i$ . Let  $A_i$  be a maximal  $\mathcal{O}_{C_i}$ -order in  $\Delta_i$ . Then  $(A_i)_{c_i} = \Delta_i$ , and therefore there is a Zariski open neighborhood  $U_i$  of  $c_i$  in  $C_i$  on which  $A_i$  is Azumaya: take the complement of the zero set of the discriminant. Since the pullback morphism  $\mathrm{Br}'(X) \rightarrow \mathrm{Br}(K_i)$  factors through the injection  $0 \rightarrow \mathrm{Br}'(U_i) \rightarrow \mathrm{Br}(K_i)$  it follows that the pullback of  $d$  to  $\mathrm{Br}'(U_i)$  lands in the image of  $\mathrm{Br}(U_i)$ . Hence the pullback of  $d$  to  $\mathrm{Br}'(U)$  for  $U = \bigcup_i U_i$  lands in the image of  $\mathrm{Br}(U)$ . Since  $U$  is dense in  $X$ , this proves our claim.

So far, we have shown that  $d$  is represented on a dense open subscheme  $U$  of  $X$  by an Azumaya algebra  $A$  on  $U$ . If we can show that  $U$  can be taken to be all of  $X$ , then we are done! Let  $V$  be a largest dense open subscheme of  $X$  on which  $d$  is represented by a sheaf of Azumaya algebras (this makes sense since the complement of  $U$  is finitely many points). Suppose  $x \in X \setminus V$ . By the previous paragraph, we know that  $x$  is a closed point of  $X$ , and therefore  $R = \mathcal{O}_{X,x}$  is a one-dimensional local ring. Set  $V_x = \mathrm{spec}(R) \cap V = \mathrm{spec}(R) \setminus \{x\}$ . The completion  $\hat{R}$  of  $R$  is a Henselian local ring, and therefore  $\mathrm{Br}(\hat{R}) = \mathrm{Br}'(\hat{R})$ . Hence the pullback of  $d$  to  $\mathrm{spec}(\hat{R})$  may be represented by an Azumaya  $\hat{R}$ -algebra  $\hat{B}$ , which must be equivalent to  $A$  on  $\hat{V}_x = \mathrm{spec}(\hat{R}) \setminus \{x\}$ . Hence there exist integers  $s, t$  such that  $M_s(A) = M_t(\hat{B})$  on  $\hat{V}_x$ . Then there exists a unique Azumaya  $R$ -algebra  $B$  such that  $B \otimes \hat{R} \cong M_t(\hat{B})$ . Gluing



$B$  and  $A$  together on  $\{x\} \cup V$ , extends  $A$  to an Azumaya algebra on  $V \cup \{x\}$ . This contradicts the maximality of  $V$ , and hence  $V = X$ . This completes the proof.

□

Furthermore, if  $X$  is a nonsingular, integral surface then  $\mathrm{Br}(X) = \mathrm{Br}(X')$ . We will prove this result in the next section.

### 3 Classical Theory of Maximal Orders

#### 3.1 Basic Definitions and Facts

Let  $R$  be a noetherian integral domain with fraction field  $K$ , and let  $\Delta$  be a central simple  $K$ -algebra, ie. a simple  $K$ -algebra with center  $K$ .

**Definition 3.1.** An  $R$ -order in  $\Delta$  is an  $R$ -subalgebra  $A$  of  $\Delta$ , finitely generated as an  $R$ -module, such that  $A \otimes_R K = \Delta$ . The set  $\underline{\text{Ord}}_{R,\Delta}$  forms a poset with ordering defined by inclusion. A  $R$ -order in  $\Delta$  is *maximal* if it is not properly contained in any  $R$ -order in  $\Delta$ .

Note that it follows from the definition that if  $A$  is an  $R$ -order in  $\Delta$ , then the center of  $A$  contains  $R$ .

**Lemma 3.1.** Let  $S \subseteq R \setminus \{0\}$  be a multiplicative subset of  $R$ . Then there is a morphism of posets

$$S^{-1} : \underline{\text{Ord}}_{R,\Delta} \rightarrow \underline{\text{Ord}}_{S^{-1}R,\Delta}, \quad A \mapsto S^{-1}A$$

which is surjective and sends maximal orders to maximal orders.

*Proof.* If  $A$  is an  $R$ -order in  $\Delta$ , then the fact that  $S^{-1}A$  is an  $S^{-1}R$ -order in  $\Delta$  follows from the exactness of localization.

Now suppose that  $A'$  is an  $S^{-1}R$ -order in  $\Delta$ , and let  $a_1, \dots, a_m$  be a set of generators for  $A'$  as an  $S^{-1}R$ -module. Set  $M = Ra_1 + \dots + Ra_m$  and let

$$A = \{a \in A' : Ma \subseteq M\}.$$

It is clear from the definition that  $A$  is an  $R$ -subalgebra of  $A'$  and that  $S^{-1}A = A'$ ; in particular this also tells us that  $A \otimes_R K = \Delta$ . Thus to show that  $A$  is an  $R$ -order in  $\Delta$ , all that is left to show is that  $A$  is finitely generated as an  $R$ -module. To see this, note that the action of  $A$  on  $M$  on the right induces an  $R$ -module monomorphism of  $A$  into the finitely generated  $R$ -module  $\text{End}_R(M)$ . Since  $R$  is noetherian, it follows that  $A$  is finitely generated.

Lastly, suppose that  $A$  is a maximal  $R$ -order in  $\Delta$ , and let  $A' = S^{-1}A$ , and suppose that  $A'$  is contained in another  $S^{-1}R$ -order  $B'$ . Then there exists a nonzero  $r \in R$  such that  $rB' \subseteq A'$ . Define  $I = A \cap rB'$ , then  $I$  is a two-sided ideal of  $A$  with  $S^{-1}I = rB'$ . Define  $B = \{b \in B' : Ib \subseteq I\}$ . By a similar argument to the one of the previous paragraph,  $B$  is an  $R$ -order in  $\Delta$  and  $S^{-1}B = B'$ . Furthermore,  $B$  contains  $A$  and therefore by maximality  $B = A$ . Hence  $B' = S^{-1}B = S^{-1}A = A'$ .  $\square$

**Corollary 3.2.** Let  $A$  be an  $R$ -order in  $\Delta$ . Then the following are equivalent.

- (a)  $A$  is maximal
- (b)  $A_{\mathfrak{p}}$  is maximal for every prime ideal  $\mathfrak{p}$  of  $R$

If in addition  $R$  is integrally closed in its fraction field, then (a) or (b) is also equivalent to

(c)  $A$  is a reflexive  $R$ -module and  $A_{\mathfrak{p}}$  is maximal for all  $\mathfrak{p} \in \text{spec}(R)$  with  $\text{ht}(\mathfrak{p}) = 1$ .

*Proof.* The fact that (a) implies (b) follows immediately from the previous lemma. Conversely, assume (b) and suppose that  $B$  is any  $R$ -order containing  $A$ . Then  $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$  for all prime ideals  $\mathfrak{p}$ , and therefore by maximality,  $A_{\mathfrak{p}} = B_{\mathfrak{p}}$  for all prime ideals  $\mathfrak{p}$ . It follows that  $A = B$ , and hence  $A$  is maximal.

Next suppose that  $R$  is also integrally closed. Note that  $A^{**}$  is an  $R$ -order containing  $A$ , and therefore if  $A$  is maximal  $A$  is reflexive. Thus it follows from the previous lemma that (a) implies (b). Conversely, assume (c). Suppose that  $B$  is any  $R$ -order containing  $A$ . Then by maximality at height one primes, we have that  $A_{\mathfrak{p}} = B_{\mathfrak{p}}$  for all  $\mathfrak{p}$  with  $\text{ht}(\mathfrak{p}) = 1$ . Then since  $R$  is integrally closed,

$$A^{**} = \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}} = \bigcap_{\text{ht}(\mathfrak{p})=1} B_{\mathfrak{p}} = B^{**}.$$

Since  $A$  is reflexive, this shows that  $A = B^{**}$ . However,  $B^{**}$  is an  $R$ -order containing  $B$ , and therefore  $A$  contains  $B$ . Hence  $A = B$ .  $\square$

**Definition 3.2.** Let  $A$  be an  $R$ -order in  $\Delta$ . We define the *reduced norm*  $N_{\text{red}}(a)$ , *reduced trace*  $\text{Tr}_{\text{red}}(a)$ , and *reduced characteristic polynomial*  $\text{Ch}_{\text{red}}(x; a)$  to be the corresponding data as elements of the Azumaya algebra  $\Delta$ .

**Lemma 3.3.** Suppose that  $R$  is integrally closed and  $A$  is an  $R$ -order in  $\Delta$ . Then for all  $a \in A$ , the reduced characteristic polynomial  $\text{Ch}_{\text{red}}(x; a)$  is in  $R[x]$ . Consequently, the reduced norm and reduced trace are also both in  $R$ .

*Proof.* Let  $a \in \Delta$ . Then since  $A$  is finitely generated over  $R$ ,  $a$  is integral over  $R$ . Let  $f(x) \in K[x]$  be the minimal polynomial of  $a$ ; since  $R$  is integrally closed,  $f(x) \in R[x]$ . Note that  $f(x)$  and  $\text{Ch}_{\text{red}}(x; a)$  have the same roots over the algebraic closure  $\bar{K}$  of  $K$  (not counting algebraic multiplicity), since they are exactly the eigenvalues of the left action of  $a \otimes 1$  on  $\Delta \otimes_K \bar{K}$ . This means that the roots of  $\text{Ch}_{\text{red}}(x; a)$  are all algebraic over  $R$ . Hence the coefficients of  $\text{Ch}_{\text{red}}(x; a)$  are algebraic over  $R$ . Since  $\text{Ch}_{\text{red}}(x; a) \in K[x]$  and  $R$  is algebraically closed in  $K$ , this implies that  $\text{Ch}_{\text{red}}(x; a) \in R[x]$ .  $\square$

**Proposition 3.4.** Let  $R$  be integrally closed. Then every  $R$ -order  $A$  in  $\Delta$  is contained in a maximal  $R$ -order in  $\Delta$ .

*Proof.* Let  $B$  be any  $R$ -order in  $\Delta$ . Then the reduced trace defines a nondegenerate bilinear form on  $B$ . Therefore the map  $B \mapsto B^*$  defined by

$$b \mapsto b^* : x \mapsto \text{Tr}_{\text{red}}(xb)$$

is an injective  $R$ -module homomorphism.

Let  $\Omega \subseteq \underline{\text{Ord}}_{R, \Delta}$  be the set of all  $R$ -orders in  $\Delta$  containing  $A$ . Then  $\Omega$  is nonzero, since it contains  $A$ , moreover if  $B_1 \subseteq B_2 \subseteq \dots$  is a chain in  $\Omega$ , then  $B := \bigcup_i B_i$  is an  $R$ -subalgebra of  $A$ . Moreover the  $R$ -module monomorphisms  $B_i \rightarrow B_i^* \subseteq A^*$  induce an  $R$ -module monomorphism  $B \rightarrow A^*$ . Since  $A^*$  is a

finitely generated  $R$ -module and  $R$  is noetherian, it follows that  $B$  is finitely generated. Hence  $B \in \Omega$ . Thus Zorn's lemma tells us that  $A$  is contained in a maximal  $R$ -order in  $\Delta$ .  $\square$

**Proposition 3.5.** *If  $A$  is an Azumaya algebra on  $R$ , then  $A$  is a maximal  $R$ -order in  $\Delta$ .*

*Proof.* We claim that the natural map  $A \mapsto A^*$  given by  $a \mapsto a^* : x \mapsto \text{Tr}_{\text{red}}(xa)$  is an isomorphism. Note that by the nondegeneracy of the reduced trace on  $\Delta$ , this map is injective, so it suffices to show that it is surjective. To see this, choose a basis  $a_1, \dots, a_N$  of  $A$  as a free  $R$ -module and consider the matrix  $T = \text{Tr}_{\text{red}}(a_i a_j) \in M_N(R)$ . Since  $A$  is an Azumaya algebra, the discriminant  $d = \det(T)$  of  $A$  is a unit. Therefore  $T$  is invertible in  $R$ . Hence for  $\vec{r} = T^{-1} \vec{e}_1$ , and  $a = \sum_i a_i r_i$ , we have

$$\text{Tr}_{\text{red}}(a_1 a) = \vec{e}_1 \cdot T \vec{r} = 1.$$

Thus  $A$  contains an element with reduced trace 1, namely  $y$ . For any  $\varphi \in \text{Hom}_R(A, R)$ , define  $\psi \in \text{End}_R(A)$  by  $\psi(x) = \varphi(x)y$ . Then since  $A$  is Azumaya, the natural map  $A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$  is an isomorphism, and therefore there exists  $u, v \in A$  such that  $\psi(x) = uxv$ . It follows that

$$\begin{aligned} \varphi(x) &= \varphi(x) \text{Tr}_{\text{red}}(y) = \text{Tr}_{\text{red}}(\varphi(x)y) \\ &= \text{Tr}_{\text{red}}(\psi(x)) = \text{Tr}_{\text{red}}(uxv) = \text{Tr}_{\text{red}}(xvu) = (vu)^*(x). \end{aligned}$$

This proves that  $A \mapsto A^*$  is an isomorphism. Now if  $B$  is any  $R$ -order in  $\Delta$  containing  $A$ , then  $A \subseteq B \rightarrow B^* \subseteq A^*$ , from which it follows that  $A = B$ .  $\square$

*Example 9.* Let  $R = k[x]$ , and let  $\Delta = M_N(k(x))$ . Then

$$A = \begin{pmatrix} R & R \\ xR & R \end{pmatrix}$$

is an  $R$ -order in  $\Delta$ . It is not maximal, since it is contained in the  $R$ -order  $M_N(R)$ . Furthermore,  $M_N(R)$  is an Azumaya algebra, and therefore is a maximal  $R$ -order in  $\Delta$  containing  $A$ . Note that it is not the only  $R$ -order in  $\Delta$  containing  $A$ , since for example  $A$  is contained in the  $R$ -order

$$B = \begin{pmatrix} R & x^{-1}R \\ xR & R \end{pmatrix}.$$

Note that  $A$  and  $B$  are isomorphic under an inner automorphism of  $\Delta$ , and therefore  $B$  is also a maximal  $R$ -order in  $\Delta$ .

Note that in the previous example, both of the maximal  $R$ -orders in  $\Delta$  are Azumaya algebras – and in fact all of the maximal  $R$ -orders in  $\Delta$  are Azumaya. This is indicative of the following more general phenomenon, which we discuss in greater detail in the section below on ramification.

### 3.2 Maximal Orders over a complete DVR

Let  $R$  be a complete DVR with uniformizer  $\pi$ , residue field  $k$ , and fraction field  $K$ . Let  $\Delta$  be a central simple  $K$ -algebra; by Wedderburn's Theorem, we may take  $\Delta = M_N(D)$ , for  $D$  a division algebra with center  $K$ . In this section, we will show that the maximal  $R$ -order in  $\Delta$  is essentially unique. More specifically, we will show that there exists a unique maximal  $R$ -order  $A$  in  $D$ , and that any maximal  $R$ -order in  $\Delta$  is isomorphic to  $M_N(A)$  via an inner automorphism of  $\Delta$ . The exposition we provide is based on the exposition found in Reiner (citation), which in turn is based on the exposition of Hasse (citation).

**Lemma 3.6.** *Suppose  $f \in K[x]$  is an irreducible polynomial with*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_1 x + a_0.$$

*Then for all  $0 \leq i \leq n$ , we have that*

$$\nu(a_i) \geq \min\{\nu(a_0), \nu(a_n)\}.$$

*Proof.* We will assume that there exists  $i$  such that  $\nu(a_i) < \min\{\nu(a_0), \nu(a_n)\}$  and will arrive at a contradiction. Let  $t = \min_i \nu(a_i)$ , and set

$$m = \min\{0 \leq i \leq n : \nu(a_i) = t\}.$$

Note that  $a_m \neq 0$ . Then since  $f$  is irreducible in  $K[x]$ , so too is the polynomial

$$g(x) := a_m^{-1} f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0,$$

for  $b_i = a_i/a_m$ . Note that by definition  $m \geq 1$  and for all  $0 \leq i < m$ , we have  $b_i \in \pi R$ . Therefore reducing  $g$  modulo the maximal ideal of  $R$ , we find

$$\bar{g}(x) = x^m (\bar{b}_n x^{n-m} + \cdots + \bar{b}_{m+1} x + \bar{b}_m).$$

By Hensel's lemma, it follows that  $g(x)$  is reducible, a contradiction. This proves our lemma.  $\square$

**Lemma 3.7.** *Let  $a \in D$  and let  $f(x) \in K[x]$  be its minimal polynomial. Then*

$$w(a) = \frac{1}{[K(a) : K]} \nu(f(0)).$$

*Consequently,  $w(a) \geq 0$  if and only if  $a$  is integral over  $R$ .*

*Proof.* Let  $g(x)$  be the characteristic polynomial of  $a$ , viewed as an endomorphism of  $D$ , and let

$$g(x) = x^m + \cdots + (-1)^m \det(a).$$

and must divide a power of  $f(x)$ . Hence  $g(x) = f(x)^{m/r}$ , from which it follows that

$$(-1)^m \det(a) = f(0)^{m/r},$$

and therefore

$$w(a) = \frac{1}{m} \nu(\det(a)) = \frac{1}{r} \nu(f(0)) = \frac{1}{[K(a) : K]} \nu(f(0)).$$

If  $a$  is integral over  $R$ , then  $g(x) \in R[x]$ , and therefore  $\det(a) \in R$ . Hence  $w(a) \geq 0$ . Conversely, suppose that  $w(a) \geq 0$ , then  $\nu(f(0)) > 0$  and therefore by the previous lemma  $f(x) \in R[x]$ . Consequently,  $a$  is integral over  $R$ .  $\square$

We are now in a position to show that  $w$  defines a discrete valuation on  $D$ .

**Theorem 3.8.** *Let  $D$  be a division  $K$ -algebra with  $\dim_K(D) = m < \infty$ . Then the valuation  $\nu$  extends to a discrete valuation  $w$  on  $D$ , defined by*

$$w(x) = \frac{1}{m} \nu(\det(x)), \quad \forall x \in D$$

where  $\det(x)$  is the determinant of  $x$  viewed as an endomorphism of  $D$  by left multiplication.

*Proof.* It's clear from the definition that  $w|_K = \nu$ . To prove that  $w$  is a discrete valuation on  $D$ , we must verify the following

- (a)  $w(x) = \infty$  if and only if  $x = 0$
- (b)  $w(ab) = w(a) + w(b)$  for all  $a, b$
- (c)  $w(D) \cong \mathbb{Z}$
- (d)  $w(a + b) \geq \min\{w(a), w(b)\}$

Note that (a), (b) follow immediately from the definition and the fact that  $D$  is a division algebra. Furthermore,  $w(D)$  is an infinite subgroup of  $\mathbb{Z}$ , and therefore (c) holds. Thus it suffices to prove (d).

Suppose that  $w(a) \geq 0$ , and let  $f(x)$  be the minimal polynomial of  $a$ . Then by the previous lemma  $f(x) \in R[x]$ , and since  $f(x-1)$  is the minimal polynomial of  $a+1$ , it follows that  $a+1$  is integral over  $R$ . Hence by the previous lemma,  $\nu(a+1) \geq 0$ . Thus

$$\nu(a+1) \geq 0 = \min\{\nu(a), \nu(1)\}.$$

Now take  $b, c \in K$  and without loss of generality assume  $\nu(b) \leq \nu(c)$ . Then

$$\nu(c/b + 1) \geq \min\{\nu(c/b), \nu(1)\}$$

and by adding  $\nu(b)$  to both sides, we obtain (d).  $\square$

As it turns out, the way that we defined  $w$  is in fact the only way that one may extend the valuation  $\nu$  to  $D$ . To prove this, we establish the following lemma.

**Lemma 3.9.** *Suppose  $f \in K[x]$  is an irreducible monic polynomial, with*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + xa_1 + a_0.$$

*Then for all  $0 \leq i \leq n-1$ ,*

$$\nu(a_i) \geq \frac{n-i}{n}\nu(a_0).$$

*Proof.* Let  $L$  be the splitting field of  $f$ , and let  $\nu'$  be the extension of  $\nu$  to  $L$  given by  $\nu'(x) = \frac{1}{n}\nu(\det(x))$ . Let  $b_1, \dots, b_n \in L$  be the roots of  $f$ . Then for all  $i$ ,  $f(x)$  is the characteristic polynomial of  $b_i$ ; hence  $a_0 = f(0) = (-1)^n \det(b_i)$ , and therefore  $\nu'(b_i) = \frac{1}{n}\nu(a_0)$ . Moreover  $f(x) = \prod_i (x - b_i)$ , making the coefficients of  $f$  the various elementary symmetric polynomials in  $b_i$  and therefore

$$\nu(a_i) = \nu'(a_i) \geq \frac{n-i}{n}\nu(a_0).$$

□

**Corollary 3.10.** *The valuation  $w$  is the unique extension of  $\nu$  to  $D$ .*

*Proof.* Suppose that  $w'$  is another extension of  $\nu$  to  $D$  that doesn't agree with  $w$ . Then there exists  $a \in D$  with  $w'(a) > w(a)$ . Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

be the minimal polynomial of  $f(x)$  (note that  $a_n = 1$ ). Then we have that

$$a_0 = - \sum_{i=1}^n a_i a^i,$$

and therefore

$$\nu(a_0) = w'(a_0) \geq \min_i \{\nu(a_i) + i w'(a)\} > \min_i \{\nu(a_i) + i w(a)\}.$$

Furthermore, by the previous lemma

$$\nu(a_i) \geq \frac{n-i}{n}\nu(a_0) = (n-i)w(a),$$

and therefore the above inequality tells us

$$\nu(a_0) > n w(a),$$

which is a contradiction. Thus  $w = w'$ . □

Using the valuation  $w$ , it is now easy to explicitly define the maximal  $R$ -order in  $D$ . It is determined by

$$A = \{a \in D : w(a) \geq 0\}.$$

It's clear from the definition that  $A$  is an  $R$ -subalgebra of  $D$ , and that  $A \otimes_R K = D$ . Furthermore, if  $B$  is any other  $R$ -order in  $D$ , then every  $b \in B$  is integral over  $R$ . This implies that  $w(b) \geq 0$ , and therefore that  $b \in A$ . Thus  $A$  contains all other  $R$ -orders in  $D$ . Thus to show that  $A$  is a maximal  $R$ -order in  $D$ , it suffices to prove that  $A$  is a finitely generated  $R$ -module. To prove this, we need to more finely probe the structure of  $A$ .

We fix an element  $\pi_D \in A$  satisfying  $\nu_D(\pi_D) = 1$ .

**Proposition 3.11.** *Let  $A$  be the  $R$ -algebra defined above. Then*

- (a)  $J(A) = \pi_D A$  is the unique maximal left (or right or two-sided) ideal of  $A$
- (b)  $J(A) \cap R = \pi R$
- (c)  $A/J(A)$  is a division  $k$ -algebra
- (d) if  $I$  is a nonzero left (or right) ideal of  $A$ , then  $I = J(A)^m = \pi_D^m A$  for some  $m$

*Proof.* Each of these statements follows from mucking about with  $v_D$ .  $\square$

**Definition 3.3.** Let  $A$  be a maximal  $R$ -order in  $D$ . We define the *ramification index*  $e = e(D/R)$  of  $D$  over  $K$  to be the unique integer  $e$  satisfying  $J(A)^e = \pi A$ . We define the *inertial degree*  $f = f(D/R)$  to be  $f = \dim_k(A/\pi A)$ .

**Theorem 3.12.** *The  $R$ -algebra  $A$  is a finitely generated free  $R$ -module of rank  $n = \dim_K(D)$ . Moreover, if  $e$  and  $f$  are the ramification index and inertial degree, respectively, then  $ef = n$ .*

*Proof.* Let  $e$  be the ramification index. Choose  $a_1, \dots, a_f \in A$  with  $\bar{a}_1, \dots, \bar{a}_f \in A/\pi A$  a  $k$ -basis. We claim that the set

$$\{a_i \pi^j : 1 \leq i \leq f, 0 \leq j \leq e-1\},$$

is  $K$ -linearly independent. To see this, suppose otherwise. Then there exist  $\alpha_{ij} \in K$  with

$$\sum_{ij} \alpha_{ij} a_i \pi^j = 0.$$

For each  $j$  set  $n_j = \min_i \nu(\alpha_{ij})$ , and define  $\beta_{ij}$  by  $\alpha_{ij} = \pi^{n_j} \beta_{ij}$ . Then  $\beta_{ij} \in R$  for all  $i, j$ , and for all  $i$  there exists  $j$  with  $\beta_{ij} \notin \pi R$ . Define  $b_j = \pi^{n_j} \pi_D^j$  and  $\sigma$  be a permutation of  $\{0, \dots, e-1\}$  such that  $\nu_D(b_{\sigma(0)}) < \dots < \nu_D(b_{\sigma(e-1)})$ , set  $c_j = b_{\sigma(j)}/b_{\sigma(0)}$  and  $\gamma_{ij} = \beta_{i\sigma^{-1}(j)}$ . Then  $\sum_{ij} \gamma_{ij} a_i b_j = 0$ , with  $\gamma_{ij} \in R$  for all  $i, j$ ,  $\gamma_{i1} \notin \pi R$  for some  $i$  and  $c_j \in \pi A$  for all  $j \geq 2$ . Therefore reducing modulo  $\pi A$  we obtain  $\sum_i \bar{\gamma}_{i1} \bar{a}_i = 0$ , which contradicts the assumption of  $k$ -linear independence.

Lastly, we claim that  $A = R\{a_i \pi_D^j\}$ . To see this, suppose that  $x \in A$ , and let  $\nu_D(x) = ke + j$  for integer  $j, k$  with  $0 \leq j < e$ . Then we may write  $x = \pi^k \pi_D^j u$  for some  $u \in A^\times$ . Therefore by definition of the  $a_i$ 's, there exist  $r_i \in R$  such that  $u = \sum_i r_i a_i$  modulo  $\pi A$ . Setting  $x_1 = \pi^k \pi_D^j (u - \sum_i r_i a_i)$  we have that



$x = y_1 + x_1$  for some  $y_1 \in R\{a_i\pi_D^j\}$  and  $x_1 \in A$  with  $\nu_D(x_1) > \nu_D(x)$ . Repeating this process, we obtain a sequence  $x_i = y_{i+1} + x_{i+1}$  with  $\nu_D(x_{i+1}) > \nu_D(x_i)$  and for all integers  $m > 0$

$$x = \sum_{\ell=1}^m y_\ell + x_m = \sum_{i,j} r_{ijm} a_i \pi_D^j + x_m.$$

For each  $i, j$  the sequence of coefficients  $r_{ijm}$  is a Cauchy sequence in  $R$ , and since  $R$  is complete, it converges to a limit  $r_{ij}$ . It follows that  $x = \sum_{i,j} r_{ij} a_i \pi_D^j$ , proving our claim.  $\square$

**Corollary 3.13.** *The algebra  $A$  is the unique maximal  $R$ -order in  $D$ .*

*Proof.* Note that  $A$  is the integral closure of  $R$  in  $\Delta$ . The previous theorem shows that  $A$  is finitely generated as an  $R$ -module, and it is clear from the definition that  $A$  is an  $R$ -subalgebra of  $\Delta$  with  $A \otimes_R K = \Delta$ . Therefore  $A$  is an  $R$ -order in  $\Delta$ , and since every  $R$ -order in  $\Delta$  consists of elements integral over  $R$ , it is necessarily maximal.  $\square$

**Theorem 3.14.** *The matrix algebra  $M_N(A)$  is a maximal  $R$ -order in  $\Delta$ . Furthermore*

- (a)  $J(A) = \pi_D M_N(A)$  is the unique maximal two-sided ideal of  $A$
- (b) if  $I$  is any two-sided ideal of  $A$ , then  $I = J(A)^n = \pi_D^n M_N(A)$  for some  $n$
- (c) if  $B$  is any other maximal  $R$ -order in  $A$ , then  $B$  is of the form  $uM_N(A)u^{-1}$  for some invertible  $u \in \Delta$

### 3.3 Maximal orders in DVRs

In this section, let  $R$  be an arbitrary DVR with uniformizer  $\pi$  and fraction field  $K$ . In the last section, we determined a great many structural theorems regarding maximal orders over complete DVRs. In this section, we will show how the theory of maximal orders over  $R$  may be related to the theory of maximal orders over their completion  $\hat{R}$ . For notational convenience, in this section  $\Delta$  will be a CSA/ $K$ ,  $\hat{K}$  will be the fraction field of  $\hat{R}$ , and  $\hat{\Delta}$  will be  $\Delta \otimes_K \hat{K}$ . In particular,  $\hat{\Delta}$  is a CSA/ $\hat{K}$ , and  $\Delta$  is a division algebra if and only if  $\hat{\Delta}$  is a division algebra.

The basic idea of relating the theory of maximal orders on  $R$  to the theory of maximal orders on  $\hat{R}$  is based on fpqc descent. In particular, take  $X = \text{spec}(R)$ ,  $U_1 = \text{spec}(\hat{R})$  and  $U_2 = \text{spec}(K)$ . Then the two natural maps  $f_i : U_i \rightarrow X$  form an fppf covering of  $X$ . Descent data on this cover consists of a sheaves  $F_i$  on  $U_i$  with isomorphisms  $\varphi_{ij} : p_{ij1}^* F_i \rightarrow p_{ij2}^* F_j$  for all  $1 \leq i \leq j \leq 2$ , where  $p_{ijk} : U_i \times_X U_j \rightarrow U_k$  is the natural projection. In our case, we have canonical identifications  $U_1 \times_X U_1 = U_1$ ,  $U_2 \times_X U_2 = U_2$  and  $U_1 \times_X U_2 = \text{spec}(\hat{K})$  with  $p_{iij} = \text{id}_{U_i}$  and  $p_{12j} : \text{spec}(\hat{K}) \rightarrow U_j$  the natural map. Therefore the only

nontrivial descent data comes from the isomorphisms  $\varphi_{ij}$  for  $i \neq j$ . Unwinding definitions, descent theory then tells us that

$$M \mapsto (M \otimes_R \hat{R}, M \otimes_R K)$$

defines a bijective correspondence between  $R$ -modules  $M$  and pairs  $(\hat{M}, V)$  with  $\hat{M}$  an  $\hat{R}$ -module and  $V$  a  $K$ -vector space satisfying  $\hat{M} \otimes_{\hat{R}} \hat{K} = V \otimes_K \hat{K}$ .

**Proposition 3.15.** *Let  $\hat{\Delta} = \Delta \otimes_K \hat{K}$ . Then  $\hat{\Delta}$  is a central simple  $\hat{K}$ -algebra and there is a bijective morphism of posets*

$$\text{Ord}_{R, \Delta} \rightarrow \text{Ord}_{\hat{R}, \hat{\Delta}}, \quad A \mapsto A \otimes_R \hat{R}.$$

*In particular  $A$  is a maximal  $R$ -order in  $\Delta$  if and only if  $A \otimes_R \hat{R}$  is a maximal  $\hat{R}$ -order in  $\hat{\Delta}$ .*

*Proof.* Taking  $V = \Delta$  in the above fpqc correspondence shows that  $M \mapsto M \otimes_R \hat{R}$  is a bijective correspondence between  $R$ -modules  $M$  satisfying  $M \otimes_R K = \Delta$  and  $\hat{R}$ -modules  $\hat{M}$  satisfying  $\hat{M} \otimes_{\hat{R}} \hat{K} = \hat{\Delta}$ . This restricts to the desired poset isomorphism.  $\square$

Thus the study of maximal orders on  $R$  is closely tied to the study of maximal  $\hat{R}$ -orders on  $\hat{\Delta}$ . Using this, we deduce some of the structure of maximal  $R$ -orders in  $\Delta$ .

**Corollary 3.16.** *If  $A$  is a maximal  $R$ -order in  $\Delta$ , then  $A$  is left and right hereditary.*

**Proposition 3.17.** *Let  $A$  be an  $R$ -order in  $\Delta$ , and let  $\bar{A} = A/\pi A$ . Then*

$$A/J(A) \cong \bar{A}/J(\bar{A}) \cong (\hat{A})/J(\hat{A}).$$

*Proof.* By elementary ring theory,  $A/J(A) \cong \bar{A}/J$  and  $\hat{A}/J(\hat{A}) \cong \bar{\hat{A}}/J(\bar{\hat{A}})$ , where  $\bar{\hat{A}} = \hat{A}/\pi\hat{A}$ . Therefore it suffices to show that  $\bar{A} \cong \bar{\hat{A}}$ . This in turn follows from the fact that  $\pi\hat{A} \cong \pi A \otimes \hat{R}$  and the exactness of  $- \otimes \hat{R}$ .  $\square$

**Theorem 3.18.** *Let  $A$  be a maximal  $R$ -order in  $\Delta$ . The ideal  $J(A)$  is the unique maximal left (or right) ideal of  $A$ , and furthermore*

$$J(\hat{A}) = J(A) \otimes \hat{R}, \quad J(\hat{A}) \cap A = J(A).$$

*In fact, every nontrivial two-sided ideal of  $A$  is a power of  $J(A)$ .*

*Proof.* We have a natural bijective correspondence

$$\{\text{maximal left ideals of } A\} \longleftrightarrow \{\text{maximal left ideals of } A/J(A)\}$$

and

$$\{\text{maximal left ideals of } \hat{A}\} \longleftrightarrow \{\text{maximal left ideals of } \hat{A}/J(\hat{A})\}$$

so by the previous proposition, there is a bijective correspondence between maximal left ideals of  $A$  and maximal left ideals of  $\hat{A}$ . Since  $\hat{A}$  has exactly one maximal left ideal, it follows that  $A$  has exactly one also. Then, since  $J(A)$  is the intersection of all maximal left ideals,  $J(A)$  must be the maximal left ideal of  $A$ . Furthermore, tracing the above correspondence, we see that  $J(A) \otimes \hat{R} = J(\hat{A})$  and  $J(A) = J(\hat{A}) \cap A$ .

Now if  $I$  is a nontrivial two-sided ideal of  $A$ , then  $I \otimes K$  is a nontrivial two-sided ideal of  $\Delta$ . Hence  $I \otimes K = \Delta$ , and therefore there exists  $a_i \in I$  and  $r_i \in K$  with  $\sum_i a_i r_i = 1$ . Choose  $r \in R$  such that  $rr_i \in R$  for all  $i$ , and set  $s = \nu(r)$ . Then  $r = \sum_i a_i (rr_i) \in I$  and therefore  $rA = \pi^s A \subseteq I$  for all  $I$ . Therefore  $I$  is the preimage of a two-sided ideal of  $A/\pi^s A \cong \hat{A}/\pi^s \hat{A}$ . Therefore  $I \otimes \hat{R}$  is a preimage of a two-sided ideal of  $\hat{A}/\pi^s \hat{A}$ , and therefore  $I \otimes \hat{R} = \pi^t \hat{A} = J(\hat{A})^t$  for some  $t \leq s$ . This tells us that  $I = (I \otimes \hat{R}) \cap A = J(\hat{A})^t \cap A = J(A)^t$ .  $\square$

**Lemma 3.19.** *Let  $A$  be a maximal  $R$ -order in  $\Delta$ , and let  $M$  be a finitely generated left  $A$ -module. Then*

- (a)  $A$  is a projective  $A$ -module
- (b)  $A$  is a free  $R$ -module
- (c) if  $B$  is another finitely generated free  $R$ -module, then  $A \cong B$  as  $A$ -modules if and only if  $A$  and  $B$  have the same rank as  $R$ -modules

**Theorem 3.20.** *Let  $A$  be a maximal  $R$ -order in  $\Delta$ . Then*

- (a) Every one-sided ideal of  $A$  is principal.
- (b) If  $B$  is another maximal  $R$ -order in  $\Delta$  then  $B = uAu^{-1}$  for some  $u \in \Delta^\times$ .
- (c) Let  $\hat{\Delta} = M_t(\hat{D})$  for  $\hat{D}$  a division algebra over  $\hat{K}$ , and let  $\hat{C}$  be a unique maximal  $R$ -order in  $\hat{D}$ . Then  $\hat{C}/J(\hat{C})$  is a division algebra over  $k$  and  $A/J(A) = M_t(\hat{C}/J(\hat{C}))$ .

### 3.4 Ramification of Maximal Orders in DVRs

In this section  $R$  will always be a DVR with fraction field  $K$ , uniformizer  $\pi$  and residue field  $k$ , and  $\Delta$  will represent a central simple  $K$ -algebra of rank  $n$ .

**Definition 3.4.** Let  $A$  be a maximal  $R$ -order in  $\Delta$ . The central simple algebra  $\Delta$  is said to *ramify* over  $R$  if  $A$  is not an Azumaya algebra on  $R$ . We define the *algebraic ramification index*  $e = e(\Delta/R)$  to be the unique integer satisfying  $J(A)^e = \pi J(A)$ , and the *cohomological ramification index*  $e' = e'(\Delta/R)$  to be  $e' = \dim_k(Z(A/J(A)))$ . We define the *inertia degree*  $f = f(\Delta/R)$  to be the dimension of  $A/J(A)$  over its center.

Note that since all of the maximal  $R$ -orders in  $\Delta$  are conjugate, whether or not  $\Delta$  ramifies and the value of its ramification index is independent of the choice of maximal  $R$ -order in  $\Delta$ .

**Lemma 3.21.** *Let  $A$  be a maximal  $R$ -order in  $\Delta$ , with algebraic and cohomological ramification index  $e$  and  $e'$ , respectively, and with inertia degree  $f$ . Then  $ee'f^2 = n^2$ .*

*Proof.* Since  $A$  is a free  $R$ -module of rank  $n^2$ , we have

$$n^2 = \dim_k(\overline{A}) = e \dim_k(\overline{A}/J(\overline{A})) = e \dim_k(A/J(A)) = ee'f^2.$$

□

**Proposition 3.22.** *Let  $A$  be a maximal  $R$ -order in  $\Delta$ , with algebraic and cohomological ramification index  $e$  and  $e'$ , respectively. Then the following are equivalent*

- (a)  $\Delta$  is ramified over  $R$
- (b)  $e > 1$  or  $e' > 1$

*Proof.* If  $\Delta$  is unramified over  $R$ , then  $A$  is an Azumaya algebra and therefore  $\overline{A} = A/\pi A$  is an Azumaya algebra on  $k$ . In particular, this means  $\overline{A}$  is a CSA over  $k$ , so that  $J(\overline{A}) = 0$  and  $Z(\overline{A}) = k$ . Since  $A/J(A) \cong \overline{A}/J(\overline{A})$ , this implies that  $A/J(A) \cong \overline{A}$ . Hence  $\dim_k(Z(A/J(A))) = 1$  and  $J(A) = \pi A$ . Therefore (b) implies (a).

If  $\Delta$  is ramified over  $R$ , then  $A$  is not Azumaya, and therefore  $\overline{A}$  is not Azumaya. If  $e = 1$ , then  $\overline{A}$  is simple with center a nontrivial cyclic extension of  $k$ , and therefore  $e' > 1$ . Hence (a) implies (b). □

**Lemma 3.23.**

- (a)  $e(\Delta/R) = e(\widehat{\Delta}/\widehat{R}) = e(\widehat{D}/\widehat{R})$
- (b)  $e'(\Delta/R) = e'(\widehat{\Delta}/\widehat{R}) = e'(\widehat{D}/\widehat{R})$
- (c)  $e|e'$

## 4 Maximal Orders on Integral Schemes

### 4.1 Basic Definitions and Facts

### 4.2 Orders in Quaternion Algebras on Surfaces

## 5 Brauer Group of a Nonsingular Surface

### 5.1 Some Etale Cohomology

In this section, we turn our attention to the Brauer group of a nonsingular integral scheme  $X$ , focusing primarily on the case of a nonsingular surface. In the nonsingular case, the cohomological Brauer group of  $X$  injects into the Brauer group of its function field  $K$ . Furthermore, coprime to the characteristic of  $K$ , the cokernel of this may be described by certain ramification data on the divisors of  $X$ . Specifically, we have the following main theorem

**Theorem 5.1.** *Let  $X$  be a nonsingular, excellent, integral scheme. If  $\ell$  is coprime to the characteristic of  $K$ , then we have an  $\ell$ -exact sequence*

$$0 \rightarrow \mathrm{Br}'(X) \rightarrow \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X^1} H^1(X, i_*(\mathbb{Q}/\mathbb{Z})) \rightarrow H^3(X^1, \mathbb{G}_m) \rightarrow H^3(X, \mathbb{G}_m).$$

Here  $X^1$  represents the set of codimension 1 points of  $X$ , and by  $H^p(X^1, \mathbb{G}_m)$ , we mean  $\lim H^p(U, \mathbb{G}_m)$  where the limit is taken over all complements  $U$  of closed subsets of  $X$  with codimension at least two. In other words, we are taking cohomology viewing  $X^1$  as a pro-object in our category. The map  $\rho$  has an interesting algebraic interpretation. In particular, given a central simple  $K$ -algebra  $\Delta$ , there exists a nonempty open subset of  $X$  to which  $\Delta$  extends to an Azumaya algebra. A Zorn's lemma argument then tells us that there exists a largest such open set  $U$ . In fact, if we let  $\mathcal{A}$  be a maximal  $\mathcal{O}_X$ -algebra in  $\Delta$ , then the set of  $x \in X$  for which  $\mathcal{A}_x$  is Azumaya is exactly  $U$ . One may show that  $X \setminus U$  is the union of the closures  $C_x$  of the  $x \in X^1$  for which  $\rho([\Delta])_x$  is nonzero.

The relationship extends even further than this. The cohomology group  $H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$  classifies cyclic extensions of  $\kappa(x)$ . Since  $X$  is nonsingular, for each  $x \in X^1$  the local ring  $\mathcal{O}_{X,x}$  is a DVR with valuation  $\nu_x$  and residue field  $\kappa(x)$ . Since  $\mathcal{A}_x$  is a maximal order on  $\mathcal{O}_{X,x}$  the theory of maximal orders on DVRs tells us that  $\mathcal{A}_x/J(\mathcal{A}_x)$  is a central simple algebra, whose center is a cyclic extension of  $k(x)$ , and therefore represented up to isomorphism by an element of  $H^1(\kappa(x), \mathbb{Q}/\mathbb{Z})$ . One may show that this element is  $\rho([\Delta])_x$ .

To prove our main theorem, we start by recognizing that the fact that for nonsingular  $X$  the sheaf of Cartier divisors  $D$  has a simple description.

**Lemma 5.2.** *Let  $D$  be the sheaf of Cartier divisors on  $X$ . Then there is an isomorphism of sheaves on the small Zariski/etale site of  $X$ :*

$$D \cong \sum_{x \in X^1} i_{x*} \mathbb{Z}.$$

*Proof.* By definition,  $D = i_{\eta*} \mathbb{G}_m / \mathbb{G}_m$ . Therefore it suffices to show that the sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow i_{\eta*} \mathbb{G}_m \xrightarrow{\rho} \sum_{x \in X^1} i_{x*} \mathbb{Z} \rightarrow 0$$

is exact on the small Zariski site of  $X$ , where here  $\rho$  is the map defined by  $s \mapsto (\nu_x(s))_x$ . This in turn is verified by checking stalks.  $\square$

As a consequence of the previous lemma, we have a long exact sequence in cohomology

$$\bigoplus_{x \in X^1} H^1(X, i_{x*}\mathbb{Z}) \rightarrow H^2(X, \mathbb{G}_m) \rightarrow H^2(X, i_{\eta*}\mathbb{G}_m) \rightarrow \bigoplus_{x \in X^1} H^2(X, i_{x*}\mathbb{Z}) \rightarrow \dots \quad (1)$$

The majority of the rest of this section is devoted to showing that this exact sequence may be wrangled to give us the long exact sequence in the statement of Theorem FIXME.

**Lemma 5.3.** *Let  $Y$  be an integral scheme with generic point  $\eta$ , and let  $f : \eta \rightarrow \mathbb{Z}$  be the inclusion. Then*

- (a)  $H^1(Y, f_*\mathbb{Z}) = 0$
- (b)  $H^i(Y, f_*\mathbb{Q}) = 0$  for all  $i > 0$
- (c)  $H^i(Y, f_*\mathbb{Z}) \cong H^{i-1}(Y, f_*(\mathbb{Q}/\mathbb{Z}))$  for all  $i > 1$

*Proof.*

- (a) The Leray spectral sequence for  $f$  shows that there is an injection  $H^1(Y, f_*\mathbb{Z}) \rightarrow H^1(\eta, \mathbb{Z})$ . Moreover,  $H^1(\eta, \mathbb{Z}) \cong H^1(K(Y), \mathbb{Z})$  and since all finite groups map trivially into  $\mathbb{Z}$ , this latter cohomology group is trivial. Hence  $H^1(Y, f_*\mathbb{Z})$  is trivial.  $\blacksquare$
- (b) For all  $j > 0$  and all  $y \in Y$ , we have  $(R^j f_*\mathbb{Q})_x \cong H^j(K(\widetilde{\mathcal{O}_{Y,y}}), \mathbb{Q})$ , where here  $\widetilde{\mathcal{O}_{Y,y}}$  represents the étale stalk of  $\mathcal{O}_Y$  at  $y$ . Note that for any finite group  $G$ , the group cohomology  $H^j(G, \mathbb{Q})$  is 0 for  $j > 0$ , because the inflation/restriction sequence shows that  $|G|$  kills all the homology groups, and is also invertible. Therefore  $R^j f_*\mathbb{Q} = 0$  for all  $j > 0$ , and the Leray spectral sequence for  $f$  tells us that  $H^j(Y, f_*\mathbb{Q}) \cong H^j(K(Y), \mathbb{Q})$ . By the same argument,  $H^j(K(Y), \mathbb{Q}) = 0$  and therefore  $H^j(Y, f_*\mathbb{Q}) = 0$ .
- (c) The short exact sequence of sheaves on  $Y$

$$0 \rightarrow f_*\mathbb{Z} \rightarrow f_*\mathbb{Q} \rightarrow f_*(\mathbb{Q}/\mathbb{Z})$$

induces a long exact sequence in cohomology, which by the result of (b) gives us isomorphisms  $H^i(Y, f_*\mathbb{Z}) \cong H^{i-1}(Y, f_*(\mathbb{Q}/\mathbb{Z}))$  for all  $i > 1$ .  $\square$

**Corollary 5.4.** *Let  $X$  be a nonsingular integral scheme. Then for all codimension 1 points  $x \in X$*

- (a)  $H^1(X, (i_x)_*\mathbb{Z}) = 0$

(b)  $H^i(X, (i_x)_*\mathbb{Z}) \cong H^{i-1}(X, (i_x)_*\mathbb{Q}/\mathbb{Z})$  for all  $i > 1$

*Proof.* Let  $C$  be the irreducible codimension 1 subscheme of  $X$  corresponding to  $x$ . Since  $X$  is nonsingular, for any abelian group  $A$ , we have  $(i_x)_*A \cong j_*f_*A$ , where  $j : C_x \rightarrow X$  is the inclusion of the irreducible, codimension 1 subscheme  $C_x$  of  $X$  for which  $x$  is the generic point, and  $f : x \rightarrow C_x$  is the inclusion of the generic point. The corollary then follows from the previous lemma.  $\square$

As a consequence of the previous corollary, we now have that the long exact sequence obtained from the sheaf of Cartier divisors looks like

$$0 \rightarrow \mathrm{Br}'(X) \rightarrow H^2(X, i_{\eta*}\mathbb{G}_m) \rightarrow \bigoplus_{x \in X^1} H^1(X, i_{x*}(\mathbb{Q}/\mathbb{Z})) \rightarrow H^3(X, \mathbb{G}_m) \rightarrow H^3(X, i_{\eta*}\mathbb{G}_m) \quad (2)$$

To finish the proof of the main theorem, it would be nice to obtain an  $\ell$ -exact isomorphism  $H^i(X, (i_{\eta})_*\mathbb{G}_m) \cong H^i(K, \mathbb{G}_m)$ . However, this is not actually the case! The real key to the rest of the argument lies in the fact that elements of  $H^2(X, \mathbb{G}_m)$  are determined by their behavior in codimension 0 and 1. More specifically, we have the following lemma:

**Lemma 5.5.** *Let  $X$  be a nonsingular, excellent, integral scheme with fraction field  $K$ . Then  $H^2(X, \mathbb{G}_m) \cong H^2(X^1, \mathbb{G}_m)$ .*

*Proof.* This is a little weird...  $\square$

Therefore, by replacing  $X$  with  $U$  in our long exact sequence, and taking the limit over all complements of closed subschemes of  $X$  with codimension at least two, we obtain a long exact sequence

$$0 \rightarrow \mathrm{Br}'(X) \rightarrow H^2(X^1, i_{\eta*}\mathbb{G}_m) \rightarrow \bigoplus_{x \in X^1} H^1(X, i_{x*}(\mathbb{Q}/\mathbb{Z})) \rightarrow H^3(X^1, \mathbb{G}_m) \rightarrow H^3(X^1, i_{\eta*}\mathbb{G}_m) \quad (3)$$

where we have used the fact that  $i_{x*}(\mathbb{Q}/\mathbb{Z})$  is supported in codimension 0 and 1, so that  $H^1(X, i_{x*}(\mathbb{Q}/\mathbb{Z})) = H^1(X^1, i_{x*}(\mathbb{Q}/\mathbb{Z}))$ . Thus to finish the proof of our big theorem, it remains to prove that  $H^p(X^1, i_{\eta*}\mathbb{G}_m) \cong H^j(K, \mathbb{G}_m)$  for  $j > 1$ .

**Lemma 5.6.** *Let  $X$  be a nonsingular integral scheme, with generic point  $\eta$  and fraction field  $K$ . Let  $\ell$  be a prime different from the characteristic of  $K$ . Then for  $j > 0$  the sheaf  $R^j(i_{\eta})_*\mathbb{G}_m$  has support in codimension  $\geq 2$ .*

*Proof.* It suffices to show that the (etale) stalks at each codimension 1 point of the sheaf  $R^j(i_{\eta})_*\mathbb{G}_m$  are trivial. Suppose that  $x \in X$ . Then there is an isomorphism

$$(R^j(i_{\eta})_*\mathbb{G}_m)_x \cong H^j(K(\widehat{\mathcal{O}_{X,x}}), \mathbb{G}_m).$$

If  $x$  is a point of codimension 1, then  $K(\widehat{\mathcal{O}_{X,x}})$  is the fraction field of an irreducible curve over a separably closed field, and hence has  $\ell$ -cohomological dimension 1. Hence the  $\ell$ -principal part of  $(R^j(i_{\eta})_*\mathbb{G}_m)_x$  is 0 for  $j > 1$ . Moreover, this also shows that  $R^1(i_{\eta})_*\mathbb{G}_m = 0$  by Hilbert's Theorem 90.  $\square$



**Lemma 5.7.** *Let  $X$  be a nonsingular, integral scheme with fraction field  $K$ . Let  $\ell$  be a prime number different from the characteristic of  $K$ . Then*

$$H^j(X^1, (i_\eta)_* \mathbb{G}_m)(\ell) \cong H^j(K, \mathbb{G}_m)(\ell).$$

*Proof.* Let  $\eta$  be the generic point of  $X$ . Then since  $R^1(i_\eta)_* \mathbb{G}_m = 0$  and  $R^j(i_\eta)_* \mathbb{G}_m(\ell)$  is supported in codimension  $\geq 2$  for all  $j > 1$ , the Leray spectral sequence for  $i_\eta : \eta \rightarrow X^1$  tells us that  $H^j(X^1, \mathbb{G}_m)(\ell) \cong H^j(K, \mathbb{G}_m)(\ell)$  for all  $j > 1$ .  $\square$

Putting this all together, the main theorem follows immediately.

*Proof of the Main Theorem.* Clear from the above discussion.  $\square$

As a consequence of the main theorem, we see that there are canonical inclusions

$$\mathrm{Br}(X) \subseteq \mathrm{Br}'(X) \subseteq H^2(X, (i_\eta)_* \mathbb{G}_m) \subseteq \mathrm{Br}(K).$$

In the case that  $X$  is a nonsingular, 2-dimensional integral scheme, the first inclusion is actually an isomorphism

**Proposition 5.8.** *Let  $X$  be a nonsingular, 2-dimensional integral scheme. Then the canonical inclusion  $\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$  is an isomorphism.*

*Proof.* Suppose that  $[d] \in \mathrm{Br}'(X)$ , and let  $[\Delta]$  be its image in  $\mathrm{Br}(K)$ . Let  $\mathcal{A}$  be a maximal  $\mathcal{O}_X$ -order in  $\Delta$ . Then  $\mathcal{A}_x$  is a maximal  $\mathcal{O}_{X,x}$ -order in  $\Delta$ . Since the latter factors through  $\mathrm{Br}(X) \rightarrow \mathrm{Br}(\mathcal{O}_{X,x})$ , it follows that  $\mathcal{A}_x$  is Azumaya. Furthermore, since  $\mathcal{A}$  is maximal, it must be reflexive. Hence  $\mathcal{A}$  is Azumaya, and  $[\mathcal{A}] \mapsto [\Delta] \in \mathrm{Br}(K)$ . Hence  $[\mathcal{A}] \mapsto [d] \in \mathrm{Br}'(X)$ .  $\square$

## 5.2 Artin-Mumford Spectral Sequence

In this section, we prove the following main theorem of Artin and Mumfords paper.

**Theorem 5.9** (Artin-Mumford). *Let  $X$  be a simply connected smooth projective surface over an algebraically closed field  $k$ . Then for any prime integer  $\ell$  invertible in  $K$ , there is an  $\ell$ -exact sequence*

$$0 \rightarrow \mathrm{Br}(X) \rightarrow \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \xrightarrow{a} \bigoplus_{x \in X^2} \mu^{-1} \xrightarrow{\Sigma} \mu^{-1} \rightarrow 0,$$

where  $X^i$  represents the set of codimension  $i$  points of  $X$ .

**Lemma 5.10.** *Let  $f : X^1 \rightarrow X$ . Then there is an exact sequence*

$$0 \rightarrow H^3(X, \mathbb{G}_m) \rightarrow H^3(X^1, \mathbb{G}_m) \rightarrow \bigoplus_{x \in X_0} (j_x)_* \mu^{-1} \rightarrow H^4(X, \mathbb{G}_m) \rightarrow 0.$$

*Proof.* Since  $R^0 f_* \mathbb{G}_m = \mathbb{G}_m$ ,  $R^3 f_* \mathbb{G}_m = \bigoplus_{x \in X_0} (j_x)_* \mu^{-1}$  and  $R^q f_* \mathbb{G}_m = 0$  otherwise, this just falls out of the Leray spectral sequence for  $f$ . There is a minor point about  $H^4(X^1, \mathbb{G}_m) = 0$  FIXME.  $\square$

*Proof of Theorem 5.9.* First note that since  $X$  is simply connected,  $H^3(X, \mathbb{G}_m) = 0$ . Also, since  $K$  is the fraction field of a nonsingular surface over an algebraically closed field  $k$ , the  $\ell$ -cohomological dimension of  $K$  is two. In particular  $H^3(K, \mathbb{G}_m) = 0$ . Taking the  $\ell$ -exact sequence

$$0 \rightarrow \mathrm{Br}(X) \rightarrow \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X_0^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \rightarrow H^3(X^1, \mathbb{G}_m) \rightarrow \overbrace{H^3(K, \mathbb{G}_m)}^{=0}$$

and combining it with the exact sequence of the previous lemma, we have the  $\ell$ -exact sequence

$$0 \rightarrow \mathrm{Br}(X) \rightarrow \mathrm{Br}(K) \xrightarrow{\rho} \bigoplus_{x \in X_0^1} H^1(\kappa(x), \mathbb{Q}/\mathbb{Z}) \rightarrow \bigoplus_{x \in X_0} (j_x)_* \mu^{-1} \rightarrow H^4(X, \mathbb{G}_m) \rightarrow 0.$$

The last part then follows from  $H^4(X, \mathbb{G}_m) = \mu^{-1}$ .  $\square$

## 6 Brauer-Severi Schemes

Let  $X$  be an integral scheme over an algebraically closed field  $k$ . We have already pointed out that the set  $H^1(X, \mathbb{PGL}_n)$  classifies twisted forms of two different algebraic structure over  $X$ :

$$\begin{array}{ccc} & H^1(X, \mathbb{PGL}_n) & \\ \swarrow & & \searrow \\ \text{Az}_n(X) & \text{-----} & \underline{\text{Twist}}(\mathbb{P}_X^{n-1}) \end{array}$$

Hence rank  $n$  Azumaya algebras on  $X$  and twisted forms of  $\mathbb{P}_X^{n-1}$  are in bijection. One way of explicitly obtaining this bijection is by means of generating the Brauer-Severi scheme associated to an Azumaya algebra, which we now define.

Let  $\mathcal{A}$  a sheaf of  $\mathcal{O}_X$ -algebras, locally free as an  $\mathcal{O}_X$ -module. We define a functor  $\mathfrak{BS}(\mathcal{A}) : \underline{\text{Sch}}_X \rightarrow \underline{\text{Sets}}$  by

$$\mathfrak{BS}(\mathcal{A}) : (T \rightarrow X) \mapsto \{\text{quotient left } \mathcal{A}_T\text{-modules of } \mathcal{A}_T \text{ locally free of rank } n\}.$$

**Lemma 6.1.** *The functor  $\mathfrak{BS}(\mathcal{A})$  is representable by a quasi-projective  $k$ -scheme.*

*Proof.* Define a subfunctor  $\mathfrak{Q}_n$  of the Quot functor  $\mathfrak{Q} := \mathfrak{Quot}_{\mathcal{A}/X/k}$  by

$$\mathfrak{Q}_n : (T \rightarrow X) \mapsto \{(\mathcal{F}, q) \in \mathfrak{Q}(T) : \mathcal{F} \text{ is locally free of rank } n\}.$$

We claim that  $\mathfrak{Q}_n$  is an open subfunctor of  $\mathfrak{Q}$ . To see this, suppose that  $T \in \underline{\text{Sch}}_k$  and that we have a natural transformation  $h_T \rightarrow \mathfrak{Q}$ . Since  $\mathfrak{Q}$  is represented by a scheme  $Q$ , there exists a natural isomorphism  $\eta : \text{Hom}(-, Q) \rightarrow \mathfrak{Q}$ . Furthermore, Yoneda tells us  $h_T \rightarrow \mathfrak{Q}$  corresponds to a unique morphism  $f : T \rightarrow Q$ , thereby a unique element  $(\mathcal{F}, q) := \eta(T)(f) \in \mathfrak{Q}(T)$ . Note that since  $\mathcal{F}$  is flat over  $T$ , it is locally free over  $T$ . Let  $U \subseteq T$  be the set of points  $t \in T$  for which  $\mathcal{F}_t$  has rank  $n$ . Then  $U$  is an open subset of  $T$ . Thus to prove that  $\mathfrak{Q}_n$  is an open subfunctor of  $\mathfrak{Q}$  it suffices to show that  $\mathfrak{Q}_n \times_{\mathfrak{Q}} h_T = h_U$ .

To see this, suppose that we have a commutative diagram

$$\begin{array}{ccc} h_{T'} & \longrightarrow & \mathfrak{Q}_n \\ \downarrow & & \downarrow \\ h_T & \longrightarrow & \mathfrak{Q} \end{array}$$

Then we have a natural transformation  $h_{T'} \rightarrow \mathfrak{Q}$ , corresponding to a morphism  $f' : T' \rightarrow Q$ , and determining an element  $(\mathcal{F}', q') := \eta(T')(f') \in \mathfrak{Q}(T')$ . Since  $h_{T'} \rightarrow \mathfrak{Q}$  factors through  $\mathfrak{Q}_n$ , this tells us  $\mathcal{F}'$  is locally free of rank  $n$ . Since  $h_{T'} \rightarrow \mathfrak{Q}$  factors through  $h_T$ , this tells us that  $f'$  factors through  $f$ ; meaning that there is a morphism  $g : T' \rightarrow T$  with  $f' = gf$ . By functoriality, it follows that  $\eta(T)(f') = \eta(g)(f) = (g^*\mathcal{F}, g^*q)$ . Thus  $g^*\mathcal{F}$  is locally free of rank  $n$ . Since

pulling back locally free modules doesn't affect the rank, the image of  $g$  must lie in  $U$ . Hence  $T' \rightarrow T$  factors through  $U \rightarrow T$ , giving us a unique morphism  $h_{T'} \rightarrow h_U$ . Thus  $h_U$  satisfies the universal property of the pullback, and this proves our claim. In particular, this shows that  $\Omega_n$  is representable by an open subscheme of  $Q$ .

Next, define a subfunctor  $\Omega'$  of  $\Omega$  by

$$\Omega' : (T \rightarrow X) \mapsto \{(\mathcal{F}, q) \in \Omega(T) : \mathcal{F} \text{ is a left } \mathcal{A}\text{-module}\}.$$

We claim that  $\Omega'$  is a locally closed subfunctor of  $\Omega$ . To see this, again consider a natural transformation  $h_T \rightarrow \Omega$ , let  $(\mathcal{F}, q)$  be the corresponding element of  $\Omega(T)$ , and let  $\mathcal{I}$  be the kernel of  $\mathcal{A}_T \rightarrow \mathcal{F}$ . We define

$$Z = \{t \in T : a\mathcal{I}_t \subseteq \mathcal{I}_t, \forall a \in A\}.$$

Then  $Z$  is a locally closed subscheme of  $Y$ . To check this, we may immediately reduce to the case where  $T = \operatorname{spec}(R)$  for finitely generated  $k$ -algebra  $R$  and  $\mathcal{A}$  is generated by a free  $R$ -module  $A$ . Here, it's easy to express  $A_{\mathfrak{p}} I_{\mathfrak{p}} \subseteq I_{\mathfrak{p}}$  in terms of finitely many relations on  $R$ . By a similar argument to before, we prove that  $h_Z \cong h_T \times_{\Omega} \Omega'$ , and therefore that  $\Omega'$  is locally closed subfunctor of  $h_Z$ . In particular it is representable.

Noting that  $\mathfrak{BS}(\mathcal{A})$  is the intersection of  $\Omega'$  and  $\Omega_n$ , we see that  $\mathfrak{BS}(\mathcal{A})$  is representable by a locally closed subscheme  $\operatorname{BS}(\mathcal{A})$  of  $Q$ . Since  $Q$  is projective over  $k$ , this also shows us that  $\operatorname{BS}(\mathcal{A})$  is quasiprojective over  $k$ .  $\square$

**Definition 6.1.** We denote the  $X$ -scheme representing the functor  $\mathfrak{BS}(\mathcal{A})$  as  $\operatorname{BS}(\mathcal{A})$ , and call it the *Brauer-Severi scheme of  $\mathcal{A}$* . In the special case that  $X = \operatorname{spec}(K)$  for some finitely generated field extension  $K/k$ , and  $\Delta$  is a central simple  $K$ -algebra, the Brauer-Severi scheme is also called the *Brauer-Severi variety*.

**Proposition 6.2.** *Let  $\mathcal{A}$  be an Azumaya algebra. Then Brauer-Severi scheme  $\operatorname{BS}(\mathcal{A})$  is étale-locally isomorphic to  $\mathbb{P}_X^{n-1}$ , and the functor  $\mathcal{A} \mapsto \operatorname{BS}(\mathcal{A})$  induces a bijective correspondence*

$$\operatorname{Az}_n(X) \longleftrightarrow \{\text{iso classes of twisted forms of } \mathbb{P}_K^{n-1}\},$$

through which the canonical bijection  $\operatorname{Az}_n(X) \rightarrow H^1(X, \mathbb{PGL}_n)$  factors.

*Example 10.* Let  $k$  be a field and  $\Delta = M_n(k)$ . Then  $\operatorname{BS}(\Delta) \cong \mathbb{P}_k^{n-1}$ .

*Example 11.* Let  $k$  be a field, and  $a, b \in k^*$  not squares, and  $\Delta = (a, b)$  the quaternion algebra

$$(a, b) = k\langle x, y \rangle / (x^2 - a, y^2 - b, xy + yx).$$

Then  $\operatorname{BS}(\Delta) = V(X^2 - aY^2 - bZ^2) \subseteq \mathbb{P}_k^2$ .

*Example 12.* Let  $k$  be a field, and  $\Delta \subseteq M_2(k)$  be

$$\Delta = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$

Then  $\text{BS}(\Delta) \cong \mathbb{P}_k^1 \vee \mathbb{P}_k^1$

**Lemma 6.3.** *Let  $f : Y \rightarrow X$  be a morphism of schemes. Then*

$$\text{BS}(\mathcal{A}) \times_X Y \cong \text{BS}(f^*\mathcal{A}).$$

*Proof.* Let  $F : \underline{\text{Sch}}_Y \rightarrow \underline{\text{Sch}}_X$  be the natural map induced by  $X$ . Then for any  $V \in \underline{\text{Sch}}_X$ ,

$$h_V \circ F = h_{V \times_X Y}.$$

Thus to show that

$$\text{BS}(\mathcal{A}) \times_X Y \cong \text{BS}(f^*\mathcal{A}),$$

by Yoneda, it suffices to prove that

$$\mathfrak{BS}(\mathcal{A}) \circ F = \mathfrak{BS}(f^*\mathcal{A}).$$

Take  $T \in \underline{\text{Sch}}_Y$  with  $\pi : T \rightarrow Y$ . Then

$$\begin{aligned} \mathfrak{BS}(\mathcal{A})(F(T)) &= \{\mathcal{I} \subseteq (f\pi)^*\mathcal{A} : \mathcal{I} \text{ loc. free rk. 2 ideal}\} \\ &= \{\mathcal{I} \subseteq \pi^*f^*\mathcal{A} : \mathcal{I} \text{ loc. free rk. 2 ideal}\} \\ &= \mathfrak{BS}(f^*\mathcal{A})(T). \end{aligned}$$

□

## 7 Maximal Orders in Quaternion Algebras on Surfaces

In this section, let  $S$  be a complete, non-singular algebraic surface over an algebraically closed field  $k$  with characteristic different from 2. We explore the presentation of maximal  $\mathcal{O}_S$ -orders in quaternion  $K(S)$ -algebras.

**Proposition 7.1.** *Let  $\Delta$  be a quaternion algebra over  $K(S)$  with nonsingular ramification curve  $C$ . Let  $\mathcal{A}$  be a maximal  $\mathcal{O}_S$ -order in  $\Delta$ . Then for any  $s \in C$ ,*

$$\mathcal{A}_s \cong \mathcal{O}_{S,s} \langle x, y \rangle / (x^2 - a, y^2 - bt, xy + yx),$$

where  $a, b \in \mathcal{O}_{S,s}^*$ , with  $a$  not a square (modulo  $t$ ), and  $t \in \mathcal{O}_{S,s}$  is a local equation for  $C$ .

*Proof.* Since  $C$  is nonsingular, it is a disjoint union of nonsingular algebraic curves  $C_1, \dots, C_n$  in  $S$ . Let  $c_i$  be the generic point of  $C_i$ . Then  $\mathcal{O}_{S,c_i}$  is a DVR  $(R_i, \nu_i)$  with fraction field  $K(S)$  and residue field  $K(C_i)$ . Furthermore,  $A_i := \mathcal{A}_{c_i}$  is a maximal  $R_i$ -order in  $\Delta$ . Let  $t_i$  be a uniformizer of  $R_i$ , and let  $J_i$  be the Jacobson radical. From the classical theory of maximal orders,  $L_i := A_i/J_i$  is a cyclic extension of  $K(C_i)$ . Furthermore, since  $[\Delta]$  has order 2 in  $\text{Br}(K(S))$ ,  $L_i$  is a degree 2 extension and  $J_i^2 = A_i t_i$  FIXME.

Let  $\text{Tr}_{\text{red}} : A_i \rightarrow R_i$  and  $\det : A_i \rightarrow R_i$  represent the reduced norm and trace on  $A_i$ . Choose  $x_0 \in A_i$  such that  $x_0$  reduces to a cyclic generator  $\bar{x}_0$  of  $L_i$  modulo  $J_i$ . Then  $\bar{x}_0^2 \in K(L_i)$ , and since  $x_0$  also satisfies  $x_0^2 - \text{Tr}_{\text{red}}(x_0)x_0 + \det(x_0) = 0$ , it follows that  $\text{Tr}_{\text{red}}(x_0) \in J_i$ . Then  $x := x_0 - \text{Tr}_{\text{red}}(x_0)$  reduces to the cyclic generator  $\bar{x}_0$  modulo  $J_i$  and has reduced trace 0, so that  $x^2 = -\det(x) = a \in R_i^*$ .

Next note that  $A_i$  comes equipped with an involution  $z \mapsto z^* := -z + \text{Tr}_{\text{red}}(z)$ . Since  $J_i$  is the unique prime ideal containing  $t$  and  $t^* = t$ ,  $J_i$  will be preserved by  $*$ . Therefore for all  $z \in J_i$ ,  $\text{Tr}_{\text{red}}(z) = z + z^* \in J_i \cap R_i = tR_i$ . Furthermore, since  $J_i^2 = tA_i$ , it follows that  $z$  is equivalent to  $z'$  modulo  $J_i^2$ , where

$$z' = z - \frac{1}{2}\text{Tr}_{\text{red}}(z) - \frac{1}{2a}\text{Tr}_{\text{red}}(xz),$$

satisfies  $\text{Tr}_{\text{red}}(z') = \text{Tr}_{\text{red}}(xz') = 0$ . Thus by means of the  $L_i$ -linear homomorphism

$$J_i/J_i^2 \xrightarrow{\cong} J \otimes_R (L_i) \xrightarrow{\subseteq} A \otimes_R L_i \xrightarrow{\cong} M_2(L_i),$$

we may identify  $J_i/J_i^2$  with the two-dimensional subspace of  $M_2(L_i)$  consisting of matrices  $z$  satisfying  $\text{Tr}_{\text{red}}(z) = \text{Tr}_{\text{red}}(xz) = 0$  (where here  $x$  is identified with its image in  $M_2(L_i)$ ). In particular  $J_i/J_i^2$  is two-dimensional as a vector space over  $L_i$ . Take  $y \in J_i$  nonzero with  $\text{Tr}_{\text{red}}(y) = \text{Tr}_{\text{red}}(xy) = 0$ . Then  $\{y, xy\}$  is a  $L_i$ -linearly independent set, and hence a basis for  $J_i/J_i^2$ . Nakayama's lemma then tells us that  $\{y, xy\}$  generates  $J_i$  as an  $R_i$ -module. Consequently  $\{1, x, y, xy\}$  generates  $A_i$  as an  $R_i$ -module. Note that since  $S$  is smooth,  $A_i$  is free of rank 4, and therefore this is in fact an  $R_i$ -module basis. Note also that we can calculate the discriminant of  $A_i$  using this basis to be  $-16a^2b^2$ .

Now since  $x, y, xy$  all have zero reduced trace,

$$-xy = (xy)^* = y^*x^* = (-y)(-x) = yx.$$

Furthermore, since  $y^2 = -\det(y) \in J_i \cap R = At$ , we may write  $y^2 = bt$  for some  $b \in A$ . Thus we may define a map

$$R\langle x, y \rangle / (x^2 - a, y^2 - bt, xy + yx) \rightarrow A_i,$$

and since the  $\{1, x, y, xy\}$  form an  $R_i$ -module basis, this is an  $R_i$ -algebra isomorphism. Since  $x$  module  $J_i$  generates  $L_i$ , it's clear that  $x$  is a unit and  $x$  is not a square modulo  $t$ . Moreover,  $b$  nonzero since  $\Delta$  has no nilpotent elements. If  $b$  is not a unit, then we can write  $b = b_0 t^m$  for some  $m$ , in which case  $y \mapsto ty$  includes  $A_i$  into the larger  $R$ -order  $R\langle x, y \rangle / (x^2 - a, y^2 - b_0 t^{m-2}, xy + yx)$ , contradicting the maximality of  $A_i$ . This proves that  $\mathcal{A}_s$  has the required presentation in the case that  $s$  is a generic point.

Next suppose that  $s$  is a closed point of  $C_i$ , and let  $A = \mathcal{A}_s$ . Again by smoothness,  $A$  is a free  $\mathcal{O}_{X,s}$ -module of rank 4. Let  $J$  be the kernel of the natural  $\mathcal{O}_{X,s}$ -module map

$$A \rightarrow \mathcal{A}_i \rightarrow L_i.$$

This identifies the quotient  $A/J$  with a subring of  $L_i$ . Hence  $A/J$  is a finitely generated  $\mathcal{O}_{C,s}$ -submodule of  $L$  (hence torsion free), and since  $\mathcal{O}_{C,s}$  is a PID  $A/J$  is free. Moreover, since  $(A/J) \otimes_{\mathcal{O}_{C,s}} K(C_i) = L_i$ ,  $A/J$  has rank 2. Again as above, we may choose  $x \in A$  such that  $x^2 = a \in \mathcal{O}_{S,s}$  (ie. so that  $x$  has reduced trace 0) and such that  $x$  reduces to a generator for  $A/J$ , and  $u, v \in J$  such that  $\{1, x, u, v\}$  is a  $\mathcal{O}_{S,s}$ -module basis for  $A$ . Using the same strategy as before, we may choose  $u$  and  $v$  to have reduced trace 0.

Since  $A$  is a maximal  $\mathcal{O}_{X,x}$ -order in  $\Delta$ , it should be Azumaya everywhere except on  $C$ . Therefore the bilinear form defined by the reduced trace on  $A$  degenerates exactly on  $C$ , meaning that the discriminant of  $A$  should be of the form  $\varepsilon t^r$  for some integer  $r > 0$  and unit  $\varepsilon \in A^*$ . Moreover, locally at the generic point of  $C_i$  the discriminant should agree with the discriminant of  $A_i$ , so  $r = 2$ . However, calculating the discriminant with respect to the basis  $\{1, x, u, v\}$ , the fact that  $\text{Tr}_{\text{red}}(x^2) = 2a$  and  $\text{Tr}_{\text{red}}(ux), \text{Tr}_{\text{red}}(vx), \text{Tr}_{\text{red}}(u^2), \text{Tr}_{\text{red}}(v^2), \text{Tr}_{\text{red}}(uv)$  are all in  $J \cap \mathcal{A}_s = t\mathcal{A}_s$ , we see that the discriminant is of the form  $4a\xi t^2 + \eta t^3$  for some  $\xi, \eta \in \mathcal{O}_X$ , from which it follows that  $4a\xi = \varepsilon$  and therefore  $a$  is a unit.

Now since  $a$  is a unit  $A/J = \mathcal{O}_{C,s}[x]$  is a semilocal Dedekind domain, in particular a PID. Thus  $J/At$ , being a  $\mathcal{O}_{C,s}[x]$  submodule of the free module  $J_i/tA_i$ , is also free. Since  $(J_i/tA_i) \otimes L_i \cong J_i/J_i^2$  is a free rank 1 module over  $L_i[x]$ , we see that  $J_i/tA_i$  has rank 1 over  $\mathcal{O}_{C,s}[x]$ . Let  $y$  be a generator; as before we may choose  $y$  such that  $\text{Tr}_{\text{red}}(y) = \text{Tr}_{\text{red}}(xy) = 0$ . Then again  $y^2 = bt$  for some  $b \in \mathcal{O}_{X,s}$ ,  $xy = -yx$  and computing the discriminant with respect to this basis we find  $-16a^2b^2t^2 = \varepsilon t^2$ , and therefore  $b$  is a unit. This verifies the desired presentation in the case that  $s$  is a closed point.  $\square$

**Proposition 7.2.** *Let  $[\Delta]$  be the class of a quaternion algebra  $\Delta$  over  $K(S)$  with nonsingular ramification curve  $C$ , and let  $\mathcal{A}$  be a maximal  $\mathcal{O}_S$ -order in  $\Delta$ . Then  $\pi : \text{BS}(\mathcal{A}) \rightarrow S$  satisfies the following properties*

- (a)  $\mathrm{BS}(\mathcal{A})$  is nonsingular (over  $k$ ?) and  $\pi$  is proper and flat  
 (b) if  $s \in S$  is a geometric point then

$$\mathrm{BS}(\mathcal{A})_s = \begin{cases} \mathbb{P}_k^1, & s \notin C \\ \mathbb{P}_k^1 \vee \mathbb{P}_k^1, & s \in C \end{cases}$$

- (c) if  $c_i$  is a generic point of an irreducible component  $C_i$  of  $C$ , then the fraction fields each of the irreducible components of  $\mathrm{BS}(\mathcal{A})_c$  define quadratic extensions of  $K(C_i)$

*Proof.* Let  $i : \mathrm{spec}(\kappa(s)) \rightarrow S$ . By Lemma FIXME,  $\mathrm{BS}(\mathcal{A})_s = \mathrm{BS}(i^*\mathcal{A})$ .

- (a)  $\pi : \mathrm{BS}(\mathcal{A}) \rightarrow S$  is projective, and  $S$  is nonsingular over  $k$   
 (b) Since  $\mathcal{A}$  is Azumaya at  $s$ ,  $i^*\mathcal{A} \cong M_2(k)$ . Therefore by Example FIXME, we have that

$$\mathrm{BS}(\mathcal{A})_s = \mathrm{BS}(M_2(k)) = \mathbb{P}_k^1.$$

- (c) From the local presentation at  $s \in C_i$ , we have that

$$\mathcal{A}_s \otimes \kappa(s) \cong \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$

which has corresponding Brauer-Severi variety  $\mathbb{P}_k^1 \vee \mathbb{P}_k^1$  by Example FIXME.

- (d) Let  $R_i = \mathcal{O}_{S, c_i}$ , and let  $j : \mathrm{spec}(R_i) \rightarrow S$ . Then  $i$  factors through  $j$  and therefore

$$\mathrm{BS}(\mathcal{A})_{c_i} = \mathrm{BS}(i^*\mathcal{A}) = \mathrm{BS}(j^*\mathcal{A})_{c_i}.$$

Moreover the local presentation of  $\mathcal{A}_s$  and Example FIXME

$$\mathrm{BS}(j^*\mathcal{A}) = V(X^2 - aY^2 - bZ^2) \subseteq \mathbb{P}_{R_i}^2,$$

where  $a, b \in R_i^*$  with  $a$  not a square and  $t$  generates the the unique maximal ideal of  $R_i$ . It follows that  $\mathrm{BS}(\mathcal{A})_{c_i} = V(X^2 - aY^2) \subseteq \mathbb{P}_{K(C_i)}^2$ , which has two irreducible components whose residues define the quadratic extension  $K(C_i)[\sqrt{a}]$  of  $K(C_i)$ .

□

**Proposition 7.3.** *Let  $\Delta$  be a quaternion algebra over  $K(S)$  with nonsingular ramification curve  $C$ , and let  $\mathcal{A}$  be a maximal  $\mathcal{O}_S$ -order in  $\Delta$ . If  $C$  is disconnected, then  $\mathrm{BS}(\mathcal{A})$  has 2-torsion in  $H^4(\mathrm{BS}(\mathcal{A}), \mathbb{Z}_2)$ .*



## 8 A Nonrational Unirational Variety

In this section, we construct a surface  $S$  with quotient field  $K = k(x, y)$  and quaternion algebra  $\Delta$  over  $K$  whose ramification locus is a union of two non-singular, disjoint curves  $C_1, C_2$  on  $S$  such that the Brauer-Severi scheme of a maximal order  $\mathcal{A}$  of  $\Delta$  over  $S$  is unirational. By Proposition FIXME, this tells us that  $\text{BS}(\mathcal{A})$  is a nonrational unirational variety.

Fix a nonsingular conic in  $Z = V(\alpha) \subseteq \mathbb{P}_k^2$ , and choose six distinct points  $P_i^{(j)} \in Z$ ,  $i = 1, 2$ ,  $j = 1, 2, 3$ .

**Lemma 8.1.** *There exist nonsingular cubic curves  $E_i = V(\beta_i) \subseteq \mathbb{P}_k^2$  such that*

(a)  *$E_i$  has double intersection with  $Z$  at  $P_i^{(j)}$  for  $j = 1, 2, 3$*

(b)  *$E_i$  and  $E_j$  transversally at nine distinct points  $O_1, \dots, O_9$*

*Proof.* Fix an isomorphism  $Z \cong \mathbb{P}_k^1$ . Then the corresponding closed immersion  $\mathbb{P}_1 \cong Z \rightarrow \mathbb{P}_2$  corresponds to a degree 2 homogeneous polynomial  $f \in \Gamma(\mathbb{P}_2, \mathcal{O}(2)) = S(2)$  for  $S = k[X, Y, Z]$ . Then the short exact sequence of graded  $S$ -modules

$$0 \rightarrow S(-2) \xrightarrow{f} S \rightarrow S/(f) \rightarrow 0$$

induces a short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}(-2) \rightarrow \mathcal{O} \rightarrow f_*\mathcal{O} \rightarrow 0.$$

Tensoring with  $\mathcal{O}(3)$ , we obtain a short exact sequence

$$0 \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}(3) \rightarrow f_*\mathcal{O}(6) \rightarrow 0.$$

Since  $H^1(\mathbb{P}^2, \mathcal{O}(1)) = 0$ , this gives us a short exact sequence on global sections

$$0 \rightarrow \Gamma(\mathbb{P}^2, \mathcal{O}(1)) \rightarrow \Gamma(\mathbb{P}^2, \mathcal{O}(3)) \rightarrow \Gamma(\mathbb{P}^1, \mathcal{O}(6)) \rightarrow 0.$$

Let  $g_i \in \Gamma(\mathbb{P}^1, \mathcal{O}(6))$  with  $(g_i) = 2 \sum_j P_i^{(j)}$ . Then the above sequence shows that the subspace  $L_i$  of  $\Gamma(\mathbb{P}^2, \mathcal{O}(3))$  of functions whose restriction to  $Z$  is  $g_i$  is a three-dimension  $k$ -vector space. Sweet! Fix a member  $\ell_i \in L_i$  with  $\ell_i \mapsto g_i$  for  $i = 1, 2$ .

Let  $E_i = \{(\ell, x) \in L_i \times \mathbb{P}^2 : \ell(x) = 0\}$  and  $B_i = \{(\ell, x) \in E_i : \nabla \ell(x) = 0\}$ , and consider the map

$$B_i \rightarrow E_i \rightarrow L_i \times \mathbb{P}^2 \rightarrow L_i.$$

We claim that  $B_i$  does not map onto  $L_i$ . To see this, we can consider the other map

$$B_i \rightarrow E_i \rightarrow L_i \times \mathbb{P}^2 \rightarrow \mathbb{P}^2.$$

Let  $y \in \mathbb{P}^2 \setminus Z$ , and suppose  $(\ell, y) \in B_i$ . Then  $\ell = \ell_i + \alpha p$  for some degree 1 homogeneous polynomial  $p$ , and we have the two equations

$$\begin{aligned} 0 &= \ell(y) = \ell_i(y) + \alpha(y)p(y) \\ 0 &= \nabla \ell(y) = \nabla \ell_i(y) + p(y)\nabla \alpha(y) + \alpha(y)\nabla p(y). \end{aligned}$$

In particular, these together along with the fact that  $y \cdot \nabla p(y) = p(y)$  tell us

$$0 = \alpha(y)y \cdot \nabla \ell(y) = \alpha(y)y \cdot \nabla \ell_i(y) - \ell_i(y)y \cdot \nabla \alpha(y) + \alpha(y)\ell_i(y).$$

Thus  $y$  sits inside a codimension 1 subspace of  $\mathbb{P}^2$ . Moreover, the value of  $p(y)$  is uniquely determined by the requirement that

$$0 = \alpha(y)\nabla \ell_i(y) - \ell_i(y)\nabla(y) + \alpha(y)^2\nabla p(y).$$

Therefore the projection  $(B_i \setminus \{L_i \times \{P_i^{(j)}\}\}) \rightarrow \mathbb{P}^2 \setminus Z$  maps injectively onto a codimension 1 subspace of  $\mathbb{P}^2$ . Thus it does not map surjectively onto  $L_i$ . Hence we may take the  $\ell_i$  to be nonsingular. By wiggling this argument, we can take  $\ell_1$  and  $\ell_2$  such that both intersect transversally.  $\square$

Let  $q$  be the rational function on  $\mathbb{P}_k^2$  whose divisor is  $Z - 2L$ , with  $L$  the line at infinity.

**Lemma 8.2.** *The restriction  $\bar{q}$  of  $q$  to  $E_i$  is not a square in  $K(E_i)$  for  $i = 1, 2$ .*

*Proof.* Let  $\mathcal{L}$  be the line bundle on  $\mathbb{P}_k^2$  corresponding to the divisor  $L$ . We claim that if  $f \in \Gamma(E_i, \mathcal{L}|_{E_i})$ , then  $f$  has at most two zeros on  $Z \cap E_i$ . To see this, let  $f \in \Gamma(E_i, \mathcal{L}|_{E_i})$ . The natural map

$$\Gamma(Z, \mathcal{L}) \rightarrow \Gamma(E_i, \mathcal{L}|_{E_i})$$

is surjective, and therefore  $f$  may be extended to a global section  $\tilde{f}$  of the sheaf  $\mathcal{L}$ . Therefore  $(\tilde{f}) + L$  is a degree 1 effective divisor, so we may choose a line  $L'$  such that  $L' = (\tilde{f}) + L$ . It follows that the zero set of  $f = \tilde{f}|_{E_i}$  is exactly  $L' \cap E_i$ , and therefore the zeros of  $f$  on  $Z \cap E_i$  are contained in  $Z \cap L'$ , this latter set having only two elements. This proves our claim.

Now suppose that  $\bar{q} = f^2$  for some  $f \in K(E_i)$ . Then  $f$  has the same number of zeros on  $Z \cap E_i$  as  $\bar{q}$ , namely 3. Furthermore, since  $(q) = Z - 2L$  we know that  $(\bar{q}) + 2E_i \cdot L$  is effective, and therefore  $(f) + E_i \cdot L$  is effective. Hence  $f \in \Gamma(E_i, \mathcal{L}|_{E_i})$ . By the argument of the previous paragraph, this is a contradiction.  $\square$

**Lemma 8.3.** *Let  $L_i := K(E_i)[\bar{q}^{1/2}]$  and let  $E'_i$  be the normalization of  $E_i$  in  $L_i$ . Then  $E'_i \rightarrow E_i$  is unramified.*

*Proof.* Since  $E'_i \rightarrow E_i$  is locally of finite type, it suffices to show that for any point  $P' \in E'_i$  with image  $P \in E_i$ , the induced map of residue fields  $\kappa(P) \rightarrow \kappa(P')$  is a finite, separable field extension. Then the local ring  $R = \mathcal{O}_{E_i, P}$  is a DVR  $(R, \nu)$  with residue field  $k$  and fraction field  $K(L_i)$ . Let  $t$  be a uniformizer for  $R$ . Depending on whether or not  $P$  is one of the intersection points  $P_i^{(j)}$  of  $E_i$  and  $Z$ , the value of  $\nu(\bar{q})$  is either 0 or 2, meaning we may write  $\bar{q} = ut^{2j}$  for some  $u \in R^*$  and integer  $j$ . Note that  $R[u^{1/2}]$  is a semilocal Dedekind domain, hence a PID, and therefore integrally closed in its fraction field  $L_i$ . Therefore since  $u^{1/2} \in L_i$  is integral over  $R$ , we have  $\mathcal{O}_{E'_i, P'} = R[u^{1/2}]$ . Furthermore  $R[u^{1/2}] = R[x]/(x^2 - u)$  is unramified since  $u$  is a unit.  $\square$

Define  $S$  to be the blow up of  $\mathbb{P}_k^2$  at the intersection points of  $O_1, \dots, O_9$  of  $E_1$  and  $E_2$ , and let  $C_i$  be the proper transform of  $E_i$  for  $i = 1, 2$ . Then  $S$  is a simply connected rational surface, on which the nonsingular curves  $C_1$  and  $C_2$  are disjoint. Note that since  $S$  is rational,  $\text{Br}(S) = 0$  by Proposition FIXME. Therefore the Artin-Mumford sequence tells us there exists a unique  $\Delta \in \text{Br}_2(K)$  whose ramification locus is  $C = C_1 \cup C_2$  and whose ramification data on  $C_i$  is the nowhere-ramified extension  $L_i$  for  $i = 1, 2$ .

Choose a cubic curve  $E_0 = V(\beta) \subseteq \mathbb{P}_k^2$  such that the divisor  $E_0$  pulls back on  $Z$  to the divisor  $\sum_{ij} P_i^{(j)}$ . In particular, this means that  $\beta_1\beta_2 - \beta$  is identically zero on  $\alpha$ , so that there exists a homogeneous polynomial  $\gamma$  of degree 4 satisfying  $4\alpha\gamma = (\beta_1\beta_2 - \beta)$ . Consider the field  $L := K[q^{1/2}]$ , and let  $Y_0$  be the quartic in  $\mathbb{P}_k^3$  defined by

$$Y_0 = V(\alpha(X_0, X_1, X_2)X_3^2 + \beta(X_0, X_1, X_2)X_3 + \gamma(X_0, X_1, X_2)) \subseteq \mathbb{P}_k^3.$$

Note that  $Q_0 = [0 : 0 : 0 : 1]$  is a singular point of  $Y_0$ . Moreover, The projection  $Y_0 \setminus \{Q_0\} \rightarrow \mathbb{P}_k^2$  is a double cover of  $\mathbb{P}_k^2$  ramified at  $V(\beta^2 - 4\alpha\gamma) = V(\beta_1\beta_2) = E_1 \cup E_2$ . The remaining singular points of  $Y_0$  must occur at ramification points, and therefore correspond to the singular points of  $E = E_1 \cup E_2$ , ie. the points  $Q_i$  mapping to the intersection points  $O_i$  of the curves  $E_1$  and  $E_2$ .

Let  $Y$  be the blow up of  $Y_0$  at  $Q_0, \dots, Q_9$ . Then  $Y_0$  is the normalization of  $\mathbb{P}_k^2$  in  $L$ , so  $Y$  is the desingularization of the normalization of  $S$  in  $L$ . Furthermore,  $Y$  is rational since it is birational to  $Y_0$ , which itself is rational because it is a quadric in  $\mathbb{P}_k^3$ . Thus if we consider the commutative diagram induced by the morphism  $f : Y \rightarrow S$  and the Artin-Mumford exact sequences for  $S$  and  $Y$

$$\begin{array}{ccccc} \text{Br}(S) & \longrightarrow & \text{Br}(K) & \longrightarrow & \bigoplus_{C \subseteq S} K(C)^*/K(C)^{*2} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Br}(Y) & \longrightarrow & \text{Br}(L) & \longrightarrow & \bigoplus_{C \subseteq Y} K(C)^*/K(C)^{*2} \end{array}$$

The image of  $[\Delta]$  in  $\text{Br}(L)$  is  $[\Delta_L]$ , and the ramification data becomes trivial. Furthermore, the fact that  $Y$  is rational implies that  $\text{Br}(Y) = 0$ . Thus the class  $[\Delta_L]$  is trivial. Since the Brauer class of  $\Delta$  splits over a quadratic extension, we conclude that  $\Delta$  may be taken to be a quaternion algebra.

**Lemma 8.4.** *Let  $\Delta$  be the quaternion algebra defined above, and let  $\mathcal{A}$  be a maximal  $\mathcal{O}_S$ -order in  $\Delta$ . Then the Brauer-Severi variety  $\text{BS}(\mathcal{A})$  of  $\mathcal{A}$  is unirational.*

*Proof.* Using the same  $f : Y \rightarrow S$  as above, we have that

$$\text{BS}(\mathcal{A}) \times_S Y \cong \text{BS}(f^*\mathcal{A}) \cong \mathbb{P}_Y^1.$$

Projection onto the first factor then gives us a surjection  $\mathbb{P}_Y^1 \rightarrow \text{BS}(\mathcal{A})$ , and since  $Y$  is rational this proves that  $\text{BS}(\mathcal{A})$  is unirational.  $\square$