MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

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Outline

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1. Real Analysis Lecture 3
   - Decimal expansions
   - The Triangle Inequality
   - Sets, Relations, Functions

Decimal approximations

A **finite decimal expansion** is an expression

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{10^3} + \cdots + \frac{a_n}{10^n},$$

where $a_0 \in \mathbb{Z}_+$ and $0 \leq a_k \leq 9$ for $1 \leq k \leq n$.

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Notation:

$$a_0.a_1 a_2 a_3 \ldots a_n.$$

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Any positive real number $x > 0$ can be approximated by a finite decimal expansion.

Decimal approximations

Theorem (Apostol Theorem 1.20)

For any real $x > 0$ and $n \in \mathbb{Z}_+$, there exists a finite decimal expansion $r_n = a_0.a_1a_2\dots a_n$ with

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

Decimal approximations

Proof.

Consider the set

$$A = \{a \in \mathbb{Z} : a \leq x\}.$$

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Define $a_1, a_2, a_3, \dots$ and $x_1, x_2, x_3, \dots$ recursively by

$$a_k = \max\{a \in \mathbb{Z}_+ : a \leq 10x_k\}$$

and $x_{k+1} = 10x_k - a_k$.

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and $x_{k+1} = 10x_k - a_k$. Then $0 \leq a_k \leq 9$ for all $k \geq 1$ and
$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_n}{10^n} \leq x < a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_n + 1}{10^n}$$
$\square$

Decimal expansions

We say that $x > 0$ has the decimal expansion $a_0.a_1a_2a_3\ldots$ and write

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$$1 \not< 0 + \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9+1}{10^n} = 1.$$

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Absolute value

The **absolute value** of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

In particular,

$$0 \leq |x|$$

and also

$$-|x| \leq x \leq |x|.$$

Challenge!

Problem

Prove Apostol Theorem 1.21 that if $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.

Solution

If $|x| \leq a$ then $-a \leq -|x|$

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. Conversely, if $-a \leq x \leq a$ then

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$$x \geq 0 \Rightarrow |x| = x \leq a$$

$$x \leq 0 \Rightarrow |x| = -x \leq -(-a) = a$$

Basic triangle inequality

Theorem (Triangle inequality)

For any real numbers $x, y \in \mathbb{R}$ we have

$$|x + y| \leq |x| + |y|.$$

Proof.

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|$$

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Adding these together, we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

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It follows from the previous theorem that

$$|x + y| \leq |x| + |y|$$

Advanced triangle inequality

Theorem (Triangle inequality)

For any real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ we have

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

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Proof.

Induction. $\square$

Higher-dimensional triangle inequality

Theorem (Cauchy-Schwartz Inequality (Apostol Theorem 1.23))

If $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are real numbers, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)$$

If y_k isn't always zero, then equality holds if and only if there exists $t \in \mathbb{R}$ with $x_k = ty_k$ for all k.

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Vector version:

$$(\vec{x} \cdot \vec{y})^2 \leq |\vec{x}|^2 |\vec{y}|^2.$$

Higher-dimensional triangle inequality

Proof.

$$\sum_{k=1}^{n}(x_k + ty_k)^2 \geq 0, \quad \text{with equality iff all terms zero}$$

$$\sum_{k=1}^{n} x_k^2 + 2t\sum_{k=1}^{n} x_k y_k + t^2\sum_{k=1}^{n} y_k^2 \geq 0$$

Take $t = -\left(\sum_{k=1}^{n} x_k y_k\right) / \left(\sum_{k=1}^{n} y_k^2\right)$:

$$\sum_{k=1}^{n} x_k^2 - \frac{\left(\sum_{k=1}^{n} x_k y_k\right)^2}{\left(\sum_{k=1}^{n} y_k^2\right)} \geq 0.$$

$\square$

Higher-dimensional triangle inequality

Theorem (Minkowski inequality)

For any real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $y_1, y_2, \ldots, y_n \in \mathbb{R}$ we have

$$\left(\sum_{k=1}^{n}(x_k + y_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} y_k^2\right)^{1/2}$$

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A higher dimensional triangle inequality!

Higher-dimensional triangle inequality

Proof.

By the triangle inequality:

$$\begin{aligned}\sum_{k=1}^{n}(x_k+y_k)^2 &\leq \sum_{k=1}^{n}(|x_k|+|y_k|)|x_k+y_k| \\ &= \sum_{k=1}^{n}|x_k|\cdot|x_k+y_k| + \sum_{k=1}^{n}|y_k|\cdot|x_k+y_k|\end{aligned}$$

Now applying the Cauchy-Schwartz inequality:

$$\begin{aligned}&\leq \left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2} \\ &= \left[\left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}\right]\left(\sum_{k=1}^{n}|x_k+y_k|^2\right)^{1/2}\end{aligned}$$

$\square$

Outline

Set basics

Intuitively, a **set** A is a "collection of things" which we call the **elements** of A.

- in practice, this is a bad definition (Russell's Paradox)
- true set formulation: Zermelo-Frankel Axioms

Examples:

- $\mathbb{R}$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{Q}$
- $(1, 5]$, $(0, \infty)$
- empty set $\varnothing$
- $\{\heartsuit, \text{Fall}, \{\varnothing\}\}$
- $\{n \in \mathbb{Z} : n \text{ is prime}\}$

Everything is a set

In the true minimalistic philosophy of mathematics, we want everything to be a set.

- integers
$$0 = \varnothing, 1 = \{\varnothing\}, 2 = \{\varnothing, \{\varnothing\}\}, \ldots$$
- ordered pairs
$$(a, b) = \{a, \{a, b\}\}$$
- even relations and functions are sets

Challenge!

Problem

Prove that for ordered pairs (a, b) and (c, d) that

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = c \text{ and } b = d$$

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$$\begin{aligned} \text{Case I:} \quad & a = c \ \text{ and } \ \{a, b\} = \{c, d\} \\ \text{Case II:} \quad & a = \{c, d\} \ \text{ and } \ \{a, b\} = c \end{aligned}$$

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Since $\{a, b\} = \{c, d\}$, we know $b \in \{c, d\}$.

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Since $\{a, b\} = \{c, d\}$, we know $b \in \{c, d\}$.
Therefore $b = c$ or $b = d$.
If $b = d$, we're done!

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Therefore $b = c$ or $b = d$.
If $b = d$, we're done! ... so assume instead that $b = c$

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If $b = d$, we're done! ... so assume instead that $b = c$
Then $a = c$ implies $a = b$.

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Therefore $\{c, d\} = \{a, b\} = \{a, a\} = \{a\}$.

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If $b = d$, we're done! ... so assume instead that $b = c$
Then $a = c$ implies $a = b$.
Therefore $\{c, d\} = \{a, b\} = \{a, a\} = \{a\}$.
It follows that $d = c = b = a$.

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Prove that for ordered pairs (a, b) and (c, d) that

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Solution

$$\text{Case II:} \quad a = \{c, d\} \;\text{ and }\; \{a, b\} = c$$

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Prove that for ordered pairs (a, b) and (c, d) that

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Solution

$$\text{Case II:} \quad a = \{c, d\} \ \text{ and } \ \{a, b\} = c$$

This would imply that $c \in a$ and $a \in c$.

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This can be shown to contradict the ZF Axioms of Set Theory.
Specifically the regularity axiom for the set $\{a, c\}$...

Relations

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If a relation $\mathcal{R}$ is a function, we usually use a symbol like f.

NOTATION: $f : A \to B$ means f is a function from A to B

NOTATION: $f(a) = b$ means $(a, b) \in f$.

The set

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Solution

$f = \mathcal{R}$ must be reflexive, so $x\mathcal{R}x$ for all x
This means $f(x) = x$ for all x.
Thus the only function which is an equivalence relation is the identity function

$$f(x) = x.$$

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If it satisfies both properties, it is called **bijective.**

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If it is, we call it the **inverse** of f.

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$$(g \circ f)(x) = g(f(x)).$$