MATH 350-2 Advanced Calculus

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Outline

1 Real Analysis Lecture 7
- Open Balls and Open Sets

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   - Open Balls and Open Sets

Euclidean space

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(f) $|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$

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(a) (scaling) $|c\vec{x}| = |c|\,|\vec{x}|$
(b) (Cauchy-Schwartz) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}|\,|\vec{y}|$

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A point $\vec{a}$ in a subset $A \subseteq \mathbb{R}^n$ is called an **interior point** of A if there exists $r > 0$ with $B(\vec{a}; r) \subseteq A$. If every point of A is an interior point, then A is called an **open set**.

Challenge!

Problem

Prove that the empty set $\varnothing$ and the whole space $\mathbb{R}^n$ are open.

Challenge!

Problem

Let $\vec{a} \in \mathbb{R}^n$. Show that the singleton set

$$A = \{\vec{a}\}$$

is not open.

Challenge!

Problem

Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Prove that the **open square**

$$(a, b) \times (c, d) = \{(x, y) : a < x < b, \ \ c < x < d\}$$

is an open set

Challenge!

Problem

Prove that an open ball is an open set.

Unions of open sets are open

Theorem (Open Union Theorem)

Suppose that $\{U_i : i \in I\}$ is an arbitrary family of open sets. Then $\bigcup_{i \in I} U_i$ is open.

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Since $\vec{x}$ is arbitrary, this proves that $\bigcup_{i \in I} U_i$ is open. $\square$

Finite intersections of open sets are open

Theorem (Open Intersection Theorem)

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Take $r = \min\{r_i : 1 \leq i \leq n\}$.

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Thus $\vec{x}$ is an interior point of $\bigcap_{i=1}^n U_i$.
Since $\vec{x}$ is arbitrary, this proves that $\bigcap_{i=1}^n U_i$ is open. $\square$

Challenge

Problem

Show that the infinite intersection

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)$$

is not open.

Component intervals

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Definition

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- $(-\infty, 0)$ is also a component interval of $\mathbb{R}\backslash\{0, 1, 2, 3\}$
- we will show all open sets of $\mathbb{R}$ are made of component intervals!

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Lemma

If I_1 and I_2 are two component intervals of an open subset $U \subseteq \mathbb{R}$, then $I_1 = I_2$ or $I_1 \cap I_2 = \varnothing$.

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Also $I_1 \subseteq U$ and $I_2 \subseteq U$, so $J \subseteq U$.

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In particular $I_1 = J = I_2$. □

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If B is not bounded above, let $b = \infty$. Otherwise, let $b = \sup(B)$.

Component intervals

Theorem (Apostol 3.10)

If $U \subseteq \mathbb{R}$ is open and $x \in U$, then there is a unique component interval of U containing x.

Proof.

Uniqueness follows from previous Lemma, so we only need existence.
Suppose that $x \in U$ and consider

$$A = \{a : (a, x) \subseteq S\}, \quad \text{and} \quad B = \{b : (x, b) \subseteq S\}$$

If A is not bounded below, let $a = -\infty$. Otherwise, let $a = \inf(A)$.
If B is not bounded above, let $b = \infty$. Otherwise, let $b = \sup(B)$.
Claim: (a, b) is a component interval of U containing x. □

Challenge

Problem

Prove that (a, b) is a component interval of U.

Open subsets of $\mathbb{R}$

Theorem (Representation Theorem for Open Intervals in $\mathbb{R}$)

If $U \subseteq \mathbb{R}$ is open, then U is the union of a countable family of disjoint open intervals.