Math 350 Section 1
Fall 2023
Exam III
November 15, 2023
Time Limit: 1 Hour 50 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) There exists a function which is discontinuous at every real number

   TRUE

   (b) The radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is $R = \infty$

   TRUE

   (c) If $f(x)^2$ is continuous at $x = a$, then $f(x)$ is continuous at $x = a$

   FALSE

   (d) The function $f(x) = \frac{1}{x^2+1}$ is continuous at every real number

   TRUE

   (e) If $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist, then $\lim_{x \to a} f(x)$ exists

   FALSE

2. (10 points)

For each of the following, either give an example or explain why an example does not exist.

(a) A function which is continuous at $x = 0$ but NOT differentiable at $x = 0$

$f(x) = |x|$

(b) A power series whose radius of convergence is $R = 3$

$\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$

(c) A function which is simultaneously discontinuous at every rational number and continuous at every irrational number.

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q}, \ \gcd(p,q) = 1 \end{cases}$$

(d) A function which is continuous on $[0, 1]$ but NOT uniformly continuous on $[0, 1]$.

DNE, continuity on $[a,b]$ $\Rightarrow$ uniform continuity on $[a,b]$

(e) A sequence of polynomials which converge pointwise to $f(x) = \frac{1}{1-x}$ on $(-1, 1)$.

$f_n(x) = \sum_{k=1}^{n} x^k$ satisfies $\lim_{n\to\infty} f_n(x) = \frac{1}{1-x}$

(Geometric Series)

3. (10 points)

Consider the function

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

(a) Write down the ϵ,δ-definition of $\lim_{x\to a} f(x)$

$\lim_{x\to a} f(x) = L$ means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon \quad \forall x \in Dom(f)$

(b) Let $a \in \mathbb{R}$ with $a \neq 0$. Prove that $\lim_{x\to a} f(x)$ does not exist

Let (s_n) and (t_n) be sequences of rational and irrational numbers, resp., both of which converge to a. Then since $a \neq 0$,

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} s_n = a \neq 0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} f(t_n)$$

This implies $\lim_{x\to a} f(x)$ DNE

(c) Prove that $\lim_{x\to 0} f(x)$ exists and is equal to 0

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$.

Then for $|x - 0| < \delta$, we consider two cases:

Case 1: Suppose $x \in \mathbb{Q}$. Then

$$|f(x) - 0| = |x - 0| < \delta = \varepsilon.$$

Case 2: Suppose $x \notin \mathbb{Q}$. Then

$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon.$$

In either case $|f(x) - 0| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves $\lim_{x\to 0} f(x) = 0$.

4. (10 points)

For each of the following power series, determine the radius of convergence. Show your work.

(a) $\sum_{n=0}^{\infty} n^n x^n$

$\beta = \limsup \sqrt[n]{n^n} = \limsup n = \infty$, so $R = 0$.

(b) $\sum_{n=0}^{\infty} n^3 x^n$

$\lim \left| \frac{(n+1)^3 x^{n+1}}{n^3 x^n} \right| = \lim \left| \left(1+\frac{1}{n}\right)^3 x \right| = (1+0)|x| = |x| < 1$

So $R = 1$.

(c) $\sum_{n=0}^{\infty} 4^n x^{2n}$

$\lim_{n\to\infty} \frac{4^{n+1} x^{2n+2}}{4^n x^{2n}} = \lim_{n\to\infty} 4x^2 = 4x^2 < 1 \Rightarrow x^2 < \frac{1}{4}$

$\Rightarrow |x| < \frac{1}{2}$

$R = \frac{1}{2}$

(d) $\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{2n} x^n$

$\beta = \lim \sqrt[n]{\left(1+\frac{1}{n}\right)^{2n}} = \lim \left(1+\frac{1}{n}\right)^2 = 1+0 = 1$

So $R = \frac{1}{\beta} = 1$

5. (10 points) Consider the sequence of functions

$$f_n(x) = e^{-nx^2}$$

and the function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

(a) Write down the definition of a sequence of functions $(f_n(x))$ converging uniformly to a function $f(x)$ on a set S

$\forall \varepsilon > 0 \ \exists N$ s.t. $n > N \Rightarrow$

$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in S$.

(b) Prove that the sequence of functions above converges pointwise to $f(x)$.

Fix x.

If $x \neq 0$, $e^{x^2} > 1$ and

$$\lim_{n\to\infty} e^{-nx^2} = \lim_{n\to\infty} \frac{1}{(e^{x^2})^n} = 0 = f(x)$$

If $x = 0$, $e^{-x^2} = 1$ and

$$\lim_{n\to\infty} e^{-nx^2} = \lim_{n\to\infty} 1 = 1 = f(x)$$

(c) Prove that the sequence of functions above does NOT converge uniformly to $f(x)$. If you use a theorem from class, make sure to carefully state it.

Each $f_n(x)$ is continuous on $\mathbb{R}$,

so if we have uniform convergence,

the limit $f(x)$ must be continuous.

Since $f(\mathbb{R}) = \{0, 1\}$ isn't an interval

it's not, so no uniform convergence.

6. (10 points)

(a) Write down the ϵ,δ-definition of continuity.

Let $a \in Dom(f)$. Then f is continuous at $x=a$ if $\forall\, \varepsilon>0\ \exists\, \delta>0$ s.t.

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon \quad \forall\, x \in Dom(f).$$

(b) Suppose that $f(x)$ is a continuous function on $\mathbb{R}$ with the property that $f(x) = f(x/2)$ for all real numbers x. Prove that $f(x)$ must be a constant function.

Induction proves $f(x) = f\left(\frac{x}{2^n}\right)$ for all $n \in \mathbb{N}$.

Therefore by continuity,

$$f(x) = \lim_{n\to\infty} f\left(\frac{x}{2^n}\right) = f(0) \quad \text{for all } x.$$

$\therefore$ $f(x)$ is constant.

(c) Show by example that (b) is false if we drop the continuity assumption.

Consider

$$f(x) = \begin{cases} 1, & x=0 \\ 0, & x \neq 0 \end{cases}$$

Then if $x=0$, $\frac{x}{2}=x$ and so $f\left(\frac{x}{2}\right)=f(x)$

if $x \neq 0$, then $\frac{x}{2} \neq 0$ so

$$f\left(\frac{x}{2}\right) = 0 = f(x)$$

Thus $f\left(\frac{x}{2}\right) = f(x) \quad \forall\, x$.

7. (10 points)

(a) State the definition of $f(x)$ being uniformly continuous on a subset $S \subseteq \mathrm{Dom}(f)$

$\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ s.t.

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$\forall \ x \in \mathrm{Dom}(f), \ a \in S$

(b) Using ϵ and δ, prove that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on the interval $[2, \infty)$.

Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{4}{3}\varepsilon\}$

Then if $x \in \mathrm{Dom}(f)$, $a \in [2,\infty)$ with $|x-a| < \delta$ we have $a \geq 2$ and so $x > a - 1 \geq 1$.

Thus $\frac{1}{|x|} < 1$, $\frac{1}{|a|} \leq \frac{1}{2}$ and $\frac{1}{|x|a^2} + \frac{1}{|a|x^2} < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, making

$|f(x) - f(a)| = \left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{x^2 - a^2}{x^2a^2}\right| = |x-a| \cdot \frac{|x+a|}{x^2a^2} \leq |x-a|\left(\frac{1}{|x|a^2} + \frac{1}{|a|x^2}\right) < \frac{3}{4}\delta$
$= \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we are done!

(c) Explain why $f(x)$ from part (b) is NOT uniformly continuous on $(0, \infty)$.

If it is, then $f(x)$ must extend to a continuous function on $[0, \infty)$.

In particular, $f((0,1))$ would be bounded.

However $f((0,1)) = (1, \infty)$ is unbounded.