Recap from last time:

$$\det(A) \neq 0 \iff A \text{ invertible}$$

$$\boxed{\text{Formula: } \quad A^{-1} = \frac{1}{\det(A)} \operatorname{cof}(A)^T}$$

$\det(A) = 0$ means A singular

$\det(A) \neq 0$ means A ~~plural~~ nonsingular

Theorem: A $n \times n$ matrix. The following are equivalent:

(1) $\det(A) \neq 0$

(2) A is invertible

(3) the homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution

(4) for any $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution

Observation: if A is invertible, then the solution of $A\vec{x} = \vec{b}$ is

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

One issue: I am too lazy to calculate A^{-1} ...

Would have to either calculate $\operatorname{cof}(A)$,

or alternatively, put $[A \mid I]$ in RREF.

because the RREF is $[I \mid A^{-1}]$

To be lazy, use Cramer's Rule.

This rule uses $n+1$ determinants to solve the problem!

Cramer's Rule

$A = [\vec{a}_1 \vec{a}_2 \vec{a}_3 \dots \vec{a}_n]$, $\vec{a}_1, \dots, \vec{a}_n, \vec{b}$ column vectors of length n

so A is an nxn matrix!

The solution of $A\vec{x} = \vec{b}$ is given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_j = \frac{\det[\vec{a}_1 \vec{a}_2 \dots \vec{a}_{j-1} \vec{b} \vec{a}_{j+1} \dots \vec{a}_n]}{\det(A)}$$

Ex: $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Cramers rule: $x = \dfrac{\det\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}$, $y = \dfrac{\det\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}}{\det\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}}$

Double-check

$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓

$= \begin{bmatrix} -1+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$x = \dfrac{4}{-4}$ $y = \dfrac{-4}{-4}$

$x = -1$, $y = 1$

Ex: $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 5 \end{bmatrix}$

$$x = \frac{\det\begin{bmatrix} 8 & 3 & 1 \\ 3 & 2 & 1 \\ 5 & 0 & 3 \end{bmatrix}}{\det\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}} = \frac{(5 \cdot 1 + 3 \cdot 7)}{2 \cdot 6 + 1} = \frac{26}{13} = 2.$$

$$y = \frac{\det\begin{bmatrix} 2 & 8 & 1 \\ 0 & 3 & 1 \\ 1 & 5 & 3 \end{bmatrix}}{\det\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}} = \frac{2\cdot 4 + 1\cdot 5}{13} = 1 \quad \checkmark$$

$$z = \frac{\det\begin{bmatrix} 2 & 3 & 8 \\ 0 & 2 & 3 \\ 1 & 0 & 5 \end{bmatrix}}{\det\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}} = \frac{2\cdot 10 + 1(-7)}{13} = 1$$

Solution is $x=2$, $y=1$, $z=1$.

Eigenvalues and Eigenvectors

Def: Let A be an $n\times n$ matrix. An eigenvector $\vec{v}$ of A with eigenvalue λ is a non-zero vector satisfying

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

Ex: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Therefore $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector w/ eigenvalue 3.

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 2.

$$A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 42\begin{bmatrix}0\\0\end{bmatrix}$$

still not eigenvector or eigenvalue because it's the zero vector!

<u>Ex</u>: $A = \begin{bmatrix}1 & 3\\2 & 2\end{bmatrix}$, eigenvectors?

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \neq \lambda\begin{bmatrix}1\\0\end{bmatrix}$$ not an eigenvector!

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix}$$ so $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of A w/ eigenvalue 4.

<u>Quest</u>: how can we find the eigenvalues and eigenvectors of a matrix A?

$$A\vec{x} = \lambda\vec{x} \quad \Rightarrow \quad A\vec{x} = \lambda I\vec{x}$$
$$\Rightarrow \quad A\vec{x} - \lambda I\vec{x} = \vec{0}$$
$$\Rightarrow \quad \underbrace{(A-\lambda I)}_{\text{matrix}}\underset{\text{vector}}{\vec{x}} = \underset{\text{zero vector}}{\vec{0}}$$

homogeneous system!

Eigenvectors are not $\vec{0}$, so we want $(A-\lambda I)\vec{x} = \vec{0}$ to have a nontrivial solution

This happens exactly when $A-\lambda I$ is singular, i.e. when $\det(A-\lambda I) = 0$.

<u>Theorem</u>: The eigenvalues of A are the values of λ for which $\det(A-\lambda I) = 0$.

$\underline{\text{Ex}}$: $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$. We found 4 is an eigenvalue of A. What are all the eigenvalues?

When is $\det(A - \lambda I) = 0$?

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} \right)$$

$$= (3-\lambda)(2-\lambda) - 1 \cdot 2 = \lambda^2 - 5\lambda + 4$$

$$= (\lambda - 4)(\lambda - 1)$$

Eigenvalues of A are 4 and 1 !!!

$\underline{\text{Ex}}$:

$$A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad \text{eigenvalues of } A?$$

Find λ with $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2-\lambda & 0 & 5 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{bmatrix} = 0$$

$$= (2-\lambda)\left[(3-\lambda)(4-\lambda) - 1 \right] = 0$$

$$= \underline{(2-\lambda)} \left[\underline{\lambda^2 - 7\lambda + 11} \right] = 0$$

$\lambda = 2$ $\underline{\underline{\text{or}}}$ $\lambda^2 - 7\lambda + 11 = 0$

$$\lambda = \frac{-(-7) \pm \sqrt{(-7)^2 - 4\cdot(1)\cdot 11}}{2\cdot 1} = \frac{7 \pm \sqrt{5}}{2}$$

Eigenvalues are $2, \frac{7+\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}$.

<u>Theorem</u>: An $n\times n$ A has exactly n eigenvalues, counting multiplicity over $\mathbb{C}$.

<u>Ex</u>: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm\sqrt{-1}$$
$$\lambda = \pm i$$

<u>Def</u>: The determinant $\det(A-\lambda I)$ is a polynomial in λ of degree n, called the <u>characteristic polynomial</u> of the $n\times n$ matrix A.

What about eigenvectors??

<u>Ex</u>: $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, eigenvalues are $1, 4$.

How do we find an eigenvector w/ eigenvalue λ?

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A-\lambda I)\vec{v} = \vec{0}$$ solve this system!

Eigenvector with eigenvalue 1: $(A-1I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

not invertible!

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\boxed{x + \frac{1}{2}y = 0}$

$0 = 0$ ✓

$x = -\frac{1}{2}y$

Eigenvectors w/ eigenvalue 1:

$$\left\{ \begin{bmatrix} -a/2 \\ a \end{bmatrix} \;\middle|\; a \neq 0 \right\}$$

Specific example: $a = 1 \rightsquigarrow \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ eigenvector!

Double-check:

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 + 1 \\ -1 + 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

Eigenvectors w/ eigenvalue 4?

$$\begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + 2R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[-R_1]{} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x - y = 0$

$0 = 0$ ✓

$x=y$

Eigenvectors w/ eigenvalue 4 : $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \neq 0 \right\}$

infinitely many eigenvectors w/ eigenvalue 4.

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ eigenvectors and eigenvalues?

First calculate the characteristic poly!

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 5 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left[(3-\lambda)^2 - 1\right] = (2-\lambda)\left[\lambda^2 - 6\lambda + 8\right]$$

$$= (2-\lambda)(\lambda-4)(\lambda-2)$$

happened twice!

Eigenvalues are: 2, 2, 4.

Eigenvectors w/ eigenvalue 2:

$$\begin{bmatrix} 0 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y = 0$

$z = 0$

$x =$ anything!

$$\boxed{\text{Eigenvectors are } \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \,\middle|\, a \neq 0 \right\}}$$