MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 3
   - Decimal expansions
   - The Triangle Inequality
   - Sets, Relations, Functions

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Decimal approximations

A **finite decimal expansion** is an expression

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n},$$

where $a_0 \in \mathbb{Z}_+$ and $0 \leq a_k \leq 9$ for $1 \leq k \leq n$.

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Notation:

$$a_0.a_1a_2a_3\ldots a_n.$$

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Any positive real number $x > 0$ can be approximated by a finite decimal expansion.

Decimal approximations

Theorem (Apostol Theorem 1.20)

For any real $x > 0$ and $n \in \mathbb{Z}_+$, there exists a finite decimal expansion $r_n = a_0.a_1a_2\dots a_n$ with

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

Decimal approximations

Proof.

Consider the set

$$A = \{a \in \mathbb{Z} : a \leq x\}.$$

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Define $a_1, a_2, a_3, \dots$ and $x_1, x_2, x_3, \dots$ recursively by

$$a_k = \max\{a \in \mathbb{Z}_+ : a \leq 10x_k\}$$

and $x_{k+1} = 10x_k - a_k$.

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$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n} \leq x < a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n + 1}{10^n}$$

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We say that $x > 0$ has the decimal expansion $a_0.a_1 a_2 a_3 \dots$ and write

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Absolute value

The **absolute value** of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In particular,

$$0 \leq |x|$$

and also

$$-|x| \leq x \leq |x|.$$

Challenge!

Problem

Prove Apostol Theorem 1.21 that if $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.

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Conversely, if $-a \leq x \leq a$ then

$$x \geq 0 \Rightarrow |x| = x \leq a$$

$$x \leq 0 \Rightarrow |x| = -x \leq -(-a) = a$$

Basic triangle inequality

Theorem (Triangle inequality)

For any real numbers $x, y \in \mathbb{R}$ we have

$$|x + y| \leq |x| + |y|.$$

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Adding these together, we get

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It follows from the previous theorem that

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Advanced triangle inequality

Theorem (Triangle inequality)

For any real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ *we have*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

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Proof.

Induction. $\square$

Higher-dimensional triangle inequality

Theorem (Cauchy-Schwartz Inequality (Apostol Theorem 1.23))

If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are real numbers, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$$

If y_k isn't always zero, then equality holds if and only if there exists $t \in \mathbb{R}$ with $x_k = t y_k$ for all k.

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If y_k isn't always zero, then equality holds if and only if there exists $t \in \mathbb{R}$ with $x_k = ty_k$ for all k.

Vector version:

$$(\vec{x} \cdot \vec{y})^2 \leq |\vec{x}|^2 |\vec{y}|^2.$$

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$$\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{k=1}^{n} x_k y_k\right)^2}{\left(\sum_{k=1}^{n} y_k^2\right)} \geq 0.$$

$\square$

Higher-dimensional triangle inequality

Theorem (Minkowski inequality)

For any real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ *and* $y_1, y_2, \ldots, y_n \in \mathbb{R}$ *we have*

$$\left(\sum_{k=1}^{n}(x_k + y_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}$$

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A higher dimensional triangle inequality!

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Now applying the Cauchy-Schwartz inequality:

$$\le \left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2}$$

$$= \left[\left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\right]\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2}$$

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- $\{n \in \mathbb{Z} : n \text{ is prime}\}$

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$$(a, b) = \{a, \{a, b\}\}$$
- even relations and functions are sets!

Challenge!

Problem

Prove that for ordered pairs (a, b) and (c, d) that

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If $b = d$, we're done!

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It follows that $d = c = b = a$.

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This can be shown to contradict the ZF Axioms of Set Theory.

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Specifically the regularity axiom for the set $\{a, c\}$...

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Give an example of a relation on $\mathbb{R}$ which is symmetric and transitive but not reflexive.

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If a relation $\mathcal{R}$ is a function, we usually use a symbol like f.

NOTATION: $f : A \to B$ means f is a function from A to B

NOTATION: $f(a) = b$ means $(a, b) \in f$.

The set

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This means $f(x) = x$ for all x.
Thus the only function which is an equivalence relation is the identity function

$$f(x) = x.$$

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If it satisfies both properties, it is called **bijective.**

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If it is, we call it the **inverse** of f.

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