Linear Systems of Equations

Def: A linear equation is an equation of the form

variables (unknown)

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

known

coefficients (known)

Ex:

$$2x - 3y + 4z = 7$$
$$x = 3y - 4z$$
$$2x_1 + 3x_2 - x_3 + x_4 = 0$$
$$x_1 - x_5 = x_3$$

all examples of linear equations!

Ex:

$$x^2 + 3y - z = 2$$
$$\sin(x) + y = 4$$
$$x + xy + z = 0$$
$$x + y + \sqrt{z} = w$$

not a linear equation!

Try it yourself!

a) $2x + 3y - 7z = 11$ linear

b) $y_1 + 14^2 y_2 + 4 = y_2 + 13 - y_1$ linear

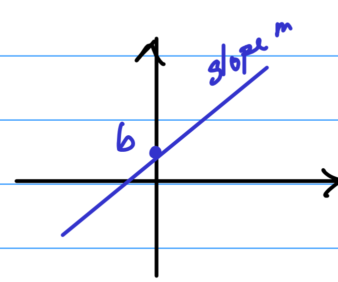
c) $\ln y = 2x$ nonlinear

d) $3x + \pi \sin(2\pi/3)\, y = z - 11x$ linear

e) $x^2 + x + 1 = 3y$ nonlinear

Special case of linear equations:

$$y = mx + b$$



Ex:

- $4xy = 0$ nonlinear
- $x_1 + \frac{7}{2}x_2 + x_3 - x_4 + 17x_5 = \sqrt[3]{-10}$ linear
- $3^x + 4 = y$ nonlinear
- $\sqrt{7}\, r + \pi s + \frac{3t}{5} = \cos(\pi/4)$ linear
- $6y + 3z = \cos(x/4)$ not linear

Systems of linear equations

Ex: $\begin{cases} 2x + 3y - 4z = 0 \\ x - 2y + z = 1 \end{cases}$

Ex: $\begin{cases} x + y + z = 1 \\ x - 2z = 0 \\ 3x + y + z = 2 \end{cases}$

Def: A system of linear equations is something of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

System with m equations and n unknowns (variables)

A solution to a system of equations consists of values of the variables making all equations satisfied simultaneously!

Ex: $\begin{cases} x + 2y = 1 \\ 2x + y = 1 \end{cases}$

$\frac{1}{3} + 2\cdot\frac{1}{3} = 1$ ✓

$2\cdot\frac{1}{3} + \frac{1}{3} = 1$

A solution is $x = \frac{1}{3}$, $y = \frac{1}{3}$.

Where do linear equations come from?

Ex: Find constants a, b with $a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}4\\7\end{pmatrix}$

$$\begin{pmatrix}2a+b\\a+3b\end{pmatrix} = \begin{pmatrix}4\\7\end{pmatrix} \rightsquigarrow \begin{cases} 2a+b = 4 \\ a+3b = 7 \end{cases}$$

Solution of this linear system is $a=1, b=2$

$2(1) + 2 = 4$ ✓

$1 + 3(2) = 7$

Real-world example:

A jar full of spare change has pennies, nickels, and quarters. There are 100 total coins and twice as many pennies as quarters. If the total monetary value is \$8.60. How many of each coin is there?

Set up as a linear system of equations!

p = # pennies

n = # nickels

q = # quarters

$$\begin{cases} p + n + q = 100 \\ p = 2q \\ 0.01\cdot p + 0.05n + 0.25q = 8.60 \end{cases}$$

Solution?

$p = 60$, $n = 10$, $q = 30$

we'll find out how soon!

Applications:

- engineering
- physics
- mathematics
- business
- economics
- nearly everything ...

More important than calculus!

Row Reduction

Given a linear system of equations

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + y - z = 0 \\ 3x + 4y + 0z = 4 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 4 & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 4 & 0 & 4 \end{array}\right]$$

coefficient matrix

augmented matrix

where it's at!

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 2x + y - z \\ 3x + 4y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\boxed{A\vec{x} = \vec{b}}$$

Ex: Write the augmented matrix for the linear system

$$\begin{cases} 4x - 7 + y = 0 \\ 2 - 5z + 3y = x \\ z = 4x \end{cases}$$

$$\text{A}\quad\begin{cases} 4x + y + 0z = 7 \\ -x + 3y - 5z = -2 \\ -4x + 0y + z = 0 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & 0 & 7 \\ -1 & 3 & -5 & -2 \\ -4 & 0 & 1 & 0 \end{array}\right]$$

Ex:
$$\begin{cases} 3x + 4y = -x \\ z - 3 = y \end{cases}$$

$$\downarrow$$

$$\begin{cases} 4x + 4y = 0 \\ -y + z = 3 \end{cases} \longrightarrow \begin{cases} 4x + 4y + 0z = 0 \\ 0x - y + z = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & -1 & 1 & 3 \end{array}\right]$$

Ex:
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 3 & 9 \end{array}\right] \longrightarrow \begin{cases} x + 2y = 3 \\ -x + 3y = 9 \end{cases}$$

Ex:
$$\left[\begin{array}{ccc|c} 8 & 6 & 7 & 5 \\ 5 & 3 & 0 & 4 \\ 9 & 1 & 2 & 3 \end{array}\right] \longrightarrow \begin{cases} 8x + 6y + 7z = 5 \\ 5x + 3y = 4 \\ 9x + y + 2z = 3 \end{cases}$$

Convert from word problems to linear systems

Ex:

The LA Zoo sells tickets for \$17 for children, \$22 for adults, and \$19 for seniors. Attendance on a certain day had 4000 people with a total revenue of \$60000. If there were twice as many children's tickets as adult tickets purchased, how many of each type of ticket were sold?

a = # adult tickets sold
c = # children's tickets sold
s = # senior's tickets sold

$$\begin{cases} c + a + s = 4000 \\ 17c + 22a + 19s = 60000 \\ c = 2a \end{cases}$$

$$\boxed{\begin{cases} c + a + s = 4000 \\ 17c + 22a + 19s = 60000 \\ c - 2a = 0 \end{cases}}$$