MATH 350-2 Advanced Calculus

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Outline

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1. Real Analysis Lecture 6
   - Functions
   - Cardinality

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The set

$$\text{img}(f) = \{f(a) : a \in A\}$$

is called the **range** or **image** of f

Challenge!

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$f = \mathcal{R}$ must be reflexive, so $x\mathcal{R}x$ for all x
This means $f(x) = x$ for all x.
Thus the only function which is an equivalence relation is the identity function

$$f(x) = x.$$

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If it satisfies both properties, it is called **bijective.**

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If it is, we call it the **inverse** of f.

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$$(g \circ f)(x) = g(f(x)).$$

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NOTATION: $\{f_{k(n)}\}$ or $\{f_{k_n}\}$ both really mean $f \circ k$

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Theorem (Cantor-Schroeder-Bernstein Theorem)

The following are equivalent

1. (i) $|A| \leq |B|$ *and* $|B| \leq |A|$
2. (ii) $|A| \geq |B|$ *and* $|B| \geq |A|$
3. (iii) $|A| \leq |B|$ *and* $|A| \geq |B|$
4. (iv) $|A| = |B|$

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Hint: consider $f : \mathbb{Z}_+ \to \mathbb{Z}$

$$f(n) = \begin{cases} n/2 & n \text{ is even} \\ -(n-1)/2 & n \text{ is odd} \end{cases}$$

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$$\mathbb{R} \text{ has larger cardinality than } \mathbb{Z}_+.$$

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Suppose there were a bijection

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f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\dots \\
f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\dots \\
f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\dots \\
f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}\dots \\
f(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55}\dots \\
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Theorem (Cantor's Theorem)

$\mathbb{R}$ *is uncountable*