Last Time : Principal Component Analysis (PCA)

Today : More applications of eigenvectors + eigenvalues

- diagonal decompositions
- singular value decompositions (SVD's)

Matrix Decompositions :

Idea : to solve many problems w/ matrices, one tactic is to decompose as a product of "nice matrices"

Ex : LU~factorization

$$A = \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

USEFUL FOR SOLVING LINEAR SYSTEMS

Ex : LDL - factorization, Cholesky factorization, QR decomposition.

We will focus on diagonalization and SVD.

Diagonalization :

Relate a matrix to a diagonal matrix.

Def : A diagonal matrix is a matrix D where
$D(i,j) = 0$ if $i \neq j$

Ex :

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is diagonal}$$

<u>Def</u>: A diagonalization of an $n \times n$ matrix A is an $n \times n$ invertible matrix P and diagonal matrix D satisfying $\boxed{A = PDP^{-1}}$

Lots of good applications!

Method to diagonalize $n \times n$ matrix A:

(1) calculate the n different eigenvalues of A
$\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct)

(2) For each λ_j, find an eigenvector $\vec{v}_j$ with eigenvalue λ_j (for repeated λ_j, might need to be careful about <u>which</u> eigenvectors we pick)

(3) $P = [\vec{v}_1 \vec{v}_2 \ldots \vec{v}_n]$ $\quad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

As long as P is invertible, P and D will define a diagonalization of A $(A = PDP^{-1})$

<u>Matlab code</u>: `[P,D] = eig(A);`

Automatically produces a diagonalization

WARNING: It's possible that no matter which eigenvectors we choose, P won't be invertible.
Some matrices just don't have diagonalizations!

<u>Def</u>: A matrix which has a diagonalization is called <u>non-degenerate</u>. Otherwise it is called <u>degenerate</u>.

Diagonalization of Symmetric Matrices

Def: An $n \times n$ matrix A is symmetric if $A = A^T$.
A real $n \times n$ matrix U is called orthogonal (or unitary) if $U^T = U^{-1}$.

Theorem: If A is an $n \times n$ symmetric matrix then there exists an $n \times n$ unitary matrix U and a diagonal matrix D satisfying

$$A = UDU^{-1} = UDU^T$$

In particular, A has a diagonalization.

MATLAB: [P,D] = eig(A);

When $A = A^T$, this automatically returns a unitary P.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \det(A - \lambda I) = \det \begin{bmatrix} (1-\lambda) & 1 \\ 0 & (2-\lambda) \end{bmatrix}$

$$= \underline{(1-\lambda)(2-\lambda)}$$

Eigenvalues of A? $1, 2$

1 has eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

2 has eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\boxed{A = PDP^{-1}}$$

Double-check: $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = A$$

Diagonalization can help us compute complicated expressions, like powers.

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$A^3 = AAA = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 8 \end{bmatrix}$$

$$A^{100} = AAAAAAAA\ldots A. = ???$$

Make this doable with diagonalization!

$$A = PDP^{-1}$$

$$\underline{A^2} = PD\cancel{P^{-1}P}DP^{-1} = PDIDP^{-1} = \underline{PD^2P^{-1}}$$

$$\underline{A^3} = PDP^{-1}(PD^2P^{-1}) = PD\cancel{P^{-1}P}D^2P^{-1} = \underline{PD^3P^{-1}}$$

$$\boxed{A^n = PD^nP^{-1}}$$

$$A^{100} = PD^{100}P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{100}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2^{100} \\ 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2^{100}-1 \\ 0 & 2^{100} \end{bmatrix}}$$

Fibonacci Sequence:

$$\begin{array}{cccccccccccc} 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & \dots \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & & & & & & \\ f_1 & f_2 & f_3 & f_4 & f_5 & \dots & & & & & \end{array}$$

$$\boxed{\text{Recursive Formula}: \quad f_{n+1} = f_n + f_{n-1}}$$

Use diagonalization to get a closed-form formula for the nth Fibonacci number f_n.

Q: f_{100} ← how can we get this ??

$$\boxed{\text{The matrix} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{satisfies} \quad A\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}}$$

$$A\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n-1}+f_n \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A^3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}A^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\boxed{A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}}$$

Use diagonalization to calculate A^n, and then use it to get an expression for f_n

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 1$$

Roots $\lambda = \dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$

Golden Ratio $\quad \gamma_+ \quad \gamma_-$

Eigenvector for A w/ eigenvalue $\gamma_\pm$ is $\begin{bmatrix} \gamma_\pm - 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} \gamma_+ - 1 & \gamma_- - 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \gamma_+ & 0 \\ 0 & \gamma_- \end{bmatrix}$$

Then

$$A = PDP^{-1}$$

$$A^n = PD^nP^{-1}$$

$$= \begin{bmatrix} \gamma_+ - 1 & \gamma_- - 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_+^n & 0 \\ 0 & \gamma_-^n \end{bmatrix} \begin{bmatrix} 1 & 1-\gamma_- \\ -1 & \gamma_+ - 1 \end{bmatrix} \frac{1}{(\gamma_+ - \gamma_-)}$$

$$= \frac{1}{(\gamma_+ - \gamma_-)} \begin{bmatrix} \gamma_+^{n+1} - \gamma_-^{n+1} + \gamma_-^n - \gamma_+^n & \gamma_+^{n+1} - \gamma_-^{n+1} + \gamma_-^n - \gamma_+^n - \gamma_+^{n+1}\gamma_- + \gamma_-^{n+1}\gamma_+ + \gamma_+^n\gamma_- - \gamma_-^n\gamma_+ \\ \gamma_+^n - \gamma_-^n & \gamma_+^n - \gamma_-^n + \gamma_-^n\gamma_+ - \gamma_+^n\gamma_- \end{bmatrix}$$

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}$$

$$\frac{1}{(\gamma_+ - \gamma_-)} \begin{bmatrix} \gamma_+^{n+1} - \gamma_-^{n+1} + \gamma_-^n - \gamma_+^n & \gamma_+^{n+1} - \gamma_-^{n+1} + \gamma_-^n - \gamma_+^n - \gamma_+^{n+1}\gamma_- + \gamma_-^{n+1}\gamma_+ + \gamma_+^n\gamma_- - \gamma_-^n\gamma_+ \\ \gamma_+^n - \gamma_-^n & \gamma_+^n - \gamma_-^n + \gamma_-^n\gamma_+ - \gamma_+^n\gamma_- \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{f_{n+2} = \frac{2\gamma_+^n - 2\gamma_-^n + \gamma_-^n\gamma_+ - \gamma_+^n\gamma_-}{\gamma_+ - \gamma_-}} \qquad \gamma_\pm = \frac{1 \pm \sqrt{5}}{2}$$

Take $n = 98$

$$\boxed{f_{100} = 354224848179261915075}$$

Singular Value Decomposition:

A $m \times n$ matrix

$A = UDV^T$, $\quad U$ $m \times m$ unitary $(U^{-1} = U^T)$
$\quad V$ $n \times n$ unitary $(V^{-1} = V^T)$
$\quad D$ $m \times n$ diagonal

This is a generalization of diagonalization.

To compute:

(1) $A^T A$ ← diagonalize this symmetric matrix by a unitary matrix
$(n \times m)(m \times n) = n \times n$

Find an $n \times n$ unitary matrix V w/ $A^T A = VDV^{-1}$ for some $n \times n$ diagonal matrix D

(2) Similarly, diagonalize AA^T: ← $m \times m$

Find a unitary ($m \times m$) matrix U w/ $AA^T = UEU^{-1}$

for some diagonal matrix E $(m \times m)$

(3) Set $D = U^T A V$ ($m \times n$ diagonal)

This automatically makes

$$\boxed{A = UDV^T}$$

Big application of SVD: approximating matrices with matrices of lower rank.