Integrals of Trig Functions

As we saw with trig. sub we end up with integrals of interesting expressions of trig. functions.

Powers of $\sin(x)$ and $\cos(x)$: $\int \sin^m(x)\cos^n(x)\,dx$

Powers of $\sec(x)$ and $\tan(x)$: $\int \sec^m(x)\tan^n(x)\,dx$

Ex: $\int \cos^3(x)\,dx = \int (1-\sin^2(x))\cos(x)\,dx$

$u=\sin(x)$, $du=\cos(x)dx$ $\quad = \int 1-u^2\,du$

$= u-\frac{1}{3}u^3 + C = \sin(x) - \frac{1}{3}\sin^3(x) + C$

Ex: $\int \sin^5(x)\,dx = \int (1-\cos^2(x))^2\sin(x)\,dx$

$u=\cos(x)$, $du=-\sin(x)$ $\quad = -\int (1-u^2)^2\,du = -\frac{1}{5}u^5 + \frac{2}{3}u^3 - u + C$

$= -\frac{1}{5}\cos^5(x) + \frac{2}{3}\cos^3(x) - \cos(x) + C$

Ex: $\int \sin^4(x)\,dx = \int \left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)^2 dx$

$= \int \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\,dx$

$= \int \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{8} - \frac{1}{8}\cos(4x)\,dx$

$= \frac{1}{4}x - \frac{1}{4}\sin(2x) + \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

Ex: $\int \tan^6(x)\sec^4(x)\,dx = \int \tan^6(x)(1+\tan^2(x))\sec^2(x)\,dx$

$u = \tan(x)$, $du = \sec^2(x)dx$ $\quad = \int u^6(1+u^2)\,du = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C$

$= \frac{1}{7}\tan^7(x) + \frac{1}{9}\tan^9(x) + C$

Ex: $\int \tan^5\theta \sec^7\theta\, d\theta = \int \tan^4\theta \sec^6\theta \sec\theta\tan\theta\, d\theta$

$u = \sec\theta$
$du = \sec\theta\tan\theta\,d\theta$

$= \int (\sec^2\theta - 1)^2 \sec^6\theta \sec\theta\tan\theta\,d\theta$

$= \int (u^2-1)^2 u^6\,du = \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C$

$= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C$

Unfortunately, things get weird when m and n are not both even or odd.

Ex: $\int \sec(x)\,dx = \int \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x)+\tan(x)}\,dx$

$u = \sec(x) + \tan(x)$
$du = \sec(x)\tan(x) + \sec^2(x)$

$= \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|\sec(x) + \tan(x)| + C.$

Integrals of Rational Functions

Basic Forms: $\int \frac{1}{x+a}\,dx = \ln|x+a| + C$

$$\int \frac{1}{x^2+bx+c}\,dx = \frac{1}{\sqrt{c-b^2/4}} \tan^{-1}\left(\frac{x+b/2}{\sqrt{c-b^2/4}}\right) + \text{CONST}$$

$$\int \frac{2x+b}{x^2+bx+c}\,dx = \ln|x^2+bx+c| + \text{CONST}$$

Strategy: decompose rational functions into irreducible atoms

$$\frac{1}{x}, \quad \frac{1}{x-3}, \quad \frac{1}{(x+2)^3}, \quad \frac{2x-1}{x^2+4}, \quad \frac{x-5}{(x^2+4)^2}$$

Ex: $\displaystyle\int \frac{x-3}{x(x+2)}\,dx = \ ?$

$$\frac{x-3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow x-3 = A(x+2) + Bx$$

$@x=0: \quad -3 = 2A \to A = -\frac{3}{2}$

$@x=-2: \quad -5 = -2B \to B = \frac{5}{2}$

$$\int \frac{x-3}{x(x+2)}\,dx = \int \frac{-3/2}{x} + \frac{5/2}{x+2}\,dx$$

$$= -\frac{3}{2}\ln|x| + \frac{5}{2}\ln|x+2| + C$$