Math 350 Section 1	Name (Print):	
Fall 2023		
Exam I	Student ID:	
September 19, 2023		
Time Limit: 1 Hour 50 Minutes		

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

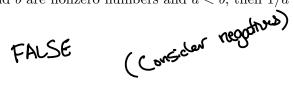
You are required to show your work on each problem on this exam. The following rules apply:

- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- Box Your Answer where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Do not write in the table to the right.

Points	Score
10	
10	
10	
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70	
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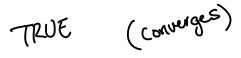
- 1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.
  - (a) If a and b are nonzero numbers and a < b, then 1/a > 1/b.



(b) Any set of positive integers has a minimum value

TRUE

(c) The sequence  $s_n = n/(2n+3)$  is Cauchy



(d) If  $A \subseteq B$  then  $\sup(A) \le \sup(B)$ 

TRUE

(e) If a bounded sequence satisfies  $s_n > 0$  for all n, then  $\limsup s_n > 0$ 

FALSE (need Z)

## 2. (10 points)

Let  $F_n$  be the sequence of Fibonacci numbers defined by  $F_1 = 1$ ,  $F_2 = 1$ ,  $F_3 = 2$  and for  $n \ge 2$ 

$$F_{n+1} = F_n + F_{n-1}$$
.

Use induction to prove that for all  $n \in \mathbb{N}$ 

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$

Base case (n=1): 
$$F_3F_1-F_2^2=2\cdot 1-1^2=1=(-1)^{1+1}$$

As an inductive assumption, assume

Then

$$F_{n+1}F_{n+3} - F_{n+2}^{2} = F_{n+1}(F_{n+2}+F_{n+1}) - F_{n+2}^{2} \qquad (recursion)$$

$$= F_{n+1}^{2} + (F_{n+1}-F_{n+2})F_{n+2} \qquad (algebra)$$

$$= F_{n+1}^{2} + (-F_{n})F_{n+2} \qquad (recursion)$$

$$= -(-1)^{n+1} \qquad (ind. hyp.)$$

Hence by PMI, our result is true for all nEIN

- 3. (10 points)
  - (a) Write down the definition of a lower bound of a set A

(b) Write down the definition of the infimum of a set A

(c) Determine the infimum of the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

0

(d) Prove your answer for part (c).

All terms on A are positive, so O is a lower bound.

If L>O, then Archamedian Principle > 3 re IN st.

N>/L, and therefore to L. Thus L.

Connot be a lower bound. Thus O is the

gratest of the lower bounds, ie. the infimum.

- 4. (10 points)
  - (a) Write down the definition of a sequence  $(s_n)$  converging to s

$$\forall \ \varepsilon > 0 \ \exists \ N \ \Rightarrow \ |s_n - s| \ \langle \varepsilon. \quad \forall \ n \in \mathbb{N}.$$

(b) Suppose that  $(s_n)$  is a sequence of positive numbers converging to a real number s. Prove that  $(\sqrt{s_n})$  converges to  $\sqrt{s}$ .

Proof:

Let 8>0.

First suppose 
$$8=0$$
.

Then  $\exists \ N \ Sit$ .  $N>N \Rightarrow |S_n-0| < \varepsilon^2$ .

Therefore  $N>N \Rightarrow |V_{3n}-0| = |V_{5}| < \varepsilon$ .

Alt. Suppose  $S \neq 0$ . Then  $S \neq 0$ , Since it is the limit of a sequence of positive numbers.

Then  $\exists \ N \ St$ .  $N>N \Rightarrow |S_n-S| < \varepsilon \sqrt{S}$ .

Therefore  $N>N \Rightarrow |S_n-S| < \varepsilon \sqrt{S}$ .

Therefore  $N>N \Rightarrow |S_n-S| < \varepsilon \sqrt{S}$ .

 $|V_{5n}-V_{5}| = |(V_{5n}-(S))\frac{V_{5n}+V_{5}}{V_{5n}+V_{5}}|$ 
 $= |S_n-S| < \varepsilon \sqrt{S}$ .

 $|V_{5n}-V_{5}| = |S_n-S| < \varepsilon \sqrt{S}$ .

Since 870 was arbitrary, thus proves

5. (10 points)

Define a sequence  $(s_n)$  recursively by letting  $s_1 = 2023$  and more generally

$$s_{n+1} = \frac{1}{2}s_n + \frac{5}{2s_n}.$$

(a) Prove that the sequence is monotone.

First note 
$$S_{N} \ge 0$$
 for all  $n$  and thurspec  $(S_{N+1} - (5)) = \frac{1}{2} S_{N} + \frac{5}{2 S_{N}} - \sqrt{5} = \frac{S_{N}^{2} - 2 S_{N} \sqrt{5} + 5}{2 S_{N}} = \frac{(S_{N} - \sqrt{5})^{2}}{2 S_{N}} \ge 0$ . Thus  $S_{N} \ge \sqrt{5}$  for all  $n \in \mathbb{N}$ .

Therefore  $S_{N} - S_{N+1} = \frac{1}{2} S_{N} - \frac{5}{2 S_{N}} = \frac{S_{N}^{2} - 5}{2 S_{N}} \ge 0$  for all  $n \in \mathbb{N}$ .

Thus  $S_{N+1} \le S_{N}$  making the sequence monotone decreasing.

(b) Prove that the sequence converges.

It is manatom decreating and bounded below by 0, 80 it converges.

(c) Determine the limit of the sequence.

Let  $8 = 6m \cdot 5n$ . Taking the limit of both sades of the update formula  $S = \frac{1}{2}S + \frac{5}{2S}$  Solving for S we get

Since 5 20 (limit of poo #1s) S=15

## 6. (10 points)

(a) Prove that there is a sequence of rational numbers which converges to  $\pi$ .

Let kell By Dersity of Rationals 3 rotional 
$$T_{k}$$
 with  $tr-\frac{1}{k} \times T_{k} \times T_{k} + t\bar{k}$  Note that  $\lim_{k \to \infty} \tau_{1} \pm \frac{1}{k} = \tau_{1}$ , so by the Squeze Theorem law  $r_{k}$  exists and is equal to  $\tau_{k}$ .

(b) Prove that if  $(s_n)$  is a monotone increasing sequence which is bounded above, then  $s_n$  converges.

Set 
$$S = \sup S_n : n \in \mathbb{N}_s^n$$
.

Let  $e > 0$ . Then  $s - e$  is not an upper bound of  $\{s_n : n \in \mathbb{N}\}^n$ .

Therefore them exists  $N \in \mathbb{N}$   $s : 1$ .  $s_N > s - e$ .

Since  $(s_n)$  is monotone increasing,  $s_n > s_N > s - e$   $\forall n > N$ .

Hence  $n > N \Rightarrow$ 
 $s - e < s_n < s$ 

which implies

 $[s_n - s] < e$ .

Since  $e > s_n < s$ 

Lim  $s_n = s_n < s$ 
 $e > s_n < s_n < s_n < s_n$ 

- 7. (10 points) Let  $(a_n)$  and  $(b_n)$  be bounded sequences.
  - (a) Define  $\limsup a_n$  and  $\liminf a_n$

low sup 
$$a_n = \lim_{N\to\infty} M_N$$
 for  $M_n = \sup_{N\to\infty} \{a_n : n \ge N\}$   
low sup  $a_n = \lim_{N\to\infty} m_N$  for  $m_n = \inf_{N\to\infty} \{a_n : n \ge N\}$ 

(b) Prove

(c) Give an explicit example of two sequences  $(a_n)$  and  $(b_n)$  for which the above two inequalities both turn out to be equalities.

Take an=0 and bn=0 Yn & W.