Determinants

2x2 determinants:

Def: The determinant of a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is $\det(A) = ad - bc$

$|A| = ad - bc$ (different notation for same thing)

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (yet another notation)

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det(A) = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2$

What is the determinant good for?

(1) $\det(A) \neq 0 \iff A^{-1}$ exists!

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

need this $\neq 0$.

(2) $\det(A)$ tells us some really cool stuff about the transformation of $\mathbb{R}^2$ given by A.

- $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotation

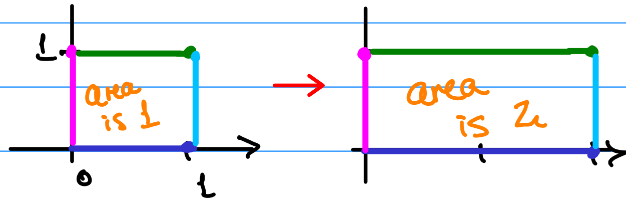
  $\det(A) = \cos^2\theta + \sin^2(\theta) = 1$!!

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection across $y=x$

$$\det(A) = 0\cdot 0 - 1\cdot 1 = -1$$

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ stretches by a factor of 2 in x-direction

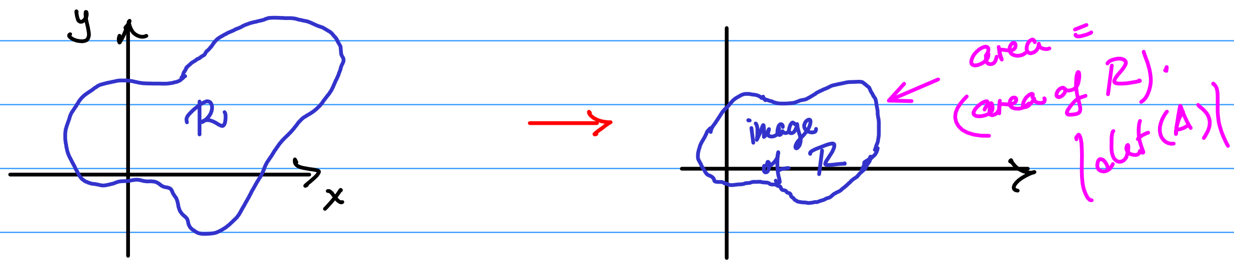
$$\det(A) = 2\cdot 1 - 0\cdot 0 = 2$$



<u>Theorem</u>: Let A be a 2×2, then

$|\det(A)|$ = area of the image of the unit square

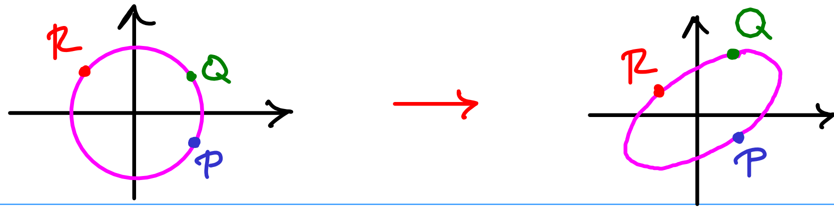
In fact, for any region $R \subseteq \mathbb{R}^2$,
area of the image of R = (area of R) $\cdot \det(A)$



<u>Note</u>: if we rotate, it doesn't change the area so it makes sense that $\det(A)=1$ for rotation matrices!

<u>Quest</u>: where does the sign of the determinant come from?

<u>Def</u>: A 2×2 matrix A is <u>orientation-preserving</u> if sends a circle with three points to a curve with those points in the same order. Otherwise it is <u>orientation-reversing</u>



orientation-preserving

orientation-reversing.

<u>Theorem</u>: Suppose A 2×2 matrix and $\det(A) \neq 0$
Then $\det(A) > 0 \iff A$ is orientation-preserving.

<u>Problem</u>: $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ $\quad \det(A) = 0\cdot 0 - (-1)(-1) = -1$

What kind of transformation is A?

$|\det(A)| = 1 \rightsquigarrow$ transformation preserves area

reflection ~~rotation~~

$\det(A) < 0 \rightsquigarrow$ transformation does not preserve orientation!

<u>How to calculate determinants</u>:

$$A = \begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det(A) = ?$$

Do a row expansion or column expansion:

$$\det(A) = a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\cdot\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\cdot\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Ex:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\det(A) = 1\cdot\det\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 1\cdot\det\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} + (-1)\det\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= 1\cdot(-2) - 1(-4) + (-1)(-2) = 4$$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \qquad \det(A) = ?$$

$$\det(A) = 1\cdot\det\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} - 2\cdot\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 3\cdot\det\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= -3 - 2 + 6 = 1 \quad ✓$$

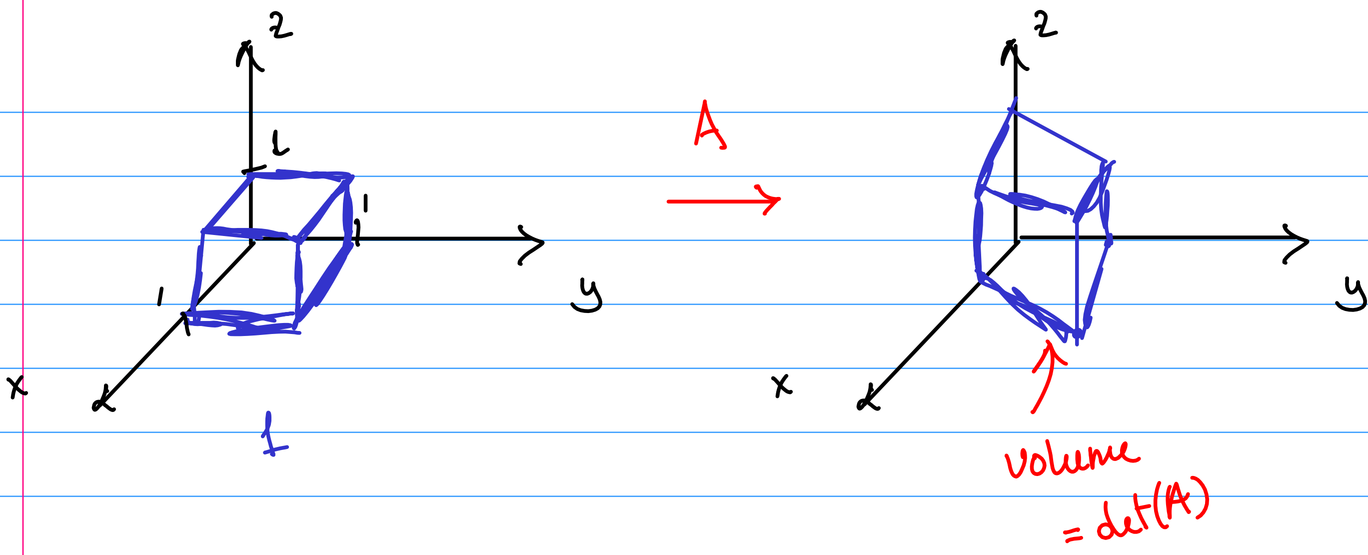
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21}\cdot\det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31}\cdot\det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 3 & 100 & 4 \\ 0 & 7 & 21 \\ 0 & 0 & 3 \end{bmatrix} \qquad \det(A) = ?$$

$$\det(A) = 3\cdot\det\begin{pmatrix} 7 & 21 \\ 0 & 3 \end{pmatrix} = 3\cdot 7\cdot 3 = 63$$



Properties of Determinants :

- A, B be $n \times n$ matrices
- $c \in \mathbb{C}$ complex number

★ (1) A is invertible if and only if $\det(A) \neq 0$

(2) $\det(AB) = \det(A)\det(B)$

(3) $\det(AB) = \det(BA)$

(4) if A^{-1} exists, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

(5) the system of linear equations $A\vec{x} = \vec{b}$ has a unique solution $\Leftrightarrow$ $\det(A) \neq 0$.

Def: An $n \times n$ matrix A is called <u>singular</u> if $\det(A) = 0$. Otherwise A is called <u>non-singular</u>.

Ex:

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \det(A) = ?$$

$$\det(A) = 1 \cdot \det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} - 0 \cdot \det\begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \det\begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= 0 - 0 + 1\cdot 1 - 2\cdot 1 = -1$

Cofactor matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

transpose of cofactor matrix

Generalize this formula for larger matrices

Def: The cofactor matrix $\mathrm{cof}(A)$ of A is the $n\times n$ matrix whose ij^{th} entry is

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} \det(A_{ij})$$

$A_{ij} = (n-1)\times(n-1)$ matrix obtained by deleting i^{th} row and j^{th} column

Ex: $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad \mathrm{cof}(A) = \begin{bmatrix} -1 & -1 & 1 \\ +1 & 3 & -3 \\ 1 & -1 & -1 \end{bmatrix}$

$$\det(A) = 3\det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} - 1 \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -3+1 = -2$$

$$\boxed{\text{Theorem: } A^{-1} = \frac{1}{\det(A)} \mathrm{cof}(A)^T}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & -3 \\ 1 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$