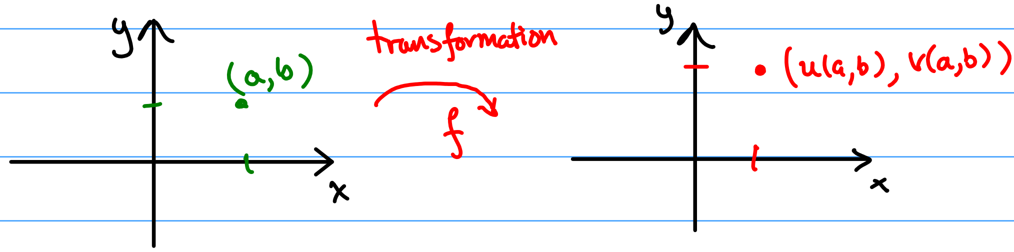
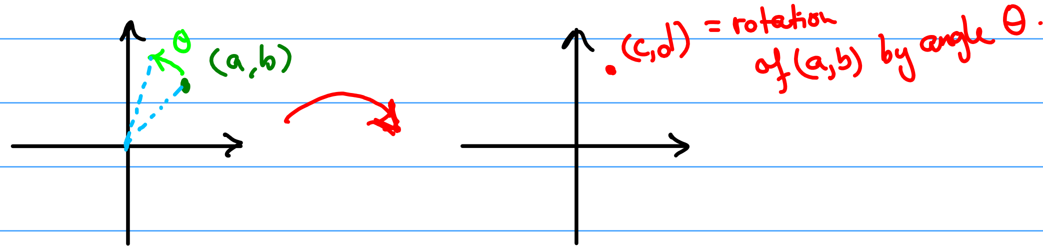
Matrix Multiplication and Transformations

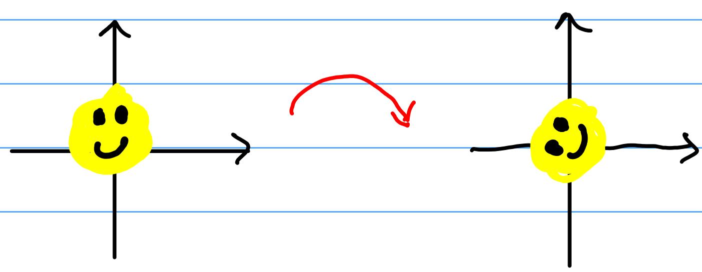
Transformations of the plane:



Examples of transformations:

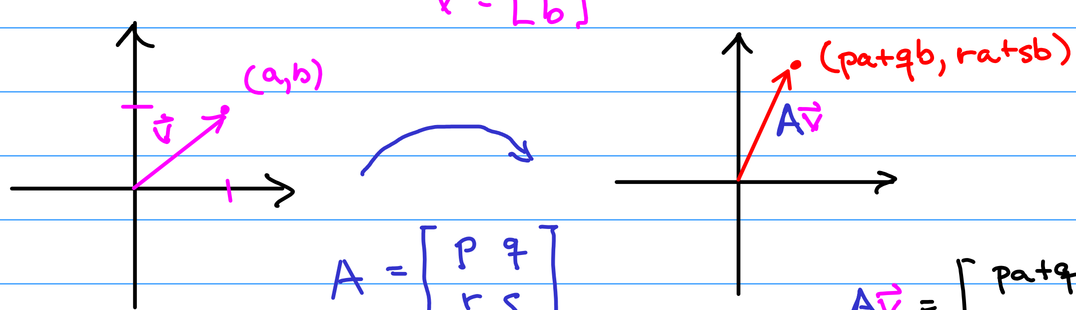
Rotation:





Any 2x2 matrix defines a transformation of the plane!

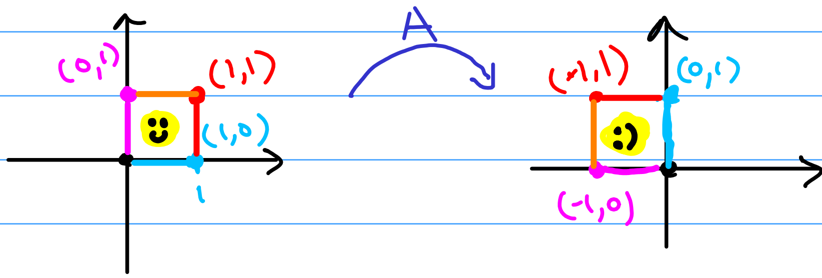
$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$



$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} pa+qb \\ ra+sb \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

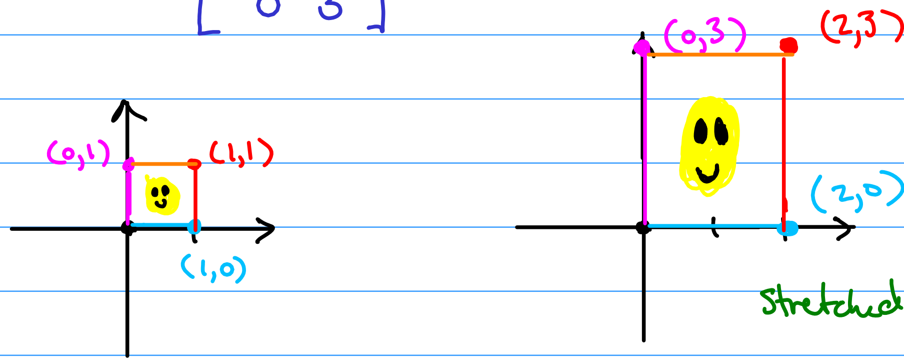


rotation counter-clockwise by $90°$

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



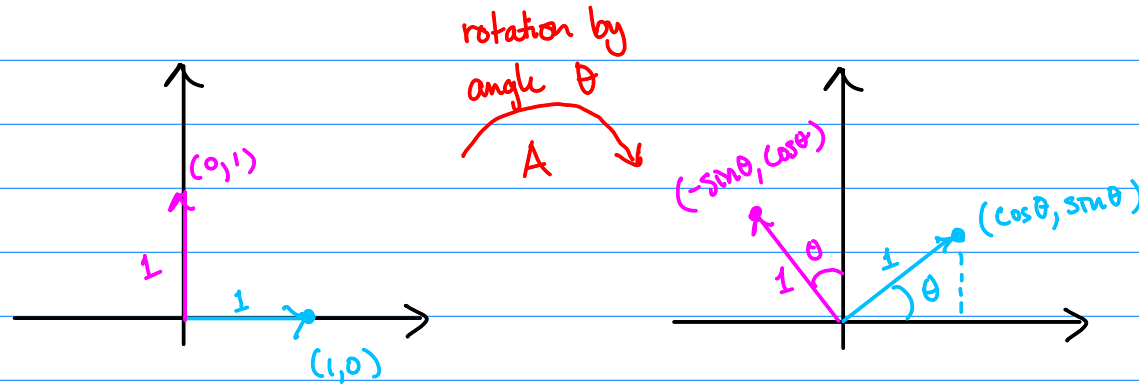
Stretched 2x in x-dir. and 3x in y-dir

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Remark: transformations could stretch or squeeze!

Quest: Can any rotation be represented as a matrix?

Answer: Yes!!



$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

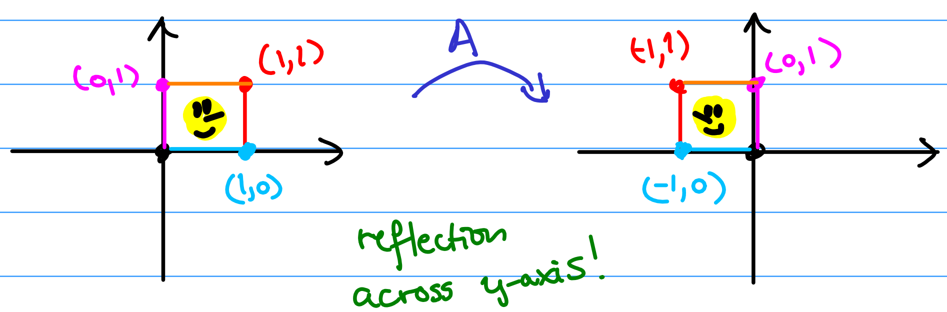
$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

<u>Theorem</u>: The matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ defines a counter-clockwise rotation by θ radians.

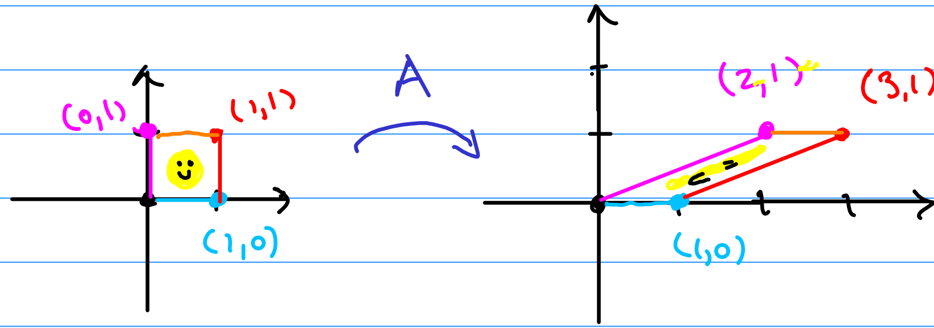
<u>Def</u>: The matrices from the previous theorem are called <u>rotation matrices</u>.

<u>Ex</u>: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



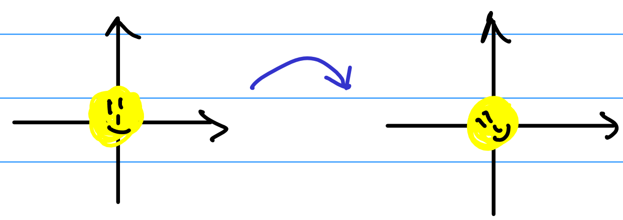
reflection across y-axis!

Ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

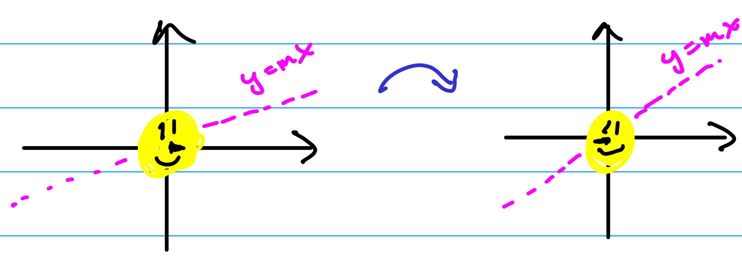


Four basic types of transformations of the plane:

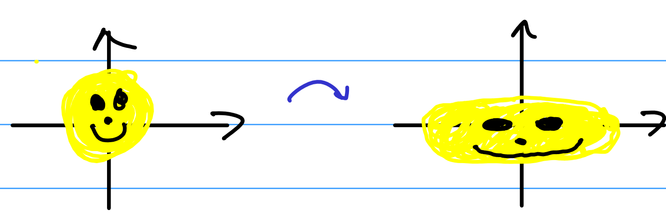
Rotations: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Rotation counter-clockwise by θ radians



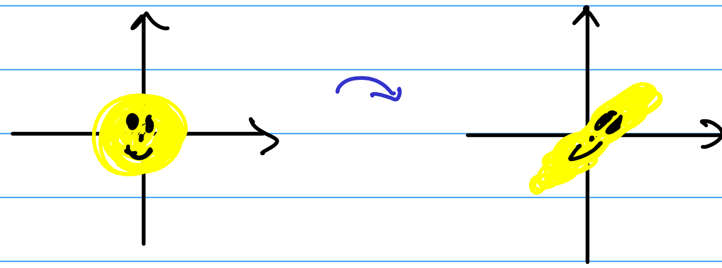
Reflections: $\begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{bmatrix}$ Reflection across $y=mx$



Shrinking/Stretching: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $a,b > 0$



Shear: $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$



Products of Matrices and Transformations:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a\cos\theta & -b\sin\theta \\ a\sin\theta & b\cos\theta \end{bmatrix}$$

The transformation defined by the product is: