MATH 350-2 Advanced Calculus

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Outline

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Warm-up Challenge

Problem

Show that the closed ball

$$\overline{B}(\vec{x}; r) = \{\vec{y} \in \mathbb{R}^n : |\vec{x} - \vec{y}| \leq r\}$$

is a closed set.

Point set topology

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- kinds of functions: continuous, homeomorphism

Answers must be in terms of open sets

Open covers

Definition

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- $\{(0, 1/2), (1/2, 2)\}$ is an open cover of the interval $(0, 1]$
- $\{(\frac{1}{n}, \frac{n+1}{n}) : n \in \mathbb{Z}_+\}$ is an open cover of $(0, 1]$

Lindelöf Covering Theorem

Theorem (Lindelöf Covering Theorem)

Let $A \subseteq \mathbb{R}^n$ be a set and suppose $\{U_i : i \in I\}$ is an open covering of A. Then there exists a countable subcover $\{U_j : j \in J\}$.

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A set $A \subseteq \mathbb{R}^n$ is called **compact** if every open cover of A has a *finite* subcover.

Challenge

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Show that a singleton set $A = \{\vec{x}\}$ is compact.

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Solution

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Then $A \subseteq \bigcup_{i \in I} U_i$.
This means that $\vec{x} \in \bigcup_{i \in I} U_i$.

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Therefore $\vec{x} \in U_j$ for some $j \in I$.

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It follows that $\{U_j\}$ is a subcover of A consisting of a single set.

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It follows that $\{U_j\}$ is a subcover of A consisting of a single set.
Thus every open cover has a finite subcover, making A compact!

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Proof.

Suppose A is a compact set.
Consider the family of sets $\{U_i : i \in I\}$ where $I = \mathbb{Z}_+$ and $U_i = B(\vec{0}; i)$.

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Moreover, if $\vec{x} \in \mathbb{R}^n$, then we can choose $N \in \mathbb{Z}_+$ such that $N > |\vec{x}|$.
Therefore $\vec{x} \in B(\vec{0}; N)$ and it follows $\vec{x} \in \bigcup_{i \in I} U_i$.
Thus $\mathbb{R}^n = \bigcup_{i \in I} U_i$, and since $A \subseteq \mathbb{R}^n$, we see $A \subseteq \bigcup_{i \in I} U_i$.

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Solution

There is a finite subset $\{i_1, i_2, \ldots, i_m\} \subseteq I$ with $A \subseteq \bigcup_{k=1}^{m} U_{i_k}$.

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Without loss of generality, $i_1 < i_2 < \dots i_m$. Then

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Consider the family of sets $\{U_i : i \in I\}$ where $I = \mathbb{Z}_+$ and $U_i = \mathbb{R}^n \backslash \overline{B}(\vec{x}; 1/i)$. □

Challenge

Problem

Prove that $\bigcup_{i \in I} U_i = \mathbb{R}^n \backslash \{\vec{x}\}$.

Challenge

Problem

Explain why $\{U_i : i \in I\}$ is an open cover of A.

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Problem

Show that if $\{U_i : i \in I\}$ has a finite subcover, then $\vec{x}$ can't be an adherent point of $\vec{A}$.

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Theorem (Bolzano-Weierstrass Theorem)

A bounded set $A \subseteq \mathbb{R}^n$ with infinitely many points must have an accumulation point.

Cantor Intersection Theorem

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Suppose that $C_1, C_2, C_3, \ldots$ are non-empty closed, bounded sets with $C_{i+1} \subseteq C_i$ for all $i \in I = \mathbb{Z}_+$. Then $\bigcap_{i \in I} C_i$ is also non-empty.

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The set $A = \{\vec{x}_i : i \in \mathbb{Z}_+\}$ is infinite and bounded (because it is contained in C_1).

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Therefore the Bolzano-Weierstrass Theorem tells us it has an accumulation point $\vec{x}$.

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In fact, if we let $A_m = \{\vec{x}_i : i \in \mathbb{Z}_+,\ i \geq m\}$, then $\vec{x}$ is an accumulation point of A_m for all m.
Moreover, $A_m \subseteq C_m$ so $\vec{x}$ is an accumulation point of C_m for all m.

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Since C_m is closed, it follows $\vec{x} \in C_m$, and thus $\vec{x} \in \bigcap_{i \in I} C_i$. In particular the intersection is non-empty! $\square$

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Define closed sets $C_1, C_2, C_3, \dots$ by

$$C_m = A \cap \left(\mathbb{R}^n \backslash \bigcup_{i=1}^{m} U_i\right).$$

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This is a contradiction. $\square$