MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 7
   - Open Balls and Open Sets

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(f) $|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$

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(a) (scaling) $|c\vec{x}| = |c|\ |\vec{x}|$
(b) (Cauchy-Schwartz) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}|\ |\vec{y}|$

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A point $\vec{a}$ in a subset $A \subseteq \mathbb{R}^n$ is called an **interior point** of A if there exists $r > 0$ with $B(\vec{a}; r) \subseteq A$. If every point of A is an interior point, then A is called an **open set.**

Challenge!

Problem

Prove that the empty set $\varnothing$ and the whole space $\mathbb{R}^n$ are open.

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Problem

Let $\vec{a} \in \mathbb{R}^n$. Show that the singleton set

$$A = \{\vec{a}\}$$

is not open.

Challenge!

Problem

Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Prove that the **open rectangle**

$$(a, b) \times (c, d) = \{(x, y) : a < x < b, \quad c < y < d\}$$

is an open set

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Prove that an open ball is an open set.

Unions of open sets are open

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Therefore there exists $r > 0$ such that $B(\vec{x}; r) \subseteq U_j$.
This means $B(\vec{x}; r) \subseteq \bigcup_{i\in I} U_i$.
Thus $\vec{x}$ is an interior point of $\bigcup_{i\in I} U_i$.
Since $\vec{x}$ is arbitrary, this proves that $\bigcup_{i\in I} U_i$ is open. $\square$