Existence and Uniqueness of Solutions

Ex:
$$\begin{cases} x+y+z=3 \\ x \quad -z=0 \\ x+2y+3z=6 \end{cases}$$
!!

Step 1:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 6 \end{array}\right]$$

Step 2: Put in RREF!

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 6 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -3 \\ 1 & 2 & 3 & 6 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3: Back to equations!

Two kinds of variables

FREE DEPENDENT

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{cases} x - z = 0 \\ y + 2z = 3 \end{cases}$$
!!

Big idea: row reducing changes the equations but <u>not</u> the solutions!

$$\boxed{\begin{aligned} x &= z \\ y &= 3-2z \\ z &= \text{anything!} \end{aligned}}$$

We get infinitely many solutions, one for each z value.

Ex: $z = 0$ Then $x=0$, $y=3$

$$\begin{cases} x+y+z=3 \\ x \quad -z=0 \\ x+2y+3z=6 \end{cases} \qquad \begin{matrix} 0+3+0=3 \checkmark \\ 0-0=0 \checkmark \\ 0+2(3)+3(0)=6 \checkmark \end{matrix}$$

$z=8$. Then $x=8$, $y=3-16=-13$

$$\begin{cases} x+y+z=3 \\ x \quad -z=0 \\ x+2y+3z=6 \end{cases} \qquad \begin{matrix} 8+(-13)+8=3 \checkmark \\ 8-8=0 \checkmark \\ 8+2(-13)+3(8)=6 \checkmark \end{matrix}$$

Big example #2:

$$\begin{cases} x+y=2 \\ x+y=1 \end{cases} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & -1 \end{array}\right]$$

$1 = x+y = 2$

$1=2$???

$$\begin{cases} x+y=2 \\ 0=-1 \end{cases}$$

WHAT???

NO SOLUTION!

Definition: A linear system of equations is <u>consistent</u> if it has at least one solution. It is <u>inconsistent</u> if it has no solutions.

Theorem: A linear system of equations either has

- no solutions
- a unique solution (every variable is dependent)
- or infinitely many solutions (there is at least one free variable)

Ex: Consider the system $\begin{cases} 2x+y=1 \\ 4x+2y=2 \end{cases}$

Step 1: $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 2 \end{array}\right]$

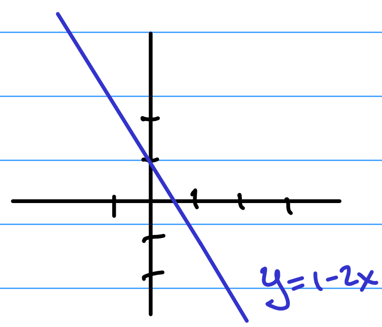
Step 2: $\left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 4 & 2 & 2 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \end{array}\right] \end{array}$

Step 3:

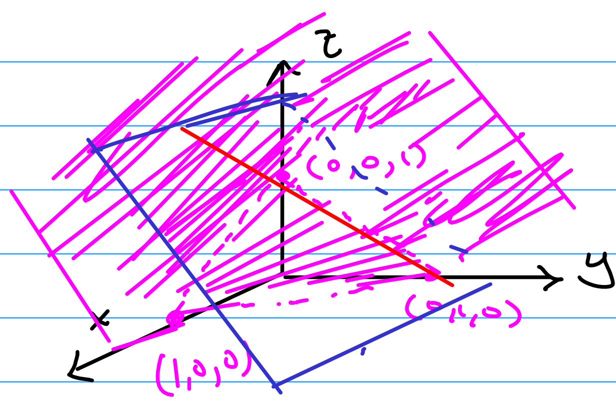
$\begin{cases} x+\frac{1}{2}y=\frac{1}{2} \end{cases}$ line $y=1-2x$

$x=\frac{1}{2}-\frac{1}{2}y$

Tons of solutions! One for each value of y...



Ex: $\begin{cases} x+y+z=1 \\ y-z=0 \end{cases}$



Try it yourself:

$$\begin{cases} 2x+3y+2z=3 \\ 4x-5y+5z=-7 \\ -3x+7y-2z=5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 3 \\ 4 & -5 & 5 & -7 \\ -3 & 7 & -2 & 5 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & -2 \\ -3 & 7 & -2 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 2 & 3 & 2 & 3 \\ -3 & 7 & -2 & 5 \end{array}\right]$$

$$\xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -1 & -4 & 7 \\ -3 & 7 & -2 & 5 \end{array}\right] \xrightarrow{R_3+3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & -1 & -4 & 7 \\ 0 & 13 & 7 & -1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -2 \\ 0 & 1 & 4 & -7 \\ 0 & 13 & 7 & -1 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 12 \\ 0 & 1 & 4 & -7 \\ 0 & 13 & 7 & -1 \end{array}\right] \xrightarrow{R_3-13R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 12 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -45 & 90 \end{array}\right] \xrightarrow{\frac{1}{-45}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 12 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1+5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2-4R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

<u>Step 3</u>

$$\begin{cases} x = 2 \\ y = 1 \\ z = -2 \end{cases} \quad ✓$$

<u>Ex</u>:

$$\begin{cases} x + 3y = 1 \\ z = -2 \\ 2x+6y+z = 0 \end{cases}$$

TAK:

$$\begin{cases} x+3y=1 \\ 2x+6y=2 \end{cases}$$

Good tactic!

<u>Step 1</u>:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 2 & 6 & 1 & 0 \end{array}\right]$$

Practice me!

Step 2:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 2 & 6 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \leadsto \begin{cases} x + 3y = 1 \\ z = -2 \end{cases}$$

Solutions $\quad x = 1 - 3y$

$\qquad\qquad\quad z = -2$

Different solution for each value of y!

Ex: $y = 0$ solution $x = 1,\ y = 0,\ z = -2$

$y = 1$ solution $x = -2,\ y = 1,\ z = -2$

$y = -2$ solution $x = 7,\ y = -2,\ z = -2$

$\vdots$