MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

October 2, 2024

Outline

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1. Real Analysis Lecture 9
   - More on Open Sets
   - Closed Sets
   - Compactness

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

Suppose that $U_1, U_2, \ldots, U_n$ are open sets. Then $\bigcap_{i=1}^{n} U_i$ is open.

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Then for all i, $\vec{x} \in U_i$.

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Suppose that $\vec{x} \in \bigcap_{i=1}^n U_i$.
Then for all i, $\vec{x} \in U_i$.
Since U_i is open, this means $\vec{x}$ is an interior point of U_i.

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Therefore there exists $r_i > 0$ such that $B(\vec{x}; r_i) \subseteq U_i$.

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Therefore $B(\vec{x}; r) \subseteq \bigcap_{i=1}^n U_i$.
Thus $\vec{x}$ is an interior point of $\bigcap_{i=1}^n U_i$.
Since $\vec{x}$ is arbitrary, this proves that $\bigcap_{i=1}^n U_i$ is open. $\square$

Challenge

Problem

Show that the infinite intersection

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

is not open.

Component intervals

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- $(-\infty, 0)$ is also a component interval of $\mathbb{R} \backslash \{0, 1, 2, 3\}$
- we will show all open sets of $\mathbb{R}$ are made of component intervals!

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Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

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Then $J := I_1 \cup I_2$ is an interval.

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Also $I_1 \subseteq U$ and $I_2 \subseteq U$, so $J \subseteq U$.

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In particular $I_1 = J = I_2$. □

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If A is not bounded below, let $a = -\infty$. Otherwise, let $a = \inf(A)$.
If B is not bounded above, let $b = \infty$. Otherwise, let $b = \sup(B)$.
Claim: (a, b) is a component interval of U containing x. □

Challenge

Problem

Prove that (a, b) is a component interval of U.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Sets in $\mathbb{R}$)

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Uh oh ... this isn't a **countable** union. $\square$

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If $x \in U$, then $x \in I_x \subseteq U$.
Any interval contains a rational number $r \in I_x$.

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Any interval contains a rational number $r \in I_x$.
Since different component intervals must be disjoint, $I_x = I_r$.
Thus $x \in \bigcup_{r \in U \cap \mathbb{Q}} I_r \subseteq U$.
Since x was arbitrary,

$$U = \bigcup_{r \in U \cap \mathbb{Q}} I_r$$

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- singleton sets!

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Examples:

- singleton sets!
- products of closed intervals

Challenge!

Problem

Prove that a singleton set

$$A = \{\vec{a}\}$$

in $\mathbb{R}^n$ is closed.

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Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.

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Solution

We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$.

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We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.

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Take $r = |\vec{x} - \vec{a}|$.

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We must show $U = \mathbb{R}^n \backslash \{\vec{a}\}$ is open.
Suppose $\vec{x} \in U$. We must show $\vec{x}$ is an interior point of U.
Take $r = |\vec{x} - \vec{a}|$. Then $B(\vec{x}; r)$ does not contain $\vec{a}$.

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Therefore $B(\vec{x}; r) \subseteq U$, so that $\vec{x}$ is an interior point.

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Take $r = |\vec{x} - \vec{a}|$. Then $B(\vec{x}; r)$ does not contain $\vec{a}$.
Therefore $B(\vec{x}; r) \subseteq U$, so that $\vec{x}$ is an interior point.
Since $\vec{x}$ was arbitrary, this proves U is open.

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Prove that the **closed square**

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

is a closed set.

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$$((-\infty, a) \times \mathbb{R}) \cup ((b, \infty) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, c)) \cup (\mathbb{R} \times (d, \infty))$$

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Union of open sets is open!

Adherent and Accumulation Points

Definition

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Examples:

- every point in A is adherent
- accumulation points are adherent points
- -1 is an accumulation point of $(-1, 1)$.
- suprema and infima are accumulation points!
- 0 is an accumulation point of $\{1/1, 1/2, 1/3, \dots\}$