Math 350 Section 1
Fall 2023
Exam II
October 17, 2023
Time Limit: 1 Hour 50 Minutes

Name (Print): _____

Student ID: _____

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) If $\lim a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges

   (b) If $\limsup a_n = \liminf a_n$ then a_n is the constant sequence

   (c) A bounded sequence must have a Cauchy subsequence

   (d) An absolutely convergent series must be convergent

   (e) There exists a sequence (s_n) where the value of every real number appears at least once

2. (10 points)

Let (a_n) and (b_n) be bounded sequences.

(a) Prove

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

(b) Prove

$$\limsup a_n + \liminf b_n \leq \limsup(a_n + b_n)$$

(c) Give an explicit example of two sequences (a_n) and (b_n) for which

$$\limsup(a_n + b_n) \neq \limsup a_n + \limsup b_n.$$

3. (10 points)

Let $r \in \mathbb{R}$.

(a) Write down a closed form expression for the sum

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1}.$$

(b) Use (a) to prove that if $|r| < 1$ then

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

(c) Find an exact fractional expression for

$$0.2023202320232023\cdots = 0.\overline{2023}$$

4. (10 points)

(a) Write down the definition of $\sum_{n=1}^{\infty} a_n$ converging.

(b) Let $k \in \mathbb{R}$ and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} ka_n$ converges.

(c) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Prove $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

(d) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent. Prove $\sum_{n=1}^{\infty} a_n b_n$ converges.

5. (10 points)

For each of the following series, determine if it diverges, converges, or converges absolutely. Carefully justify your answer.

(a)

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

(b)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

(c)

$$\sum_{n=1}^{\infty} \frac{5^n}{n^n}$$

(d)

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$$

(e)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

6. (10 points)

Let $\mathbb{I}$ be the set of all irrational numbers in the interval $[0, 1]$. Consider the function

$$f : I \to [0, 1]$$

defined via decimal expansions by

$$f(0.d_1d_2d_3d_4d_5d_6\dots) = 0.d_1d_3d_5d_7\dots$$

(a) Prove that $f(x)$ is not injective.

(b) Prove that $f(x)$ is surjective.

(c) If we replace I with the interval $[0, 1]$ above, prove that the function is no longer well-defined. Carefully explain.

7. (10 points)

(a) Write down the ϵ,δ-definition of continuity.

(b) Consider the function

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Let $a \in \mathbb{R}$ with $a \neq 0$. Prove that $f(x)$ is not continuous at $x = a$.

(c) Prove that $f(x)$ is continuous at $x = 0$.