MATH 350-2 Advanced Calculus

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Outline

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Adherent and Accumulation Points

Definition

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- suprema and infima are accumulation points!
- 0 is an accumulation point of $\{1/1, 1/2, 1/3, \dots\}$

Characterizing accumulation points

Theorem (Apostol Theorem 3.17)

A point $\vec{x}$ is an accumulation point of A if and only if for all $r > 0$, the ball $B(\vec{x}; r)$ contains infinitely many points of A.

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Obviously, if for all $r > 0$, the ball $B(\vec{x}; r)$ contains infinitely many points of A, then it contains at least one point of A different from $\vec{x}$, so it's an accumulation point.

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Suppose instead that $\vec{x}$ is an accumulation point and let $r > 0$. It suffices to show that the set $C = (B(\vec{x}; r) \backslash \{\vec{x}\}) \cap A$ is infinite. $\square$

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Then the set $\{|\vec{y} - \vec{x}| : \vec{y} \in C\}$ is also finite.

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However, $\vec{y} \in C$ and $|\vec{y} - \vec{x}| < s$, contradicting the minimality of s.
We conclude that C is infinite. $\square$

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Theorem (Apostol Theorem 3.18, 3.20, 3.22)

A set is closed if and only if it contains all of its adherent points (ie. $A = \overline{A}$), or equivalently if and only if A contains all of its accumulation points.

Closure of a set

Proof.

Suppose that A is closed.
Then $\mathbb{R}^n \backslash A$ is open.
If $\vec{x} \in \mathbb{R}^n \backslash A$, then there exist $r > 0$ such that $B(\vec{x}; r) \subseteq \mathbb{R}^n \backslash A$.
This means that $B(\vec{x}; r) \cap A = \varnothing$.
Thus $\vec{x}$ is not an adherent point of A.
Thus every adherent point of A is an element of A. $\square$

Closures are closed

Theorem

If $A \subseteq \mathbb{R}^n$ is any set, then $\overline{A}$ is closed.

Proof.

Exercise! $\square$

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- sequences have convergent subsequences
- Cauchy sequences converge
- open covers have finite subcovers
- in $\mathbb{R}^n$: compact = closed and bounded

Open covers

Definition

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- $\{(\frac{1}{n}, \frac{n+1}{n}) : n \in \mathbb{Z}_+\}$ is an open cover of $(0, 1]$

Lindelöf Covering Theorem

Theorem

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

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A set $A \subseteq \mathbb{R}^n$ is called **compact** if every open cover of A has a *finite* subcover.

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2. Then use Bolzano-Weierstrass to prove the Cantor Intersection Theorem
3. We use the Cantor Intersection Theorem to prove the Heine-Borel Theorem

Bolzano-Weierstrass Theorem

Theorem (Bolzano-Weierstrass Theorem)

A bounded set $A \subseteq \mathbb{R}^n$ with infinitely many points will contain an accumulation point.