

MATH 350-2 Advanced Calculus

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Outline

- 1 Real Analysis Lecture 12
 - Metric Spaces
 - Subspaces

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 - Subspaces

Warm-up Challenge

Problem

Write down each of the following:

- the definition of an open cover of a set A

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- Cantor Intersection Theorem

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- Heine-Borel Theorem

Metric space

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Definition

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- A4 **triangle inequality:** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$

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- A4 **triangle inequality:** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$

The value $d(x, y)$ is called a **metric** and describes the “distance” between x and y .

Open balls

Open balls

Definition

Let (M, d) be a metric space. The open ball of radius $r > 0$ centered at $x \in M$ is

$$B_M(x; r) = \{y \in M : d(x, y) < r\}.$$

Examples of metrics on \mathbb{R}^2

$$d_{\text{eucl}}(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Examples of metrics on \mathbb{R}^2

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- Euclidean metric (2-norm)

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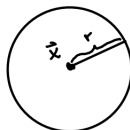
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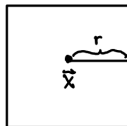
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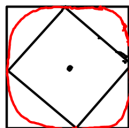
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Challenge!

Let M be a nonempty set and define $d : M \times M \rightarrow \mathbb{R}$ by

$$d_{\text{disc}}(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

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Prove that d_{disc} is a metric.

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$$d_{\text{disc}}(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Problem

Prove that d_{disc} is a metric.

This is called the **discrete metric**.

Challenge!

Problem

What do the open balls with the discrete metric look like?

Open sets

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The set $\text{int}(A)$ of interior points of A is called the **interior** of A .

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The set A is **open** if every point in A is an interior point, or equivalently $\text{int}(A) = A$.

Challenge!

Problem

Consider the metric space $(\mathbb{R}, d_{\text{disc}})$ where d_{disc} is the discrete metric.

What sets are open sets?

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A point $x \in M$ is an **adherent point** if for all $r > 0$ the ball $B_M(x; r)$ contains an element of M .

A point $x \in M$ is an **accumulation point** if for all $r > 0$ the ball $B_M(x; r)$ contains an element of M different from x .

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A point $x \in M$ is an **accumulation point** if for all $r > 0$ the ball $B_M(x; r)$ contains an element of M different from x .

The set \overline{M} of all adherent points of M is called the **closure** of M .

Challenge!

Problem

Consider the metric space $(\mathbb{R}, d_{\text{disc}})$ where d_{disc} is the discrete metric.

Which sets are closed?

Challenge!

Problem

Prove the following (Apostol Theorem 3.36). If (M, d) is a metric space, $U \subseteq M$ is open, and $C \subseteq M$ is closed, then $U \setminus C$ is open and $C \setminus U$ is closed.

Characterizing closed sets

Theorem (Apostol Theorem 3.37)

Let (M, d) be a metric space and $A \subseteq S$. Then the following are equivalent:

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- (d) $A = \overline{A}$

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Identical to the proof for \mathbb{R}^n . □

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The space (S, d) with the relative metric is called a **metric subspace**.

Open balls in S :

$$B_S(x; r) = B_M(x; r) \cap S.$$

Open sets in subspaces

Theorem

Let (M, d) be a metric space and let (S, d) be a subspace. Then V is an open subset of S if and only if there exists an open subset U of M with $V = U \cap S$.

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Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.

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This is a union of open sets, and is therefore open.

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Furthermore $U \cap S = V$.



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We need to show that $V := U \cap S$ is open.

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If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.

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If $x \in V$, then $x \in U$ and there exists $r > 0$ such that

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It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.

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Since $x \in V$ was arbitrary, this shows that V is open in S . □

Closed sets in subspaces

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Closed sets in subspaces

Theorem

Let (M, d) be a metric space and let (S, d) be a subspace. Then B is a closed subset of S if and only if there exists a closed subset A of M with $B = A \cap S$.

Proof.

Let $B \subseteq S$.

B is closed in S if and only if $S \setminus B$ is open in S

This is true if and only if there exists an open subset U of M with $S \setminus B = U \cap S$.

This is true if and only if there exists an open subset U of M with $B = (M \setminus U) \cap S$.

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- $(1/2, 1]$ is open in S , but not in M