Probability Density Functions

A probability density function on the real line is a function $f(x)$ which is never negative and satisfies

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Then the probability x is between a and b is

$$P(a \le x \le b) = \int_a^b f(x)\,dx.$$

Note: $P(x=a) = 0$. How come?

cte^{-ct}

Important special cases:

- Exponential density $f(x) = \begin{cases} \frac{1}{\mu} e^{-t/\mu}, & t \ge 0 \\ 0, & t < 0 \end{cases}$, μ mean

- Gaussian/normal density $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, μ mean, σ std. dev.

Why the normal distribution?

Flipping coins: H → WIN 1 dollar, T → LOSE 1 dollar. x = dollars won/lost after $2n$ flips

$$P(x=2k) = \binom{2n}{n+k} \Big/ 2^{2n}$$

$$= \frac{(2n)!/2^{2n}}{(n+k)!\,(n-k)!}$$

Sterling's Approx: $m! \approx \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$, $n \gg 1$

$$P(X=2k) \approx \frac{\sqrt{4\pi n}/2^{2n}}{2\pi\sqrt{(n-k)(n+k)}} \left(\frac{2n}{e}\right)^{2n} \left(\frac{e}{n-k}\right)^{n-k} \left(\frac{e}{n+k}\right)^{n+k}$$

$$= \sqrt{\frac{n}{\pi(n^2-k^2)}} \left(\frac{n^2}{n^2-k^2}\right)^{n} \left(\frac{n-k}{n+k}\right)^{k}$$

$$= \frac{1}{\sqrt{\pi}}\, e^{\left[(2n+\frac{1}{2})\ln(n) - (n+\frac{1}{2}-k)\ln(n-k) - (n+\frac{1}{2}+k)\ln(n+k)\right]}$$

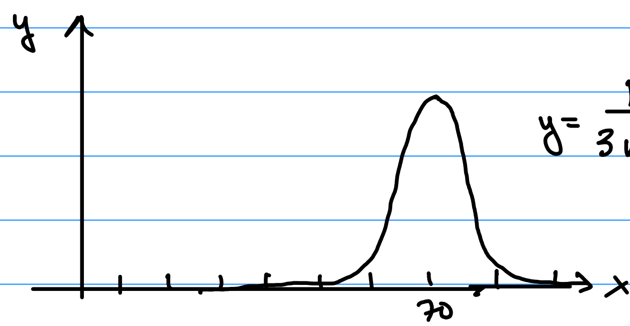
$$\approx \frac{1}{\sqrt{\pi n}}\, e^{-k^2/n}$$

!!!

Same as the normal distribution with $\mu = 0$, $\sigma = \sqrt{n/2}$.

Lots of real-world processes naturally resemble normal density.

<u>Ex</u>: The height of males in the US is normally distributed w/ mean 70 in and std. dev. 3 in.



$$y = \frac{1}{3\sqrt{2\pi}}\, e^{-(x-70)^2/18}$$

Probability of height $\geq$ 6 ft?

$$P(x \geq 72) = \int_{72}^{\infty} \frac{1}{3\sqrt{2\pi}}\, e^{-(x-70)^2/18}\, dx$$

Okay to assume $x \geq 96$ (8 ft tall) negligible

$$\int_{72}^{96} \frac{1}{3\sqrt{2\pi}}\, e^{-(x-70)^2/18}\, dx$$

Approx. w/ calculator or Midpoint Rule.

$$P(x \geq 72) \approx 0.25249 \qquad (\approx 25\%)$$

Average / Expected Value

The <u>average</u> or <u>expected value</u> of a function $g(x)$ of a random value x w/ dist $f(x)$ is

$$\bar{g} = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

<u>Ex</u>: If $f(x) = \frac{1}{\mu} e^{-x/\mu}$, $x \geq 0$ (0 otherwise)

$$\bar{x} = \int_0^{\infty} x f(x)\, dx = \mu.$$

<u>Ex</u>: If $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$,

$$\bar{x} = \int_{-\infty}^{\infty} x f(x)\, dx = \mu.$$

<u>Ex</u>: If $f(x) = \frac{1/\pi}{1+x^2}$, then

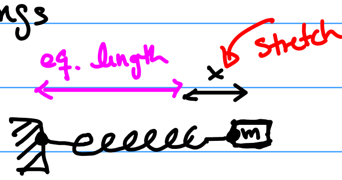
$$\bar{x} = \int_{-\infty}^{\infty} x \frac{1}{\pi} \frac{1}{1+x^2}\, dx = 0.$$

Differential Equations

Bacteria growth model (basic) $\qquad \dfrac{dP}{dt} = kP$ — growth proportional to current pop.

Bacteria growth model (advanced) $\qquad \dfrac{dP}{dt} = kP(C-P)$ — limit introduced by environment

Springs



Hooke: $F = -kx$

Newton: $F = m\dfrac{d^2x}{dt^2}$

$$m\frac{d^2x}{dt^2} + kx = 0$$

In general: a differential equation relates a function and its derivatives

A <u>solution</u> is a function satisfying the equation

<u>Ex</u>: $y = \sin(x)$ is a solution of $y'' + y = 0$.

How do we solve differential equations?

Classification:

- separable ☆
- exact
- homogeneous nonlinear
- linear
- wilder stuff