MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

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Outline

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1. Real Analysis Lecture 14
   - More on limits
   - Cauchy sequences

Existence of limits

Theorem

If a sequence $\{x_n\}$ in a metric space (M, d) converges to a value $L \in M$, then the range

$$X = \{x_1, x_2, \dots\} \subseteq M$$

is a bounded set and L is an adherent point of X.

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When the range of a sequence is bounded, we call the sequence $\{x_n\}$ bounded.

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then $d(x_n, L) \leq R$ for all N.
Therefore $X \subseteq B_M(L, R)$ and in particular X is bounded. $\square$

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If S is a subset of a metric space (M, d) and $L \in M$ is an adherent point of S, then there exists a sequence $\{x_n\}$ of elements of S which converges to L.

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It follows that for all $n \geq N$

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Since $\epsilon > 0$ was arbitrary, this proves convergence. □

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A sequence $\{x_n\}$ in a metric space (M, d) converges to a value $L \in M$ if and only if every subsequence $\{x_{k(n)}\}$ also converges to L.

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Let $\{x_{k(n)}\}$ be a subsequence of $\{x_n\}$.

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To show it limits to L, let $\epsilon > 0$.

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To show it limits to L, let $\epsilon > 0$.

Then there exists $N \in \mathbb{Z}_+$ such that $n \geq N$ implies $d(x_n, L) < \epsilon$.

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Then there exists $N \in \mathbb{Z}_+$ such that $n \geq N$ implies $d(x_n, L) < \epsilon$.

Moreover, $k(n) \geq n$ so $k(n) \geq N$ and therefore $d(x_{k(n)}, L) < \epsilon$.

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Since $\epsilon > 0$ was arbitrary, this proves convergence. □

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It is called **strictly decreasing** if $x_n > x_{n+1}$ for all n.
In any of these cases, we call the sequence **monotone**.

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A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \dots\}$.

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Theorem (Monotone convergence theorem)

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A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \ldots\}$.
A monotone decreasing sequence $\{x_n\}$ which is bounded below converges to $L = \inf\{x_1, x_2, \ldots\}$.

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Let $\epsilon > 0$.
Then $L - \epsilon$ is not a supremum of X, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.

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Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \le x_n < L$.

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Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \leq x_n < L$.

It follows that

$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

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$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves x_n converges to L. $\square$

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- is every Cauchy sequence convergent?

Challenge!

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then $d(x_n, x_N) \leq 2024$ for all $n \geq 1$.
Hence $X = \{x_1, x_2, \ldots\} \subseteq B_M(x_N, R)$, and is bounded.