

Math 350 Section 1

Fall 2023

Exam III

November 15, 2023

Time Limit: 1 Hour 50 Minutes

Name (Print): _____

Student ID: _____

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

(a) There exists a function which is discontinuous at every real number

(b) The radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is $R = \infty$

(c) If $f(x)^2$ is continuous at $x = a$, then $f(x)$ is continuous at $x = a$

(d) The function $f(x) = \frac{1}{x^2+1}$ is continuous at every real number

(e) If $\lim_{x \rightarrow a+} f(x)$ and $\lim_{x \rightarrow a-} f(x)$ exist, then $\lim_{x \rightarrow a} f(x)$ exists

2. (10 points)

For each of the following, either give an example or explain why an example does not exist.

- (a) A function which is continuous at $x = 0$ but NOT differentiable at $x = 0$

- (b) A power series whose radius of convergence is $R = 3$

- (c) A function which is simultaneously discontinuous at every rational number and continuous at every irrational number.

- (d) A function which is continuous on $[0, 1]$ but NOT uniformly continuous on $[0, 1]$.

- (e) A sequence of polynomials which converge pointwise to $f(x) = \frac{1}{1-x}$ on $(-1, 1)$.

3. (10 points)

Consider the function

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(a) Write down the ϵ, δ -definition of $\lim_{x \rightarrow a} f(x)$

(b) Let $a \in \mathbb{R}$ with $a \neq 0$. Prove that $\lim_{x \rightarrow a} f(x)$ does not exist

(c) Prove that $\lim_{x \rightarrow 0} f(x)$ exists and is equal to 0

4. (10 points)

For each of the following power series, determine the radius of convergence. Show your work.

(a) $\sum_{n=0}^{\infty} n^n x^n$

(b) $\sum_{n=0}^{\infty} n^3 x^n$

(c) $\sum_{n=0}^{\infty} 4^n x^{2n}$

(d) $\sum_{n=0}^{\infty} \left(1 + \frac{1}{n}\right)^{2n} x^n$

5. (10 points) Consider the sequence of functions

$$f_n(x) = e^{-nx^2}$$

and the function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

- (a) Write down the definition of a sequence of functions $(f_n(x))$ converging uniformly to a function $f(x)$ on a set S
- (b) Prove that the sequence of functions above converges pointwise to $f(x)$.
- (c) Prove that the sequence of functions above does NOT converge uniformly to $f(x)$. If you use a theorem from class, make sure to carefully state it.

6. (10 points)

(a) Write down the ϵ, δ -definition of continuity.

(b) Suppose that $f(x)$ is a continuous function on \mathbb{R} with the property that $f(x) = f(x/2)$ for all real numbers x . Prove that $f(x)$ must be a constant function.

(c) Show by example that (b) is false if we drop the continuity assumption.

7. (10 points)

(a) State the definition of $f(x)$ being uniformly continuous on a subset $S \subseteq \text{Dom}(f)$

(b) Using ϵ and δ , prove that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on the interval $[2, \infty)$.

(c) Explain why $f(x)$ from part (b) is NOT uniformly continuous on $(0, \infty)$.