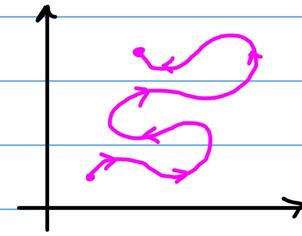
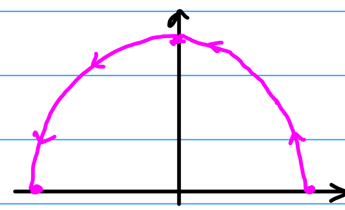
Parametric Equations

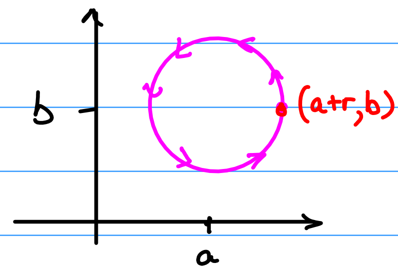
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t_0 \le t \le t_1$$



Ex: $$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}, \quad 0 \le t \le \pi$$

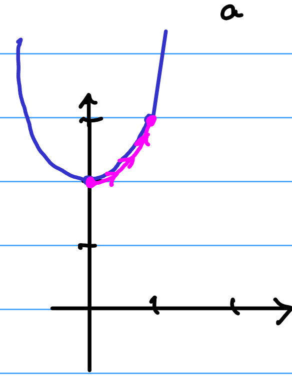


Ex: $$\begin{cases} x = r\cos t + a \\ y = r\sin t + b \end{cases}, \quad 0 \le t \le 2\pi$$



Ex: $$\begin{cases} x = \sqrt{t} \\ y = t+2 \end{cases}, \quad 0 \le t \le 1$$

$y = x^2 + 2$



Calculus with parametric equations

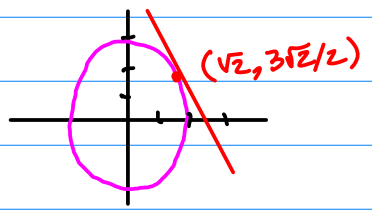
Slope of tangent line to curve?

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt}$$

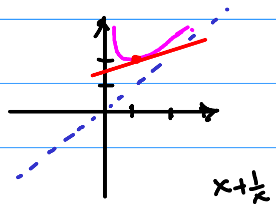
Ex: Slope of tangent line to the curve $\begin{cases} x = 2\cos(t) \\ y = 3\sin(t) \end{cases}$, $0 \le t \le 2\pi$ at $t = \frac{\pi}{4}$.



$$\frac{dx}{dt} = -2\sin(t), \quad \frac{dy}{dt} = 3\cos(t)$$

$$\frac{dy}{dx} = \frac{3\cos(t)}{-2\sin(t)} \quad @\, t = \frac{\pi}{4} : -\frac{3}{2}$$

Ex: Find the slope of the tangent line at the point $(1,2)$ $\begin{cases} x = e^t \\ y = e^t + e^{-t} \end{cases}$, $-1 \le t \le 1$



$$\frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t} = 1 - e^{-2t}$$

$$@\, t=0 : 0$$

Lengths of curves:

$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{t_0}^{t_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$\boxed{s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.}$$

Ex: Find the length of the curve $\begin{cases} x = t^2 \\ y = 2t \end{cases}$, $0 \le t \le 2$

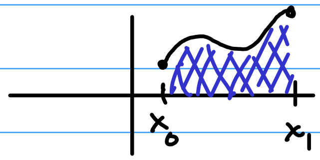
$$s = \int_0^2 \sqrt{(2t)^2 + (2)^2}\, dt = 2\int_0^2 \sqrt{1+t^2}\, dt$$

$$= \left(t\sqrt{1+t^2} + \ln\left|\sqrt{1+t^2} + t\right|\right)\Big|_0^2$$

$$= 2\sqrt{5} + \ln(\sqrt{5}+2)$$

Area under the curve:

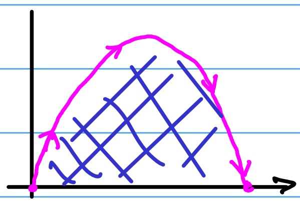
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t_0 \le t \le t_1$$



$$A = \int_{x_0}^{x_1} y \, dx = \int_{t_0}^{t_1} y \frac{dx}{dt} dt$$

$$\boxed{A = \int_{t_0}^{t_1} g(t) f'(t) dt}$$

Ex: Area under the curve $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$, $0 \le t \le 2\pi$



$$\int_0^{2\pi} (1-\cos t)(t-\sin t)' dt$$

$$= \int_0^{2\pi} (1-\cos t)^2 dt = \int_0^{2\pi} 1 - 2\cos t + \cos^2 t \, dt$$

$$= \int_0^{2\pi} \frac{3}{2} + \frac{1}{2}\cos(2t) - 2\cos t \, dt = 3\pi$$

Surface Area

Rotated around x-axis: $\int 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$