Math 350 Section 1
Fall 2023
Exam II
October 17, 2023
Time Limit: 1 Hour 50 Minutes

Name (Print): ______________________________

Student ID: ______________________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) If $\lim a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges

   F

   (b) If $\limsup a_n = \liminf a_n$ then a_n is the constant sequence

   F

   (c) A bounded sequence must have a Cauchy subsequence

   T

   (d) An absolutely convergent series must be convergent

   T

   (e) There exists a sequence (s_n) where the value of every real number appears at least once

   F

2. (10 points)

   Let (a_n) and (b_n) be bounded sequences.

   (a) Prove

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

Let $A_N = \sup\{a_n : n \geq N\}$ and $B_n = \sup\{b_n : n \geq N\}$ and $C_N = \sup\{a_n + b_n : n \geq n\}$. Then

$a_n \leq A_N$ and $b_n \leq B_N$ $\forall$ $n \geq N$ $\Rightarrow$

$a_n + b_n \leq A_N + B_N$ $\forall n \geq N$. Thus $A_N + B_N$ is an upper bound of $\{a_n + b_n : n \geq N\}$. Since C_N is the <u>least</u> upper bound, $C_N \leq A_N + B_N$. Taking the limit:

$\limsup(a_n + b_n) = \lim C_N \leq \lim A_N + \lim B_N = \limsup(a_n) + \limsup(b_n)$.

   (b) Prove

$$\limsup a_n + \liminf b_n \leq \limsup(a_n + b_n)$$

Let $\tilde{a}_n = a_n + b_n$ and $\tilde{b}_n = -b_n$. Then from (a) we have

$\limsup(\tilde{a}_n + \tilde{b}_n) \leq \limsup(\tilde{a}_n) + \limsup(\tilde{b}_n)$.

Therefore

$\limsup(a_n) \leq \limsup(a_n + b_n) + \limsup(-b_n)$

and so

$\limsup(a_n) \leq \limsup(a_n + b_n) - \liminf(b_n)$

Adding $\liminf(b_n)$ to both sides finishes the proof.

   (c) Give an explicit example of two sequences (a_n) and (b_n) for which

$$\limsup(a_n + b_n) \neq \limsup a_n + \limsup b_n.$$

Take

$(a_n) = (1, 0, 1, 0, 1, 0, \dots)$

and

$(b_n) = (0, 1, 0, 1, 0, 1, \dots)$

$\limsup(a_n + b_n) = 1 \neq 2 = \limsup(a_n) + \limsup(b_n)$.

3. (10 points)

Let $r \in \mathbb{R}$.

(a) Write down a closed form expression for the sum

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1}.$$

$$\frac{1-r^n}{1-r}$$

(b) Use (a) to prove that if $|r| < 1$ then

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} r^k = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r}$$

(c) Find an exact fractional expression for

$$0.2023202320232023\cdots = 0.\overline{2023}$$

$$0.\overline{2023} = \sum_{n=1}^{\infty} 2023 \left(\frac{1}{10}\right)^{-4n} = \frac{2023}{10^4} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^{-4n}$$

$$= \frac{2023}{10^4} \left(\frac{1}{1-\left(\frac{1}{10}\right)^{-4}}\right)$$

$$= \frac{2023}{10^4 - 1} = \frac{2023}{9999}$$

4. (10 points)

(a) Write down the definition of $\sum_{n=1}^{\infty} a_n$ converging.

The sequence of partial sums converges:

$\lim_{N\to\infty} S_N$ exists for $S_N = \sum_{n=1}^{N} a_n$.

(b) Let $k \in \mathbb{R}$ and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} ka_n$ converges.

Let $S_N = \sum_{n=1}^{N} a_n$. Then $\sum_{n=1}^{N} ka_n = kS_N$.

Since S_N converges, linearity $\Rightarrow$ kS_N converges.

Thus since the sequence of partial sums converges,

$\sum_{n=1}^{\infty} kS_n$ converges.

(c) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Prove $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

Let $S_N = \sum_{n=1}^{N} a_n$ and $t_N = \sum_{n=1}^{N} b_n$. Then

$\sum_{n=1}^{N}(a_n+b_n) = S_N + t_N$. Since S_N and t_N converge,

$S_N + t_N$ converges by linearity!

Thus $\sum_{n=1}^{\infty} S_n + t_n$ converges.

(d) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent. Prove $\sum_{n=1}^{\infty} a_n b_n$ converges.

Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$ and so a_n

is bounded. Therefore $\exists K > 0$ s.t. $|a_n| \le K$ $\forall n \in \mathbb{N}$.

Thus $|a_n b_n| \le K|b_n|$. Since (b_n) is absolutely convergent

part (b) implies $\sum_{n=1}^{\infty} K|b_n|$ converges. Therefore the

Comparison Test $\Rightarrow$ $\sum_{n=1}^{\infty} |a_n b_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n b_n$

is absolutely convergent and therefore converges.

5. (10 points)

   For each of the following series, determine if it diverges, converges, or converges absolutely. Carefully justify your answer.

   (a)
   $$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

   $\frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the original series converge by the Comparison Test.

   (b)
   $$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

   Converges by Ratio Test

   (c)
   $$\sum_{n=1}^{\infty} \frac{5^n}{n^n}$$

   Converges by Root Test

   (d)
   $$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

   Converges by AST

   (e)
   $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

   $$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n} = \frac{1}{\lim(1+\frac{1}{n})^n} = \frac{1}{e}$$

   and so we converge by Ratio Test.

6. (10 points)

Let $\mathbb{I}$ be the set of all irrational numbers in the interval $[0, 1]$. Consider the function

$$f : I \to [0, 1]$$

defined via decimal expansions by

$$f(0.d_1d_2d_3d_4d_5d_6\ldots) = 0.d_1d_3d_5d_7\ldots$$

(a) Prove that $f(x)$ is not injective.

Let $\sqrt{2} = 0.a_1a_2a_3a_4\ldots$ and $\sqrt{3} = 0.b_1b_2b_3b_4\ldots$

$$\left.\begin{aligned} x &= 0.d_1a_1d_3a_2d_5a_3d_7a_4d_9a_5\ldots \\ y &= 0.d_1b_1d_3b_2d_5b_3d_7b_4d_9b_5\ldots \end{aligned}\right\} \in \mathbb{I}\cap[0,1]$$ since decimal is non-repeating

Then $x \neq y$ but $f(x) = f(y)$. Thus f is not injective!

(b) Prove that $f(x)$ is surjective.

Let (a_n) be as in part (a) Then for any $z \in [0,1]$ write $z = 0.d_1d_2d_3d_4d_5\ldots$ and take

$x = 0.d_1a_1d_3a_2d_5a_3d_7a_4\ldots$ This is in $\mathbb{I}\cap[0,1]$

Then $f(x) = z$. $\therefore f$ is surjective.

(c) If we replace I with the interval $[0, 1]$ above, prove that the function is no longer well-defined. Carefully explain.

$$f(0.0099999) = 0.099999\ldots$$

$$f(0.0100000) = 0.000000\ldots$$

however $0.0099999 = 0.0100000 \Rightarrow\Leftarrow$.

So $f(x)$ isn't well-defined.

7. (10 points)

(a) Write down the ϵ,δ-definition of continuity.

$f(x)$ is continuous at a if $\forall\ \varepsilon>0\ \exists\ \delta>0$ s.t. $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$.

(b) Consider the function

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Let $a \in \mathbb{R}$ with $a \neq 0$. Prove that $f(x)$ is not continuous at $x = a$.

If $f(x)$ is continuous at $x=a$, then for all sequences (a_n) with $\lim_{n\to\infty} a_n = a$ we must have $\lim_{n\to\infty} f(a_n) = f(a)$.

If $a \in \mathbb{Q}$, take $a_n = a + \frac{\sqrt{2}}{n}$. Then $\lim f(a_n) = 0 \neq a = f(a)$

If $a \notin \mathbb{Q}$, take (a_n) to be a sequence of rationals with $\lim a_n = a$. Then $\lim f(a_n) = \lim a_n = a \neq 0 = f(a)$.

In either case, we have a contradiction

(c) Prove that $f(x)$ is continuous at $x = 0$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Then for $|x-0|<\delta$

we have two cases:

(I) Assume $x \in \mathbb{Q}$. Then $|f(x)-f(0)| = |x-0| < \delta = \varepsilon$.

(II) Assume $x \notin \mathbb{Q}$. Then $|f(x)-f(0)| = |0-0| = 0 < \varepsilon$.

In either case $|f(x)-f(0)|<\varepsilon$. Since $\varepsilon>0$ was arbitrary, this proves $f(x)$ is continuous at $x=0$.