MATH 350-2 Advanced Calculus

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August 26, 2024

Outline

1. Real Analysis Lecture 2
   - Types of real numbers
   - Integers
   - Upper Bound and Supremum
   - Decimal expansions

Outline

Real numbers

The **real numbers** are the unique field $\mathbb{R}$ satisfying Axiom 1-Axiom 9, plus an extra axiom called the completeness axiom.

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- Intuitively, it represents the fact that the real line has no holes or gaps.

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Examples:

$$\mathbb{R}, \quad \mathbb{Q}, \quad (0, \infty), \quad \mathbb{Z}, \quad \mathbb{N}, \ldots$$

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- rationals are

$$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z},\ b \neq 0\}.$$

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Proof:

Let

$$S = \{n \in \mathbb{Z}_+ : 2|n(n+1)\}$$

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. We see that $1 \in S$ because 2 divides $1(1+1)$.

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Now suppose $x \in S$. (This is our usual inductive assumption).

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This means $(x+1)(x+2) = x(x+1) + 2(x+1) = 2(k+x+1)$.

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By the Principle of Induction, this means $\mathbb{Z}_+ \subseteq S$.
In other words $n \in S$ for every positive integer n.
Hence 2 divides $n(n+1)$ for every positive integers n.

Prime numbers

A positive integer p is **prime** if its only positive divisors are 1 and p.

Theorem (Apostol Theorem 1.5)

Every integer is prime or a product of primes

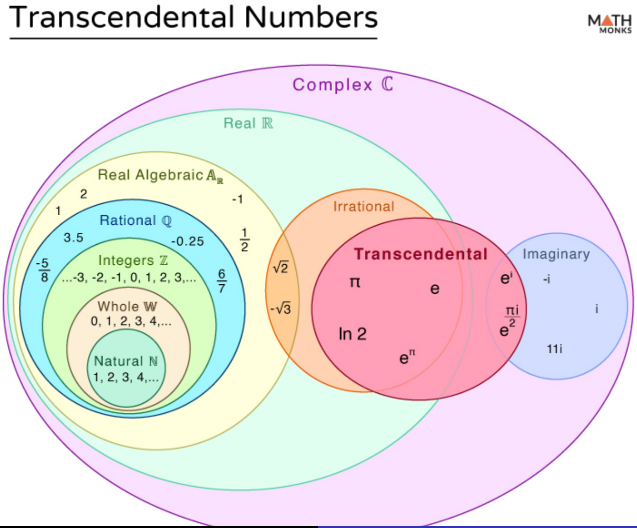
Theorem (Apostol Theorem 1.8)

If p is prime and p divides ab, then p divides a or p divides b.

Theorem (Fundamental Theorem of Arithmetic (Apostol Theorem 1.9))

Every integer $n > 1$ has a unique factorization as a product of primes, up to reordering.

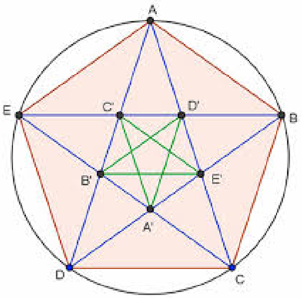
Types of numbers



Irrational numbers

Numbers which are not of the form a/b with $a, b \in \mathbb{Z}$ are called **irrational**.

- Discovered by Hippasus, a pythagorean (a student of Pythagoras)



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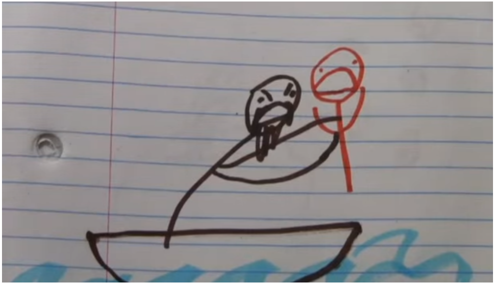

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3. transcendentals are mysterious ... but most real numbers are transcendental!

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- $[3, 7)$ has an upper bound but no maximal element

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Suppose that b_1 and b_2 are both maximal elements of S. Then $x \leq b_1$ and $x \leq b_2$ for all $x \in S$.

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Suppose that b_1 and b_2 are both maximal elements of S.
Then $x \leq b_1$ and $x \leq b_2$ for all $x \in S$.
Moreover, $b_1 \in S$ and $b_2 \in S$.

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By the trichotomy, we find $b_1 = b_2$.

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Now we can say *the* maximum, $\max(S)$

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In other words

a supremum is a **least upper bound**

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Show that if S has a supremum, then it is unique.

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Suppose that b_1 and b_2 are both suprema of S.

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Suppose that b_1 and b_2 are both suprema of S.
Then b_1 and b_2 are both upper bounds of S.

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Suppose that b_1 and b_2 are both suprema of S.
Then b_1 and b_2 are both upper bounds of S.
Since b_1 is a least upper bound, $b_1 \leq b_2$.

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Since b_1 is a least upper bound, $b_1 \leq b_2$.
Since b_2 is also a least upper bound, $b_2 \leq b_1$.

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Since b_1 is a least upper bound, $b_1 \leq b_2$.
Since b_2 is also a least upper bound, $b_2 \leq b_1$.
By the trichotomy, we find $b_1 = b_2$.

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Now we can say *the* supremum, $\sup(S)$

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Let $b = \max(S)$.
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Let $b = \max(S)$.
Then b is an upper bound of S and $b \in S$.
If b' is another upper bound of S, then by definition $b \leq b'$.

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If b' is another upper bound of S, then by definition $b \leq b'$.
Thus b is the least upper bound of S.

Completeness Axiom

Now we have the machinery in place to state the Completeness Axiom.

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$$\left\{ \frac{n}{n+1} : n \in \mathbb{Z}_+ \right\}$$

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- 3 is the supremum of $(0, 3)$
- 1 is the supremum of

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- π is the supremum of

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \dots\}.$$

Challenge

Problem

Prove that if A is a set of integers that is bounded above, then A has a maximum.

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Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.

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The set A is bounded above, so it has a supremum $b = \sup(A)$. Since $b - 1$ is not an upper bound, so there exists $a \in A$ with $b - 1 < a$.

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Therefore $k \leq 0$ and $a' \leq a$.

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Since b is an upper bound, $b \geq a + k > b - 1 + k$, making $0 > k - 1$.
Therefore $k \leq 0$ and $a' \leq a$.
It follows that a is an upper bound of A, and since $a \in A$ it is a maximum.

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

$$b \leq x \quad \text{for all } x \in S.$$

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In other words

an infimum is a **greatest lower bound**

Properties of Suprema

The first property of suprema is that they must be arbitrarily close to elements of the set.

Theorem (Approximation Property)

Let $S \subseteq \mathbb{R}$ be bounded above, and let $b = \sup(S)$. Then for all $\epsilon > 0$, there exists $a \in S$ with

$$b - \epsilon < a < b.$$

Properties of Suprema

Proof.

Since $b - \epsilon < b$, the definition of a supremum implies $b - \epsilon$ cannot be an upper bound.

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Since $b - \epsilon < b$, the definition of a supremum implies $b - \epsilon$ cannot be an upper bound.
Therefore thre must exist $a \in S$ with $a > b - \epsilon$. $\square$

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

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For every n,

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 3 - \frac{1}{2^n} < 3$$

so S is bounded above by 3.

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so S is bounded above by 3. Therefore S has a supremum, $\sup(S)$.

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We've just shown that $e^1 = e$ exists.

Properties of Suprema

Also, suprema play nicely with addition.

Theorem (Additive Property)

Let $A, B \subseteq \mathbb{R}$ be bounded above set

$$C = \{x + y : x \in A,\ y \in B\}.$$

Then C is bounded above and

$$\sup(C) = \sup(A) + \sup(B).$$

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Proof.

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Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.

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We will prove $c \leq a + b$ and then $a + b \leq c$.
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We will prove $c \leq a + b$ and then $a + b \leq c$.
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Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.

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If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.

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Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.

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Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.
Therefore $c - y$ is an upper bound of A and $a \leq c - y$.
It follows that $y \leq c - a$ for all $y \in B$.
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Thus $c - a$ is an upper bound of B, and it follows $b \leq c - a$.
Therefore $a + b \leq c$. $\square$

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Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

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Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.

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This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.

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This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.

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The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.
This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.
However, $n + 1 \in \mathbb{Z}$, so this contradicts b being an upper bound.

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

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For every $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{Z}_+$ with $y < nx$.

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Replace x with y/x in the previous theorem. □

Outline

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A **finite decimal expansion** is an expression

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n},$$

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Any positive real number $x > 0$ can be approximated by a finite decimal expansion.

Decimal approximations

Theorem (Apostol Theorem 1.20)

For any real $x > 0$ and $n \in \mathbb{Z}_+$, there exists a finite decimal expansion $r_n = a_0.a_1a_2\dots a_n$ with

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

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$$1 \not< 0 + \frac{9}{\ } + \frac{9}{\ } + \cdots + \frac{9+1}{\ } = 1.$$