MATH 350-2 Advanced Calculus

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Outline

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Metric subspace

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The space (S, d) with the relative metric is called a **metric subspace.**

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The space (S, d) with the relative metric is called a **metric subspace.**

Open balls in S:

$$B_S(x; r) = B_M(x; r) \cap S.$$

Example

Let $M = \mathbb{R}$ and d be the Euclidean metric.

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Open sets in subspaces

Theorem

Let (M, d) be a metric space and let (S, d) be a subspace. Then V is an open subset of S if and only if there exists an open subset U of M with $V = U \cap S$.

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$$U = \bigcup_{x \in V} B_M(x; r_x).$$

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This is a union of open sets, and is therefore open.

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This is a union of open sets, and is therefore open.
Furthermore $U \cap S = V$. $\square$

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$\Longleftarrow$: Assume U is an open subset of M.
We need to show that $V := U \cap S$ is open.

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If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.
It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.

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This shows that x is an interior point of S in the metric subspace (S, d).

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It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.
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Since $x \in V$ was arbitrary, this shows that V is open in S. $\square$

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- $(1/2, 1]$ is open in S, but not in M

Outline

1. Real Analysis Lecture 13
   - Metric subspaces
   - Limits of sequences

Definition of a limit

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We say $\{x_n\}$ **converges** to L and L is the **limit of the sequence.**

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Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to 1. $\square$

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Suppose that it converges to a number L.

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Proof.

Suppose that it converges to a number L.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.

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Proof.

Suppose that it converges to a number L.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.
Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.

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Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.
With the discrete metric $d(x_n, L) < 1$ implies $x_n = L$.

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Consider the metric space (M, d) defined by $M = \mathbb{R}$ with the discrete metric d.
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Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.
Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.
With the discrete metric $d(x_n, L) < 1$ implies $x_n = L$.
This means $x_n = 1/n = L$ for all $n \geq N$, which is a contradiction! $\square$

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If a sequence $\{x_n\}$ in M converges to L and converges to M, then $L = M$.

Proof.

Suppose that $\{x_n\}$ converges to L and M.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.

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Let (M, d) be a metric space.
If a sequence $\{x_n\}$ in M converges to L and converges to M, then $L = M$.

Proof.

Suppose that $\{x_n\}$ converges to L and M.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.

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Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists N_2 such that $d(x_n, L) < \epsilon$ for $n > N_2$.

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Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists N_2 such that $d(x_n, L) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

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Let (M, d) be a metric space.
If a sequence $\{x_n\}$ in M converges to L and converges to M, then $L = M$.

Proof.

Suppose that $\{x_n\}$ converges to L and M.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists N_2 such that $d(x_n, L) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

$$d(L, M) \leq d(L, x_n) + d(x_n, M) < 2\epsilon = d(L, M).$$

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$$d(L, M) \leq d(L, x_n) + d(x_n, M) < 2\epsilon = d(L, M).$$

This is a contradiction. $\square$

Existence of limits

Theorem

If a sequence $\{x_n\}$ in a metric space (M, d) converges to a value $L \in M$, then the range

$$X = \{x_1, x_2, \ldots\} \subseteq M$$

is a bounded set and L is an adherent point of X.

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is a bounded set and L is an adherent point of X.

When the range of a sequence is bounded, we call the sequence $\{x_n\}$ bounded.

Challenge!

Try to prove the previous theorem.

Existence of limits

Sort of a converse of the previous theorem.

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Theorem

If S is a subset of a metric space (M, d) and $L \in M$ is an adherent point of S, then there exists a sequence $\{x_n\}$ of elements of S which converges to L.

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Recall a **subsequence** of a sequence $\{x_n\}$ is a sequence of the form $\{x_{k(n)}\}$, for $k(1) < k(2) < \ldots$ positive integers.

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Theorem

A sequence $\{x_n\}$ in a metric space (M, d) converges to a value $L \in M$ if and only if every subsequence $\{x_{k(n)}\}$ also converges to L.

Challenge!

Try to prove the previous theorem.

Monotone sequences

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It is called **strictly decreasing** if $x_n > x_{n+1}$ for all n.
In any of these cases, we call the sequence **monotone**.

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A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \ldots\}$.

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Theorem (Monotone convergence theorem)

Consider the metric space formed by $\mathbb{R}$ with Euclidean distance.
A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \dots\}$.
A monotone decreasing sequence $\{x_n\}$ which is bounded below converges to $L = \inf\{x_1, x_2, \dots\}$.

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Let $\epsilon > 0$.

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Then $L - \epsilon$ is not a supremum of X, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.

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Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \leq x_n < L$.

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It follows that

$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

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Since $\epsilon > 0$ was arbitrary, this proves x_n converges to L. $\square$