MATH 107 Lecture 11

Matrix Multiplication

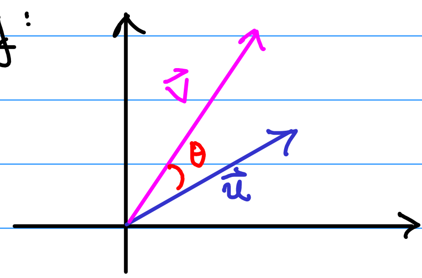
Def: Let $\vec{u} = [u_1\ u_2\ \dots\ u_n]$ and $\vec{v} = [v_1 v_2 \dots v_n]$. Then the dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Ex: $\vec{u} = [1\ 2\ 3]$, $\vec{v} = [4\ 0\ {-1}]$

$$\vec{u} \cdot \vec{v} = 1 \cdot 4 + 2 \cdot 0 + 3 \cdot (-1) = 1$$

Geometric meaning:



$\vec{u} = [u_1\ \ u_2]$

$\vec{v} = [v_1\ \ v_2]$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$

Dot product–angle relationship: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$

Ex: What is the angle between $\vec{u} = [1\ 0]$ and $\vec{v} = [1\ 1]$?

$$\|\vec{u}\| = \sqrt{1^2 + 0^2} = 1, \quad \|\vec{v}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\vec{u} \cdot \vec{v} = 1 \cdot 1 + 0 \cdot 1 = 1$$

$$1 \cdot \sqrt{2} \cdot \cos\theta = 1$$

$$\cos\theta = 1/\sqrt{2} = \sqrt{2}/2$$

$$\theta = \cos^{-1}(\sqrt{2}/2) = \pi/4$$

Picture:

Definition: Let A be an $m \times n$ matrix and $\vec{v}$ be a $n \times 1$ column vector. The **matrix product** of A and $\vec{v}$ is

$$A\vec{v} = \begin{bmatrix} \vec{a}_1 \cdot \vec{v} \\ \vec{a}_2 \cdot \vec{v} \\ \vdots \\ \vec{a}_m \vec{v} \end{bmatrix}$$

where $\vec{a}_j = j^{th}$ row vector of A for $1 \le j \le m$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} [1\ 2] \cdot [-1\ 2] \\ [0\ 1] \cdot [-1\ 2] \\ [3\ -1] \cdot [-1\ 2] \end{bmatrix} = \begin{bmatrix} 1\cdot(-1) + 2\cdot 2 \\ 0\cdot(-1) + 1\cdot 2 \\ 3\cdot(-1) + (-1)\cdot 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

General observation: If A is $m \times n$ and $\vec{v}$ is $n \times 1$, then $A\vec{v}$ is $m \times 1$.

Ex: $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\quad 3\times 3 \qquad\qquad 3 \times 1$

$A\vec{v} = 3 \times 1$ vector!

$$A\vec{v} = \begin{bmatrix} [3\,0\,1] \cdot [1\ 2\ 3] \\ [0\,1\,0] \cdot [1\ 2\ 3] \\ [-1\ 2\ -1] \cdot [1\ 2\ 3] \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 0\cdot 2 + 1\cdot 3 \\ 0\cdot 1 + 1\cdot 2 + 0\cdot 3 \\ (-1)\cdot 1 + 2\cdot 2 + (-1)\cdot 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

Matrix product : $\left.\begin{array}{l} A \quad m\times n \text{ matrix} \\ B \quad n\times \ell \text{ matrix} \end{array}\right\}$

$$AB = \left[A\vec{b}_1 \ A\vec{b}_2 \ \cdots \ A\vec{b}_\ell \right] , \quad \vec{b}_j = j^{th} \text{ column vector of } B$$
$$\text{for } 1 \le j \le \ell$$

Ex :
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$AB = \left[A\begin{bmatrix} -1 \\ 2 \end{bmatrix} \ A\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} [1\,2]\cdot[-1\,2] & [1\,2]\cdot[0\,1] \\ [3\,4]\cdot[-1\,2] & [3\,4]\cdot[0\,1] \\ [5\,6]\cdot[-1\,2] & [5\,6]\cdot[0\,1] \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 7 & 6 \end{bmatrix}$$

General observation: If A is $m\times n$ and B is $n\times \ell$, then AB is $m\times \ell$.

Ex :
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$2\times 3 \qquad\qquad\qquad 3\times 3$$

$$AB = 2\times 3$$

$$AB = \begin{bmatrix} [1\,0\,1]\cdot[1\,0\,0] & [1\,0\,1]\cdot[0\,1\,0] & [1\,0\,1]\cdot[0\,0\,1] \\ [2\,{-1}\,0]\cdot[1\,0\,0] & [2\,{-1}\,0]\cdot[0\,1\,0] & [2\,{-1}\,0]\cdot[0\,0\,1] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

Definition: An $n \times n$ matrix of the form

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is called the <u>$n \times n$ identity matrix</u>.

Theorem: Let A be an $m \times n$ matrix, Then

$$I_m A = A \qquad A I_n = A$$

Warning: dimensions must match up!

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

AB does not make any sense!

$\underset{3\times 2}{B}\ \underset{2\times 2}{A} = 3 \times 2$ matrix

so the dimensions match up!

MATLAB:

By default * performs matrix multiplication on matrices!

Algebraic properties of matrix multiplication:

- Associativity $\quad A \; m\times n, \; B \; n\times l, \; C \; l\times p$

$$(AB)C = A(BC)$$

- Distributivity $\quad A \; m\times n, \; B \; n\times l, \; C \; n\times l$

$$A(B+C) = AB + AC$$

Matrix multiplication is NOT commutative!

$A \; n\times n, \; B \; n\times n$

$AB \neq BA$

Powers of Matrices: $\quad A =$ $n\times n$ square matrix

$A^2 = AA$
$A^3 = AAA$
$A^4 = AAAA$
$\vdots$

Ex: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B^3 \quad \text{for } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 = B^3$ but $A \neq B$

Matrix powers and the Fibonacci sequence:

Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A^n \vec{v}$ are related to the Fibonacci sequence!

$n=1$: $\quad A\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$n=2$: $\quad A^2\vec{v} = A(A\vec{v}) = A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$n=3$: $\quad A^3\vec{v} = A(A^2\vec{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$n=4$ $\quad A^4\vec{v} = A(A^3\vec{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Theorem: The second entry of $A^n\vec{v}$ gives the Fibonacci seq.