MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

September 4, 2024

Outline

Outline

Completeness Axiom

Last time, we finished with the Completeness Axiom:

Completeness Axiom

Last time, we finished with the Completeness Axiom:

Completeness Axiom

Last time, we finished with the Completeness Axiom:

- A10 **completeness axiom**: if S is any subset of real numbers which is bounded above, then it has a supremum $\sup(S)$

Completeness Axiom

Last time, we finished with the Completeness Axiom:

- A10 **completeness axiom**: if S is any subset of real numbers which is bounded above, then it has a supremum $\sup(S)$

Examples:

Completeness Axiom

Last time, we finished with the Completeness Axiom:

A10 **completeness axiom**: if S is any subset of real numbers which is bounded above, then it has a supremum $\sup(S)$

Examples:

- 3 is the supremum of $(0, 3)$

Completeness Axiom

Last time, we finished with the Completeness Axiom:

A10 **completeness axiom**: if S is any subset of real numbers which is bounded above, then it has a supremum $\sup(S)$

Examples:

- 3 is the supremum of $(0,3)$
- 1 is the supremum of

$$\left\{\frac{n}{n+1} : n \in \mathbb{Z}_+\right\}$$

Completeness Axiom

Last time, we finished with the Completeness Axiom:

A10 **completeness axiom**: if S is any subset of real numbers which is bounded above, then it has a supremum $\sup(S)$

Examples:

- 3 is the supremum of $(0, 3)$
- 1 is the supremum of

$$\left\{\frac{n}{n+1} : n \in \mathbb{Z}_+\right\}$$

- π is the supremum of

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \dots\}.$$

Challenge

Problem

Prove that if A is a set of integers that is bounded above, then A has a maximum.

Challenge

Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.

Challenge

Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$. Since $b - 1$ is not an upper bound, so there exists $a \in A$ with $b - 1 < a$.

Challenge

Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.
Since $b - 1$ is not an upper bound, so there exists $a \in A$ with $b - 1 < a$.
If $a' \in A$, then a' is an integer and therefore $a' = a + k$ for $k \in \mathbb{Z}$.

Challenge

Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.
Since $b - 1$ is not an upper bound, so there exists $a \in A$ with $b - 1 < a$.
If $a' \in A$, then a' is an integer and therefore $a' = a + k$ for $k \in \mathbb{Z}$.
Since b is an upper bound, $b \geq a + k > b - 1 + k$, making $0 > k - 1$.

Challenge

Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.
Since $b - 1$ is not an upper bound, so there exists $a \in A$ with $b - 1 < a$.
If $a' \in A$, then a' is an integer and therefore $a' = a + k$ for $k \in \mathbb{Z}$.
Since b is an upper bound, $b \geq a + k > b - 1 + k$, making $0 > k - 1$.
Therefore $k \leq 0$ and $a' \leq a$.

Challenge

Solution

The set A is bounded above, so it has a supremum $b = \sup(A)$.
Since $b - 1$ is not an upper bound, so there exists $a \in A$ with $b - 1 < a$.
If $a' \in A$, then a' is an integer and therefore $a' = a + k$ for $k \in \mathbb{Z}$.
Since b is an upper bound, $b \geq a + k > b - 1 + k$, making $0 > k - 1$.
Therefore $k \leq 0$ and $a' \leq a$.
It follows that a is an upper bound of A, and since $a \in A$ it is a maximum.

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

$$b \leq x \quad \text{for all } x \in S.$$

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

$$b \leq x \quad \text{for all } x \in S.$$

In this case, we say S is **bounded below** by b.

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

$$b \leq x \quad \text{for all } x \in S.$$

In this case, we say S is **bounded below** by b.
If $b \in S$ also, then b is called a **minimal element** of S.

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

$$b \leq x \quad \text{for all } x \in S.$$

In this case, we say S is **bounded below** by b.
If $b \in S$ also, then b is called a **minimal element** of S. An **infimum** of a set S of real numbers is a real number $b \in \mathbb{R}$ such that

- b is a lower bound of S
- if $b < b'$, then b' is not a lower bound of S

Lower bounds and infima

A **lower bound** for a set $S \subseteq \mathbb{R}$ is a number b such that

$$b \leq x \quad \text{for all } x \in S.$$

In this case, we say S is **bounded below** by b.
If $b \in S$ also, then b is called a **minimal element** of S. An **infimum** of a set S of real numbers is a real number $b \in \mathbb{R}$ such that

- b is a lower bound of S
- if $b < b'$, then b' is not a lower bound of S

In other words

an infimum is a **greatest lower bound**

Properties of Suprema

The first property of suprema is that they must be arbitrarily close to elements of the set.

Theorem (Approximation Property)

Let $S \subseteq \mathbb{R}$ be bounded above, and let $b = \sup(S)$. Then for all $\epsilon > 0$, there exists $a \in S$ with

$$b - \epsilon < a \leq b.$$

Properties of Suprema

Proof.

Since $b - \epsilon < b$, the definition of a supremum implies $b - \epsilon$ cannot be an upper bound.

Properties of Suprema

Proof.

Since $b - \epsilon < b$, the definition of a supremum implies $b - \epsilon$ cannot be an upper bound.
Therefore thre must exist $a \in S$ with $a > b - \epsilon$. $\square$

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Notice that

$$s_1 < s_2 < s_3 < s_4 < \ldots$$

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Notice that

$$s_1 < s_2 < s_3 < s_4 < \dots$$

Consider the set

$$S = \{s_n : n \in \mathbb{Z}_+\}.$$

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Notice that

$$s_1 < s_2 < s_3 < s_4 < \ldots$$

Consider the set

$$S = \{s_n : n \in \mathbb{Z}_+\}.$$

For every n,

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 3 - \frac{1}{2^n} < 3$$

so S is bounded above by 3.

Important example: The number e

For each n, let s_n denote the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Notice that

$$s_1 < s_2 < s_3 < s_4 < \ldots$$

Consider the set

$$S = \{s_n : n \in \mathbb{Z}_+\}.$$

For every n,

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 3 - \frac{1}{2^n} < 3$$

so S is bounded above by 3. Therefore S has a supremum, $\sup(S)$.

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

Important example: The number *e*

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

This says $\sup(S)$ is *really* close to s_N.

Important example: The number *e*

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

This says $\sup(S)$ is *really* close to s_N. Taking ϵ smaller and smaller

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

This says $\sup(S)$ is *really* close to s_N. Taking ϵ smaller and smaller

$$\sup(S) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

This says $\sup(S)$ is *really* close to s_N. Taking ϵ smaller and smaller

$$\sup(S) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

If we remember Taylor series

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

This says $\sup(S)$ is *really* close to s_N. Taking ϵ smaller and smaller

$$\sup(S) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

If we remember Taylor series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Important example: The number e

Even if ϵ is very small ($\epsilon = 0.000000001$), $\sup(S) - \epsilon$ is not an upper bound of S.
So there exists N with $s_N > \sup(S) - \epsilon$.

$$s_1 < s_2 < s_3 < s_4 < \cdots < \sup(S) - \epsilon < s_N < \cdots < \sup(S).$$

This says $\sup(S)$ is *really* close to s_N. Taking ϵ smaller and smaller

$$\sup(S) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

If we remember Taylor series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

We've just shown that $e^1 = e$ exists.

Properties of Suprema

Also, suprema play nicely with addition.

Theorem (Additive Property)

Let $A, B \subseteq \mathbb{R}$ be bounded above set

$$C = \{x + y : x \in A,\ y \in B\}.$$

Then C is bounded above and

$$\sup(C) = \sup(A) + \sup(B).$$

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.
This means $c \leq a + b$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.
This means $c \leq a + b$.
Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.
This means $c \leq a + b$.
Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.
Therefore $c - y$ is an upper bound of A and $a \leq c - y$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.
This means $c \leq a + b$.
Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.
Therefore $c - y$ is an upper bound of A and $a \leq c - y$.
It follows that $y \leq c - a$ for all $y \in B$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.
This means $c \leq a + b$.
Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.
Therefore $c - y$ is an upper bound of A and $a \leq c - y$.
It follows that $y \leq c - a$ for all $y \in B$.
Thus $c - a$ is an upper bound of B, and it follows $b \leq c - a$.

Properties of Suprema

Proof.

Let $a = \sup(A)$, $b = \sup(B)$, and $c = \sup(C)$.
We will prove $c \leq a + b$ and then $a + b \leq c$.
First, note a is an upper bound of A, so $x \leq a$ for all $x \in A$.
Also b is an upper bound of b, so $y \leq b$ for all $y \in B$.
If $z \in C$, then $z = x + y$ for some $x \in A$ and $y \in B$, and therefore $z = x + y \leq a + b$.
Therefore $a + b$ is an upper bound of C.
This means $c \leq a + b$.
Next, note for all $x \in A$ and $y \leq B$ that $x \leq c - y$.
Therefore $c - y$ is an upper bound of A and $a \leq c - y$.
It follows that $y \leq c - a$ for all $y \in B$.
Thus $c - a$ is an upper bound of B, and it follows $b \leq c - a$.
Therefore $a + b \leq c$. □

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.
This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.
This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.

Challenge

Problem

Use the completeness axiom to prove that the $\mathbb{Z}_+$ is not bounded above. (Apostol Theorem 1.17)

Hint: assume it is and consider $\sup(\mathbb{Z}_+) - 1$

Solution

Assume it is.
The completeness axiom implies that $b = \sup(\mathbb{Z}_+)$ exists
The number b is an upper bound and if $b' < b$, then b' is not.
Then $b - 1 < b$ by Axiom 7, so $b - 1$ cannot be an upper bound.
This means there exists $n \in \mathbb{Z}_+$ with $b - 1 < n$.
It follows from Axiom 7 that $b < n + 1$.
However, $n + 1 \in \mathbb{Z}$, so this contradicts b being an upper bound.

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

Proof.

If not, then x is an upper bound of $\mathbb{Z}_+$. $\square$

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

Proof.

If not, then x is an upper bound of $\mathbb{Z}_+$. $\square$

Theorem (Archimedian Property of Reals)

For every $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{Z}_+$ with $y < nx$.

Archimedian property

Theorem (Apostol Theorem 1.18)

For every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}_+$ with $n > x$.

Proof.

If not, then x is an upper bound of $\mathbb{Z}_+$. □

Theorem (Archimedian Property of Reals)

For every $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{Z}_+$ with $y < nx$.

Proof.

Replace x with y/x in the previous theorem. □