MATH 350-2 Advanced Calculus

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Outline

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1. Real Analysis Lecture 12
   - Metric Spaces
   - Subspaces

Warm-up Challenge

Problem

Write down each of the following:

- the definition of an open cover of a set A

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- Cantor Intersection Theorem
- Heine-Borel Theorem

Metric space

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Definition

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- A4 **triangle inequality:** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$

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The value $d(x, y)$ is called a **metric** and describes the “distance" between x and y.

Open balls

Open balls

Definition

Let (M, d) be a metric space. The open ball of radius $r > 0$ centered at $x \in M$ is

$$B_M(x; r) = \{y \in M : d(x, y) < r\}.$$

Examples of metrics on $\mathbb{R}^2$

$$d_{\mathrm{eucl}}(\vec{x},\vec{y}) = |\vec{x}-\vec{y}| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

Examples of metrics on $\mathbb{R}^2$

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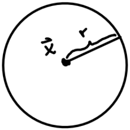

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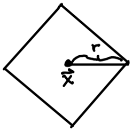

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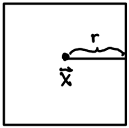

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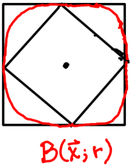

Challenge!

Let M be a nonempty set and define $d : M \times M \to \mathbb{R}$ by

$$d_{\text{disc}}(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

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Prove that d_{disc} is a metric.

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Problem

Prove that d_{disc} is a metric.

This is called the **discrete metric.**

Challenge!

Problem

What do the open balls with the discrete metric look like?

Open sets

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A point $x \in A$ is called an **interior point** if there exists $r > 0$ such that $B_M(x; r) \subseteq A$.

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A point $x \in A$ is called an **interior point** if there exists $r > 0$ such that $B_M(x; r) \subseteq A$.
The set $\text{int}(A)$ of interior points of A is called the **interior** of A.

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The set A is **open** if every point in A is an interior point, or equivalently $\text{int}(A) = A$.

Challenge!

Problem

Consider the metric space $(\mathbb{R}, d_{\text{disc}})$ where d_{disc} is the discrete metric.
What sets are open sets?

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A point $x \in M$ is an **accumulation point** if for all $r > 0$ the ball $B_M(x; r)$ contains an element of A different from x.

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A point $x \in M$ is an **accumulation point** if for all $r > 0$ the ball $B_M(x; r)$ contains an element of A different from x.
The set $\overline{A}$ of all adherent points of A is called the **closure** of A.

Challenge!

Problem

Consider the metric space $(\mathbb{R}, d_{\text{disc}})$ where d_{disc} is the discrete metric.
Which sets are closed?

Challenge!

Problem

Prove the following (Apostol Theorem 3.36). If (M, d) is a metric space, $U \subseteq M$ is open, and $C \subseteq M$ is closed, then $U \backslash C$ is open and $C \backslash U$ is closed.

Characterizing closed sets

Theorem (Apostol Theorem 3.37)

Let (M, d) be a metric space and $A \subseteq S$. Then the following are equivalent:

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(d) $A = \overline{A}$

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Let (M, d) be a metric space and $\{U_i : i \in I\}$ be a family of open sets in M. The union $\bigcup_{i \in I} U_i$ is open.

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Identical to the proof for $\mathbb{R}^n$. $\square$

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The space (S, d) with the relative metric is called a **metric subspace.**

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The space (S, d) with the relative metric is called a **metric subspace.**

Open balls in S:

$$B_S(x; r) = B_M(x; r) \cap S.$$

Open sets in subspaces

Theorem

Let (M, d) be a metric space and let (S, d) be a subspace. Then V is an open subset of S if and only if there exists an open subset U of M with $V = U \cap S$.

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$\Longrightarrow$: Assume V is an open subset of S.
Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.

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Define
$$U = \bigcup_{x \in V} B_M(x; r_x).$$

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This is a union of open sets, and is therefore open.

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This is a union of open sets, and is therefore open.
Furthermore $U \cap S = V$. $\square$

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We need to show that $V := U \cap S$ is open.

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If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.
It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.

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Since $x \in V$ was arbitrary, this shows that V is open in S. $\square$

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This is true if and only if there exists an open subset U of M with $S \backslash B = U \cap S$.

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B is closed in S if and only if $S \backslash B$ is open in B
This is true if and only if there exists an open subset U of M with $S \backslash B = U \cap S$.
This is true if and only if there exists an open subset U of M with $B = (M \backslash U) \cap S$.

Closed sets in subspaces

Theorem

Let (M, d) be a metric space and let (S, d) be a subspace. Then B is a closed subset of S if and only if there exists a closed subset A of M with $B = A \cap S$.

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Consider the metric space $(\mathbb{R}, d)$ with $d = d_{\text{eucl}}$ the Euclidean metric, and the metric subspace (S, d) for $S = [0, 1]$.

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