Last Time :

- linear equations
- systems of linear equations
- story problem $\longrightarrow$ system of linear equations

Today is all about solving systems of linear equations

- RREF row-reduced echelon form } see both
  reduced row echelon form }
- Gaussian elimination
  - elementary row operations
- solving linear systems via Gaussian elimination

Row-reduced echelon form (RREF)

Def: An $m \times n$ matrix is in row echelon form if it satisfies the following properties

(1) • the first nonzero entry of every row is a 1 (leading 1)

(2) • the leading 1 of any row must occur to the right of the leading 1's of rows above

(3) • any rows of zeros are at the bottom of the matrix

if in addition we satisfy the property

(4) • any column with a leading 1 has no other nonzero entries

then we say the matrix is in RREF

Ex:
$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) is satisfied!
(2) is satisfied! } row echelon form!!
(3) is satisfied!
(4) is satisfied! RREF

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(1) is satisfied!
(2) not satisfied !! ⌢
not row echelon form
not RREF!

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(1) is satisfied
(2) is satisfied
(3) is not satisfied !! ⌢
not row echelon form!

Ex: $\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(1) is not satisfied !! ⌢
not RREF!

Ex: $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$

(1) satisfied
(2) satisfied
(3) satisfied
(4) satisfied
matrix is in RREF!

Which of the following are in RREF?

a) $\boxed{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$ RREF!

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ not!

c) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ row echelon
not RREF!!

d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ RREF!

e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ not!

f) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ row echelon NOT RREF

Gaussian elimination

Elementary row operations

- swapping two rows $R_i \leftrightarrow R_j$
- multiplying a row by a nonzero constant cR_i
- adding a constant multiple of on row to another $R_i + cR_j$

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 6 & 7 \\ 5 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 8 & 6 & 7 \\ 15 & 9 & 0 \\ 9 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ \frac{1}{7}(7 & 7) \end{bmatrix} \xrightarrow{R_1 - \frac{1}{7}R_3} \begin{bmatrix} 1 & 0 \\ 3 & 4 \\ 7 & 7 \end{bmatrix}$$

Gaussian elimination:

$$\begin{bmatrix} -1 & 2 & 5 & 3 \\ 1 & 0 & -6 & 1 \\ -4 & 2 & 2 & -2 \end{bmatrix} \xrightarrow{-1 \cdot R_1} \begin{bmatrix} 1 & -2 & -5 & -3 \\ 1 & 0 & -6 & 1 \\ -4 & 2 & 2 & -2 \end{bmatrix}$$

$[1\ 0\ -6\ 1] - [1\ -2\ -5\ -3]$
$= [0\ 2\ -1\ 4]$

$\xrightarrow{R_2 - R_1}$

$$\begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 2 & -1 & 4 \\ -4 & 2 & 2 & -2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 2 & -1 & 4 \\ -2 & 1 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 2 & -1 & 4 \\ 0 & -3 & -9 & -7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & -3 & -9 & -7 \end{bmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{21}{2} & -1 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & -\frac{21}{2} & -1 \end{bmatrix}$$

$$\xrightarrow{-\frac{2}{21}R_3} \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & \frac{2}{21} \end{bmatrix} \xrightarrow{R_1 + 6R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & \frac{2}{21} \end{bmatrix}$$

$$\xrightarrow{R_2 + \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{43}{21} \\ 0 & 0 & 1 & \frac{2}{21} \end{bmatrix}$$

RREF!

Ex:
$$\boxed{\begin{cases} x + y = 2 \\ y + z = 0 \\ x + y + z = 1 \end{cases}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \longrightarrow \boxed{\begin{cases} x = 1 \\ y = 1 \\ z = -1 \end{cases}}$$

Theorem: If we row reduce, the solutions of the linear system are unchanged!

$x = 1, \quad y = 1, \quad z = -1$

Double-check!

complicated

$$\boxed{\begin{cases} x + y = 2 \\ y + z = 0 \\ x + y + z = 1 \end{cases}}$$

$1 + 1 = 2$ ✓

$1 + (-1) = 0$ ✓

$1 + 1 + (-1) = 1$ ✓

Special cases:

(1) $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 1 \end{array}\right] \rightsquigarrow \begin{cases} x = 3 \\ 0 = 1 \end{cases}$

inconsistent system of equations! no solutions

(2) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \boxed{\begin{cases} x + 2y = 3 \\ 0 = 0 \end{cases}}$

multiple solutions!

extra flexibility $y=0 \rightarrow x+2(0)=3$

$x=3$

$x=3, y=0$

$y=1 \rightarrow x+2(1)=3$

$x=1$

$x=1, y=1$