MATH 350-2 Advanced Calculus

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Outline

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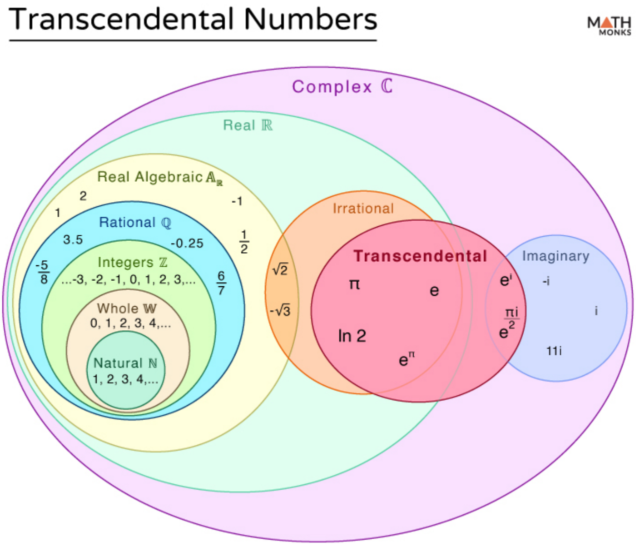
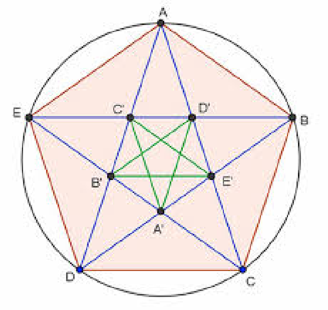
Types of numbers



Irrational numbers

Numbers which are not of the form a/b with $a, b \in \mathbb{Z}$ are called **irrational.**

- Discovered by Hippasus, a pythagorean (a student of Pythagoras)



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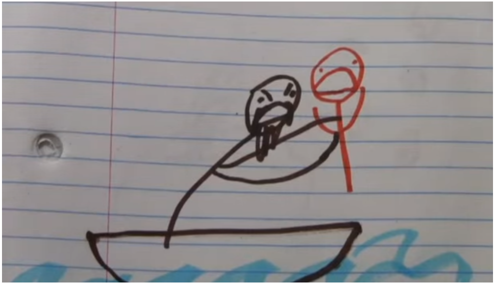

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Challenge!

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Can you prove that if $n \in \mathbb{Z}_+$ is not a perfect square, then $\sqrt{n}$ is irrationial? (Apostol Theorem 1.10)

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3. transcendentals are mysterious ... but most real numbers are transcendental!

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- $[3, 7)$ has an upper bound but no maximal element

Challenge!

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Show that if S has a maximal element, then it is unique.

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Suppose that b_1 and b_2 are both maximal elements of S.
Then $x \leq b_1$ and $x \leq b_2$ for all $x \in S$.
Moreover, $b_1 \in S$ and $b_2 \in S$.

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By the trichotomy, we find $b_1 = b_2$.

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Now we can say *the* maximum, $\max(S)$

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In other words

a supremum is a **least upper bound**

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Let $b = \max(S)$.
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- π is the supremum of

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \dots\}.$$

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Prove that if A is a set of integers that is bounded above, then A has a maximum.

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Therefore $k \leq 0$ and $a' \leq a$.

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Therefore $k \leq 0$ and $a' \leq a$.
It follows that a is an upper bound of A, and since $a \in A$ it is a maximum.

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an infimum is a **greatest lower bound**

Properties of Suprema

The first property of suprema is that they must be arbitrarily close to elements of the set.

Theorem (Approximation Property)

Let $S \subseteq \mathbb{R}$ be bounded above, and let $b = \sup(S)$. Then for all $\epsilon > 0$, there exists $a \in S$ with

$$b - \epsilon < a < b.$$

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Proof.

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Since $b - \epsilon < b$, the definition of a supremum implies $b - \epsilon$ cannot be an upper bound.
Therefore thre must exist $a \in S$ with $a > b - \epsilon$. $\square$

Important example: The number e

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For every n,

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 3 - \frac{1}{2^n} < 3$$

so S is bounded above by 3.

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$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 3 - \frac{1}{2^n} < 3$$

so S is bounded above by 3. Therefore S has a supremum, $\sup(S)$.

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We've just shown that $e^1 = e$ exists.

Properties of Suprema

Also, suprema play nicely with addition.

Theorem (Additive Property)

Let $A, B \subseteq \mathbb{R}$ be bounded above set

$$C = \{x + y : x \in A,\ y \in B\}.$$

Then C is bounded above and

$$\sup(C) = \sup(A) + \sup(B).$$

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Proof.

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If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.

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Thus $c - a$ is an upper bound of B, and it follows $b \leq c - a$.
Therefore $a + b \leq c$. □

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Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

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Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.

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It follows from Axiom 7 that $b < n + 1$.

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This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.
However, $n + 1 \in \mathbb{Z}$, so this contradicts b being an upper bound.

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

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Proof.

If not, then x is an upper bound of $\mathbb{Z}_+$. $\square$

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Theorem (Archimedian Property of Reals)

For every $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{Z}_+$ with $y < nx$.

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Proof.

Replace x with y/x in the previous theorem. □

Outline

Decimal approximations

A **finite decimal expansion** is an expression

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n},$$

where $a_0 \in \mathbb{Z}_+$ and $0 \leq a_k \leq 9$ for $1 \leq k \leq n$.

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Any positive real number $x > 0$ can be approximated by a finite decimal expansion.

Decimal approximations

Theorem (Apostol Theorem 1.20)

For any real $x > 0$ and $n \in \mathbb{Z}_+$, there exists a finite decimal expansion $r_n = a_0.a_1a_2\dots a_n$ with

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

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Consider the set

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Define $a_1, a_2, a_3, \dots$ and $x_1, x_2, x_3, \dots$ recursively by

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