Complex Numbers

Question: What are numbers?

- We can tell the difference between a pile of three apples and five apples.

The whole numbers $(1, 2, 3, 4, 5, \ldots)$ come from this place of counting apples

Adding whole numbers is intuitive too!
Adding apples to a pile is just like adding numbers!

We can "undo" adding two apples by removing two apples from the pile $\rightarrow$ subtraction!
Leads right away to negative numbers.
The integers $(\ldots, -3, -2, -1, 0, 1, 2, 3)$ allow us to do addition and "undo" addition (subtraction).

By repeating addition, we also get multiplication!
Want to undo multiplication $\rightsquigarrow$ division!

$3 \cdot 5 = 15$

$15 \div 5 = 3$

Problem! What is $3 \div 5$??? Invent $\frac{3}{5}$

More generally, we make rational numbers
$\{a/b \mid a, b \text{ integers } b \neq 0\}$

By repeated multiplication, we get exponentiation!

$3^2 = 9$, $5^2 = 25$, $3^3 = 27$, $5^3 = 125, \ldots$

Again we want to be able to "undo" this operation

$\sqrt{9} = 3$ $\sqrt[3]{27} = 3$ We introduce radicals!

$\sqrt{3} = ???$ irrational

$\sqrt{-7} = ???$

Leads to the introduction of <u>real numbers</u>
such as $\sqrt{3}$, $\sqrt[3]{5}$, ...
and <u>imaginary numbers</u>.

Imaginary numbers:

<u>Definition</u>: We define i to be the number satisfying $i^2 = -1$

In other words $i = \sqrt{-1}$.

An <u>imaginary number</u> is anything of the form bi where b is a real number.

We can do all our usual fun operations w/ imaginary numbers!

$$i \cdot 3i = 3i^2 = 3(-1) = -3.$$

$$i^3 = iii = (-1)i = -i$$
$$i^4 = i^3 i = (-i)i = -i^2 = -(-1) = 1.$$

<u>Ex</u>: $\sqrt{-9} = \sqrt{-1}\sqrt{9} = i3 = 3i$.

Complex Numbers:

<u>Def</u>: A <u>complex number</u> is a number of the form $a+ib$ where a and b are real numbers.

<u>Ex</u>: $2+3i$, $7-4i$, $\sqrt{2}+6i$, $\pi + i$, π

Every real is complex, but not every complex is real!

Anatomy of a complex number: $z = a + ib$

- real part $\mathrm{Re}(z) = a$
- imaginary part $\mathrm{Im}(z) = b$
- modulus (a.k.a. magnitude or absolute value)
  $|z| = \sqrt{a^2 + b^2}$

Every complex number has an associated number called its <u>conjugate</u>.

<u>Def</u>: Given $z = a + ib$, the <u>conjugate</u> $\bar{z}$ is $\bar{z} = a - ib$

<u>Ex</u>: $z = 2 + 3i$

$\mathrm{Re}(z) = 2$, $\mathrm{Im}(z) = 3$

$|z| = \sqrt{13}$, $\bar{z} = 2 - 3i$

We can do all the usual algebra with complex numbers! addition, subtraction, multiplication, division

<u>addition</u>

$$(a+ib) + (x+iy) = (a+x) + i(b+y)$$

<u>subtraction</u>

$$(a+ib) - (x+iy) = (a-x) + i(b-y)$$

<u>multiplication</u>

$$\begin{aligned}(a+ib)(x+iy) &= ax + aiy + ibx + ibiy \\ &= ax + i(ay+bx) + i^2 by \\ &= ax + i(ay+bx) - by\end{aligned}$$

$$\boxed{(a+ib)(x+iy) = (ax-by) + i(ay+bx)}$$

<u>Ex</u>: $(2+3i) - (4+2i) = (2-4) + (3-2)i = -2 + i$

$$(2+3i)(4+2i) = 2\cdot 4 + 3i\cdot 4 + 2\cdot 2i + 3i\cdot 2i$$
$$= 8 + 12i + 4i + 6i^2$$
$$= 2 + 16i$$

Ex: $(2+3i)(2-3i) = 2\cdot 2 + \cancel{3i\cdot 2} - \cancel{2\cdot 3i} + 3i(-3i)$
$$= 4 - 9i^2 = 4+9 = 13$$

$|2+3i| = \sqrt{13}$

link!

Proposition: $z\bar{z} = |z|^2$

Division: $\dfrac{a+ib}{x+iy} = \dfrac{a+ib}{x+iy} \cdot \dfrac{x-iy}{x-iy} = \dfrac{(a+ib)(x-iy)}{(x+iy)(x-iy)} = \dfrac{(a+ib)(x-iy)}{x^2+y^2}$

I know how to divide by real numbers!!!

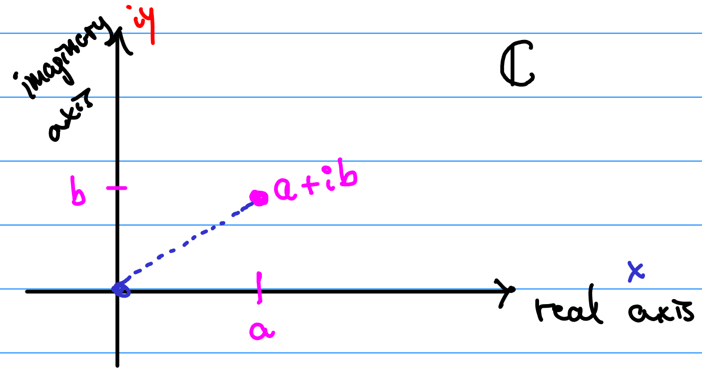
Ex: $\dfrac{2+3i}{1+i} = \dfrac{2+3i}{1+i}\,\dfrac{1-i}{1-i} = \dfrac{(2+3i)(1-i)}{(1+i)(1-i)} = \dfrac{2-2i+3i-3i^2}{2}$

$$= \frac{5+i}{2} = \boxed{\frac{5}{2} + \frac{1}{2}i}$$

Ex: $\dfrac{1+2i}{2-i} = \dfrac{1+2i}{2-i}\,\dfrac{2+i}{2+i} = \dfrac{(1+2i)(2+i)}{(2-i)(2+i)} = \dfrac{2+i+4i+2i^2}{5}$

$$= \frac{2+5i-2}{5} = \frac{5i}{5} = i$$

Complex Plane



$\mathbb{C}$

Dynamical Systems ~

z_0 ← initial point

$z_1 = z_0^2 + C$

$z_2 = z_1^2 + C$

$z_3 = z_2^2 + C$

$\vdots$