MATH 350-2 Advanced Calculus

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September 18, 2024

Outline

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Ultimate Cantor diagonalization

The **power set** $\mathcal{P}(A)$ of a set A is

$$\mathcal{P}(A) = \{S : S \subseteq A\}.$$

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Proof.

Suppose $f : A \to \mathcal{P}(A)$ is surjective.
Consider the set

$$S = \{a \in A : a \notin f(a)\}.$$

$\square$

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Problem

Consider the statement $x \in S$. What can you conclude?

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Is there a set with cardinality larger than $\mathbb{R}$?

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Show that the cardinality of the line segment $(0, 1)$ and the square $(0, 1) \times (0, 1)$ is the same.

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NOTATION:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

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NOTATION:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

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Challenge

- index set $I = (1, 2)$
- family of sets $\{A_i : i \in I\}$
- $A_i = [0, i]$

Problem

Determine $\bigcup_{i \in I} A_i$.

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- index set $I = \mathbb{Z}_+$
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- $A_i = [0, i)$

Problem

Determine $\bigcap_{i\in I} A_i$.

Countable unions

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Define $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i \in I} A_i$
$f(j, k) = a_{i_j,k}$. Surjection! $\square$

Challenge!

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Prove that the set of all algebraic numbers is countable.

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Hint: use the previous theorem!

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1. Real Analysis Lecture 7
   - More with cardinality
   - Set algebra
   - Open Balls and Open Sets

Euclidean space

n-dimensional **euclidean space** is

$$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) : a_1, \ldots, a_n \in \mathbb{R}\}.$$

Definitions:
Given $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$

(a) $\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n + y_n)$
(b) $a\vec{x} = (ax_1, \ldots, ax_n)$
(c) $\vec{x} - \vec{y} = \vec{x} + (-1)\vec{y} = (x_1 - y_1, \ldots, x_n - y_n)$
(d) $\vec{0} = (0, \ldots, 0)$
(e) $\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_y$
(f) $|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$