Integration by Parts

Integral version of the product rule:

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow \quad f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$$

Final version

$$\boxed{\int_a^b \underbrace{f(x)}_{u}\underbrace{g'(x)dx}_{dv} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v}\Big|_a^b - \int_a^b \underbrace{g(x)}_{v}\underbrace{f'(x)\,dx}_{du}}$$

Same idea with indefinite integrals!

Ex: $\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx = x\sin(x) + \cos(x) + C$

$u = x,\quad dv = \cos(x)dx$
$du = dx,\quad v = \sin(x)$

Ex: $\int_1^e \ln(x)\,dx = x\ln x\Big|_1^e - \int_1^e x\frac{1}{x}dx = (x\ln x - x)\Big|_1^e = 1$

$u = \ln x,\quad dv = dx$
$du = \frac{1}{x}dx \quad v = x$

Ex: $\int x^2 e^x\,dx = x^2e^x - \int e^x 2x\,dx$

$u = x^2 \quad dv = e^x dx$
$du = 2xd \quad v = e^x$

$= x^2e^x - 2\int xe^x\,dx$

Again!

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$

$u = x,\ dv = e^x dx$

$du = dx,\ v = e^x$

Final: $x^2e^x - 2xe^x + 2e^x + C$

Ex: $\int e^x \sin(x)\, dx = -e^x\cos(x) + \int e^x\cos(x) dx$

$u = e^x,\ dv = \sin(x)dx$

$du = e^x dx,\ v = -\cos(x)$

$\int e^x \cos(x)\, dx = e^x\sin(x) - \int e^x \sin(x)\, dx$

$u = e^x \quad dv = \cos x\, dx$

$du = e^x dx \quad v = \sin(x)$

Therefore $\int e^x \sin(x) dx = -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x) dx$

$2\int e^x \sin(x) dx = -e^x\cos(x) + e^x\sin(x)$

$\int e^x \sin(x) dx = \frac{1}{2}\left(e^x\sin(x) - e^x\cos(x)\right) + C$

Trig Substitution

Substituting $x = \cos\theta$ or $x = \tan\theta$ can be a powerful technique!

Ex: $\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\cos^2\theta}\,(-\sin\theta)\, d\theta$

$x = \cos\theta,\ dx = -\sin\theta\, d\theta$

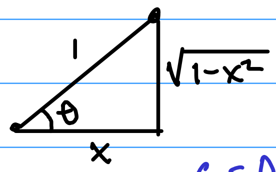
$$= \int \sqrt{\sin^2\theta}\,(-\sin\theta)\,d\theta = -\int \sin^2\theta\,d\theta$$

$$= -\int \tfrac{1}{2} - \tfrac{1}{2}\sin(2\theta)\,d\theta$$

$$= -\tfrac{1}{2}\theta + \tfrac{1}{4}\cos(2\theta) + C$$

Now write in terms of x!!

Pythagorean Theorem:



$\cos\theta = x$

$\sin\theta = \sqrt{1-x^2}$

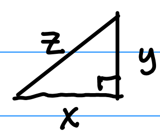
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= x^2 - (1-x^2)$$
$$= 2x^2 - 1$$

$$\theta = \cos^{-1}(x)$$

$$\int \sqrt{1-x^2}\,dx = -\tfrac{1}{2}\cos^{-1}(x) + \tfrac{1}{4}(2x^2-1) + C$$

We will use many trig. identities!

PYTHAGOREAN IDENTITIES:



$x^2+y^2=z^2$

$\Downarrow$

$\cos^2\theta + \sin^2\theta = 1$

$1+\tan^2\theta = \sec^2\theta$ $\qquad$ $\cot^2\theta + 1 = \csc^2\theta$

ANGLE ADDITION IDENTITIES

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)$$

$$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

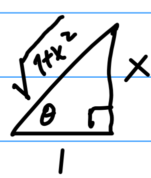
Ex:

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$= \ln|\sec\theta + \tan\theta| + C$$



$\sec\theta = \sqrt{1+x^2}$, $\tan\theta = x$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln\left|\sqrt{1+x^2} + x\right| + C$$

Ex:

$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x-1)^2}\,dx$$

Want $x = 2\cos\theta + 1$

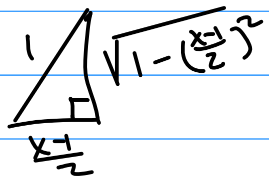
$dx = -2\sin\theta$

$$= \int \sqrt{4-4\cos^2\theta}\,(-2)\sin\theta\,d\theta$$

$$= -4\int \sin^2\theta \, d\theta = -4\int \frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta$$

$\cos\theta = \frac{x-1}{2}$

$$= -2\theta + \sin\theta + C$$



$$= -2\cos^{-1}\left(\frac{x-1}{2}\right) + \sqrt{1-\left(\frac{x-1}{2}\right)^2} + C$$