MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 5
   - Sets, Relations, Functions
   - Cardinality

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- $\{n \in \mathbb{Z} : n \text{ is prime}\}$

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- even relations and functions are sets!

Challenge!

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Prove that for ordered pairs (a, b) and (c, d) that

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It follows that $d = c = b = a$.

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This can be shown to contradict the ZF Axioms of Set Theory.
Specifically the regularity axiom for the set $\{a, c\}$...

Relations

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- $\leq = \{(x, y) : y - x \in [0, \infty)\}$ is reflexive and transitive but not symmetric on $\mathbb{R}$

Challenge!

Problem

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The set

$$\text{img}(f) = \{f(a) : a \in A\}$$

is called the **range** or **image** of f

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Thus the only function which is an equivalence relation is the identity function

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If it satisfies both properties, it is called **bijective.**

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If it is, we call it the **inverse** of f.

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If $k : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a function which is **strictly increasing**, meaning

$$m < n \quad \Rightarrow \quad k(m) < k(n),$$

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If $k : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a function which is **strictly increasing**, meaning

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NOTATION: $\{f_{k(n)}\}$ or $\{f_{k_n}\}$ both really mean $f \circ k$

Outline

1. Real Analysis Lecture 5
   - Sets, Relations, Functions
   - Cardinality

Cantor's Paradise

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