MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 9
   - More on Closed Sets

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Adherent and Accumulation Points

Definition

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- -1 is an accumulation point of $(-1, 1)$.
- suprema and infima are accumulation points!
- 0 is an accumulation point of $\{1/1, 1/2, 1/3, \dots\}$

Characterizing accumulation points

Theorem (Apostol Theorem 3.17)

A point $\vec{x}$ is an accumulation point of A if and only if for all $r > 0$, the ball $B(\vec{x}; r)$ contains infinitely many points of A.

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Obviously, if for all $r > 0$, the ball $B(\vec{x}; r)$ contains infinitely many points of A, then it contains at least one point of A different from $\vec{x}$, so it's an accumulation point.

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Suppose instead that $\vec{x}$ is an accumulation point and let $r > 0$.

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Suppose instead that $\vec{x}$ is an accumulation point and let $r > 0$. It suffices to show that the set $C = (B(\vec{x}; r)\backslash\{\vec{x}\}) \cap A$ is infinite. $\square$

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Suppose that C is finite.

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Suppose that C is finite.
Then the set $\{|\vec{y} - \vec{x}| : \vec{y} \in C\}$ is also finite.

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It is also nonempty and has only positive values.

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Since $\vec{x}$ is an accumulation point, $B(\vec{x}; s) \cap A$ has an element $\vec{y}$ different from $\vec{x}$.
However, $\vec{y} \in C$ and $|\vec{y} - \vec{x}| < s$, contradicting the minimality of s.

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Then the set $\{|\vec{y} - \vec{x}| : \vec{y} \in C\}$ is also finite.
It is also nonempty and has only positive values. This means that it has a minimum value $s > 0$.
Since $\vec{x}$ is an accumulation point, $B(\vec{x}; s) \cap A$ has an element $\vec{y}$ different from $\vec{x}$.
However, $\vec{y} \in C$ and $|\vec{y} - \vec{x}| < s$, contradicting the minimality of s.
We conclude that C is infinite. $\square$

Closure of a set

Definition

The **closure** of a set A is the set $\overline{A}$ of all adherent points of A

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Theorem (Apostol Theorem 3.18, 3.20, 3.22)

A set is closed if and only if it contains all of its adherent points (ie. $A = \overline{A}$), or equivalently if and only if A contains all of its accumulation points.

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Proof.

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If $\vec{x} \in \mathbb{R}^n \backslash A$, then there exist $r > 0$ such that $B(\vec{x}; r) \subseteq \mathbb{R}^n \backslash A$.

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Thus $\vec{x}$ is not an adherent point of A.

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Thus $\vec{x}$ is not an adherent point of A.
Thus every adherent point of A is an element of A.

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Conversely, suppse that A contains all of its accumulation points.

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If $\vec{x} \in \mathbb{R}^n \backslash A$, then $\vec{x}$ is not an accumulation point.

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This implies that $B(\vec{x}; r) \subseteq \mathbb{R}^n \backslash A$.

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If $\vec{x} \in \mathbb{R}^n \backslash A$, then $\vec{x}$ is not an accumulation point.
Therefore there exists $r > 0$ with $B(\vec{x}; r) \cap A = \varnothing$.
This implies that $B(\vec{x}; r) \subseteq \mathbb{R}^n \backslash A$.
Thus $\vec{x}$ is an interior point of $\mathbb{R}^n \backslash A$.

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If $\vec{x} \in \mathbb{R}^n \backslash A$, then $\vec{x}$ is not an accumulation point.
Therefore there exists $r > 0$ with $B(\vec{x}; r) \cap A = \varnothing$.
This implies that $B(\vec{x}; r) \subseteq \mathbb{R}^n \backslash A$.
Thus $\vec{x}$ is an interior point of $\mathbb{R}^n \backslash A$.
Since $\vec{x}$ was an arbitrary element of $\mathbb{R}^n \backslash A$, this proves $\mathbb{R}^n \backslash A$ is open.

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Thus $\vec{x}$ is not an adherent point of A.
Thus every adherent point of A is an element of A.
Conversely, suppse that A contains all of its accumulation points.
If $\vec{x} \in \mathbb{R}^n \backslash A$, then $\vec{x}$ is not an accumulation point.
Therefore there exists $r > 0$ with $B(\vec{x}; r) \cap A = \varnothing$.
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Thus $\vec{x}$ is an interior point of $\mathbb{R}^n \backslash A$.
Since $\vec{x}$ was an arbitrary element of $\mathbb{R}^n \backslash A$, this proves $\mathbb{R}^n \backslash A$ is open.
Therefore A is closed. □

Closures are closed

Theorem

If $A \subseteq \mathbb{R}^n$ is any set, then $\overline{A}$ is closed.

Proof.

Exercise! $\square$