Ratio and Root Tests

Interesting idea:

If $|a_n| \le r^n$ for n large enough, $r < 1$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \le r < 1$ for n large enough   take a limit!

Ratio Test:

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $= \infty$ $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$??? inconclusive!

Related idea:

If $|a_n| \le r^n$ for n large enough, $r < 1$

$\sqrt[n]{|a_n|} \le r < 1$ for n large enough

Root Test:

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $= \infty$ $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1 \Rightarrow$??? inconclusive!

Ex: $\sum_{n=1}^{\infty} 7\left(1-\frac{1}{n}\right)^{n^2}$?

$$\lim_{n\to\infty} \sqrt[n]{7}\left(1-\frac{1}{n}\right)^{n} = \left(\lim_{n\to\infty}\sqrt[n]{7}\right)\left(\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}\right)$$

$$= 1\cdot e^{-1} < 1 \qquad \therefore \text{ converges}$$

Ex: $\sum_{n=1}^{\infty} \frac{n}{3^n}$???

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{3^{n+1}}\,\frac{3^n}{n} = \frac{n+1}{n}\,\frac{1}{3} \longrightarrow \frac{1}{3} \quad \therefore \text{ converges}$$

Ex: $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{23}\right)}{n!}$???

$$\left|\frac{a_{n+1}}{a_n}\right| \le \left|\frac{\cos\left(\frac{(n+1)\pi}{23}\right)}{(n+1)!}\,\frac{n!}{\cos\left(\frac{n\pi}{23}\right)}\right| \le (\text{Const})\cdot\frac{1}{n+1} \to 0$$

Ex: $\sum_{n=1}^{\infty} \frac{(-7)^n}{n^2}$???

Ex: $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$???

Ex: $\sum_{n=1}^{\infty} \frac{n^n}{n!}$???

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$???

Strategy:

Classify!

- check for divergence!
- if known series ~ p-series, geo. series, telescoping — do it!
- if looks like known series ~ comparison test!
- if alternating, use alternating series test!
- factorials or products? ratio test!
- $a_n = (b_n)^n$? root test!
- If $a_n = f(n)$ with $\int f(x)\,dx$ evaluable, integral test!