MATH 350-2 Advanced Calculus

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September 16, 2024

Outline

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NOTATION: $f : A \to B$ means *f* is a function from *A* to *B*

NOTATION: $f(a) = b$ means $(a, b) \in f$.

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The set

$$\text{img}(f) = \{f(a) : a \in A\}$$

is called the **range** or **image** of *f*

Challenge!

Problem

Determine all the equivalence relations on $\mathbb{R}$ which are also functions.

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This means $f(x) = x$ for all x.

Thus the only function which is an equivalence relation is the identity function

$$f(x) = x.$$

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If it satisfies both properties, it is called **bijective**.

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If it is, we call it the **inverse** of f.

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$$(g \circ f)(x) = g(f(x)).$$

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NOTATION: $\{f_{k(n)}\}$ or $\{f_{k_n}\}$ both really mean $f \circ k$

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Philosophy: sets are the same size, or **cardinality**, if there is a bijection between them.

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Theorem (Cantor-Schroeder-Bernstein Theorem)

The following are equivalent

(i) $|A| \leq |B|$ *and* $|B| \leq |A|$

(ii) $|A| \geq |B|$ *and* $|B| \geq |A|$

(iii) $|A| \leq |B|$ *and* $|A| \geq |A|$

(iv) $|A| = |B|$

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Hint: consider $f : \mathbb{Z}_+ \to \mathbb{Z}$

$$f(n) = \begin{cases} n/2 & n \text{ is even} \\ -(n-1)/2 & n \text{ is odd} \end{cases}$$

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Cantor's discovery: there are multiple sizes of infinity!

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$\mathbb{Z}_+, \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}, \mathbb{Q},$ are all the same cardinality

$\mathbb{R}$ has larger cardinality than $\mathbb{Z}_+$.

Cantor diagonalization

Suppose there were a bijection

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$$f(3) = 0.a_{31}a_{32}a_{33}a_{34}a_{35} \ldots$$
$$f(4) = 0.a_{41}a_{42}a_{43}a_{44}a_{45} \ldots$$
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Now define $b_1, b_2, b_3 \cdots \in \{0, 9\}$ by

$$b_j = \begin{cases} 1, & a_{jj} = 0 \\ 0, & a_{jj} \neq 0 \end{cases}$$

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Theorem (Cantor's Theorem)

$\mathbb{R}$ *is uncountable*

Ultimate Cantor diagonalization

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Consider the set

$$S = \{a \in A : a \notin f(a)\}.$$

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Problem

Consider the statement $x \in S$. What can you conclude?

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NOTATION:
$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

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- family of sets $\{A_i : i \in I\}$
- $A_i = [0, i)$

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