Matrix Multiplication and Linear Systems

$$\begin{cases} c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = b_1 \\ c_{21}x_1 + c_{22}x_2 + \dots + c_{2n}x_n = b_2 \\ \vdots \\ c_{m1}x_1 + c_{m2}x_2 + \dots + c_{mn}x_n = b_m \end{cases}$$

m equations $\qquad$ n unknowns

Augmented matrix:

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & b_1 \\ c_{21} & c_{22} & \cdots & c_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} & b_m \end{array}\right]$$

A

MATLAB: rref(A)

Systems w/ hundreds of variables? Thousands?

RREF can be not the best

To find new methods to solve these systems, we need to change our perspective!

Idea: rewrite in terms of matrix multiplication

$$\begin{cases} c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = b_1 \\ c_{21}x_1 + c_{22}x_2 + \dots + c_{2n}x_n = b_2 \\ \vdots \\ c_{m1}x_1 + c_{m2}x_2 + \dots + c_{mn}x_n = b_m \end{cases}$$

$$\boxed{A\vec{x} = \vec{b}}$$

$$A = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

<u>Ex</u>:

$$\begin{cases} x + 2y + z = 1 & ✓ \\ 3x + z = 4 & ✓ \\ 3x + 2y + 4z = 5 & ✓ \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \qquad \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Double-check:

$$\begin{bmatrix} x + 2y + z \\ 3x + z \\ 3x + 2y + 4z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

<u>Recap</u>: a linear system of equations can always be written in <u>matrix form</u> $A\vec{x} = \vec{b}$

<u>Def</u>: A linear system of equations is called <u>homogeneous</u> if it is of the form:

$$\begin{cases} c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = 0 \\ c_{21}x_1 + c_{22}x_2 + \dots + c_{2n}x_n = 0 \\ \vdots \\ c_{m1}x_1 + c_{m2}x_2 + \dots + c_{mn}x_n = 0 \end{cases}$$

A homogeneous system always has the <u>trivial solution</u>

$$x_1=0,\ x_2=0,\ x_3=0,\ \ldots,\ x_n=0$$

Note: either this is the only solution, or there are infinitely many!

In <u>matrix form</u>: a homogeneous linear system of equations is one of the form $\boxed{A\vec{x}=\vec{0}}$

The matrix version allows us to make a <u>very clear relationship</u> between homogeneous and nonhomogeneous systems

<u>Def</u>: Given a linear system $A\vec{x}=\vec{b}$, we call $A\vec{x}=\vec{0}$ the <u>associated homogeneous system</u>.

<u>Theorem</u>: If $\vec{x}=\vec{c}$ and $\vec{x}=\vec{d}$ are both solutions of $A\vec{x}=\vec{b}$, then $\vec{x}=\vec{c}-\vec{d}$ is a solution of the associated homogeneous system.

<u>Proof</u>:

$\vec{x}=\vec{c}$ solves $A\vec{x}=\vec{b}$ $\Rightarrow$ $A\vec{c}=\vec{b}$

$\vec{x}=\vec{d}$ solves $A\vec{x}=\vec{b}$ $\Rightarrow$ $A\vec{d}=\vec{b}$

$\underline{x=\vec{c}-\vec{d}}$ $\quad A(\vec{c}-\vec{d})=A\vec{c}-A\vec{d}=\vec{b}-\vec{b}=\vec{0}$

So $\vec{x}=\vec{c}-\vec{d}$ solves $A\vec{x}=\vec{0}$. $\square$

<u>Big picture</u>: $A\vec{x}=\vec{b}$ — Find <u>a</u> solution (particular solution) $\qquad A\vec{x}=\vec{0}$ — Find all solutions

Then all solutions of $A\vec{x}=b$ are obtained by taking our particular solution of $A\vec{x}=\vec{b}$ and adding some solution of $A\vec{x}=\vec{0}$.

Theorem: The system of equations $A\vec{x} = \vec{b}$ has a unique solution if and only if $A\vec{x} = \vec{0}$ has only the trivial solution.

Ex: $\begin{cases} x + y + z = 3 \\ x + z = 2 \\ 4x + 2y + 4z = 10 \end{cases}$

$1 + 1 + 1 = 3$ ✓
$1 + 1 = 2$ ✓
$4 + 2 + 4 = 10$

Notice $x = 1$, $y = 1$, $z = 1$ is a solution!

All solutions??

Homogeneous equation $\begin{cases} x + y + z = 0 \\ x + z = 0 \\ 4x + 2y + 4z = 0 \end{cases}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y = 0$
$x = -z$

Solutions of homogeneous: $\begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$, for any z

A solution $\begin{cases} x + y + z = 3 \\ x + z = 2 \\ 4x + 2y + 4z = 10 \end{cases}$ is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 1-z \\ 1 \\ 1+z \end{bmatrix}$$

Ex: $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

Homogenuous systems have nicely behaved solutions:

If $\vec{x} = \vec{d}$ and $\vec{x} = \vec{c}$ are solutions of $A\vec{x} = \vec{0}$ then

- $\vec{x} = a\vec{c}$ is a solution
- $\vec{x} = \vec{c} + \vec{d}$ is a solution

This is called a vector space, specifically the nullspace.

Recap:
- write linear systems in matrix form $A\vec{x} = \vec{b}$
- related to associated homogeneous equation $A\vec{x} = \vec{0}$

Def: The nullspace of a matrix is the set

$$\text{null}(A) = \{\text{solutions to } A\vec{x} = \vec{0}\}$$

Theorem: all solutions of $A\vec{x} = \vec{b}$ differ by an element of the nullspace null(A).
There is a unique solution $\Leftrightarrow$ $\text{null}(A) = \{\vec{0}\}$.

Let's think about when $\text{null}(A) = \{\vec{0}\}$ and there's a unique solution.

$$\boxed{A\vec{x} = \vec{b}}$$

Ex: $3x = 4$ $\qquad \frac{1}{3}(3x) = \frac{1}{3}(4)$

$$x = \frac{4}{3}$$

Silly idea:

Ex: $A\vec{x} = \vec{b}$ $\qquad A^{-1}(A\vec{x}) = A^{-1}b$

$$\vec{x} = A^{-1}b$$

What the heck is that??!!

What should it do? $\qquad 3^{-1}\cdot 3 = 1$

$$A^{-1}A = I$$

Definition: Let A be an $n\times n$ matrix. The inverse of A (if it exists) is the unique matrix A^{-1} satisfying $A^{-1}A = I$ and $AA^{-1} = I$

The 2x2 case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Works as long as $ad \neq bc$.

Ex: $A = \begin{bmatrix} 7 & 15 \\ 7 & 3 \end{bmatrix}$, $\quad A^{-1} = \dfrac{1}{7\cdot 3 - 7\cdot 15}\begin{bmatrix} 3 & -15 \\ -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{3}{84} & \frac{15}{84} \\ \frac{1}{12} & -\frac{1}{12} \end{bmatrix}$$

Double-check: $A^{-1}A = I$??

$$\begin{bmatrix} -\frac{3}{84} & \frac{15}{84} \\ \frac{1}{12} & -\frac{1}{12} \end{bmatrix} \begin{bmatrix} 7 & 15 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} \left(\frac{-3}{84}\right)(7) + \left(\frac{15}{84}\right)(7) & \left(\frac{-3}{84}\right)(15) + \left(\frac{15}{84}\right)3 \\ \left(\frac{1}{12}\right)(7) + \left(-\frac{1}{12}\right)(7) & \left(\frac{1}{12}\right)(15) + \left(-\frac{1}{12}\right)(3) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4} + \frac{5}{4} & -\frac{45}{84} + \frac{45}{84} \\ \frac{7}{12} + \left(\frac{-7}{12}\right) & \frac{15}{12} - \frac{3}{12} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ identity matrix !

Ex :

$$\begin{cases} 7x + 15y = 2 \\ 7x + 3y = 3 \end{cases}$$

$$\downarrow$$

$$\begin{bmatrix} 7 & 15 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -\frac{3}{84} & \frac{15}{84} \\ \frac{1}{12} & -\frac{1}{12} \end{bmatrix} \begin{bmatrix} 7 & 15 \\ 7 & 3 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{3}{84} & \frac{15}{84} \\ \frac{1}{12} & -\frac{1}{12} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} 39/84 \\ -1/12 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 39/84 \\ -1/12 \end{bmatrix}}$$

Ex: Calculate A^{-1} for $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{(2\cdot 1)-(3\cdot 4)} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{2}{10} \end{bmatrix}$$

$$\begin{cases} 2x + 3y = 8 \\ 4x + y = 1 \end{cases}$$

Solve using inverses!

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{2}{15} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{2}{15} \end{bmatrix} \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/10 \\ 30/10 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$$!!!