MATH 350-2 Advanced Calculus

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Outline

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- $\{n \in \mathbb{Z} : n \text{ is prime}\}$

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- even relations and functions are sets!

Challenge!

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Prove that for ordered pairs (a, b) and (c, d) that

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$$\begin{aligned} \text{Case I:} &\quad a = c \ \text{ and } \ \{a, b\} = \{c, d\} \\ \text{Case II:} &\quad a = \{c, d\} \ \text{ and } \ \{a, b\} = c \end{aligned}$$

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If $b = d$, we're done!

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It follows that $d = c = b = a$.

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This would imply that $c \in a$ and $a \in c$.

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This can be shown to contradict the ZF Axioms of Set Theory.

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This can be shown to contradict the ZF Axioms of Set Theory.
Specifically the regularity axiom for the set $\{a, c\}$...

Relations

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$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

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- $\leq = \{(x, y) : y - x \in [0, \infty)\}$ is reflexive and transitive but not symmetric on $\mathbb{R}$

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