MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

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Outline

1. Real Analysis Lecture 7
   - More with cardinality
   - Set algebra
   - Open Balls and Open Sets

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Ultimate Cantor diagonalization

The **power set** $\mathcal{P}(A)$ of a set A is

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Suppose $f : A \to \mathcal{P}(A)$ is surjective.
Consider the set

$$S = \{a \in A : a \notin f(a)\}.$$

□

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Problem

Consider the statement $x \in S$. What can you conclude?

Challenge

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Is there a set with cardinality larger than $\mathbb{R}$?

Challenge

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Show that the cardinality of the line segment $(0, 1)$ and the square $(0, 1) \times (0, 1)$ is the same.

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NOTATION:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

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- index set $I = (1, 2)$
- family of sets $\{A_i : i \in I\}$
- $A_i = [0, i]$

Problem

Determine $\bigcup_{i \in I} A_i$.

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$f(j, k) = a_{i_j,k}$. Surjection! □

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Hint: use the previous theorem!

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(f) $|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$

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(a) (scaling) $|c\vec{x}| = |c|\ |\vec{x}|$
(b) (Cauchy-Schwartz) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}|\ |\vec{y}|$