Last Time

- $i = \sqrt{-1}$
- imaginary numbers ib $\quad$ b real
- complex numbers $a+ib$ $\quad$ a, b real

We did algebra with these! multiply, add, subtract, divide

Ex: $$\frac{2+i}{1-i}\left(\frac{1+i}{1+i}\right) = \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{2+2i+i+i^2}{2}$$

$$= \frac{1+3i}{2} = \boxed{\frac{1}{2}+\frac{3}{2}i}$$

Euler's Formula:

~ exponentials of imaginary numbers ~

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

linking complex #s with geometry and trigonometry!

Special cases:

$$1 = e^0 = e^{i\cdot 0} = \overset{1}{\cos(0)} + i\,\cancel{\sin(0)}^{\,0} = 1 \quad \checkmark$$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

$$e^{i\pi/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$
$$= 0 + i$$

$$\boxed{e^{i\pi/2} = i}$$

Cool application : trig identities

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos\theta\cos\phi + i\sin\theta\cos\phi + i\cos\theta\sin\phi + i^2\sin\theta\sin\phi$$
$$= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = e^{i\theta}e^{i\phi} = e^{i\theta + i\phi}$$
$$= e^{i(\theta+\phi)}$$
$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

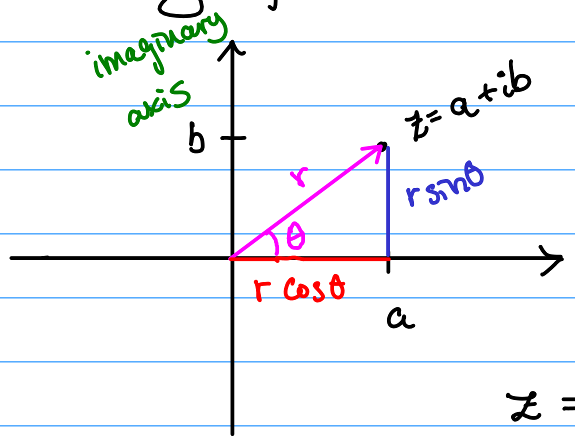
$$\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

Angle addition formulas:

"The Cheer"

- $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
- $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

Geometry of complex numbers



r = magnitude of vector

$= \sqrt{a^2 + b^2}$

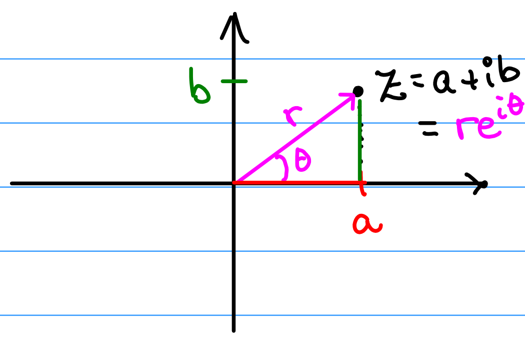
$= |z|$

$$z = a + i b$$
$$= r\cos\theta + i r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= r e^{i\theta}$$

<u>Theorem</u>: Any complex number $z = a+ib$ can be written in the form

$$\boxed{z = re^{i\theta}}$$

where $r = |z| = \sqrt{a^2+b^2}$ and θ is the angle of the associated vector counter-clockwise from the positive x-axis.



r, θ is related to a, b by

$$r = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}(b/a) \quad (\text{for QI})$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

<u>Ex</u>:

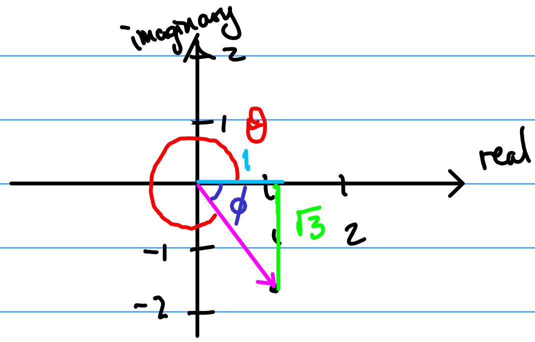
Write $z = 2e^{i\pi/4}$ in $a+ib$ form

$$z = 2e^{i\pi/4} = 2\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= 2\cos\left(\tfrac{\pi}{4}\right) + i\,2\sin\left(\tfrac{\pi}{4}\right)$$

$$= 2\left(\tfrac{\sqrt{2}}{2}\right) + i\,2\left(\tfrac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

<u>Ex</u>: $z = 1 - \sqrt{3}\,i$ write in $z = re^{i\theta}$ form



$$r = \sqrt{(1)^2 + (-\sqrt{3})^2}$$
$$= \sqrt{1+3}$$
$$= \sqrt{4} = 2$$

$$\tan\phi = \sqrt{3}/1$$

$$\tan\phi = \sqrt{3}$$
$$\phi = \pi/3$$
$$\theta = 2\pi - \pi/3 = 5\pi/3$$

$$\boxed{z = 2e^{i\frac{5\pi}{3}}}$$

Braulio: what about $2e^{i\pi/3}$
$\downarrow$
$1 + \sqrt{3}\,i$

Alternative:
$$z = 2e^{-i\pi/3}$$

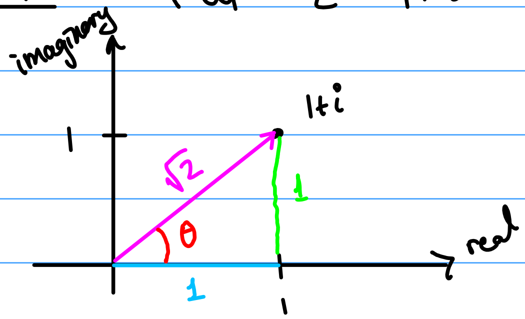
Complex conjugates of exponentials

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta$$

$$\bar{z} = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta}$$

<u>Theorem</u>: The conjugate of $re^{i\theta}$ is $re^{-i\theta}$

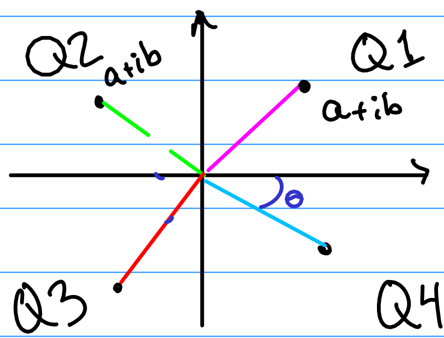
Ex: Put $z = 1+i$ in $re^{i\theta}$ form.



$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\boxed{z = \sqrt{2}\, e^{i\frac{\pi}{4}}}$$

Finding the angle θ:



$$\boxed{\begin{array}{ll} \theta = \tan^{-1}|b/a| & \text{for Q1} \\ \theta = \pi - \tan^{-1}|b/a| & \text{for Q2} \\ \theta = \pi + \tan^{-1}|b/a| & \text{for Q3} \\ \theta = 2\pi - \tan^{-1}|b/a| & \text{for Q4} \end{array}}$$

Ex: Calculate $(1+i)^{100}$

$(1+i)(1+i)(1+i)(1+i)\dots$ this stinks!

Convert to Euler form!

$$1+i = \sqrt{2}\, e^{i\pi/4}$$

$$(1+i)^{100} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{100} = \left(\sqrt{2}\right)^{100}\left(e^{i\pi/4}\right)^{100}$$

$$= 2^{50}\, e^{i\pi 100/4} = 2^{50}\, e^{i\pi 25}$$

$$= 2^{50}\left(\cos(25\pi) + i\sin(25\pi)\right)$$

$$= 2^{50}\left(\cos(\pi) + i\sin(\pi)\right)$$

$$= -2^{50}$$

De Moivre's Formula

$$\boxed{\left(\cos\theta + i\sin(\theta)\right)^n = \cos(n\theta) + i\sin(n\theta)}$$

$$\left(\cos\theta + i\sin\theta\right)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Roots of Unity:

Def: An n^{th} root of unity z is a solution of the equation $z^n = 1$

Use Euler's formula to solve $z^n = 1$

$z = re^{i\theta}$

$$\left(re^{i\theta}\right)^n = 1$$

$$r^n e^{in\theta} = 1$$

$$r^n\cos(n\theta) + i\, r^n\sin(n\theta) = 1.$$

$r^n\cos(n\theta) = 1$ $\qquad$ $r^n\sin(n\theta) = 0$

$r = 1$, $\quad n\theta = 2\pi k$

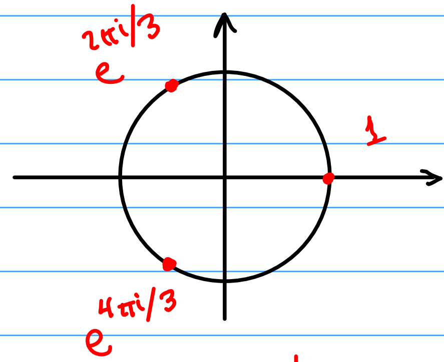
Theorem: The n^{th} roots of unity are

$0 \le k < n$

$$z = e^{2\pi i k/n} \text{ for some integer } k$$

$$e^{i\theta} = e^{i(\theta + 2\pi)}$$

Ex: Third roots of unity

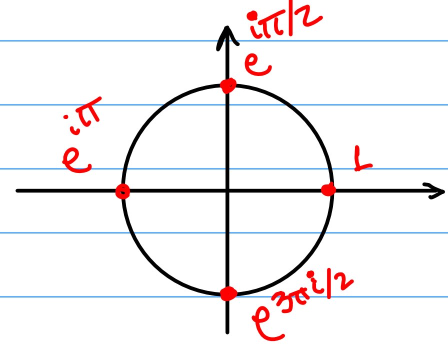


$$z^3 = 1$$

$$1, \quad e^{2\pi i/3}, \quad e^{2\pi i\, 2/3}$$

$$\boxed{e^{2\pi i k/n}}$$

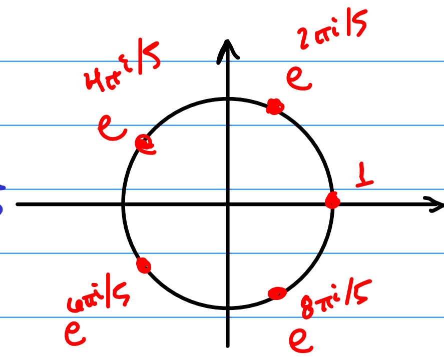
Ex: Fourth roots of unity



$$z^4 = 1$$

$$1, \quad e^{2\pi i/4}, \quad e^{2\pi i\, 2/4}, \quad e^{2\pi i\, 3/4},$$

Ex: Fifth roots of unity



$$1, \quad e^{2\pi i/5}, \quad e^{2\pi i\, 2/5}, \quad e^{2\pi i\, 3/5}, \quad e^{2\pi i\, 4/5}$$

What about more general roots?

Solve : $z^3 = -1$

Find a single solution of this
Every other solution is the same up to a multiple by a root of unity!

$$z = -1, \quad z = -1\, e^{2\pi i/3}, \quad z = -1\, e^{4\pi i/3}$$

Ex: Solving $z^4 = 16$

$z = 2$ works!

$$2, \quad 2e^{2\pi i/4}, \quad 2e^{2\pi i\, 2/4}, \quad 2e^{2\pi i\, 3/4}$$

$$2, \quad 2i, \quad -2, \quad -2i$$

Fundamental Theorem of Algebra:

A polynomial equation $p(z) = 0$ has the same number of solutions (counting multiplicity) as the degree of $p(z)$.