MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 9
   - More on Open Sets
   - Closed Sets
   - Compactness

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   - More on Open Sets
   - Closed Sets
   - Compactness

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

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Therefore $B(\vec{x}; r) \subseteq \bigcap_{i=1}^n U_i$.
Thus $\vec{x}$ is an interior point of $\bigcap_{i=1}^n U_i$.
Since $\vec{x}$ is arbitrary, this proves that $\bigcap_{i=1}^n U_i$ is open. $\square$

Challenge

Problem

Show that the infinite intersection

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is not open.

Component intervals

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- we will show all open sets of $\mathbb{R}$ are made of component intervals!

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Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

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Then $J := I_1 \cup I_2$ is an interval.

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In particular $I_1 = J = I_2$. □

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If A is not bounded below, let $a = -\infty$. Otherwise, let $a = \inf(A)$.
If B is not bounded above, let $b = \infty$. Otherwise, let $b = \sup(B)$.
Claim: (a, b) is a component interval of U containing x. □

Challenge

Problem

Prove that (a, b) is a component interval of U.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Sets in $\mathbb{R}$)

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Uh oh ... this isn't a **countable** union. □

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Any interval contains a rational number $r \in I_x$.

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Any interval contains a rational number $r \in I_x$.
Since different component intervals must be disjoint, $I_x = I_r$.
Thus $x \in \bigcup_{r \in U \cap \mathbb{Q}} I_r \subseteq U$.
Since x was arbitrary,

$$U = \bigcup_{r \in U \cap \mathbb{Q}} I_r$$

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Examples:

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- products of closed intervals

Challenge!

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Prove that a singleton set

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in $\mathbb{R}^n$ is closed.

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Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.

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Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$.

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We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.

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Therefore $B(\vec{x}; r) \subseteq U$, so that $\vec{x}$ is an interior point.

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Since $\vec{x}$ was arbitrary, this proves U is open.

Challenge!

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Prove that the **closed square**

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

is a closed set.

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Union of open sets is open!

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- suprema and infima are accumulation points!
- 0 is an accumulation point of $\{1/1, 1/2, 1/3, \dots\}$

Characterizing accumulation points

Theorem (Apostol Theorem 3.17)

A point $\vec{x}$ is an accumulation point of A if for all $r > 0$, the ball $B(\vec{x}; r)$ contains infinitely many points of A.

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Theorem (Apostol Theorem 3.18, 3.20, 3.22)

A set is closed if and only if it contains all of its adherent points (ie. $A = \overline{A}$), or equivalently if and only if A contains all of its accumulation points.

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- sequences have convergent subsequences
- Cauchy sequences converge
- open covers have finite subcovers

Intuition of compactness

We are moving toward a very important notion called **compactness.**
This important concept is captured in many different ways:

- no “missing points" in the set
- sequences have convergent subsequences
- Cauchy sequences converge
- open covers have finite subcovers
- in $\mathbb{R}^n$: compact = closed and bounded

Open covers

Definition

A **open cover** of a set $A \subseteq \mathbb{R}^n$ is a family $\{U_i : i \in I\}$ of open sets with $A \subseteq \bigcup_{i \in I} U_i$.

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- $\{(\frac{1}{n}, \frac{n+1}{n}) : n \in \mathbb{Z}_+\}$ is an open cover of $(0, 1]$

Lindelöf Covering Theorem

Theorem

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

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Question

Can we do better than this?

Definition

A set $A \subseteq \mathbb{R}^n$ is called **compact** if every open cover of A has a *finite* subcover.

Roadmap

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A set $A \subseteq \mathbb{R}^n$ is called **bounded** if there exists $\vec{a} \in \mathbb{R}^n$ and $r > 0$ with $A \subseteq B(\vec{a}; r)$.

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1. We prove the Bolzano-Weierstrass Theorem
2. Then use Bolzano-Weierstrass to prove the Cantor Intersection Theorem
3. We use the Cantor Intersection Theorem to prove the Heine-Borel Theorem

Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass Theorem)

A bounded set $A \subseteq \mathbb{R}^n$ with infinitely many points will contain an accumulation point.