Algebraic Generalizations of the Fourier Transform

William Riley Casper

1. Project Description (971 words)

Purpose and Merit (430 words) The Fourier transform is one of the most useful mathematical tools known today. It is central to our understanding of a huge spectrum of topics, from Heisenberg's uncertainty principal in quantum mechanics to the relationship between audio quality and bandwidth on your mobile phone [6]. This project will explore new generalizations of the Fourier transform which have recently played a role in solving multiple long-standing open problems in pure mathematics.

In Heisenberg's description of the quantum world, quantities that we are able to see and measure are called *observables*. Mathematically, each observable is represented by a unique, special symbol called an *operator* [5]. For example, imagine a very tiny particle wandering around a room. Our observables are things like the particle's location and momentum, represented by the special operator symbols x and ∂. Since we can add and multiply measurements, we can likewise add and multiply observables, giving new operators like $x+\partial$ and $x\partial$. In this way, the collection of all observables is a very special object, called an *operator algebra*. In this quantum point of view, the Fourier transform becomes a mathematical machine which naturally rearranges measurements, interchanging x and ∂.

In recent work, the PI and his collaborators have considered abstractions of the Fourier transform coming from other operator algebras, called *Fourier algebras*, which mimic operator algebras in quantum mechanics. Each Fourier algebra has its own generalization of a Fourier transform. These novel generalizations have proven to be extremely significant via their pivotal role in the PI's solutions of two long-standing open problems in pure mathematics, leading to publications in extremely prestigious journals [1, 2, 3, 4]. Consequently, there is exceptional merit in the continued exploration of Fourier algebras and their applications. Furthermore, while the theory motivating these solutions is deep, the Fourier algebras themselves are charmingly simple to define, requiring only some knowledge of basic calculus. Thus a hands-on exploration of Fourier algebras lends itself readily to undergraduate research.

This project proposes to expand the PI's study and application of Fourier algebras to pure mathematics, in particular to integrable systems, with an emphasis on new collaborative work with undergraduate students. The primary purpose is to explore specific instances of Fourier algebras relelvant to other areas of mathematics, particularly in representation theory and spectral theory, with the goal of generating further publications in high-level journals. The secondary purpose is to create research opportunities and coauthor publications with the joint goals of (1) increasing the advantage of CSUF students applying to graduate school, and (2) building a undergraduate mentorship track record which can be leveraged to secure external funding for further undergraduate research support.

Procedures and Methodology (292 words) The PI has developed software which can be used to explore operator algebras symbolically on the computer using Python and Sympy. Using this software, the PI has been able to identify a universal description of the algebraic structure that these algebras contain, connecting them with diverse other branches of mathematics, including algebraic geometry, representation theory, and integrable systems. Leveraging a combination of software and theory, the PI proposes to discover new, explicit expression of generalized Fourier transforms.

March-May: The PI will focus on preliminary numerical calculations of specific examples of Fourier algebras, focusing on deriving explicit formulas for generalized Fourier maps.

Simultaneously, the PI will develop instructional materials intended to jump start a student researcher with the background material necessary to explore Fourier algebras in detail, beginning with a review of Fourier transforms.

June: The student researcher and the PI will meet daily to discuss background material and begin to engage with research related software, studying examples of Fourier algebras that the PI has already explored in detail. Meanwhile the PI will work toward connecting specific results to general theorems fitting into his broader research agenda.

July: At this point the student should be competent in the software and familiar enough with the mathematical techniques to explore new examples of Fourier algebras independently. For each example, they will try to describe the associated generalized Fourier transform.

August: Students will summarize their results and generate a poster presentation in anticipation of presenting at undergraduate research conferences, including the MAA Undergraduate Research Poster Session. The PI will simultaneously prepare results to be submitted to International Math Research Notices or J. Approximation Theory.

At the start of the new term, the PI and the student researcher will meet to summarize progress and discuss future efforts.

Applicant Qualifications (249)

The PI's publication record is extremely strong, featuring publications in mathematics journals widely considered to be among the best in the world, including Crelle's Journal, the American Journal of Mathematics, and Proceedings of the National Academy of Sciences. Additionally, the PI has a strong history of research dissemination through invitations to speak at conferences and workshops, including national meetings sponsored by the AMS and AIMS, and international meetings such as ICM and CMS. The PI's research has also resulted in invitations to visit and give talks at several universities, including UC Berkeley, U Oregon, Radboud University in the Netherlands, Stockholm University in Sweden, UNAM in Mexico, and the University of Cordoba in Argentina.

The PI is actively engaged in the broader mathematical community as a referee for Communications in Mathematical Physics, the Journal of Approximation Theory, the SIAM Journal on Mathematical Analysis, and Studies in Applied Mathematics.

The PI's mentorship record includes mentoring students in both the Washington Experimental Mathematics Laboratory, and a Computational Physics Summer School at Los Alamos National Laboratory. Currently, the PI is working with four undergraduate students at CSUF on various research projects related to the proposed topic.

The undergraduate research experience and publications advance the PI's scholarly agenda by strengthening the PI's profile for applying for a research grant in Algebra and Number Theory through the NSF in the future. Simultaneously, this project bolsters PI's track record of undergraduate research mentorship in anticipation of applying for funding for further undergraduate research opportunities at CSUF.

Brief CV

William Riley Casper

1 Professional Preparation

University of Washington	Seattle, WA	Mathematics	Ph.D 2017
North Dakota State Unversity	Fargo, ND	Mathematics	M.S. 2010[1]
North Dakota State Unversity	Fargo, ND	Mathematics	B.S. 2010
North Dakota State Unversity	Fargo, ND	Physics	B.S. 2010

[1] all three degrees were completed simultaneously

2 Appointments

2020-Present	Assistant Professor of Mathematics, California State University, Fullerton, CA
2017-2020	Postdoctoral Researcher, Louisiana State University, Baton Rouge, LA
2011-2017	Graduate Student Instructor, UW Seattle (fall, winter, and spring)
2011-2017	Graduate Research Assistant, Los Alamos National Lab (summer)
2010-2011	Post-Bac Research Assistant, Los Alamos National Lab (year-long)
2007-2010	Teaching Assistant, North Dakota State University (year-long)

3 Publications

Publications Related to the Proposed Project

1. Casper, W. Riley and Milen Yakimov "The Matrix Bochner Problem," American Journal of Mathematics 2020 (to appear). arXiv:1803.04405

2. Casper, W. Riley, Stefan Kolb and Milen Yakimov "Bivariate Continuous q-Hermite Polynomials and Deformed Quantum Serre Relations," Journal of Algebra and Its Applications 2021, Vol. 20, No. 01, 2140016. arXiv:2002.07895

3. Casper, W. Riley, F. Alberto Grünbaum, Milen Yakimov, and Ignacio Zurrián, "Reflective prolate-spheroidal operators and the KP/KdV equations," Proc. Natl. Acad. of Sci. USA 2019, 116(37) 18310-18315. arXiv:1909.01448

4. Casper, W. Riley and Milen Yakimov, "Integral operators, bispectrality and growth of Fourier algebras," J. Reine Angew. Math (Crelle's Journal) 2019 doi:10.1515/crelle-2019-0031.
arxiv:1807.09314

5. Casper, W. Riley "Elementary Examples of Solutions to Bochner's Problem for Matrix Differential Operators." Journal of Approximation Theory, 2018, 229:36-71. arxiv:1509.03674

6. Casper, W. Riley "The symmetric 2×2 hypergeometric matrix differential operators." preprint, 2019, submitted for publication. arXiv:1907.12703

Other Publications

7. Casper, W. Riley "A Connection Between Orthogonal Polynomials and Shear Instabilities in the QG Shallow Water Equations." arXiv preprint 1710.02756, 2017.

8. Casper, W. Riley and Balasubramanya Nadiga "A new spectral clustering algorithm." (submitted) arXiv preprint 1710.02756, 2017.

9. Coles, Patrick J. et al "Quantum Algorithm Implementation for Beginners." arXiv preprint 1804.03719, 2018.

10. Nadiga, Balasubramanya T., W. Riley Casper, and Philip W. Jones. "Ensemble-based global ocean data assimilation." Ocean Modelling 72 (2013): 210-230.

4 Synergistic Activities

Referee/Review Work

Ongoing	Reviewer for Zentralblatt MATH
Ongoing	Referee for Communications in Mathematical Physics, Journal of Approximation Theory, SIAM Journal on Mathematical Analysis, Studies in Applied Mathematics, and the International Electronic Journal of Geometry

Mentoring/Training

2016	Graduate student mentor for the Washington Experimental Mathematics Laboratory
2016	Graduate student mentor for the Los Alamos Summer School in Computational Physics
2015-16	Volunteer college math instructor at the Washington Corrections Center for Women (WCCW) as part of the Freedom Education Project of Puget Sound (FEPPS)
2015	Organizer for a weekly algebraic geometry seminar for students at Los Alamos National Lab, a student-run seminar discussing algebraic geometry and its scientific applications
2007-19	Tutor, teacher, grader, and mentor for diverse mathematics classes
Ongoing	Speaker multiple times for Problem Solving Seminar at CSUF
Ongoing	Research mentor for multiple undergraduate research teams at CSUF

Invited Conference Talks:

2018	ICM Satellite Conference in Cusco, Peru
2018	AMS-CMS Joint Math Conference in Shanghai, China
2018	Geometry and Physics XVI in Timisoara, Romania
2019	Orthogonal Polynomials, Special Functions and Appications (OPSFA) in Hagenberg, Austria
2019	Matrix-valued Special Functions and Integrability in Nijmegen, Netherlands
2020	13th AIMS Conference on Dynamical Systems, Differential Equations and Applications: *Algebraic and Geometric Methods in Nonlinear Differential Equations* in Atlanta, GA
2020	AMS Sectional Meeting at Tufts University, Medford, MA
2020	Orthogonal Polynomials, Special Functions, Operator Theory and Applications in Kent, UK

2. Expected Outcomes and Methods of Dissemination (322 words)

The PI expects the proposed research project to result in

(a) a paper submitted to a Q1 research journal in pure mathematics, coauthored with an undergraduate research student

(b) presentations at research conferences, including the MAA Undergraduate Research Poster Session in January 2022

(c) research talks in one or more departmental seminars at CSUF, such as the problem solving seminar

Additionally, the strong results obtained from this research will bolster the PI's application for external funding via a grant in algebra and number theory from the NSF (PD 20-1264), due October 2021. In particular, the close collaboration with undergraduates will simultaneously enhance the PI's NSF grant application in the area of Broader Impacts. At the same time, this work is consistent with the University's Strategic Plan 2018-2023 to "provide a transformative educational experience and environment for all students" by prividing high-impact co-curricular experiences in the form of real research experience with publishable results. In the future, the PI will leverage this collaborative opportunity to establish a solid track record of undergraduate research essential for applying for external funding for additional undergraduate research in the future. In this way, the current project is also consistent with the university's goal to increase on-campus student employment, internships, and professional development opportunities.

Starting in Fall 2021, the PI in collaboration with other faculty from the math department will design and implement a series of research and professional development seminars aimed at junior and senior math students planning to apply for graduate school, summer research internships (eg. REUs), or positions in industry. Specifically, the seminars will include both a mix of talks by students and faculty on recent or ongoing research projects, as well as practical workshops on creating professional resumes and writing strong application essays. In the future, the PI intends to apply for external funding to establish a formal summer mathematics research program combining both student/faculty research collaboration and extended professional development opportunities.

Timeline Template

Activity	When	Responsible Party
Example:		
Creation of mentoring materials	***March-May 2021***	***PI- Casper***
Initial exploration / example calculations	***March-May 2021***	***PI-Casper***
Daily mentorship meetings covering background / software	***June 2021***	***PI – Casper and Student***
Independent research / exploration of new examples	***July 2021***	***Student***
Analysis of acquired examples / development of general theory	***July 2021***	***PI – Casper and Student***
Drafting of paper	***August 2021***	***PI – Casper***
Create poster presentation/talk	***August 2021***	***PI – Casper and Student***
Department seminar talk	***Fall 2021***	***PI – Casper and Student***
Poster presentation at AMS JMM	***January 2022***	***PI – Casper and Student***
Submit Final Report	***May 2022***	***PI – Casper***

3. Proposed Budget (299)

Budget Justification (299)

The PI's primary aim in applying for this grant is to secure funding for additional faculty summer salary (in lieu of teaching summer courses) and student summer salary to develop mentoring materials and conduct focused research during Summer 2021, in collaboration with an undergraduate research student working 10 hours a week from the beginning of June to mid-August. Presently, the PI is in his first year as a junior faculty member at CSUF and has access to some start-up funds for conference travel, but these funds are not available for additional faculty summer support. Furthermore, the PI does not expect any additional sources of funding during the upcoming summer and academic year.

The extremely fast-paced time frame outlined in the project necessitates extensive preparation during Spring 2021, in particular in the creation of background lecture material, preliminary investigation/calculation, and code development and documentation. During Summer 2021, the accelerated pace of research will require careful mentoring of the undergraduate, both mathematically and in relevant programming skill, requiring daily availability in addition to the PI's research efforts.

External sources of funding, particulary research grants, are extremely competitive in pure mathematics. Researchers who have been awarded previous research grants tend to present a much stronger case for being awarded funding in the future. If awarded, the PI intends to leverage the current grant as part of a strong grant application for an NSF research grant in the near future. Additionally, numerous faculty in the Department of Mathematics at CSUF have expressed strong interest in establishing summer research opportunites for undergraduate students such as a NSF Research Experience for Undergraduates (REU) program or a similar research opportunity. By bolstering his undergraduate mentorship track record, the PI hopes that he and other faculty can submit successful grants for undergraduate research funding in the future.

2021- 2022 Junior/Senior Budget Template

Directions:
Enter information only into green and red cells.
Course Release: Enter name, select $4747 and semester.
Faculty Salary: Enter name and amount up to $5000. Additional Rows can be added to the spreadsheet as needed. Please ensure that page settings are set for "Fit Sheet on One Page".

Check the calculations and the totals to ensure accuracy.

Course Release OR Faculty Salary	Faculty Name	Use Dropdown selection for Course Release Time (row 7 and 8) OR Enter Faculty Salary Amount (row 9)	SubTotal	Fringe Benefits/ Payroll Taxes 43%	Total	
Course Release (3WTUs) Request Use Dropdown selection		$ -	$ -	$ -	$ -	
Course Release Semester Use Dropdown selection		Semester				
Faculty Salary	W. Riley Casper	$ 2,950.00	$ 2,950	$ 324.50	$ 3,275	Faculty additional pay is charged the 11% which will be covered by the funds awarded.
Additional Faculty Salary		$ -	$ -	$ -	$ -	Faculty additional pay is charged the 11% which will be covered by the funds awarded.
Personnel SubTotal					**$ 3,275**	(WTUs ARE Included in totals)

Consultant Services (Non-CSU Employee)	Estimated Hours	Hourly Rate			Total	Budget Narrative: Relevance to Project
		$ -			$ -	

Student/ Graduate Assistants (Use Dropdown selection)	Hourly Wage ($14.00 min. wage)	# of Hours/Wk	# of Weeks	SubTotal	Payroll Taxes (7.65%)	Total	Student Assistant Pay Scale: 1. Must be hired through Auxiliary Service Corporation 2. Can work a maximum of 20 hours per week during the AY semesters. During the summer, students can work a maximum of 40 hours per week. Undergrad | Graduate
Student Assistant	$ 16.00	10	10	$ 1,600	$ 122	$ 1,722	Range I: Basic skills: $14.00 - $15.00 | $18.00 - $19.00
Graduate Assistant	$ -			$ -	$ -	$ -	Range II: Some Exp: $15.00 - $16.00 | $19.00 - $21.00
Student Assistant	$ -			$ -	$ -	$ -	Range III: Extensive Exp: $16.00 - $17.00 | $21.00 - $23.00
Graduate Assistant	$ -			$ -	$ -	$ -	Range IV: Highly Exp: $17.00 - $20.00 | $23.00 - $25.00
Other Personnel SubTotal						**$ 1,722**	

Materials and Supplies	Cost	Total	Budget Narrative: Relevance to Project
	$ -	$ -	
	$ -	$ -	
	$ -	$ -	
	$ -	$ -	
Materials and Supplies Subtotal		**$ -**	

Incentives (Cash/Gift Cards)	Cost	Quantity	Total	Budget Narrative: Relevance to Project
	$ -		$ -	

Travel	Airfare	Mileage (miles driven x $0.575)	Expenses	Hotel	Registration Fees	Total	Budget Narrative: Relevance to Project
Conference	$ -	$ -	$ -	$ -	$ -	$ -	
Workshops	$ -	$ -	$ -	$ -		$ -	
Expenses	$ -	$ -	$ -	$ -		$ -	
Travel Subtotal						**$ -**	

Mileage	Total Miles	Mileage (miles driven x $0.56)	Total	Budget Narrative: Relevance to Project
		$ 0.560	$ -	
Mileage Subtotal			**$ -**	

Budget Total	**$ 4,997**	
Remaining from maximum of $5,000	$ 3	

4. Past Intramural Awards (18 words)

The PI has never been awarded this grand and is applying for this grant for the first time.

References Cited

[1] W. Riley Casper, F. Alberto Grünbaum, Milen T. Yakimov, and Ignacio Zurrián. Reflective prolate-spheroidal operators and the KP/KdV equations. *Proc. of the Nat. Acad. of Sci. (PNAS)*, 2019.

[2] W. Riley Casper, F. Alberto Grünbaum, Milen T. Yakimov, and Ignacio Zurrián. Reflective prolate-spheroidal operators and the adelic grassmannian. *submitted, arXiv preprint arXiv:2003.11616*, 2020.

[3] W. Riley Casper and Milen T. Yakimov. Integral operators, bispectrality and growth of Fourier algebras. *J. Reine Angew. Math. (Crelle's Journal)*, doi: 10.1515/crelle-2019-0031, 2019.

[4] W. Riley Casper and Milen T. Yakimov. The matrix Bochner problem. *American Journal of Mathematics (to appear) arXiv preprint arXiv:1803.04405*, 2020.

[5] Jun John Sakurai and Eugene D Commins. Modern quantum mechanics, revised edition, 1995.

[6] Claude Elwood Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27(3):379–423, 1948.