MATH 350-2 Advanced Calculus

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Outline

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Warm-up Challenge

Problem

Show that the closed ball

$$\overline{B}(\vec{x}; r) = \{\vec{y} \in \mathbb{R}^n : |\vec{x} - \vec{y}| \leq r\}$$

is a closed set.

Point set topology

We have been studying the beginnings of **point set topology**. In topology, we start with a set X called the **space** and a collection of special subsets called **open sets**.

Fundamental questions:

- kinds of points: interior points, boundary points, accumulation points, adherent points
- kinds of sets: open, closed, connected, component, compact
- kinds of functions: continuous, homeomorphism

Answers must be in terms of open sets

Open covers

Definition

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Examples:

- $\{\mathbb{R}\}$ is an open cover of the interval $(0, 1]$

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Examples:

- $\{\mathbb{R}\}$ is an open cover of the interval $(0, 1]$
- $\{(0, 1/2), (1/2, 2)\}$ is an open cover of the interval $(0, 1]$
- $\{(\frac{1}{n}, \frac{n+1}{n}) : n \in \mathbb{Z}_+\}$ is an open cover of $(0, 1]$

Lindelöf Covering Theorem

Theorem (Lindelöf Covering Theorem)

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

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Question

Can we do better than this?

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Definition

A set $A \subseteq \mathbb{R}^n$ is called **compact** if every open cover of A has a *finite* subcover.

Challenge

Problem

Show that a singleton set $A = \{\vec{x}\}$ is compact.

Solution

Suppose that $\{U_i : i \in I\}$ is an open cover of X.
Then $A \subseteq \bigcup_{i \in I} U_i$.
This means that $\vec{x} \in \bigcup_{i \in I} U_i$.
Therefore $\vec{x} \in U_j$ for some $j \in I$.
It follows that $\{U_j\}$ is a subcover of A consisting of a single set.
Thus every open cover has a finite subcover, making A compact!

Bounded sets

We will show that the compact subsets of $\mathbb{R}^n$ are exactly the closed sets that are bounded.

Definition

A subset $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{x} \in \mathbb{R}^n$ and $r > 0$ such that $A \subseteq B(\vec{x}; r)$.

Theorem

If A is a compact set, then A must be bounded.

Proof.

Suppose A is a compact set.
Consider the family of sets $\{U_i : i \in I\}$ where $I = \mathbb{Z}_+$ and $U_i = B(\vec{0}; i)$.

$\square$

Challenge

Problem

Show that $\{U_i : i \in I\}$ is an open cover of A.

Solution

Note that U_i is an open subset for all i, so we need only show $A \subseteq \bigcup_{i \in I} U_i$.

$$\bigcup_{i \in I} U_i = \bigcup_{i=1}^{\infty} B(\vec{0}; i) \subseteq \mathbb{R}^n.$$

Moreover, if $\vec{x} \in \mathbb{R}^n$, then we can choose $N \in \mathbb{Z}_+$ such that $N > |\vec{x}|$.
Therefore $\vec{x} \in B(\vec{0}; N)$ and it follows $\vec{x} \in \bigcup_{i \in I} U_i$.
Thus $\mathbb{R}^n = \bigcup_{i \in I} U_i$, and since $A \subseteq \mathbb{R}^n$, we see $A \subseteq \bigcup_{i \in I} U_i$.

Challenge

Problem

Explain what it means that $\{U_i : i \in I\}$ has a finite subcover.

Solution

There is a finite subset $\{i_1, i_2, \ldots, i_m\} \subseteq I$ with $A \subseteq \bigcup_{k=1}^{m} U_{i_k}$.

Challenge

Problem

Show that $\bigcup_{k=1}^{m} U_{i_k}$ is an open ball, and thus A is bounded.

Solution

Without loss of generality, $i_1 < i_2 < \dots i_m$. Then

$$B(\vec{0}; i_1) \subseteq B(\vec{0}; i_2) \subseteq \cdots \subseteq B(\vec{0}; i_m).$$

This means

$$A \subseteq \bigcup_{k=1}^{m} A_{i_k} = \bigcup_{k=1}^{m} B(\vec{0}; i_k) = B(\vec{0}, i_m).$$

This shows A is bounded.

Compact implies closed

Theorem

Suppose that $A \subseteq \mathbb{R}^n$ is compact. Then A is closed.

Proof.

Suppose that A is compact and let $\vec{x} \in \mathbb{R}^n \backslash A$.
We will show $\vec{x}$ cannot be an adherent point of A and therefore A contains all of its adherent points.
Consider the family of sets $\{U_i : i \in I\}$ where $I = \mathbb{Z}_+$ and $U_i = \mathbb{R}^n \backslash \overline{B}(\vec{x}; 1/i)$. □

Challenge

Problem

Prove that $\bigcup_{i\in I} U_i = \mathbb{R}^n \backslash \{\vec{x}\}$.

Challenge

Problem

Explain why $\{U_i : i \in I\}$ is an open cover of A.

Challenge

Problem

Show that if $\{U_i : i \in I\}$ has a finite subcover, then $\vec{x}$ can't be an adherent point of $\vec{A}$.

Bolzano-Weierstrass Theorem

So far, we have seen compact sets are closed and bounded.
We want to prove the opposite is true too!
This will require obtaining some other fundamental results about real numbers.

Theorem (Bolzano-Weierstrass Theorem)

A bounded set $A \subseteq \mathbb{R}^n$ with infinitely many points will contain an accumulation point.

Cantor Intersection Theorem

Theorem (Cantor Intersection Theorem)

Suppose that $C_1, C_2, C_3, \dots$ are non-empty closed, bounded sets with $C_{i+1} \subseteq C_i$ for all $i \in I = \mathbb{Z}_+$. Then $\bigcap_{i \in I} C_i$ is also non-empty.

Proof.

If C_i is finite for some i, then the proof is simple. Assume otherwise.
Then choose $\vec{x}_i \in C_i$ for all $i \in I$ all distinct.
The set $A = \{\vec{x}_i : i \in \mathbb{Z}_+\}$ is infinite and bounded (because it is contained in C_1).
Therefore the Bolzano-Weierstrass Theorem tells us it has an accumulation point $\vec{x}$.

□

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Proof by Contradiction.

In fact, if we let $A_m = \{\vec{x}_i : i \in \mathbb{Z}_+,\ i \geq m\}$, then $\vec{x}$ is an accumulation point of A_m for all m.

Moreover, $A_m \subseteq C_m$ so $\vec{x}$ is an accumulation point of C_m for all m.

Since C_m is closed, it follows $\vec{x} \in C_m$, and thus $\vec{x} \in \bigcap_{i \in I} C_i$. In particular the intersection is non-empty! $\square$

Heine-Borel Theorem

Theorem (Heine-Borel Theorem)

A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

We already proved compact implies closed and bounded.
Assume A is closed and bounded.
Take an open cover $\{U_i : i \in I\}$.
Then by Lindelöf's Theorem, we can take a countable subcover.
Therefore without loss of generality, assume $I = \mathbb{Z}_+$.
Define closed sets $C_1, C_2, C_3, \dots$ by

$$C_m = A \cap \left(\mathbb{R}^n \backslash \bigcup_{i=1}^{m} U_i\right).$$

□

Heine-Borel Theorem

Theorem (Heine-Borel Theorem)

A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

If C_m is empty for some m, then $\{U_1, \dots, U_m\}$ covers A.
To prove this must happen, we assume otherwise.
Notice that $C_{m+1} \subseteq C_m$ for all i.
Since they are non-empty, closed and bounded, the Cantor Intersection Theorem says $\bigcap_{m=1}^{\infty} C_m$ is non-empty.
Let $\vec{x} \in \bigcap_{m=1}^{\infty} C_m$.
Since $\vec{x} \in A \subseteq \bigcup_{i \in I} U_i$, we have that $\vec{x} \in U_k$ for some $k \in I$, meaning $\vec{x} \notin C_k$.
This is a contradiction. $\square$