MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

September 30, 2024

Outline

Outline

1. Real Analysis Lecture 9
   - More on Open Sets
   - Closed Sets
   - Compactness

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^{n} U_i$ is open.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^{n} U_i$ is open.

Proof.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.
Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \dots, U_n$ are open sets. Then $\bigcap_{i=1}^{n} U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^{n} U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.
Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.
Take $r = \min\{r_i : 1 \leq i \leq n\}$.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.
Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.
Take $r = \min\{r_i : 1 \leq i \leq n\}$.
This means $B(\vec{x}; r) \subseteq B(\vec{x}; r_i) \subseteq U_i$ for all i.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.
Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.
Take $r = \min\{r_i : 1 \leq i \leq n\}$.
This means $B(\vec{x}; r) \subseteq B(\vec{x}; r_i) \subseteq U_i$ for all i.
Therefore $B(\vec{x}; r) \subseteq \bigcap_{i=1}^n U_i$.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^{n} U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^{n} U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.
Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.
Take $r = \min\{r_i : 1 \leq i \leq n\}$.
This means $B(\vec{x}; r) \subseteq B(\vec{x}; r_i) \subseteq U_i$ for all i.
Therefore $B(\vec{x}; r) \subseteq \bigcap_{i=1}^{n} U_i$.
Thus $\vec{x}$ is an interior point of $\bigcap_{i=1}^{n} U_i$.

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \dots, U_n$ are open sets. Then $\bigcap_{i=1}^n U_i$ is open.

Proof.

Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.
Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.
Take $r = \min\{r_i : 1 \leq i \leq n\}$.
This means $B(\vec{x}; r) \subseteq B(\vec{x}; r_i) \subseteq U_i$ for all i.
Therefore $B(\vec{x}; r) \subseteq \bigcap_{i=1}^n U_i$.
Thus $\vec{x}$ is an interior point of $\bigcap_{i=1}^n U_i$.
Since $\vec{x}$ is arbitrary, this proves that $\bigcap_{i=1}^n U_i$ is open. $\square$

Challenge

Problem

Show that the infinite intersection

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$$

is not open.

Component intervals

Let $U \subseteq \mathbb{R}$ be open.

Component intervals

Let $U \subseteq \mathbb{R}$ be open.

Definition

A **component interval** of U is an interval I with $I \subseteq U$ and with the property that if J is an interval and $I \subseteq J$, then $J \not\subseteq U$.

Component intervals

Let $U \subseteq \mathbb{R}$ be open.

Definition

A **component interval** of U is an interval I with $I \subseteq U$ and with the property that if J is an interval and $I \subseteq J$, then $J \not\subseteq U$.

- $(0, 1)$ is a component interval of $\mathbb{R} \backslash \{0, 1, 2, 3\}$

Component intervals

Let $U \subseteq \mathbb{R}$ be open.

Definition

A **component interval** of U is an interval I with $I \subseteq U$ and with the property that if J is an interval and $I \subseteq J$, then $J \not\subseteq U$.

- $(0, 1)$ is a component interval of $\mathbb{R} \backslash \{0, 1, 2, 3\}$
- $(-\infty, 0)$ is also a component interval of $\mathbb{R} \backslash \{0, 1, 2, 3\}$

Component intervals

Let $U \subseteq \mathbb{R}$ be open.

Definition

A **component interval** of U is an interval I with $I \subseteq U$ and with the property that if J is an interval and $I \subseteq J$, then $J \not\subseteq U$.

- $(0, 1)$ is a component interval of $\mathbb{R} \backslash \{0, 1, 2, 3\}$
- $(-\infty, 0)$ is also a component interval of $\mathbb{R} \backslash \{0, 1, 2, 3\}$
- we will show all open sets of $\mathbb{R}$ are made of component intervals!

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Proof.

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Proof.

Suppose that $I_1 \cap I_2 \neq \varnothing$.

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Proof.

Suppose that $I_1 \cap I_2 \neq \varnothing$.
Then $J := I_1 \cup I_2$ is an interval.

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Proof.

Suppose that $I_1 \cap I_2 \neq \varnothing$.
Then $J := I_1 \cup I_2$ is an interval.
Also $I_1 \subseteq U$ and $I_2 \subseteq U$, so $J \subseteq U$.

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Proof.

Suppose that $I_1 \cap I_2 \neq \varnothing$.
Then $J := I_1 \cup I_2$ is an interval.
Also $I_1 \subseteq U$ and $I_2 \subseteq U$, so $J \subseteq U$.
Since I_i is a component interval and $I_i \subseteq J \subseteq U$, we have $I_i = J$.

Component intervals

Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

Proof.

Suppose that $I_1 \cap I_2 \neq \varnothing$.
Then $J := I_1 \cup I_2$ is an interval.
Also $I_1 \subseteq U$ and $I_2 \subseteq U$, so $J \subseteq U$.
Since I_i is a component interval and $I_i \subseteq J \subseteq U$, we have $I_i = J$.
In particular $I_1 = J = I_2$. $\square$

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Proof.

Uniqueness follows from previous Lemma, so we only need existence.

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Proof.

Uniqueness follows from previous Lemma, so we only need existence.
Suppose that $x \in U$ and consider

$$A = \{a : (a, x) \subseteq S\}, \quad \text{and} \quad B = \{b : (x, b) \subseteq S\}$$

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Proof.

Uniqueness follows from previous Lemma, so we only need existence.
Suppose that $x \in U$ and consider

$$A = \{a : (a, x) \subseteq S\}, \quad \text{and} \quad B = \{b : (x, b) \subseteq S\}$$

If A is not bounded below, let $a = -\infty$. Otherwise, let $a = \inf(A)$.

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Proof.

Uniqueness follows from previous Lemma, so we only need existence.
Suppose that $x \in U$ and consider

$$A = \{a : (a, x) \subseteq S\}, \quad \text{and} \quad B = \{b : (x, b) \subseteq S\}$$

If A is not bounded below, let $a = -\infty$. Otherwise, let $a = \inf(A)$.
If B is not bounded above, let $b = \infty$. Otherwise, let $b = \sup(B)$.

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Proof.

Uniqueness follows from previous Lemma, so we only need existence.
Suppose that $x \in U$ and consider

$$A = \{a : (a, x) \subseteq S\}, \quad \text{and} \quad B = \{b : (x, b) \subseteq S\}$$

If A is not bounded below, let $a = -\infty$. Otherwise, let $a = \inf(A)$.
If B is not bounded above, let $b = \infty$. Otherwise, let $b = \sup(B)$.
Claim: (a, b) is a component interval of U containing x. □

Challenge

Problem

Prove that (a, b) is a component interval of U.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

From the previous theorem, each $x \in U$ is contained in a unique component interval I_x of U.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

From the previous theorem, each $x \in U$ is contained in a unique component interval I_x of U.
Therefore

$$U = \bigcup_{x \in U} I_x$$

is a union of open intervals!

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

From the previous theorem, each $x \in U$ is contained in a unique component interval I_x of U.
Therefore

$$U = \bigcup_{x \in U} I_x$$

is a union of open intervals!
Uh oh ... this isn't a **countable** union. □

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

Instead, we consider rational points $U \cap \mathbb{Q}$.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

Instead, we consider rational points $U \cap \mathbb{Q}$.
If $x \in U$, then $x \in I_x \subseteq U$.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

Instead, we consider rational points $U \cap \mathbb{Q}$.
If $x \in U$, then $x \in I_x \subseteq U$.
Any interval contains a rational number $r \in I_x$.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

Instead, we consider rational points $U \cap \mathbb{Q}$.
If $x \in U$, then $x \in I_x \subseteq U$.
Any interval contains a rational number $r \in I_x$.
Since different component intervals must be disjoint, $I_x = I_r$.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

Instead, we consider rational points $U \cap \mathbb{Q}$.
If $x \in U$, then $x \in I_x \subseteq U$.
Any interval contains a rational number $r \in I_x$.
Since different component intervals must be disjoint, $I_x = I_r$.
Thus $x \in \bigcup_{r \in U \cap \mathbb{Q}} I_r \subseteq U$.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.

Proof.

Instead, we consider rational points $U \cap \mathbb{Q}$.
If $x \in U$, then $x \in I_x \subseteq U$.
Any interval contains a rational number $r \in I_x$.
Since different component intervals must be disjoint, $I_x = I_r$.
Thus $x \in \bigcup_{r \in U \cap \mathbb{Q}} I_r \subseteq U$.
Since x was arbitrary,

$$U = \bigcup_{r \in U \cap \mathbb{Q}} I_r$$

Outline

1. Real Analysis Lecture 9
   - More on Open Sets
   - Closed Sets
   - Compactness

Closed sets

Definition

A set $A \subseteq \mathbb{R}^n$ is called **closed** if it is the complement $\mathbb{R}^n \backslash A$ of an open set.

Examples:

- singleton sets!
- products of closed intervals

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \setminus \{\vec{a}\}$ is open.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \setminus \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.
Take $r = |\vec{x} - \vec{a}|$.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.
Take $r = |\vec{x} - \vec{a}|$. Then $B(\vec{x}; r)$ does not contain $\vec{a}$.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.
Take $r = |\vec{x} - \vec{a}|$. Then $B(\vec{x}; r)$ does not contain $\vec{a}$.
Therefore $B(\vec{x}; r) \subseteq U$, so that $\vec{x}$ is an interior point.

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.
Take $r = |\vec{x} - \vec{a}|$. Then $B(\vec{x}; r)$ does not contain $\vec{a}$.
Therefore $B(\vec{x}; r) \subseteq U$, so that $\vec{x}$ is an interior point.
Since $\vec{x}$ was arbitrary, this proves U is open.

Challenge!

Problem

Prove that the **closed square**

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq c,\ b \leq y \leq d\}$$

is a closed set.

Challenge!

Problem

Prove that the **closed square**

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq c,\ b \leq y \leq d\}$$

is a closed set.

Solution

$\mathbb{R}^2 \backslash ([a, b] \times [c, d])$ is the same as

$$((-\infty, a) \times \mathbb{R}) \cup ((b, \infty) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, c)) \cup (\mathbb{R} \times (d, \infty))$$

Challenge!

Problem

Prove that the **closed square**

$$[a,b]\times[c,d]=\{(x,y)\in\mathbb{R}^2 : a\leq x\leq c,\ b\leq y\leq d\}$$

is a closed set.

Solution

$\mathbb{R}^2\backslash([a,b]\times[c,d])$ is the same as

$$((-\infty,a)\times\mathbb{R})\cup((b,\infty)\times\mathbb{R})\cup(\mathbb{R}\times(-\infty,c))\cup(\mathbb{R}\times(d,\infty))$$

Union of open sets is open!

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Examples:

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Examples:

- every point in A is adherent

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Examples:

- every point in A is adherent
- accumulation points are adherent points

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Examples:

- every point in A is adherent
- accumulation points are adherent points
- -1 is an accumulation point of $(-1, 1)$.

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Examples:

- every point in A is adherent
- accumulation points are adherent points
- -1 is an accumulation point of $(-1, 1)$.
- suprema and infima are accumulation points!

Adherent and Accumulation Points

Definition

A point $\vec{x} \in A$ is called an **adherent point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A.
It is called an **accumulation point** if for all $r > 0$ the ball $B(\vec{x}; r)$ contains at least one element of A different from $\vec{x}$.

Examples:

- every point in A is adherent
- accumulation points are adherent points
- -1 is an accumulation point of $(-1, 1)$.
- suprema and infima are accumulation points!
- 0 is an accumulation point of $\{1/1, 1/2, 1/3, \ldots\}$

Characterizing accumulation points

Theorem (Apostol Theorem 3.17)

A point $\vec{x}$ is an accumulation point of A if for all $r > 0$, the ball $B(\vec{x}; r)$ contains infinitely many points of A.

Closure of a set

Definition

The **closure** of a set A is the set $\overline{A}$ of all adherent points of A

Closure of a set

Definition

The **closure** of a set A is the set $\overline{A}$ of all adherent points of A

Theorem (Apostol Theorem 3.18, 3.20, 3.22)

A set is closed if and only if it contains all of its adherent points (ie. $A = \overline{A}$), or equivalently if and only if A contains all of its accumulation points.

Outline

Intuition of compactness

We are moving toward a very important notion called **compactness.**

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

- no “missing points" in the set

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

- no “missing points" in the set
- sequences have convergent subsequences

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

- no “missing points" in the set
- sequences have convergent subsequences
- Cauchy sequences converge

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

- no “missing points" in the set
- sequences have convergent subsequences
- Cauchy sequences converge
- open covers have finite subcovers

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

- no “missing points" in the set
- sequences have convergent subsequences
- Cauchy sequences converge
- open covers have finite subcovers
- in $\mathbb{R}^n$: compact = closed and bounded

Open covers

Definition

A **open cover** of a set $A \subseteq \mathbb{R}^n$ is a family $\{U_i : i \in I\}$ of open sets with $A \subseteq \bigcup_{i \in I} U_i$.

Open covers

Definition

A **open cover** of a set $A \subseteq \mathbb{R}^n$ is a family $\{U_i : i \in I\}$ of open sets with $A \subseteq \bigcup_{i \in I} U_i$.
A **subcover** is a family $\{U_j : j \in J\}$ with $J \subseteq I$ which is still covers A, ie. $A \subseteq \bigcup_{j \in J} U_j$.

Open covers

Definition

A **open cover** of a set $A \subseteq \mathbb{R}^n$ is a family $\{U_i : i \in I\}$ of open sets with $A \subseteq \bigcup_{i \in I} U_i$.
A **subcover** is a family $\{U_j : j \in J\}$ with $J \subseteq I$ which is still covers A, ie. $A \subseteq \bigcup_{j \in J} U_j$.

Examples:

Open covers

Definition

A **open cover** of a set $A \subseteq \mathbb{R}^n$ is a family $\{U_i : i \in I\}$ of open sets with $A \subseteq \bigcup_{i \in I} U_i$.
A **subcover** is a family $\{U_j : j \in J\}$ with $J \subseteq I$ which is still covers A, ie. $A \subseteq \bigcup_{j \in J} U_j$.

Examples:

- $\{\mathbb{R}\}$ is an open cover of the interval $(0, 1]$

Open covers

Definition

A **open cover** of a set $A \subseteq \mathbb{R}^n$ is a family $\{U_i : i \in I\}$ of open sets with $A \subseteq \bigcup_{i \in I} U_i$.
A **subcover** is a family $\{U_j : j \in J\}$ with $J \subseteq I$ which is still covers A, ie. $A \subseteq \bigcup_{j \in J} U_j$.

Examples:

- $\{\mathbb{R}\}$ is an open cover of the interval $(0, 1]$
- $\{(\frac{1}{n}, \frac{n+1}{n}) : n \in \mathbb{Z}_+\}$ is an open cover of $(0, 1]$

Lindelöf Covering Theorem

Theorem

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

Lindelöf Covering Theorem

Theorem

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

Question

Can we do better than this?

Lindelöf Covering Theorem

Theorem

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

Question

Can we do better than this?

Definition

A set $A \subseteq \mathbb{R}^n$ is called **compact** if every open cover of A has a *finite* subcover.

Roadmap

Definition

A set $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{a} \in \mathbb{R}^n$ and $r > 0$ with $A \subseteq B(\vec{a}; r)$.

Roadmap

Definition

A set $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{a} \in \mathbb{R}^n$ and $r > 0$ with $A \subseteq B(\vec{a}; r)$.

We wish to prove the **Heine-Borel Theorem** that closed and bounded sets are compact.

Roadmap

Definition

A set $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{a} \in \mathbb{R}^n$ and $r > 0$ with $A \subseteq B(\vec{a}; r)$.

We wish to prove the **Heine-Borel Theorem** that closed and bounded sets are compact.

1. We prove the Bolzano-Weierstrass Theorem

Roadmap

Definition

A set $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{a} \in \mathbb{R}^n$ and $r > 0$ with $A \subseteq B(\vec{a}; r)$.

We wish to prove the **Heine-Borel Theorem** that closed and bounded sets are compact.

1. We prove the Bolzano-Weierstrass Theorem
2. Then use Bolzano-Weierstrass to prove the Cantor Intersection Theorem

Roadmap

Definition

A set $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{a} \in \mathbb{R}^n$ and $r > 0$ with $A \subseteq B(\vec{a}; r)$.

We wish to prove the **Heine-Borel Theorem** that closed and bounded sets are compact.

1. We prove the Bolzano-Weierstrass Theorem
2. Then use Bolzano-Weierstrass to prove the Cantor Intersection Theorem
3. We use the Cantor Intersection Theorem to prove the Heine-Borel Theorem

Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass Theorem)

A bounded set $A \subseteq \mathbb{R}^n$ with infinitely many points will contain an accumulation point.