Eigenvalues and Eigenvectors

Recap: If A is an $n \times n$ matrix, then an eigenvector of A with eigenvalue λ is a non-zero vector $\vec{v}$ satisfying $\boxed{A\vec{v} = \lambda\vec{v}}$ (equiv. $(A-\lambda I)\vec{v} = \vec{0}$)

Theorem: The eigenvalues are all roots of the characteristic polynomial $p(x) = \det(A - xI)$

Quest: who cares?

- physics and engineering
- mathematics
- data science

What are eigenvalues?

- algebraic description ← last time
- geometric description ← goal for today

Geometric Description:

Remember that an $n \times n$ matrix A describes a linear transformation of n-dimensional space

2x2 matrix ⟷ transformation of x,y-plane

3x3 matrix ⟷ transformation of 3d-space
x,y,z-space

⋮

and so on

Transformations of the plane:

- rotation
- reflection
- stretching or shrinking
- shear

<u>Rotation</u>: $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

eigenvalues of A?

$$p(x) = \det(A - xI) = \det\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} \frac{1}{\sqrt{2}} - x & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - x \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} - x\right)^2 - \left(-\frac{1}{2}\right)$$

Eigenvalues = roots λ of $p(x)$

$$\left(\frac{1}{\sqrt{2}} - x\right)^2 + \frac{1}{2} = 0 \Rightarrow \left(\frac{1}{\sqrt{2}} - x\right)^2 = -\frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}} - x = \pm \frac{1}{\sqrt{2}} i$$

$$\Rightarrow x = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$$

Eigenvalues are $\lambda = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$\lambda = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

For rotations we will always see the presence of non-real eigenvalues.

$\underline{\text{Ex}}$: $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Try to find $\underline{\underline{\text{both}}}$ the eigenvalues $\underline{\text{and}}$ eigenvectors

Eigenvalues are roots of

$$p(x) = \det(A - xI) = \det\left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det \begin{pmatrix} -x & -1 \\ -1 & -x \end{pmatrix} = x^2 - 1$$

$$x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Eigenvectors? $(A - \lambda I)\vec{v} = 0$

$\lambda = 1$: $$\left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x + y = 0 \\ 0 = 0 \end{cases} \Rightarrow x = -y$$

$y = -x$

$$\left\{ \begin{bmatrix} -a \\ a \end{bmatrix} \;\middle|\; a \neq 0 \right\} \qquad \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} \;\middle|\; a \neq 0 \right\}$$

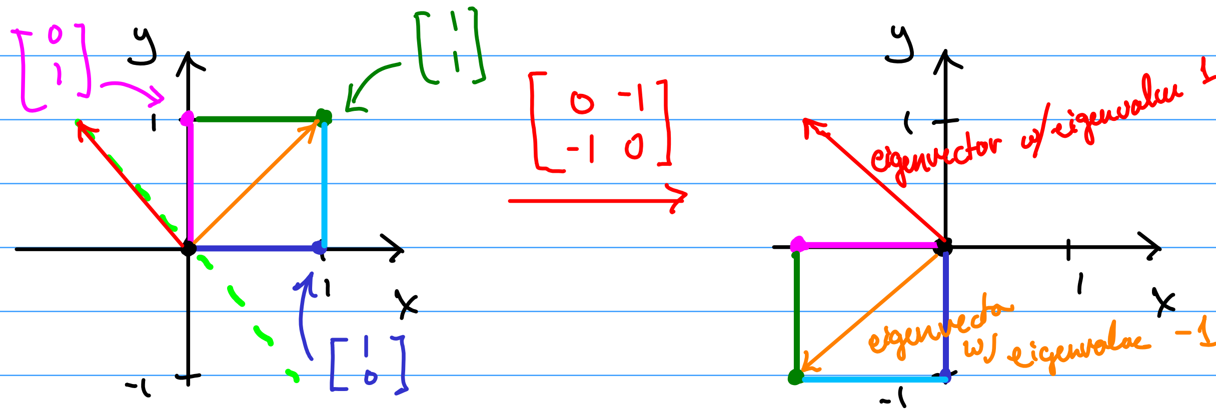
specific eigenvector: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = -1$: $$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \ a \neq 0 \right\}$$

$x - y = 0 \Rightarrow x = y$

Specific eigenvector : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Reflection across line $y = -x$
exactly the same as the direction of the eigenvector w/ eigenvalue 1
orthogonal to the direction of the eigenvector with eigenvalue -1

Ex: $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

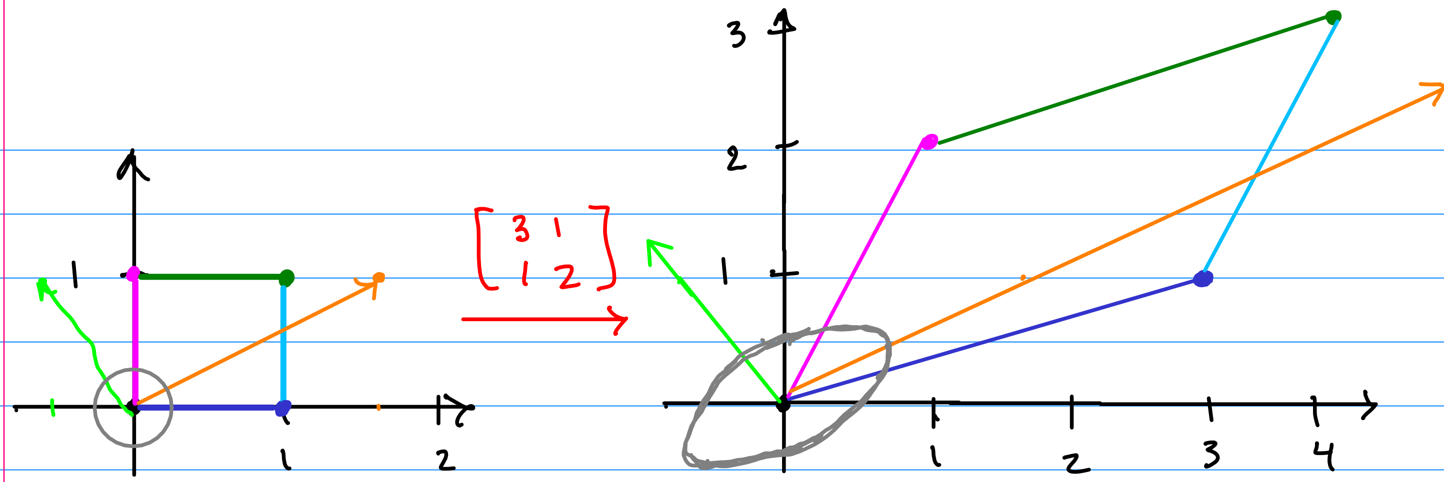
eigenvectors and eigenvalues?

char poly: $(3-x)(2-x) - 1 = x^2 - 5x + 6 - 1$
$= x^2 - 5x + 5$

$$\lambda = \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot 5}}{2} = \frac{5 \pm \sqrt{5}}{2}.$$

Eigenvectors w/ eigenvalue $\frac{5+\sqrt{5}}{2}$: $\left\{ \begin{bmatrix} (1+\sqrt{5})c \\ 2c \end{bmatrix} \middle| \ c \neq 0 \right\}$

$\frac{5-\sqrt{5}}{2}$: $\left\{ \begin{bmatrix} (1-\sqrt{5})c \\ 2c \end{bmatrix} \middle| \ c \neq 0 \right\}$



$$\begin{bmatrix} (1+\sqrt{5})/2 \\ 1 \end{bmatrix} \longmapsto \frac{5+\sqrt{5}}{2}\begin{bmatrix} (1+\sqrt{5})/2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \longmapsto \frac{5-\sqrt{5}}{2}\begin{bmatrix} (1-\sqrt{5})/2 \\ 1 \end{bmatrix}$$

The amount the circle is stretched is given by the eigenvalue. The direction it is stretched is the eigenvector!

- Geometric interpretation:
  the transformation stretches, shrinks, reflects, ... in the directions of the eigenvectors!

Two notable exceptions:

- when eigenvalues are not real
  - indicative of the presence of rotation
- when eigenvalues are repeated but the matrix is not diagonal
  - indicative of the presence of shear!

<u>Ex</u>: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ eigenvalues $= 1, 1$

eigenvectors $= \left\{ \begin{pmatrix} c \\ 0 \end{pmatrix} \middle| \, c \neq 0 \right\}$

MATLAB commands :

det(A) — calculates the determinant

eig(A) — calculates the eigenvalues

[P,D] = eig(A) — sets P = matrix whose columns are the eigenvectors and D to a diagonal matrix whose values are the corresponding eigenvalues