Recall the dot product:

$\vec{u}, \vec{v}$ vectors of length n

$$\vec{u}\cdot\vec{v} = \sum_{k=1}^{n} u_k v_k = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \vec{u}\cdot\vec{v} = 1\cdot 2 + 2\cdot 0 + 3\cdot 1 = 5$

Idea: use the dot product to multiply a vector by a matrix!

Two ways to write a matrix (A $m\times n$ matrix)

- $A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ \vec{r}_m \end{bmatrix}$ $\quad \vec{r}_1, \vec{r}_2, \dots$ are the rows of A

- $A = [\vec{c}_1\ \vec{c}_2 \dots \vec{c}_n]$ $\quad \vec{c}_1, \vec{c}_2, \dots$ are the columns of A

Def: We define the product of A times an $n\times 1$ column vector $\vec{b}$ to be

$$A\vec{b} = \begin{bmatrix} \vec{r}_1\cdot\vec{b} \\ \vec{r}_2\cdot\vec{b} \\ \vdots \\ \vec{r}_m\cdot\vec{b} \end{bmatrix}$$

Ex:

$$\underset{3\times 3}{A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \qquad \underset{3\times 1}{\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}}$$

$$A\vec{b} = \begin{bmatrix} 1\cdot 4 + 2\cdot 1 + 0\cdot 2 \\ 0\cdot 4 + 3\cdot 1 + 1\cdot 2 \\ 1\cdot 4 + 1\cdot 1 + 1\cdot 2 \end{bmatrix} = \underset{3\times 1}{\begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}} \quad ✓$$

4×3

3×1

Ex:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \\ 4 & 1 & 3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Calculate $A\vec{b}$ ...

$$A\vec{b} = \begin{bmatrix} 1\cdot 1 + 0\cdot 0 + (-1)(-1) \\ 0\cdot 1 + 1\cdot 0 + 2(-1) \\ 2\cdot 1 + 1\cdot 0 + 0(-1) \\ 4\cdot 1 + 1\cdot 0 + 3(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

4×1

Observation: if A is $m \times n$ and $\vec{b}$ is $n \times 1$, then $A\vec{b}$ is $m \times 1$

CAUTION! $A\vec{b}$ only makes sense if the number of columns of A is the same as the number of rows of $\vec{b}$

Special Case: $\vec{a} = [a_1 \ a_2 \ a_3 \dots a_n]$ $\qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$1 \times n$ $\qquad$ $n \times 1$

$$\boxed{\vec{a}\vec{b} = \vec{a} \cdot \vec{b}}$$

Product of Matrices:

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$$ $m \times n$ matrix (each $\vec{a}_j$ has length n)

$$B = [\vec{b}_1 \ \vec{b}_2 \dots \vec{b}_\ell]$$ $n \times \ell$ matrix (each $\vec{b}_j$ has length n)

Def: The **product** of A and B is

$$AB = [A\vec{b}_1 \; A\vec{b}_2 \; \dots \; A\vec{b}_\ell]$$

$$= \begin{bmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 & \dots & \vec{a}_1\cdot\vec{b}_\ell \\ \vec{a}_2\cdot\vec{b}_1 & \vec{a}_2\cdot\vec{b}_2 & \dots & \vec{a}_2\cdot\vec{b}_\ell \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_m\cdot\vec{b}_1 & \vec{a}_m\cdot\vec{b}_2 & \dots & \vec{a}_m\cdot\vec{b}_\ell \end{bmatrix}$$

$m \times \ell$ matrix

Observation: If A $m\times n$ and B $n \times \ell$, then AB $m \times \ell$

Caution: For AB to make sense, the # cols of A = # rows of B

Ex: $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}$ 3×2 $\quad B = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ 2×3

$$AB = \begin{bmatrix} 1\cdot2+3\cdot1 & 1\cdot4+3\cdot1 & 1\cdot0+3\cdot2 \\ 2\cdot2+1\cdot1 & 2\cdot4+1\cdot1 & 2\cdot0+1\cdot2 \\ 0\cdot2+4\cdot1 & 0\cdot4+4\cdot1 & 0\cdot0+4\cdot2 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 6 \\ 5 & 9 & 2 \\ 4 & 4 & 8 \end{bmatrix} \quad 3\times 3$$

$\neq$!!

Sergio predicts $AB \neq BA$

$$BA = \begin{bmatrix} 2\cdot1+4\cdot2+0\cdot0 & 2\cdot3+4\cdot1+0\cdot4 \\ 1\cdot1+1\cdot2+2\cdot0 & 1\cdot3+1\cdot1+2\cdot4 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 3 & 12 \end{bmatrix} \quad 2\times 2$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$AB = ?$

$$AB = \begin{bmatrix} 1\cdot 0+2\cdot 1 & 1\cdot 1+2\cdot 0 & 1\cdot 1+2\cdot 1 \\ 3\cdot 0+1\cdot 1 & 3\cdot 1+1\cdot 0 & 3\cdot 1+1\cdot 1 \\ 0\cdot 0+2\cdot 1 & 0\cdot 1+2\cdot 0 & 0\cdot 1+2\cdot 1 \\ 1\cdot 0+1\cdot 1 & 1\cdot 1+1\cdot 0 & 1\cdot 1+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Properties of Matrix Multiplication

A $m\times n$ matrix
B, C $n\times l$ matrices
D $l\times p$ matrix

- distributivity $\quad A(B+C) = AB+AC$
  $\qquad\qquad\qquad (B+C)D = BD+CD$
- scalars $\quad (\alpha A)B = \alpha(AB) = A(\alpha B)$
- associativity $\quad A(BD) = (AB)D$

Some properties that don't hold:

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$AB \neq BA$ non-commutativity

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

$AC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

$AB = AC$
but
$B \neq C$

If $AB = AC$, it does <u>NOT</u> mean $B = C$!!!

<u>Def</u>: A <u>square matrix</u> is a matrix w/ the same # rows and # colums.

A $n \times n$ matrix $\qquad$ AA makes sense!

AAA makes sense too!

<u>Def</u>: The k'th power of a square matrix A is

$$A^k = \underbrace{AAA \ldots A}_{k\text{-times}}$$

<u>Ex</u>: $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = AA^3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 0 \end{bmatrix}$

even though

$A \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Identity matrix :

$$I_n = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}^{n} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}\right\} n$$

matrix version of 1

Prop: If A is an $m \times n$ matrix then

$$I_m A = A \quad \text{and} \quad A I_n = A$$

Ex:

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot1+1\cdot0 & 3\cdot0+1\cdot1 \\ 2\cdot1+1\cdot0 & 2\cdot0+1\cdot1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot3+0\cdot2 & 1\cdot1+0\cdot1 \\ 0\cdot3+1\cdot2 & 0\cdot1+1\cdot1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$AI = IA$ $\quad I$ commutes with A.