MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 1
   - Origin of Real Numbers
   - Properties of real numbers
   - Integers

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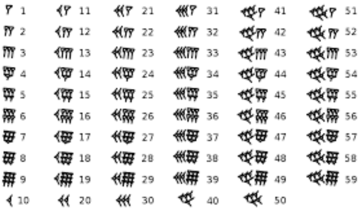
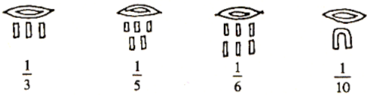
Prehistoric numbers

20,000 BC tallies on Ishango bone



3400 BC Sumerian system



1000 BC Egyptian fractions

Invention of zero

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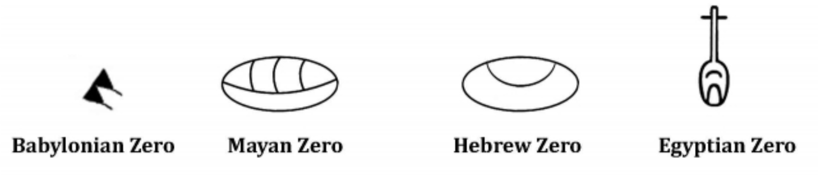
- 150 AD - Ptolemy using it in astronomy
- medieval scholars - debating the existence of 0



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- 1202,1225 - Fibonacci allows negatives as solutions for financial problems
- up to 18th century - rejected by western sources, referred to as "absurd numbers"

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- 300 BC -appear in Euclid's elements

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- 17th century - European mathematicians distinguish between transcendentals and algebraic numbers

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- instead, we will take for granted that the reals exist and describe ten fundamental rules (axioms) for how it behaves
- the integers, rationals, etc. will be defined in terms of the reals

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Special notation: $0 := x - x$ and $1 := x/x$, and $-x := 0 - x$

Challenge!

Problem

use the axioms to show that $(x + y)z = xz + yz$.

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Solution

$$(x + y)z \overset{A1}{=} z(x + y) \overset{A3}{=} zx + zy \overset{A1}{=} xz + yz.$$

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Therefore we have

$$(x+y)+z_1 \overset{A2}{=} x+(y+z_1) \overset{A1}{=} x+(z_1+y) \overset{A2}{=} (x+z_1)+y = x+y,$$

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- **field of Boolean numbers**

$$\mathbb{F}_2 = \{0, 1\}$$

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0$$

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1$$

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There aren't multiplicative inverses! For example, $1/2$ doesn't make sense because there isn't an integer z with $1 = 2z$.

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Special notation:

- $x > y$ means $y < x$
- $x \leq y$ means $x < y$ or $x = y$
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This says $a < a$, which contradicts the trichotomy. $\square$

Real numbers

The **real numbers** are the unique field $\mathbb{R}$ satisfying Axiom 1-Axiom 9, plus an extra axiom called the completeness axiom.

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- Intuitively, it represents the fact that the real line has no holes or gaps.

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- **open and closed intervals**

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Examples:

$$\mathbb{R}, \quad \mathbb{Q}, \quad (0, \infty), \quad \mathbb{Z}, \quad \mathbb{N}, \dots$$

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Example: Let's prove $n(n+1)$ is divisible by 2 for all positive integers n, using the Principle of Induction.

Induction

Principle of Induction:

$$\text{If } S \text{ is an inductive set, then } \mathbb{Z}_+ \subseteq S$$

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Let

$$S = \{n \in \mathbb{Z}_+ : 2|n(n+1)\}$$

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Now suppose $x \in S$. (This is our usual inductive assumption).

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Then 2 divides $x(x+1)$, so $x(x+1) = 2k$ for some integer k.

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Then 2 divides $x(x+1)$, so $x(x+1) = 2k$ for some integer k.
This means $(x+1)(x+2) = x(x+1) + 2(x+1) = 2(k+x+1)$.

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Since $x+1$ was arbitrary, this shows that S is an inductive set.
By the Principle of Induction, this means $\mathbb{Z}_+ \subseteq S$.
In other words $n \in S$ for every positive integer n.
Hence 2 divides $n(n+1)$ for every positive integers n.

Prime numbers

A positive integer p is **prime** if its only positive divisors are 1 and p.

Theorem (Apostol Theorem 1.5)

Every integer is prime or a product of primes

Theorem (Apostol Theorem 1.8)

If p is prime and p divides ab, then p divides a or p divides b.

Theorem (Fundamental Theorem of Arithmetic (Apostol Theorem 1.9))

Every integer $n > 1$ has a unique factorization as a product of primes, up to reordering.