MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

September 22, 2024

Outline

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1. Real Analysis Lecture 7
    - More with cardinality
    - Set algebra

Ultimate Cantor diagonalization

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$$\mathcal{P}(A) = \{S : S \subseteq A\}.$$

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Proof.

Suppose $f : A \to \mathcal{P}(A)$ is surjective.
Consider the set

$$S = \{a \in A : a \notin f(a)\}.$$

□

Challenge

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Problem

Consider the statement $x \in S$. What can you conclude?

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Is there a set with cardinality larger than $\mathbb{R}$?

Challenge

Problem

Show that the cardinality of the line segment $(0, 1)$ and the square $(0, 1) \times (0, 1)$ is the same.

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NOTATION:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

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Challenge

- index set $I = (1, 2)$
- family of sets $\{A_i : i \in I\}$
- $A_i = [0, i]$

Problem

Determine $\bigcup_{i \in I} A_i$.

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- index set $I = \mathbb{Z}_+$
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Determine $\bigcap_{i \in I} A_i$.

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Define $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i \in I} A_i$
$f(j, k) = a_{i_j, k}$. Surjection! □

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Hint: use the previous theorem!