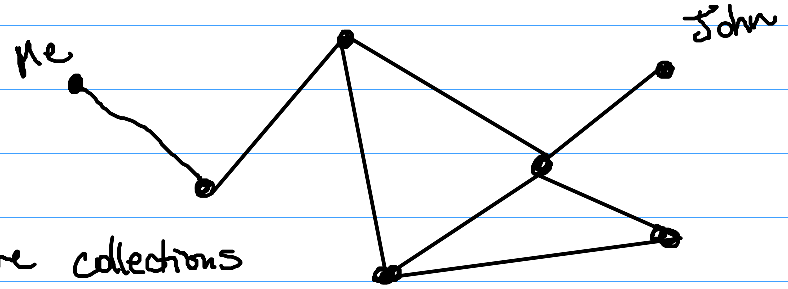
Applications of Eigenvectors and Eigenvalues to Graphs

Recall: If A $n \times n$ matrix, an eigenvector with eigenvalue λ is a nonzero vector $\vec{v}$ s.t

$$A\vec{v} = \lambda \vec{v}$$

Graphs (aka networks)

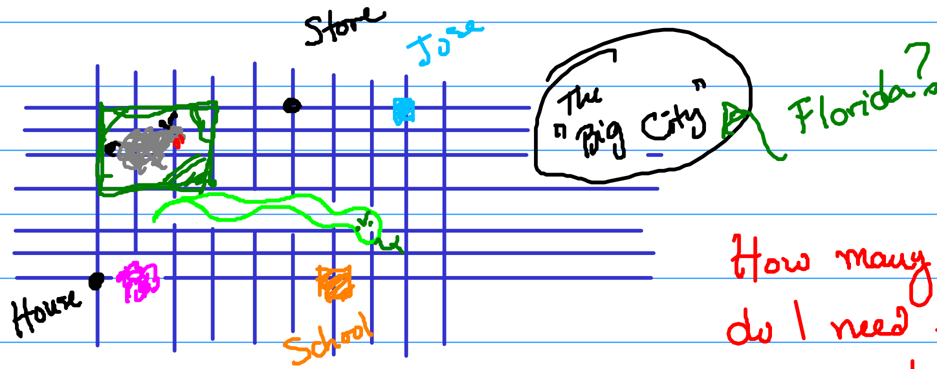


Graphs are collections of little round circles called vertices connected by curve segments called edges.

Where do graphs show up in real life?

1. Highway system — vertices = cities, edges = highways
2. River systems
3. The internet, or any computer network
4. Airplanes — nodes = cities, edges = is there a flight?
5. Math 180 — graphs coming from the possible moves in a game (Micro Robots, Chess)
6. Radio/TV network.
7. Utility systems in cities
8. Electrical networks, circuits, etc.
9. Traffic light system.
10. Walking/bike paths
11. Veins in body / neurons in brain

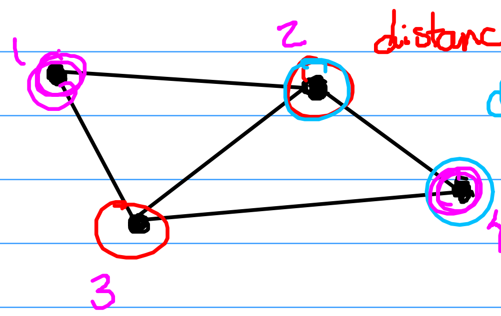
Important question: Suppose I have a graph. How far apart can two vertices be?



How many blocks do I need to walk from my house to the store?

Def: The diameter of a graph is the farthest distance between two vertices.

Ex:



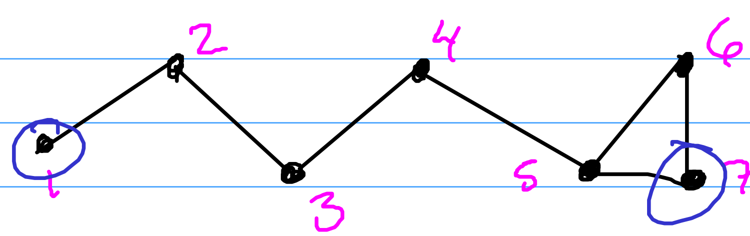
distance 1 away

distance 1 away

distance 2 away

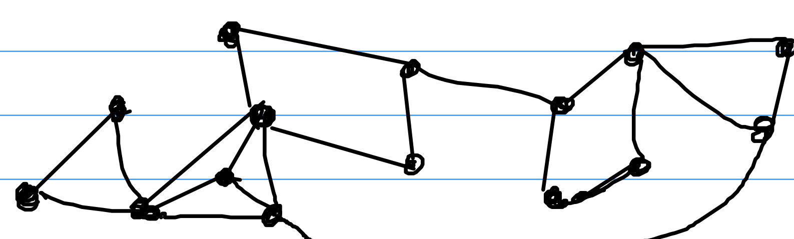
Diameter is 2.

Ex:



diameter is 5

Ex:



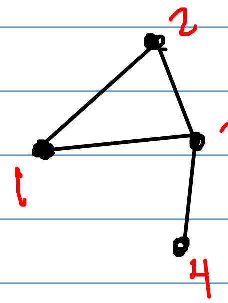
diameter ??

Idea: relate to linear algebra!

How do we go from a graph to linear algebra?

<u>Adjacency matrix!</u>

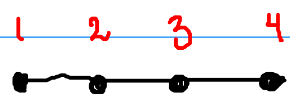
<u>Ex</u>:



$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Symmetric matrix

<u>Ex</u>:



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = A$$

What are the eigenvalues of A?

$\det(A - \lambda I) = 0$ solve for λ.

$$\det \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$$= -\lambda\left(-\lambda(\lambda^2-1)+\lambda\right) - (\lambda^2-1)$$

$$= -\lambda^2(2-\lambda^2) - \lambda^2 + 1$$

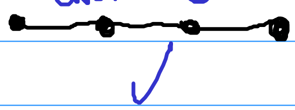
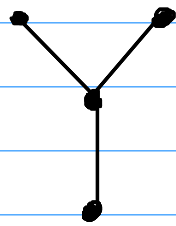
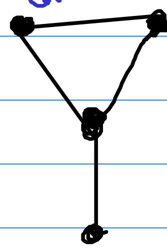
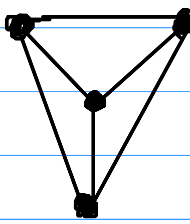
$$= \lambda^4 - 3\lambda^2 + 1 = 0$$

$$(\lambda^2)^2 - 3\lambda^2 + 1 = 0$$

$$\lambda^2 = \frac{3 \pm \sqrt{9-4}}{2}, \quad \lambda^2 = \frac{3+\sqrt{5}}{2} \text{ or } \frac{3-\sqrt{5}}{2}$$

Eigenvalues: $\sqrt{\frac{3+\sqrt{5}}{2}}, \sqrt{\frac{3-\sqrt{5}}{2}}, -\sqrt{\frac{3+\sqrt{5}}{2}}, -\sqrt{\frac{3-\sqrt{5}}{2}}$

<u>Experiment</u>: For each of the following graphs calculate the determinant and eigenvalues of the adjacency matrix.

diam = 3	diam = 2	diam = 2	diam = 1
			
Det = 1	Det = 0	Det = 1	Det = -3
Eigenvalues	Eigenvalues	Eigenvalues	Eigenvalues
0.618, −0.618, 1.618, −1.618	0, 0, 1.732, −1.732	−1, 0.311, −1.481, 2.170	−1, −1, −1, 3

Observations:
- sum of eigenvalues = 0
- product of eigenvalues = determinant
- smaller diameter ⇒ more repeated eigenvalues!

Cool idea: bound the diameter in terms of the # of unique eigenvalues.