Convergence of Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$$

Partial sums: $\quad s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, $s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}, \ldots$

$$S_n = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \frac{1}{4(4+1)} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \ldots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

Telescoping Series

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\lim_{n\to\infty} S_n = 1 \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Theorem (Test for Divergence):

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If $\{a_n\}$ diverges or $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges

Ex: $\lim_{n\to\infty} \cos(n)$ does not exist, so

$\sum_{n=1}^{\infty} \cos(n)$ diverges

Ex: $\sum_{n=1}^{\infty} \frac{n^2}{n^2+n+1}$ diverges because $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 \neq 0$.

Ex $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Geometric sums

$$S_n = 1 + r + r^2 + \dots + r^{n-1}$$

$$S_n + r^n = S_{n+1} = rS_n + 1 \quad \Rightarrow \quad S_n + r^n = rS_n + 1$$

$$\Rightarrow \quad \boxed{S_n = \frac{1-r^n}{1-r}}$$

If $|r| < 1$, then $\lim_{n\to\infty} S_n = \frac{1}{1-r}$

Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r|<1$$

DIVERGENT otherwise!

r = common ratio

Ex:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = -1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = -1 + \frac{1}{1-\frac{1}{2}} = -1+2 = 1$$

Ex:

$$\frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \frac{2}{3^5} + \frac{2}{3^6} - \dots$$

$$= -2 + \frac{2}{3} + \left(2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \dots\right)$$

$$= -2 + \frac{2}{3} + \sum_{n=0}^{\infty} 2\left(-\frac{1}{3}\right)^{n-1} = -2 + \frac{2}{3} + \frac{2}{1-\left(-\frac{1}{3}\right)} = \frac{1}{6}$$

Ex: Write $0.23145145145145\ldots$ as a fraction

$$= \frac{23}{100} + \frac{1}{100}\left(0.145145145145\ldots\right)$$

$$= \frac{23}{100} + \frac{1}{100}\left(\frac{145}{10^3} + \frac{145}{10^6} + \frac{145}{10^9} + \ldots\right)$$

$$= \frac{23}{100} + \frac{1}{100}\left(\sum_{n=1}^{\infty} 145\left(\frac{1}{10^3}\right)^{n-1} - 145\right)$$

$$= \frac{23}{100} + \frac{1}{100}\left(\frac{145}{1-\left(\frac{1}{10^3}\right)} - 145\right)$$

$$= \frac{23}{100} + \frac{145}{100}\left(\frac{1}{999}\right) = \frac{23}{100} + \frac{145}{99900}$$

$$= \frac{23122}{99900}$$

Ex: $10 - 4 + 1.6 - 0.64 + \ldots$ geometric series

$$a_n = ar^{n-1} \Rightarrow r = \frac{a_2}{a_1} = \frac{-4}{10} = -\frac{2}{5}$$

$$a = r_0 = 10 \Rightarrow a_n = 10\left(-\frac{2}{5}\right)^{n-1}$$

$$10 - 4 + 1.6 - 0.64 + \ldots = \sum_{n=1}^{\infty} 10\left(\frac{-2}{5}\right)^{n-1} = \frac{10}{1-\left(\frac{-2}{5}\right)} = \frac{50}{7}$$

<u>Properties of Convergent Series</u> Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge

- $$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

- $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n.$

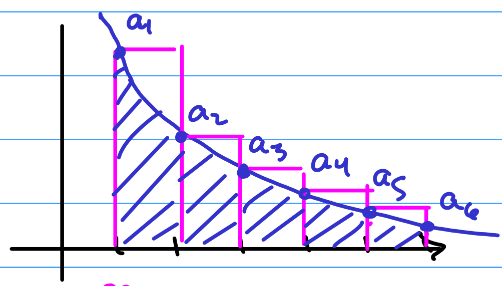
The Integral Test

Suppose $a_n = f(n)$ for some function $f(x)$ which is continuous, positive and decreasing on $[1, \infty)$.*

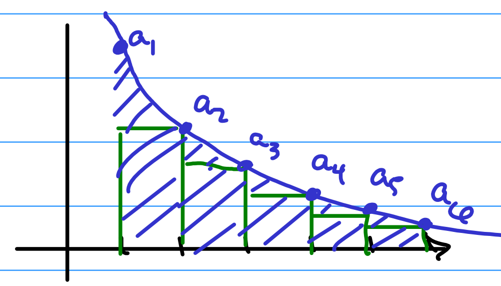
* can actually be $[M, \infty)$ for some value M

$$\int_1^{\infty} f(x)\,dx < \infty \;\Rightarrow\; \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\int_1^{\infty} f(x)\,dx = \infty \;\Rightarrow\; \sum_{n=1}^{\infty} a_n \text{ diverges}$$



$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\,dx$$



$$-a_1 + \sum_{n=1}^{\infty} a_n \leq \int_1^{\infty} f(x)\,dx$$

Ex: $\int_1^{\infty} \frac{1}{x}\,dx = \infty$ so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by Integral Test

Ex: $\int_1^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} < \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Ex: For which p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \infty \quad \text{if} \quad p \le 1$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1} < \infty \quad \text{if} \quad p > 1.$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff$ $p > 1$. by Integral Test.

Error bounds

S_n gets close to $\sum_{n=1}^{\infty} a_n$ for n large.

Error is estimated by the tail

$$\boxed{R_n := \left(\sum_{n=1}^{\infty} a_n\right) - S_n = \sum_{k=n+1}^{\infty} a_k}$$

Theorem If $f(x)$ is positive, continuous, decreasing on $[n, \infty)$.

$$\int_n^{\infty} f(x)\,dx \le R_n \le \int_{n+1}^{\infty} f(x)\,dx$$

Ex: Determine $\sum_{n=1}^{\infty} \frac{1}{n^3}$ up to 5 decimal places.

$$\int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}, \qquad \int_{n+1}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2(n+1)^2}$$

$$\Rightarrow \quad \left| s_n - \sum_{n=1}^{\infty} \frac{1}{n^3} \right| = |R_n| \leq \frac{1}{2n^2}$$

Want 5 decimals ... $\frac{1}{2n^2} < 0.000001 = 10^{-6}$

$$\Rightarrow \quad 2n^2 > 10^5 \Rightarrow n > \frac{10^{6/2}}{\sqrt{2}} = \frac{1000}{\sqrt{2}}$$

Need n at least

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{708^3} = 1.20205$$

accurate!!!

Actual value: 1.202056903159