MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 14
   - More on limits
   - Cauchy sequences

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Existence of limits

Theorem

If a sequence $\{x_n\}$ in a metric space (M, d) converges to a value $L \in M$, then the range

$$X = \{x_1, x_2, \dots\} \subseteq M$$

is a bounded set and L is an adherent point of X.

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When the range of a sequence is bounded, we call the sequence $\{x_n\}$ bounded.

Existence of limits

Proof.

Suppose that $\lim_{n\to\infty} x_n = L$.

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If we take

$$R = \max\{d(x_1, L), d(x_2, L), \ldots, d(x_{N-1}, L), 2024\},$$

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then $d(x_n, L) \leq R$ for all N.
Therefore $X \subseteq B_M(L, R)$ and in particular X is bounded. $\square$

Existence of limits

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Theorem

If S is a subset of a metric space (M, d) and $L \in M$ is an adherent point of S, then there exists a sequence $\{x_n\}$ of elements of S which converges to L.

Existence of limits

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Then for all $n \in \mathbb{Z}_+$, the ball $B_M(x, \frac{1}{n})$ contains at least one element of S.

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Then for all $n \in \mathbb{Z}_+$, the ball $B_M(x, \frac{1}{n})$ contains at least one element of S.
For each $n \in \mathbb{Z}_+$, choose $x_n \in S \cap B_M(x, \frac{1}{n})$.

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We claim $\lim x_n = L$.

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To see this, let $\epsilon > 0$.

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Then choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.

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To see this, let $\epsilon > 0$.
Then choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
It follows that for all $n \geq N$

$$d(x_n, L) < 1/n < 1/N < \epsilon.$$

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Since $\epsilon > 0$ was arbitrary, this proves convergence. □

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Theorem

A sequence $\{x_n\}$ in a metric space (M, d) converges to a value $L \in M$ if and only if every subsequence $\{x_{k(n)}\}$ also converges to L.

Existence of limits

Proof.

Suppose that every subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ converges to L.

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Let $\{x_{k(n)}\}$ be a subsequence of $\{x_n\}$.

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To show it limits to L, let $\epsilon > 0$.

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Conversely, suppose that $\{x_n\}$ converges to L.
Let $\{x_{k(n)}\}$ be a subsequence of $\{x_n\}$.
To show it limits to L, let $\epsilon > 0$.
Then there exists $N \in \mathbb{Z}_+$ such that $n \geq N$ implies $d(x_n, L) < \epsilon$.

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Let $\{x_{k(n)}\}$ be a subsequence of $\{x_n\}$.
To show it limits to L, let $\epsilon > 0$.
Then there exists $N \in \mathbb{Z}_+$ such that $n \geq N$ implies $d(x_n, L) < \epsilon$.
Moreover, $k(n) \geq n$ so $k(n) \geq N$ and therefore $d(x_{k(n)}, L) < \epsilon$.

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Since $\epsilon > 0$ was arbitrary, this proves convergence. □

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It is called **strictly decreasing** if $x_n > x_{n+1}$ for all n.
In any of these cases, we call the sequence **monotone**.

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Theorem (Monotone convergence theorem)

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Consider the metric space formed by $\mathbb{R}$ with Euclidean distance.
A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \ldots\}$.

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Bounded monotone sequences always converge (with Euclidean metric)!

Theorem (Monotone convergence theorem)

Consider the metric space formed by $\mathbb{R}$ with Euclidean distance.
A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \dots\}$.
A monotone decreasing sequence $\{x_n\}$ which is bounded below converges to $L = \inf\{x_1, x_2, \dots\}$.

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Let $\epsilon > 0$.

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Then $L - \epsilon$ is not a supremum of X, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.

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Then $L - \epsilon$ is not a supremum of X, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.
Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \leq x_n < L$.

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Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \leq x_n < L$.

It follows that

$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

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It follows that

$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves x_n converges to L. $\square$

Outline

Cauchy sequence

A sequence $\{x_n\}$ in a metric space (M, d) is called Cauchy if the terms get arbitrarily close together as n gets arbitrarily large.

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- is every Cauchy sequence convergent?

Challenge!

Problem

Prove that if $\{x_n\}$ is a Cauchy sequence in a metric space (M, d), then it is bounded.

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Suppose $\{x_n\}$ is Cauchy.
Let $\epsilon = 2024$.
Then $\exists$ $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < 2024$ for all $m, n \geq N$.
Therefore for all $n \geq N$, we have $d(x_n, x_N) \leq 2024$.

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So if we define

$$R = \max\{d(x_1, x_N), d(x_2, x_N), \dots, d(x_{n-1}, x_N), 2024\},$$

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So if we define

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then $d(x_n, x_N) \leq 2024$ for all $n \geq 1$.
Hence $X = \{x_1, x_2, \dots\} \subseteq B_M(x_N, R)$, and is bounded.

Cauchy sequence

Every convergent sequence is Cauchy.

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Theorem

Let (M, d) be a metric space and suppose that $\{x_n\}$ is a sequence in M which converges to $L \in M$.
Then $\{x_n\}$ is Cauchy.

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Let (M, d) be a metric space and suppose that $\{x_n\}$ is a sequence in M which converges to $L \in M$.
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Proof.

Let $\epsilon > 0$.

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Then there exists N such that $n \geq N$ implies $d(x_n, L) < \epsilon/2$.

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Proof.

Let $\epsilon > 0$.
Then there exists N such that $n \geq N$ implies $d(x_n, L) < \epsilon/2$.
Therefore for any $m, n \geq N$ we have

$$d(x_m, x_n) \leq d(x_m, L) + d(L, x_n) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves that $\{x_n\}$ is Cauchy. $\square$

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Partial converse.

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Theorem

Let $M = \mathbb{R}^n$ with the Eucliean metric, and suppose that $\{x_n\}$ is a Cauchy sequence in M.
Then $\{x_n\}$ converges.

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Let $M = \mathbb{R}^n$ with the Euclidean metric, and suppose that $\{x_n\}$ is a Cauchy sequence in M.
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Proof.

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Consider the range $X = \{x_1, x_2, \dots\}$ of $\{x_n\}$.

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Consider the range $X = \{x_1, x_2, \dots\}$ of $\{x_n\}$.
Since $\{x_n\}$ is Cauchy, X is bounded.

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Since $\{x_n\}$ is Cauchy, X is bounded.
We consider two cases:

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Consider the range $X = \{x_1, x_2, \dots\}$ of $\{x_n\}$.
Since $\{x_n\}$ is Cauchy, X is bounded.
We consider two cases:

- when X is finite

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We consider two cases:

- when X is finite
- when X is infinite

□

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Proof.

Case I: Assume X is finite.

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Then

$$\{d(x, y) : x, y \in X,\ x \neq y\}$$

is a finite set of positive values.

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Therefore it has a minimum $\epsilon > 0$.

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is a finite set of positive values.
Therefore it has a minimum $\epsilon > 0$.
Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.

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Proof.

Case I: Assume X is finite.
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$$\{d(x, y) : x, y \in X,\ x \neq y\}$$

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Therefore it has a minimum $\epsilon > 0$.
Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.
This implies $x_n = x_N$ for all $n \geq N$.

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is a finite set of positive values.
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Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.
This implies $x_n = x_N$ for all $n \geq N$.
Hence $\lim_{n\to\infty} x_n = x_N$. □

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Proof.

Case II: Assume X is infinite.

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Then by the Bolzano-Weierstrass Theorem, X has an accumulation point $L \in M$.

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Case II: Assume X is infinite.
Then by the Bolzano-Weierstrass Theorem, X has an accumulation point $L \in M$.
Let $\epsilon > 0$.
Since $\{x_n\}$ is Cauchy, we can choose $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon/2$ for all $m, n \geq N$.

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Moreover, the ball $B(L, \epsilon/2)$ contains infinitely many points of X, so we can choose $\ell \geq N$ with $x_\ell \in B(L, \epsilon/2)$.

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Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to L. $\square$

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- $[0, 1]$ with the Euclidean metric is complete
- $(0, 1)$ with the Euclidean metric is not complete

Challenge!

Problem

Show that the interval $(0, 1) \subseteq \mathbb{R}$ with the Euclidean metric is not a complete metric space.