MATH 350-2 Advanced Calculus

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Outline

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Decimal approximations

A **finite decimal expansion** is an expression

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n},$$

where $a_0 \in \mathbb{Z}_+$ and $0 \leq a_k \leq 9$ for $1 \leq k \leq n$.

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Notation:

$$a_0.a_1a_2a_3\ldots a_n.$$

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Any positive real number $x > 0$ can be approximated by a finite decimal expansion.

Decimal approximations

Theorem (Apostol Theorem 1.20)

For any real $x > 0$ and $n \in \mathbb{Z}_+$, there exists a finite decimal expansion $r_n = a_0.a_1a_2\dots a_n$ with

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

Decimal approximations

Proof.

Consider the set

$$A = \{a \in \mathbb{Z} : a \leq x\}.$$

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Define $a_1, a_2, a_3, \ldots$ and $x_1, x_2, x_3, \ldots$ recursively by

$$a_k = \max\{a \in \mathbb{Z}_+ : a \leq 10x_k\}$$

and $x_{k+1} = 10x_k - a_k$.

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and $x_{k+1} = 10x_k - a_k$. Then $0 \leq a_k \leq 9$ for all $k \geq 1$ and
$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n} \leq x < a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n + 1}{10^n}$$
□

Decimal expansions

We say that $x > 0$ has the decimal expansion $a_0.a_1a_2a_3\ldots$ and write

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$$1 \not< 0 + \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9+1}{10^n} = 1.$$

Challenge!

Problem

Find the decimal expansion of $1/7$.

Challenge!

Problem

Find a rational number whose decimal expansion is

$$0.45454545\ldots.$$

Challenge!

Problem

Which kinds of numbers have decimal expansions that end? (Meaning that after a while, all the decimals are zero?

Challenge!

Problem

Which kinds of numbers have decimal expansions that repeat?

Outline

1. Real Analysis Lecture 3
   - Decimal expansions
   - The Triangle Inequality
   - Sets, Relations, Functions

Absolute value

The **absolute value** of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In particular,

$$0 \leq |x|$$

and also

$$-|x| \leq x \leq |x|.$$

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Prove Apostol Theorem 1.21 that if $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.

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If $|x| \leq a$ then $-a \leq -|x|$ and therefore

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Conversely, if $-a \leq x \leq a$ then

$$x \geq 0 \Rightarrow |x| = x \leq a$$

$$x \leq 0 \Rightarrow |x| = -x \leq -(-a) = a$$

Basic triangle inequality

Theorem (Triangle inequality)

For any real numbers $x, y \in \mathbb{R}$ we have

$$|x + y| \leq |x| + |y|.$$

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Adding these together, we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

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It follows from the previous theorem that

$$|x + y| \leq |x| + |y|$$

Advanced triangle inequality

Theorem (Triangle inequality)

For any real numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ we have

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|.$$

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Proof.

Induction. $\square$

Higher-dimensional triangle inequality

Theorem (Cauchy-Schwartz Inequality (Apostol Theorem 1.23))

If $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are real numbers, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$$

If y_k isn't always zero, then equality holds if and only if there exists $t \in \mathbb{R}$ with $x_k = ty_k$ for all k.

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If y_k isn't always zero, then equality holds if and only if there exists $t \in \mathbb{R}$ with $x_k = t y_k$ for all k.

Vector version:

$$(\vec{x} \cdot \vec{y})^2 \leq |\vec{x}|^2 |\vec{y}|^2.$$

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Take $t = -\left(\sum_{k=1}^{n} x_k y_k\right) / \left(\sum_{k=1}^{n} y_k^2\right)$:

$$\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{k=1}^{n} x_k y_k\right)^2}{\left(\sum_{k=1}^{n} y_k^2\right)} \geq 0.$$

$\square$

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Theorem (Minkowski inequality)

For any real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $y_1, y_2, \ldots, y_n \in \mathbb{R}$ we have

$$\left(\sum_{k=1}^{n}(x_k + y_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}$$

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A higher dimensional triangle inequality!

Higher-dimensional triangle inequality

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$$\begin{aligned}\sum_{k=1}^{n}(x_k+y_k)^2 &\le \sum_{k=1}^{n}(|x_k|+|y_k|)|x_k+y_k| \\ &= \sum_{k=1}^{n}|x_k|\cdot|x_k+y_k| + \sum_{k=1}^{n}|y_k|\cdot|x_k+y_k|\end{aligned}$$

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$$\leq \left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2}$$

$$= \left[\left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\right]\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2}$$

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$$= \left[\left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n} |y_k|^2\right)^{1/2}\right] \left(\sum_{k=1}^{n} |x_k + y_k|^2\right)^{1/2}$$

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- $\{n \in \mathbb{Z} : n \text{ is prime}\}$

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- even relations and functions are sets!

Challenge!

Problem

Prove that for ordered pairs (a, b) and (c, d) that

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = c \text{ and } b = d$$

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Clearly, if $a = c$ and $b = d$, then $\{a, \{a, b\}\} = \{c, \{c, d\}\}$

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The tough part is the opposite direction!

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It follows that $d = c = b = a$.

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Specifically the regularity axiom for the set $\{a, c\}$...

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Challenge!

Problem

Give an example of a relation on $\mathbb{R}$ which is symmetric and transitive but not reflexive.

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The set

$$\mathrm{img}(f) = \{f(a) : a \in A\}$$

is called the **range** or **image** of f

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Determine all the equivalence relations on $\mathbb{R}$ which are also functions.

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This means $f(x) = x$ for all x.
Thus the only function which is an equivalence relation is the identity function

$$f(x) = x.$$

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If it satisfies both properties, it is called **bijective.**

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If it is, we call it the **inverse** of f.

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