Sequences and Series

Examples of sequences

- $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \dots\}$ $\quad a_n = 2^{-n}$
- $\{1, \frac{-1}{3}, \frac{1}{5}, -\frac{1}{9}, \frac{1}{11}, \dots\}$ $\quad a_n = \frac{(-1)^{n+1}}{2n-1}$
- $\{\sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \dots\}$ $\quad a_n = \sqrt{n+2}$

Fibonnaci sequence

$$f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \quad f_5 = 5$$

$$f_n = f_{n-1} + f_{n-2}$$

Examples of Series

Series are <u>sums</u> of sequences

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots \qquad \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \dots \qquad \sum_{n=1}^{\infty} \sqrt{n+2}$$

The <u>limit</u> of a sequence $\{a_n\}$ is a number L with the following property

for any $\varepsilon > 0$, we can choose N such that

$$n \geq N \Rightarrow |a_n - L| < \varepsilon$$

<u>Notation</u>: $\lim_{n \to \infty} a_n = L$

If limit exists, CONVERGENT

If no limit exists, DIVERGENT

Ex: $\left\{ 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{6}}, \dots \right\}$

This sequence is $a_n = \frac{1}{\sqrt{n}}$.

Let's show $\lim_{n\to\infty} a_n = 0$.

Let $\varepsilon > 0$. NTS we can find $N > 0$ s.t.

$$n \geq N \Rightarrow |a_n - 0| < \varepsilon.$$

$$|a_n - 0| = \left|\frac{1}{\sqrt{n}}\right| < \varepsilon \Leftrightarrow \frac{1}{n} < \varepsilon^2 \Leftrightarrow n > \frac{1}{\varepsilon^2}.$$

Choose N to be an integer greater than $\frac{1}{\varepsilon^2}$.

$$n \geq N \Rightarrow n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow |a_n - 0| < \varepsilon.$$

Properties of Limits: Assume $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_n = M$

- $\lim_{n\to\infty} c a_n = cL$
- $\lim_{n\to\infty} (a_n + b_n) = L + M$
- $\lim_{n\to\infty} (a_n - b_n) = L - M$
- $\lim_{n\to\infty} a_n b_n = LM$
- $\lim_{n\to\infty} a_n / b_n = \frac{L}{M}$ $(M \neq 0)$
- $\lim_{n\to\infty} (a_n)^r = L^r$, $a_n \geq 0$, $r > 0$

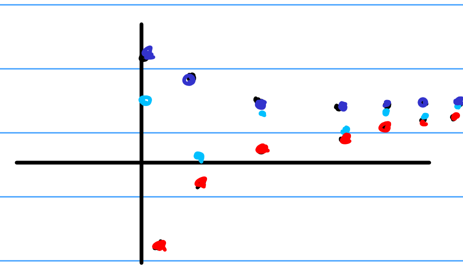
Squeeze Theorem:

If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L$

and $\lim_{n\to\infty} c_n = L$

Then

$$\lim_{n\to\infty} b_n = L$$



Ex: $\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} 1 = 1$

$$\lim_{n\to\infty} \frac{1}{n}+1 = 1$$

$$\lim_{n\to\infty} \frac{n}{1+n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+1} = \frac{1}{1} = 1$$

Theorem: If $f(x)$ is a function and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} f(n) = L$.

Ex: $\lim_{n\to\infty} \frac{\ln(n)}{n} = ?$ $\quad \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$

Therefore $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0.$

Ex: $\lim_{n\to\infty} \frac{n}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+\frac{1}{n}}$ ← goes to 0+

$= \infty.$ DIVERGENT

Ex: $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ by Squeeze!

Def: A sequence $\{a_n\}$ is bounded below if there is a k with $k \le a_n$ for all n.

It is bounded above if there is a K with $a_n \le K$ $\forall n$.

If it's bounded below and above it is called bounded.

It is called monotone increasing if

$$a_{n+1} \ge a_n \quad \text{for all } n$$

and monotone decreasing if $a_{n+1} \le a_n$ for all n

Monotone Convergence Theorem :

A bounded monotone sequence converges

Def: The limit of a series is the limit of the sequence of partial sums.

$$S_n = a_1 + a_2 + \dots + a_n .$$

If $\lim_{n \to \infty} S_n = L$ then L is the sum of the series and the series is convergent. Otherwise divergent.

Notation: $\sum_{n=1}^{\infty} a_n = L$

Most important series :

Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$

Converges $\Leftrightarrow$ $|r| < 1$.

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1}$$