Math 350 Section 1
Fall 2023
Exam I
September 19, 2023
Time Limit: 1 Hour 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) If a and b are nonzero numbers and $a < b$, then $1/a > 1/b$.

   FALSE (consider negatives)

   (b) Any set of positive integers has a minimum value

   TRUE

   (c) The sequence $s_n = n/(2n+3)$ is Cauchy

   TRUE (converges)

   (d) If $A \subseteq B$ then $\sup(A) \leq \sup(B)$

   TRUE

   (e) If a bounded sequence satisfies $s_n > 0$ for all n, then $\limsup s_n > 0$

   FALSE

   (need $\geq$)

2. (10 points)

Let F_n be the sequence of Fibonacci numbers defined by $F_1 = 1$, $F_2 = 1$, $F_3 = 2$ and for $n \geq 2$

$$F_{n+1} = F_n + F_{n-1}.$$

Use induction to prove that for all $n \in \mathbb{N}$

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$

Base case $(n=1)$: $F_3 F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^{1+1}$ ✓

As an inductive assumption, assume

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$$

Then

$$\begin{aligned}
F_{n+1} F_{n+3} - F_{n+2}^2 &= F_{n+1}(F_{n+2} + F_{n+1}) - F_{n+2}^2 && \text{(recursion)} \\
&= F_{n+1}^2 + (F_{n+1} - F_{n+2}) F_{n+2} && \text{(algebra)} \\
&= F_{n+1}^2 + (-F_n) F_{n+2} && \text{(recursion)} \\
&= -(-1)^{n+1} && \text{(ind. hyp.)} \\
&= (-1)^{n+2}
\end{aligned}$$

Hence by PMI, our result is true for all $n \in \mathbb{N}$.

$\square$

3. (10 points)

(a) Write down the definition of a lower bound of a set A

L is a lower bound if

$L \leq a$ for all $a \in A$

(b) Write down the definition of the infimum of a set A

L_0 is the infimum if L_0 is a lower bound of A and $L_0 \geq L$ for all lower bounds L of A.

(c) Determine the infimum of the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

0

(d) Prove your answer for part (c).

All terms in A are positive, so 0 is a lower bound.

If $L > 0$, then Archimedian Principle $\Rightarrow \exists\, n \in \mathbb{N}$ s.t. $n > 1/L$, and therefore $\frac{1}{n} < L$. Thus L cannot be a lower bound. Thus 0 is the greatest of the lower bounds, ie. the infimum.

4. (10 points)

(a) Write down the definition of a sequence (s_n) converging to s

$\forall \varepsilon > 0 \ \exists N$ s.t.

$n > N \Rightarrow |s_n - s| < \varepsilon. \quad \forall n \in \mathbb{N}.$

(b) Suppose that (s_n) is a sequence of positive numbers converging to a real number s. Prove that $(\sqrt{s_n})$ converges to $\sqrt{s}$.

Proof:

Let $\varepsilon > 0$.

First suppose $s = 0$.

Then $\exists N$ s.t. $n > N \Rightarrow |s_n - 0| < \varepsilon^2$.

Therefore $n > N \Rightarrow |\sqrt{s_n} - 0| = |\sqrt{s}| < \varepsilon$.

Alt. suppose $s \neq 0$. Then $s > 0$, since it is the limit of a sequence of positive numbers.

Then $\exists N$. s.t. $n > N \Rightarrow |s_n - s| < \varepsilon\sqrt{s}$.

Therefore $n > N \Rightarrow$

$$|\sqrt{s_n} - \sqrt{s}| = \left| \left(\sqrt{s_n} - \sqrt{s}\right) \frac{\sqrt{s_n} + \sqrt{s}}{\sqrt{s_n} + \sqrt{s}} \right|$$

$$= \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves

$\lim \sqrt{s_n} = \sqrt{s}$

$\square$

5. (10 points)

Define a sequence (s_n) recursively by letting $s_1 = 2023$ and more generally

$$s_{n+1} = \frac{1}{2}s_n + \frac{5}{2s_n}.$$

(a) Prove that the sequence is monotone.

First note $s_n \geq 0$ for all n and therefore

$$(s_{n+1} - \sqrt{5}) = \frac{1}{2}s_n + \frac{5}{2s_n} - \sqrt{5} = \frac{s_n^2 - 2s_n\sqrt{5} + 5}{2s_n} = \frac{(s_n - \sqrt{5})^2}{2s_n} \geq 0.$$

Thus $s_n \geq \sqrt{5}$ for all $n \in \mathbb{N}$.

Therefore $s_n - s_{n+1} = \frac{1}{2}s_n - \frac{5}{2s_n} = \frac{s_n^2 - 5}{2s_n} \geq 0$ for all $n \in \mathbb{N}$.

Thus $s_{n+1} \leq s_n$ making the sequence monotone decreasing.

(b) Prove that the sequence converges.

It is monotone decreasing and bounded below by 0, so it converges.

(c) Determine the limit of the sequence.

Let $s = \lim s_n$. Taking the limit of both sides of the update formula

$$s = \frac{1}{2}s + \frac{5}{2s}$$

Solving for s we get

$$s = \pm\sqrt{5}$$

Since $s \geq 0$ (limit of pos #'s) $s = \sqrt{5}$.

6. (10 points)

(a) Prove that there is a sequence of rational numbers which converges to π.

Let $k \in \mathbb{N}$

By Density of Rationals $\exists$ rational r_k with

$$\pi - \frac{1}{k} < r_k < \pi + \frac{1}{k}$$

Note that $\lim\limits_{k \to \infty} \pi \pm \frac{1}{k} = \pi$, so by the Squeeze Theorem $\lim r_k$ exists and is equal to π.

(b) Prove that if (s_n) is a monotone increasing sequence which is bounded above, then s_n converges.

Set $s = \sup\{s_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$. Then $s - \varepsilon$ is not an upper bound of $\{s_n : n \in \mathbb{N}\}$

Therefore there exists $N \in \mathbb{N}$ s.t. $s_N > s - \varepsilon$.

Since (s_n) is monotone increasing, $s_n > s_N > s - \varepsilon$ $\forall n > N$.

Hence $n > N \Rightarrow$

$$s - \varepsilon < s_n < s$$

which implies

$$|s_n - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves

$$\lim s_n = s$$

$\square$

7. (10 points) Let (a_n) and (b_n) be bounded sequences.

(a) Define $\limsup a_n$ and $\liminf a_n$

$\limsup a_n = \lim_{N\to\infty} M_N$ for $M_N = \sup\{s_n : n \geq N\}$

$\liminf a_n = \lim_{N\to\infty} m_N$ for $m_N = \inf\{s_n : n \geq N\}$

(b) Prove

$$\limsup a_n + \liminf b_n \leq \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

Let $M_N^a = \sup\{a_n : n \geq N\}$, $M_N^b = \sup\{b_n : n \geq N\}$, $M_N^{a+b} = \{a_n + b_n : n \in \mathbb{N}\}$.

Then for $n \geq N$, $a_n \leq M_N^a$ and $b_n \leq M_N^b$, since they are upper bounds.

Therefore $a_n + b_n \leq M_N^a + M_N^b$, and since suprema are least upper bounds $M_N^{a+b} \leq M_N^a + M_N^b$. Taking the limit $\Rightarrow$ $\limsup(a_n+b_n) \leq \limsup a_n + \limsup b_n$

Also let $m_N^b = \inf\{b_n : n \geq N\}$. Then for $n \geq N$, $b_n \geq m_N^b$.

Therefore $a_n + m_N^b \leq a_n + b_n \leq M_N^{a+b}$, giving $a_n \leq M_N^{a+b} - m_N^b$.

It follows $M_N^a \leq M_N^{a+b} - m_N^b$ and taking the limit

$\limsup a_n \leq \limsup(a_n+b_n) - \liminf(b_n)$

(c) Give an explicit example of two sequences (a_n) and (b_n) for which the above two inequalities both turn out to be equalities.

Take $a_n = 0$ and $b_n = 0$ $\forall n \in \mathbb{N}$.