Math 350 Section 1
Fall 2023
Final Exam
December 10, 2023
Time Limit: 1 Hour 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) If $f(x)/x$ is continuous at $x = 1$, then $f(x)$ is continuous at $x = 1$

   (b) If the sequential limit $\lim_{n\to\infty} f(1/n)$ is equal to $f(0)$, then $f(x)$ is continuous at $x = 0$

   (c) If $f(x)$ is uniformly continuous on $(0, 1)$, then $f(x)$ extends to a continuous function on $[0, 1]$

   (d) If $f(x)$ is a monotone increasing function on $[a, b]$, then $f(x)$ is Darboux integrable on $[a, b]$

   (e) If the domain of $f(x)$ is $\mathbb{R}$ and the range of $f(x)$ is $\{y \in \mathbb{R} : |y| > 1\}$, then $f(x)$ has a discontinuity

2. (10 points)

For each of the following, either give an example or explain why an example does not exist.

(a) A bounded sequence with no convergent subsequence

(b) A function which is not Riemann integrable

(c) A power series whose radius of convergence is $R = 0$

(d) A function $f(x)$ differentiable everywhere with $f(1) - f(0) > f'(x)$ for all $x \in (0, 1)$

(e) A sequence of functions differentiable at $x = 0$ which converge uniformly on $[-1, 1]$ to a function which is not differentiable at $x = 0$

3. (10 points) Let (a_n) and (b_n) be bounded sequences.

   (a) Define $\limsup a_n$ and $\liminf a_n$

   (b) Prove

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

   (c) Give an explicit example of two sequences (a_n) and (b_n) for which the above inequality is strict (not equal).

4. (10 points)

   (a) Write down the ϵ,δ-definition of continuity.

   (b) Suppose that $f(x)$ and $g(x)$ are continuous functions on $\mathbb{R}$ with the property that $f(x) = g(x)$ for all rational numbers x. Prove that $f(x) = g(x)$ must be true for all real numbers x.

   (c) Show by example that (b) is false if we drop the continuity assumption.

5. (10 points)

(a) Write down the definition of $f(x)$ being differentiable at $x = a$.

(b) Prove using only basic definitions and algebra that $f(x) = x^{1/3}$ is differentiable a $x = 8$.

(c) Carefully prove that $f(x) = x^{1/3}$ is not differentiable at $x = 0$

6. (10 points)

   (a) State what it means for $f(x)$ to be uniformly continuous on a subset $S \subseteq \mathrm{Dom}(f)$

   (b) State the Mean Value Theorem

   (c) Suppose that $f(x)$ is differentiable on $\mathbb{R}$ and that there exists a constant $M > 0$ such that $|f'(x)| < M$ for all x. Prove that $f(x)$ must be uniformly continuous on $\mathbb{R}$

7. (10 points)

Consider the function

$$f(x) = \begin{cases} 5, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(a) Write the definition of the upper Darboux sum $U(f,P)$ of $f(x)$ on a partition $P = \{a_0, \ldots, a_n\}$ on $[a,b]$

(b) Write the definition of the lower Darboux integral $L(f)$ of $f(x)$ on $[a,b]$

(c) Evaluate $L(f)$ and $U(f)$.