Comparison Tests

DIRECT COMPARISON:

If $0 \le a_n \le b_n$ for all n, then (actually only need for all $n \ge M$)

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{b_n} \text{ diverges}$$

$$\sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ converge or diverge?

$\displaystyle\frac{1}{n^2(n+1)} \le \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by integral test!

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$ converges.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or diverges?

$\displaystyle\frac{\ln(n)}{n} \ge \frac{1}{n}$ for $n \ge 3$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test

Direct comparison test $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

Ex Try yourself! $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

Estimating errors:

$$\sum_{n=1}^{\infty} a_n = L \quad \text{means} \quad \lim_{n\to\infty} S_N = L.$$

$$L - S_N = R_N = \sum_{n=N+1}^{\infty} a_n \qquad (\text{tail})$$

Now for two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ we have two tails

$$R_N = \sum_{n=N+1}^{\infty} a_n \quad \text{and} \quad T_N = \sum_{n=N+1}^{\infty} b_n$$

If $0 \le a_n \le b_n$ for all n, then

$$R_N \le T_N .$$

Ex: Consider $\sum_{n=1}^{\infty} a_n$ for $a_n = \frac{1}{n^2(n+1)}$.

We take $b_n = \frac{1}{n^3}$ so $a_n \le b_n$.

$$R_N \le T_N \le \int_N^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2N^2}$$

So if I want error ≤ 0.005, use

$$\frac{1}{2N^2} \le 0.005 \Rightarrow N^2 \ge 100 \Rightarrow N \ge 10,$$

Must sum at least 10 terms to get this close to actual value!

Limit Comparison Test

If $a_n \geq 0$, $b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$

Then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, or both diverge.

Ex: $\sum_{n=1}^{\infty} \frac{5n^3+2n}{\sqrt{n^7+3n^2}}$ converge or diverge?

Compare with $\sum_{n=1}^{\infty} n^{-1/2}$. $\lim_{n\to\infty} \frac{\left(\frac{5n^3+2n}{\sqrt{n^7+3n^2}}\right)}{n^{-1/2}} = 5 > 0$

Now $\sum_{n=1}^{\infty} n^{-1/2}$ diverges by integral test $\Rightarrow$ series diverges.

Alternating Series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

Alternating Series Test:

If $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all n
then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

Error bound:

$$0 \leq R_N \leq a - a_N$$

Ex: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ Converges!

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Ex: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!}$ converges!