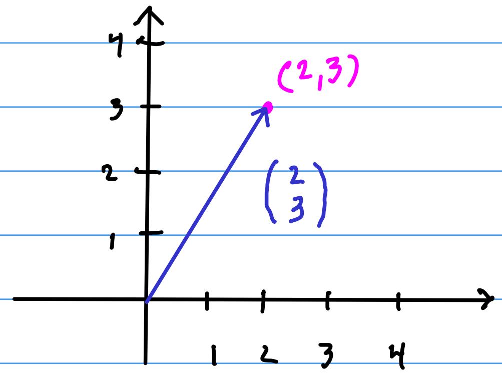
Visualizing Matrix Multiplication

Thinking about $\mathbb{R}^2$ (the xy-plane)

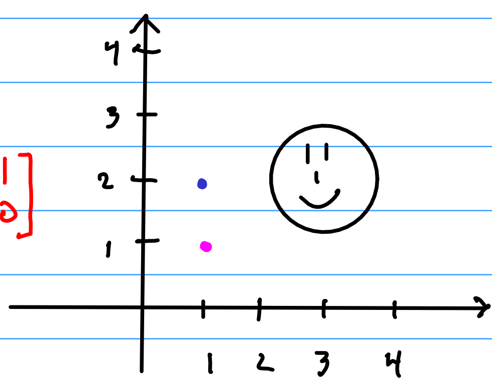
Points in $\mathbb{R}^2$ are represented by ordered pairs



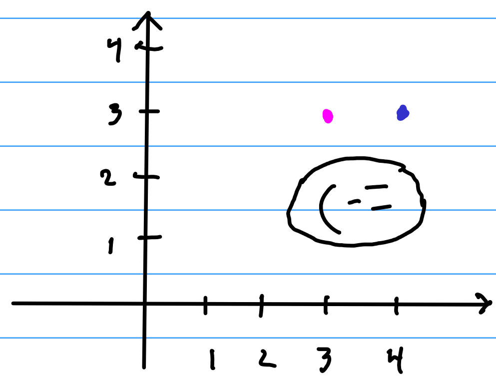
Points can also be represented by column vectors.

A 2x2 matrix defines a transformation of $\mathbb{R}^2$

$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$



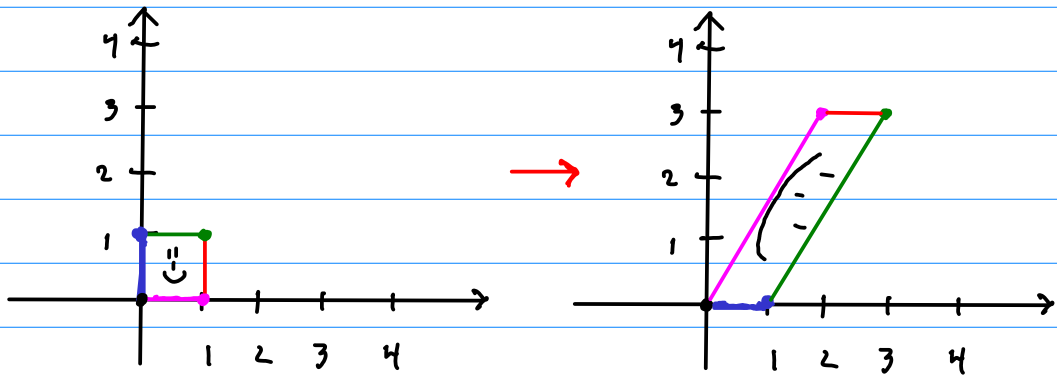
$\longrightarrow$



$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

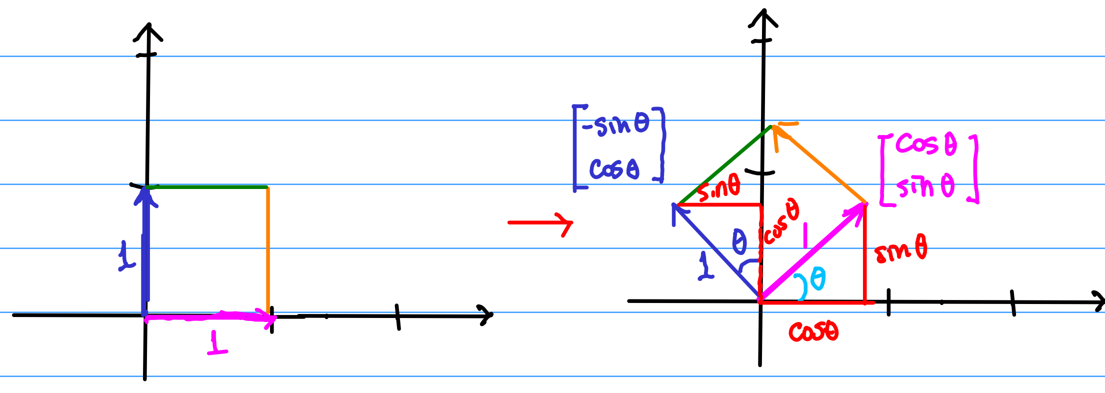
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$



Quest: Find the matrix representing rotation by θ radians counter-clockwise

In general, transformations will look like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

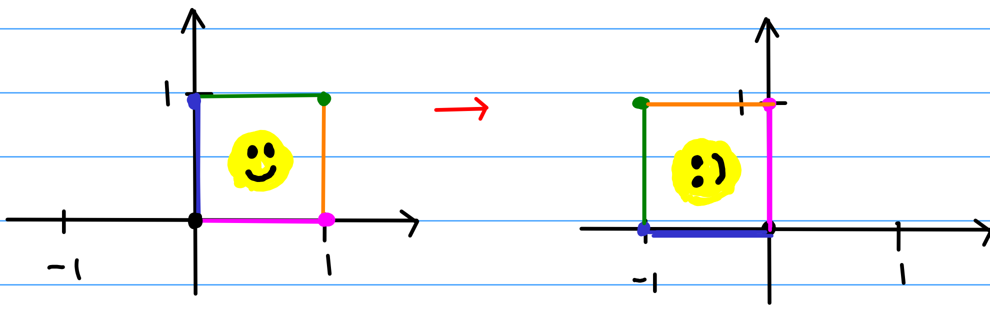
$a = \cos\theta \qquad c = \sin\theta \qquad b = -\sin\theta \qquad d = \cos\theta$

My super cool rotation matrix buddy!

Counter-clockwise rotation is represented by the 2x2 matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

__Ex__:

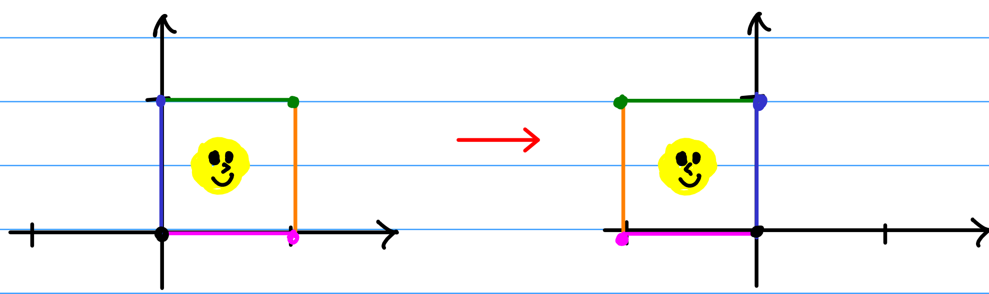
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

__Reflections__:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a = -1, \quad b = 0, \quad c = 0, \quad d = 1$

Reflection across x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection across y-axis

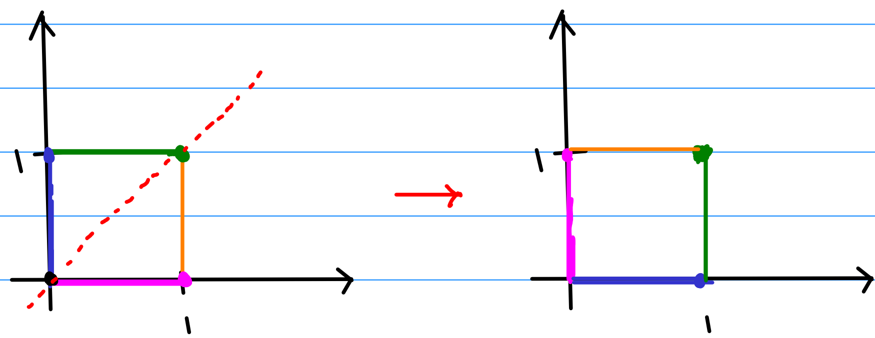
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

More generally we can reflect across <u>any</u> line passing through the origin!

$$\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2} \end{bmatrix}$$ reflects across $y = mx$

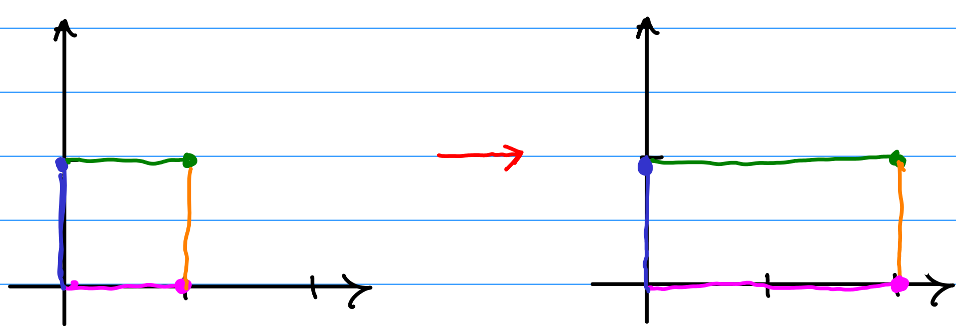
<u>Ex</u>: $m=1$ $y=x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

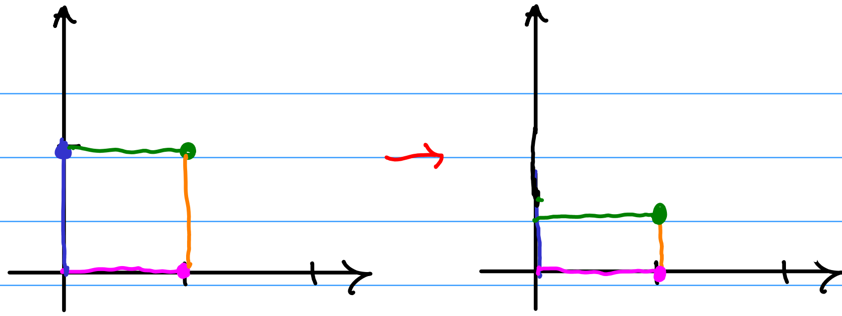


Stretching or Shrinking

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$$



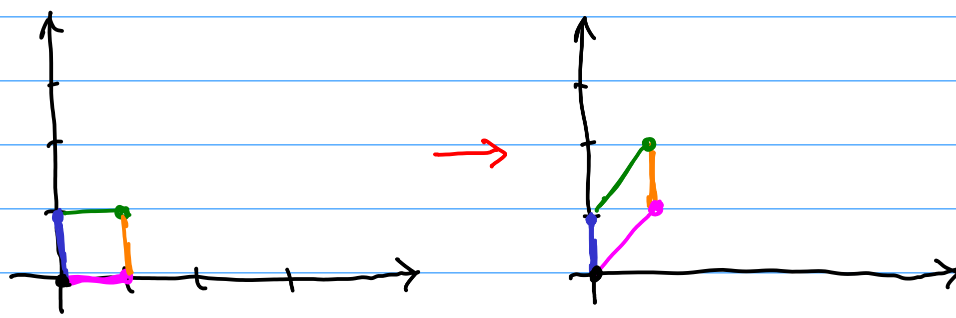
General stretch/shrink $\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

Shear

vertical shear by k: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

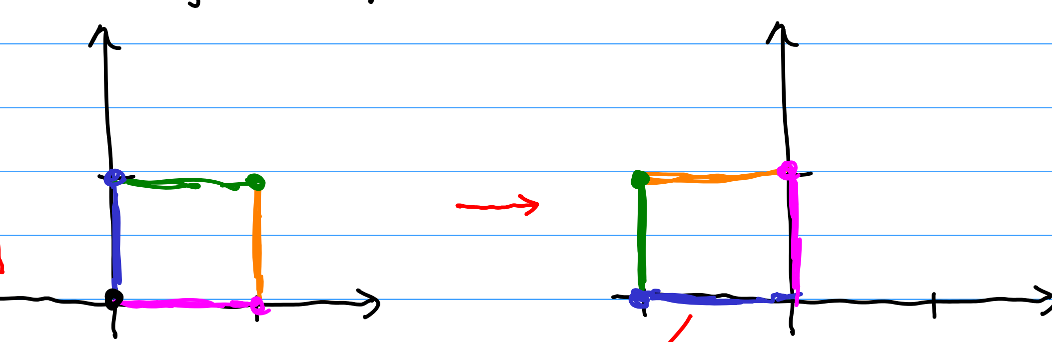
hozontal shear by k: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

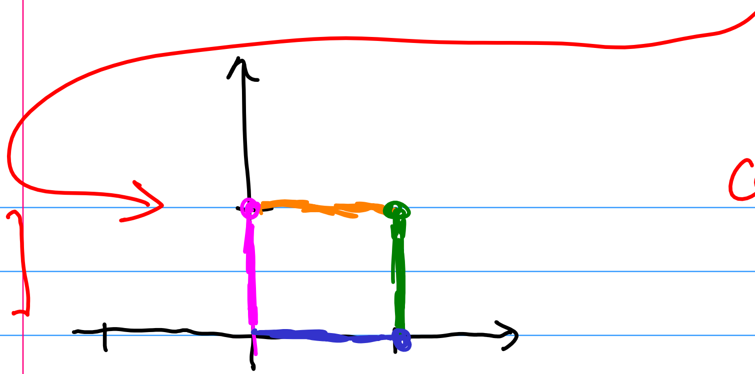


Products of Transformation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



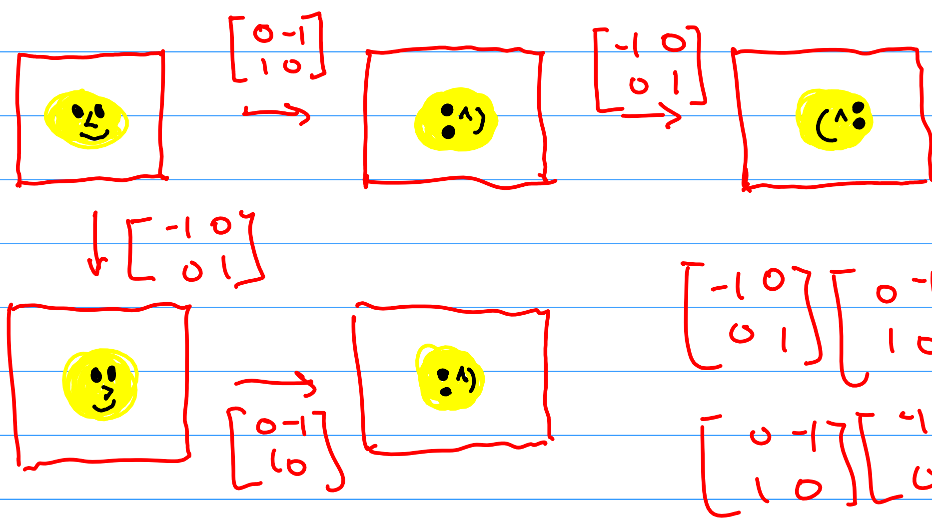
Combined transformation is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

NOTICE
$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{reflection}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad !!!$$

Theorem: If we do a transformation rep. by a matrix A, followed by another transformation represented by B, the combined transformation is represented by BA.

Remember: $AB \neq BA$



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$