Partial Fraction Decomposition

Rational Function $\quad P(x)/Q(x)$

Simplify into smallest atoms!

- $1, x-c, (x-c)^2, (x-c)^3, \dots$
- $\frac{1}{x-c}, \frac{1}{(x-c)^2}, \frac{1}{(x-c)^3}, \dots$
- $\frac{ax+b}{(x-c)^2+d^2}, \frac{ax+b}{[(x-c)^2+d^2]^2}, \frac{ax+b}{[(x-c)^2+d^2]^3}, \dots$

Theorem: Any rational function is equal to a linear combination of these basic pieces.

Ex: $$\frac{x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x+3 = A(x^2+1) + (Bx+C)x$$

@ $x=0$: $\quad 3 = A$ ✓

$$0x^2 + x + 3 = 3(x^2+1) + Bx^2 + Cx$$
$$= (3+B)x^2 + Cx + 3$$

comparing coefficients ~

$$B=-3 \checkmark, \quad C=1 \checkmark$$

$$\frac{x+3}{x(x^2+1)} = \frac{3}{x} + \frac{-3x+1}{x^2+1}$$

Ex: $$\frac{2x^2+x}{x^2+5x+6} = ?$$

First do long division!

$$\begin{array}{r} 2 \\ x^2+5x+6 \overline{) 2x^2+x+0} \\ -2x^2+10x+12 \\ \hline -9x-12 \end{array}$$

$$\frac{2x^2+x}{x^2+5x+6} = 2 - \frac{9x+12}{x^2+5x+6}$$

$$\frac{9x+12}{x^2+5x+6} = \frac{9x+12}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$9x+12 = A(x+3) + B(x+2)$$

@ $x=-2$: $-6 = A$

@ $x=-3$: $-15 = -B \Rightarrow B=15$

$$\frac{2x^2+x}{x^2+5x+6} = 2 - \left(\frac{-6}{x+2} + \frac{15}{x+3} \right)$$

Ex: $$\frac{3x}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$

$$3x = A(x-2)(x+1) + B(x+1) + C(x-2)^2$$

@ $x=2$: $6 = 3B \Rightarrow B=2$

@ $x=-1$: $-3 = 9C \Rightarrow C = -\frac{1}{3}$

@ $x=1$: $3 = -2A + 2B + C$

$3 = -2A + 4 - \frac{1}{3} \Rightarrow A = \frac{1}{3}$

$$\frac{3x}{(x-2)^2(x+1)} = \frac{1/3}{x-2} + \frac{2}{(x-2)^2} - \frac{1/3}{x+1}$$

Now our only problem is how to integrate the various atoms...

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C$$

$$\int \frac{1}{(x-a)^2}\,dx = -(x-a)^{-1} + C$$

$$\int \frac{1}{(x-a)^3}\,dx = -\frac{1}{2}(x-a)^{-2} + C$$

$$\vdots$$

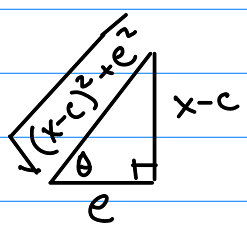
$$\int \frac{1}{(x-a)^n}\,dx = -\frac{1}{n-1}(x-a)^{-(n-1)} + C$$

For the more complicated atoms, we use trig substitution (with tangent).

$$\int \frac{ax+b}{(x-c)^2+e^2}\,dx = \int \frac{a(e\tan\theta + c)+b}{e^2\tan^2\theta + e^2}\, e\sec^2\theta\, d\theta$$

$$x-c = e\tan\theta$$
$$dx = e\sec^2\theta\, d\theta$$

$$= \int \frac{a(e\tan\theta+c)+b}{e^{\cancel{2}}\,\cancel{\sec^2}\theta}\,\cancel{e}\,\cancel{\sec^2}\theta\, d\theta$$



$$= \frac{1}{e}\int ae\tan\theta + ac + b \; d\theta$$

$$= a\ln|\sec\theta| + \frac{ac+b}{e}\theta + \text{CONST}$$

$$= a\ln\frac{\sqrt{(x-c)^2+e^2}}{e} + \left(\frac{ac+b}{e}\right)\tan^{-1}\left(\frac{x-c}{e}\right)$$

$$+ \text{CONST}$$