Strategies of Integration

For *any* integration problem, apply the following steps.

1. Algebraically simplify
2. Do obvious u-subs
3. Identify which strategy to apply
   (integration by parts, partial fractions, trig sub, ...)
4. If 3 fails, try a different strat. or else
   do something clever.

Ex: $\int \frac{\sqrt{x}}{1+x}\,dx$

1. already simplified.
2. u-sub $u=\sqrt{x}$, $dx = 2u\,du$

$= 2\int \frac{u^2}{1+u^2}\,du$

3. now its rational, so do partial fractions!

$= 2\int 1 - \frac{1}{1+u^2}\,du$

$= 2u - 2\tan^{-1}(u) + C \quad = 2\sqrt{x} - 2\tan^{-1}(\sqrt{x}) + C$

Ex: $\int \sqrt{x+2+x^{-1}}\,dx$

1. Simplify!

$x+2+x^{-1} = (x^{1/2}+x^{-1/2})^2$

$= \int \sqrt{(x^{1/2}+x^{-1/2})^2}\,dx = \int x^{1/2}+x^{-1/2}\,dx = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$

Ex: $\int e^x \sqrt{1+e^{2x}}\, dx.$

1. Already simplified
2. u-sub $u = e^x$

$$= \int \sqrt{1+u^2}\, du$$

3. Trig sub $u = \tan\theta$

$$= \int \sec^2\theta \sqrt{1+\tan^2\theta}\, d\theta$$

$$= \int \sec^3\theta\, d\theta$$

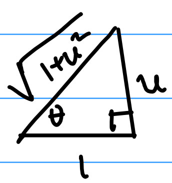
Now use integration by parts $u = \sec\theta$, $dv = \sec^2\theta\, d\theta$

$du = \sec\theta\tan\theta$, $v = \tan\theta$

$$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta\, d\theta$$

$$\Rightarrow \int \sec^3\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$



$$= \frac{1}{2}\sqrt{1+u^2}\, u + \frac{1}{2}\ln\left|\sqrt{1+u^2}+u\right| + C$$

$$= \frac{1}{2}\sqrt{1+e^{2x}}\, e^x + \frac{1}{2}\ln\left(\sqrt{1+e^{2x}}+e^x\right) + C$$

Ex: $\int x\cos^2(x)\, dx$

1. Simplify w/ trig
2. u-sub $u = 2x$

$$= \frac{1}{2}\int x + x\cos(2x)\, dx$$

$$= \frac{1}{8}\int u + u\cos(u)\, du$$

3. Integration by parts

$$= \frac{1}{16}u^2 + \frac{1}{8}u\sin(u) + \frac{1}{8}\cos(u) + C = \frac{1}{4}x^2 + \frac{1}{4}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

Ex: $\displaystyle\int \frac{1}{4+5\cos\theta}\,d\theta$

1. Already simplified
2. No obvious u-sub
3. No obvious method...
4. Clever?

VERY TRICKY!!

Use trig. identity $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$$\begin{aligned} 4+5\cos\theta &= 10\cos^2\left(\tfrac{\theta}{2}\right) - 1 \\ &= 9\cos^2\left(\tfrac{\theta}{2}\right) - \sin^2\left(\tfrac{\theta}{2}\right) \end{aligned}$$

$$\int \frac{1}{4+5\cos\theta}\,d\theta = \int \frac{1}{9\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}\,d\theta$$

$$= \int \frac{\sec^2\left(\frac{\theta}{2}\right)}{9 - \tan^2\left(\frac{\theta}{2}\right)}\,d\theta$$

Now do u-sub! $u = \tan\left(\frac{\theta}{2}\right)$

$du = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)d\theta$

$$= 2\int \frac{1}{9-u^2}\,du$$

Now do partial fractions!

$$= 2\int \frac{1/6}{3-u} + \frac{1/6}{3+u}\,du$$

$$= \frac{1}{3}\ln|3+u| - \frac{1}{3}\ln|3-u| + C$$

$$= \frac{1}{3}\ln\left|3+\tan\left(\tfrac{\theta}{2}\right)\right| - \frac{1}{3}\ln\left|3-\tan\left(\tfrac{\theta}{2}\right)\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{3+\tan\left(\frac{\theta}{2}\right)}{3-\tan\left(\frac{\theta}{2}\right)}\right| + C$$