MATH 350-2 Advanced Calculus

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Outline

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Completeness Axiom

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- 3 is the supremum of $(0, 3)$

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- 1 is the supremum of

$$\left\{ \frac{n}{n+1} : n \in \mathbb{Z}_+ \right\}$$

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- 1 is the supremum of

$$\left\{ \frac{n}{n+1} : n \in \mathbb{Z}_+ \right\}$$

- π is the supremum of

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \dots\}.$$

Challenge

Problem

Prove that if A is a set of integers that is bounded above, then A has a maximum.

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Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.

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Therefore $k \leq 0$ and $a' \leq a$.

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Since b is an upper bound, $b \geq a + k > b - 1 + k$, making $0 > k - 1$.
Therefore $k \leq 0$ and $a' \leq a$.
It follows that a is an upper bound of A, and since $a \in A$ it is a maximum.

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

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If $b \in S$ also, then b is called a **minimal element** of S.

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In other words

an infimum is a **greatest lower bound**

Properties of Suprema

The first property of suprema is that they must be arbitrarily close to elements of the set.

Theorem (Approximation Property)

Let $S \subseteq \mathbb{R}$ be bounded above, and let $b = \sup(S)$. Then for all $\epsilon > 0$, there exists $a \in S$ with

$$b - \epsilon < a < b.$$

Properties of Suprema

Proof.

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Since $b - \epsilon < b$, the definition of a supremum implies $b - \epsilon$ cannot be an upper bound.
Therefore thre must exist $a \in S$ with $a > b - \epsilon$. □

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

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Consider the set

$$S = \{s_n : n \in \mathbb{Z}_+\}.$$

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Consider the set

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For every n,

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 3 - \frac{1}{2^n} < 3$$

so S is bounded above by 3.

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so S is bounded above by 3. Therefore S has a supremum, $\sup(S)$.

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We've just shown that $e^1 = e$ exists.

Properties of Suprema

Also, suprema play nicely with addition.

Theorem (Additive Property)

Let $A, B \subseteq \mathbb{R}$ be bounded above set

$$C = \{x + y : x \in A,\ y \in B\}.$$

Then C is bounded above and

$$\sup(C) = \sup(A) + \sup(B).$$

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Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.

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If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.

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Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.

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Therefore $c - y$ is an upper bound of A and $a \leq c - y$.

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It follows that $y \leq c - a$ for all $y \in B$.

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Therefore $c - y$ is an upper bound of A and $a \leq c - y$.
It follows that $y \leq c - a$ for all $y \in B$.
Thus $c - a$ is an upper bound of B, and it follows $b \leq c - a$.

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We will prove $c \le a + b$ and then $a + b \le c$.
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Therefore $c - y$ is an upper bound of A and $a \le c - y$.
It follows that $y \le c - a$ for all $y \in B$.
Thus $c - a$ is an upper bound of B, and it follows $b \le c - a$.
Therefore $a + b \le c$. $\square$

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Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

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Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists

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The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.

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The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.

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This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.

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This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.

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The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.
This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.
However, $n + 1 \in \mathbb{Z}$, so this contradicts b being an upper bound.

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

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For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

Proof.

If not, then x is an upper bound of $\mathbb{Z}_+$. □

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Proof.

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Theorem (Archimedian Property of Reals)

For every $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{Z}_+$ with $y < nx$.

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Theorem (Apostol Theorem 1.18)

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For every $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{Z}_+$ with $y < nx$.

Proof.

Replace x with y/x in the previous theorem. □

Outline

Decimal approximations

A **finite decimal expansion** is an expression

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n},$$

where $a_0 \in \mathbb{Z}_+$ and $0 \leq a_k \leq 9$ for $1 \leq k \leq n$.

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Notation:

$$a_0.a_1a_2a_3\ldots a_n.$$

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Notation:

$$a_0.a_1 a_2 a_3 \ldots a_n.$$

Any positive real number $x > 0$ can be approximated by a finite decimal expansion.

Decimal approximations

Theorem (Apostol Theorem 1.20)

For any real $x > 0$ and $n \in \mathbb{Z}_+$, there exists a finite decimal expansion $r_n = a_0.a_1a_2\ldots a_n$ with

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

Decimal approximations

Proof.

Consider the set

$$A = \{a \in \mathbb{Z} : a \leq x\}.$$

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The set A is a set of integers which is bounded above by x, so it has a maximum a_0.

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$$a_k = \max\{a \in \mathbb{Z}_+ : a \leq 10x_k\}$$

and $x_{k+1} = 10x_k - a_k$.

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$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n} \leq x < a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n + 1}{10^n}$$

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We say that $x > 0$ has the decimal expansion $a_0.a_1a_2a_3\dots$ and write

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$$1 \not< 0 + \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9+1}{10^n} = 1.$$

Outline

Absolute value

The **absolute value** of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In particular,

$$0 \leq |x|$$

and also

$$-|x| \leq x \leq |x|.$$

Challenge!

Problem

Prove Apostol Theorem 1.21 that if $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.

Solution

If $|x| \leq a$ then $-a \leq -|x|$

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If $|x| \leq a$ then $-a \leq -|x|$ and therefore

$$-a \leq -|x| \leq x \leq |x| \leq a$$

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$$x \geq 0 \Rightarrow |x| = x \leq a$$

$$x \leq 0 \Rightarrow |x| = -x \leq -(-a) = a$$

Basic triangle inequality

Theorem (Triangle inequality)

For any real numbers $x, y \in \mathbb{R}$ we have

$$|x + y| \leq |x| + |y|.$$

Proof.

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|$$

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Adding these together, we get

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It follows from the previous theorem that

$$|x + y| \leq |x| + |y|$$

Advanced triangle inequality

Theorem (Triangle inequality)

For any real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ *we have*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Advanced triangle inequality

Theorem (Triangle inequality)

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Proof.

Induction. □

Higher-dimensional triangle inequality

Theorem (Cauchy-Schwartz Inequality (Apostol Theorem 1.23))

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are real numbers, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$$

If y_k isn't always zero, thene equality holds if and only if there exists $t \in \mathbb{R}$ with $x_k = t y_k$ for all k.

Higher-dimensional triangle inequality

Proof.

$$\sum_{k=1}^{n}(x_k + ty_k)^2 \geq 0, \quad \text{with equality iff all terms zero}$$

$$\sum_{k=1}^{n} x_k^2 + 2t\sum_{k=1}^{n} x_k y_k + t^2 \sum_{k=1}^{n} y_k^2 \geq 0$$

Take $t = -\left(\sum_{k=1}^{n} x_k y_k\right) / \left(\sum_{k=1}^{n} y_k^2\right)$:

$$\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{k=1}^{n} x_k y_k\right)^2}{\left(\sum_{k=1}^{n} y_k^2\right)} \geq 0.$$

□