# MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

October 14, 2024

# Outline

1. Real Analysis Lecture 13
   - Metric subspaces
   - Limits of sequences

# Outline

# Metric subspace

## Definition

Let $(M, d)$ be a metric space.

# Metric subspace

## Definition

Let $(M, d)$ be a metric space.
If $S \subseteq M$ is a subset, then $d : M \times M \to \mathbb{R}$ restricts to a metric $d : S \times S \to \mathbb{R}$, called the **relative metric.**

# Metric subspace

**Definition**

Let $(M, d)$ be a metric space.
If $S \subseteq M$ is a subset, then $d : M \times M \to \mathbb{R}$ restricts to a metric $d : S \times S \to \mathbb{R}$, called the **relative metric.**
The space $(S, d)$ with the relative metric is called a **metric subspace.**

# Metric subspace

## Definition

Let $(M, d)$ be a metric space.
If $S \subseteq M$ is a subset, then $d : M \times M \to \mathbb{R}$ restricts to a metric $d : S \times S \to \mathbb{R}$, called the **relative metric.**
The space $(S, d)$ with the relative metric is called a **metric subspace.**

Open balls in $S$:

$$B_S(x; r) = B_M(x; r) \cap S.$$

# Example

Let $M = \mathbb{R}$ and $d$ be the Euclidean metric.
Consider the metric subspace $(S, d)$ for $S = [-1, 1) \cup \{2\}$.
Open balls:

$$\{2\}$$

$$[-1, s), \quad -1 < s < 1$$

$$(r, s), \quad -1 < r < s < 1$$

$$[-1, s) \cup \{2\}, \quad -1 < s < 1$$

$$(r, s) \cup \{2\}, \quad -1 < r < s < 1$$

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\Longrightarrow$: Assume $V$ is an open subset of $S$.

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\implies$: Assume $V$ is an open subset of $S$.
Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\Longrightarrow$: Assume $V$ is an open subset of $S$.
Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.
This says $B_M(x; r_x) \cap S \subseteq V$.

# Open sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

## Proof.

$\Longrightarrow$: Assume $V$ is an open subset of $S$.
Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.
This says $B_M(x; r_x) \cap S \subseteq V$.
Define

$$U = \bigcup_{x \in V} B_M(x; r_x).$$

# Open sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

## Proof.

$\Longrightarrow$: Assume $V$ is an open subset of $S$.
Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.
This says $B_M(x; r_x) \cap S \subseteq V$.
Define

$$U = \bigcup_{x \in V} B_M(x; r_x).$$

This is a union of open sets, and is therefore open.

# Open sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

## Proof.

$\Longrightarrow$: Assume $V$ is an open subset of $S$.
Then for all $x \in V$, there exists $r_x > 0$ such that $B_S(x; r_x) \subseteq V$.
This says $B_M(x; r_x) \cap S \subseteq V$.
Define

$$U = \bigcup_{x \in V} B_M(x; r_x).$$

This is a union of open sets, and is therefore open.
Furthermore $U \cap S = V$. $\square$

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

# Open sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

## Proof.

$\Longleftarrow$: Assume $U$ is an open subset of $M$.

# Open sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

## Proof.

$\Longleftarrow$: Assume $U$ is an open subset of $M$.
We need to show that $V := U \cap S$ is open.

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\Longleftarrow$: Assume $U$ is an open subset of $M$.
We need to show that $V := U \cap S$ is open.
If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\Longleftarrow$: Assume $U$ is an open subset of $M$.
We need to show that $V := U \cap S$ is open.
If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.
It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\Longleftarrow$: Assume $U$ is an open subset of $M$.
We need to show that $V := U \cap S$ is open.
If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.
It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.
This shows that $x$ is an interior point of $S$ in the metric subspace $(S, d)$.

# Open sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $V$ is an open subset of $S$ if and only if there exists an open subset $U$ of $M$ with $V = U \cap S$.*

**Proof.**

$\Longleftarrow$: Assume $U$ is an open subset of $M$.
We need to show that $V := U \cap S$ is open.
If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.
It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.
This shows that $x$ is an interior point of $S$ in the metric subspace $(S, d)$.
Since $x \in V$ was arbitrary, this shows that $V$ is open in $S$. □

# Closed sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

# Closed sets in subspaces

**Theorem**

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

**Proof.**

# Closed sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

## Proof.

Let $B \subseteq S$.

# Closed sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

## Proof.

Let $B \subseteq S$.
$B$ is closed in $S$ if and only if $S \backslash B$ is open in $B$

# Closed sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

## Proof.

Let $B \subseteq S$.
$B$ is closed in $S$ if and only if $S \setminus B$ is open in $B$
This is true if and only if there exists an open subset $U$ of $M$ with $S \setminus B = U \cap S$.

# Closed sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

## Proof.

Let $B \subseteq S$.
$B$ is closed in $S$ if and only if $S \backslash B$ is open in $B$
This is true if and only if there exists an open subset $U$ of $M$ with $S \backslash B = U \cap S$.
This is true if and only if there exists an open subset $U$ of $M$ with $B = (M \backslash U) \cap S$.

# Closed sets in subspaces

## Theorem

*Let $(M, d)$ be a metric space and let $(S, d)$ be a subspace. Then $B$ is a closed subset of $S$ if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$.*

## Proof.

Let $B \subseteq S$.
$B$ is closed in $S$ if and only if $S \backslash B$ is open in $B$
This is true if and only if there exists an open subset $U$ of $M$ with $S \backslash B = U \cap S$.
This is true if and only if there exists an open subset $U$ of $M$ with $B = (M \backslash U) \cap S$.
this is true if and only if there exists a closed subset $A$ of $M$ with $B = A \cap S$. $\square$

# Example

Consider the metric space $(\mathbb{R}, d)$ with $d = d_{\text{eucl}}$ the Euclidean metric, and the metric subspace $(S, d)$ for $S = [0, 1]$.

# Example

Consider the metric space $(\mathbb{R}, d)$ with $d = d_{\text{eucl}}$ the Euclidean metric, and the metric subspace $(S, d)$ for $S = [0, 1]$.

- $[0, 1]$ is open in $S$ and closed in $S$, but it is not open in $M$

# Example

Consider the metric space $(\mathbb{R}, d)$ with $d = d_{\text{eucl}}$ the Euclidean metric, and the metric subspace $(S, d)$ for $S = [0, 1]$.

- $[0, 1]$ is open in $S$ and closed in $S$, but it is not open in $M$
- $[0, 1/2)$ is open in $S$, but not in $M$

# Example

Consider the metric space $(\mathbb{R}, d)$ with $d = d_{\text{eucl}}$ the Euclidean metric, and the metric subspace $(S, d)$ for $S = [0, 1]$.

- $[0, 1]$ is open in $S$ and closed in $S$, but it is not open in $M$
- $[0, 1/2)$ is open in $S$, but not in $M$
- $(0, 1/2)$ is open in both $S$ and in $M$

# Example

Consider the metric space $(\mathbb{R}, d)$ with $d = d_{\text{eucl}}$ the Euclidean metric, and the metric subspace $(S, d)$ for $S = [0, 1]$.

- $[0, 1]$ is open in $S$ and closed in $S$, but it is not open in $M$
- $[0, 1/2)$ is open in $S$, but not in $M$
- $(0, 1/2)$ is open in both $S$ and in $M$
- $(1/2, 1]$ is open in $S$, but not in $M$

# Outline

# Definition of a limit

# Definition of a limit

Back in Calc. I:

# Definition of a limit

Back in Calc. I:

$$\lim_{n \to \infty} x_n = L \quad \text{means}$$

# Definition of a limit

Back in Calc. I:

$$\lim_{n\to\infty} x_n = L \quad \text{means}$$

that as $n$ gets arbitrarily large, the value of $x_n$ gets arbitrarily close to $L$.

# Definition of a limit

Back in Calc. I:

$$\lim_{n\to\infty} x_n = L \quad \text{means}$$

that as $n$ gets arbitrarily large, the value of $x_n$ gets arbitrarily close to $L$.

**Formally:**

# Definition of a limit

Back in Calc. I:

$$\lim_{n\to\infty} x_n = L \quad \text{means}$$

that as $n$ gets arbitrarily large, the value of $x_n$ gets arbitrarily close to $L$.

**Formally:**

$$\forall\epsilon > 0, \quad \text{there exists } N \in \mathbb{Z}_+ \text{ such that } |x_n - L| < \epsilon \; \forall n \geq N.$$

# Definition of a limit

Back in Calc. I:

$$\lim_{n\to\infty} x_n = L \quad \text{means}$$

that as $n$ gets arbitrarily large, the value of $x_n$ gets arbitrarily close to $L$.

**Formally:**

$$\forall \epsilon > 0, \quad \text{there exists } N \in \mathbb{Z}_+ \text{ such that } |x_n - L| < \epsilon \; \forall n \geq N.$$

**Metric space version:**

# Definition of a limit

Back in Calc. I:

$$\lim_{n\to\infty} x_n = L \quad \text{means}$$

that as $n$ gets arbitrarily large, the value of $x_n$ gets arbitrarily close to $L$.

**Formally:**

$$\forall\epsilon > 0, \quad \text{there exists } N \in \mathbb{Z}_+ \text{ such that } |x_n - L| < \epsilon\, \forall n \geq N.$$

**Metric space version:**

$$\forall\epsilon > 0, \quad \text{there exists } N \in \mathbb{Z}_+ \text{ such that } d(x_n, L) < \epsilon\, \forall n \geq N.$$

# Definition of a limit

Back in Calc. I:

$$\lim_{n\to\infty} x_n = L \quad \text{means}$$

that as $n$ gets arbitrarily large, the value of $x_n$ gets arbitrarily close to $L$.

**Formally:**

$$\forall \epsilon > 0, \quad \text{there exists } N \in \mathbb{Z}_+ \text{ such that } |x_n - L| < \epsilon \; \forall n \geq N.$$

**Metric space version:**

$$\forall \epsilon > 0, \quad \text{there exists } N \in \mathbb{Z}_+ \text{ such that } d(x_n, L) < \epsilon \; \forall n \geq N.$$

We say $\{x_n\}$ **converges** to $L$ and $L$ is the **limit of the sequence.**

# Example

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

Let $\epsilon > 0$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
Then $n > N$ implies

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
Then $n > N$ implies

$$d(x_n, 0) = |1/n - 0| = 1/n < 1/N < \epsilon.$$

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
Then $n > N$ implies

$$d(x_n, 0) = |1/n - 0| = 1/n < 1/N < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to 0.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ converges to zero.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
Then $n > N$ implies

$$d(x_n, 0) = |1/n - 0| = 1/n < 1/N < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to 0. □

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = (n-1)/n$ converges to 1.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = (n-1)/n$ converges to 1.

**Proof.**

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = (n-1)/n$ converges to 1.

**Proof.**

Let $\epsilon > 0$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = (n-1)/n$ converges to 1.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = (n-1)/n$ converges to 1.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
Then $n > N$ implies

$$d(x_n, 1) = |(n-1)/n - 1| = 1/n < 1/N < \epsilon.$$

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the Euclidean metric $d$.
The sequence $\{x_n\}$ defined by $x_n = (n-1)/n$ converges to 1.

**Proof.**

Let $\epsilon > 0$.
Choose $N \in \mathbb{Z}_+$ with $N > 1/\epsilon$.
Then $n > N$ implies

$$d(x_n, 1) = |(n-1)/n - 1| = 1/n < 1/N < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to 1. $\square$

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

**Proof.**

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

**Proof.**

Suppose that it converges to a number $L$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

**Proof.**

Suppose that it converges to a number $L$.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

**Proof.**

Suppose that it converges to a number $L$.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.
Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

**Proof.**

Suppose that it converges to a number $L$.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.
Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.
With the discrete metric $d(x_n, L) < 1$ implies $x_n = L$.

# Example

Consider the metric space $(M, d)$ defined by $M = \mathbb{R}$ with the discrete metric $d$.
The sequence $\{x_n\}$ defined by $x_n = 1/n$ does not converge.

**Proof.**

Suppose that it converges to a number $L$.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.
Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.
With the discrete metric $d(x_n, L) < 1$ implies $x_n = L$.
This means $x_n = 1/n = L$ for all $n \geq N$, which is a contradiction! $\square$

# Uniqueness of limits

## Theorem (Uniqueness of limits)

# Uniqueness of limits

**Theorem (Uniqueness of limits)**

*Let $(M, d)$ be a metric space.*

# Uniqueness of limits

**Theorem (Uniqueness of limits)**

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

# Uniqueness of limits

**Theorem (Uniqueness of limits)**

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

**Proof.**

# Uniqueness of limits

## Theorem (Uniqueness of limits)

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

## Proof.

Suppose that $\{x_n\}$ converges to $L$ and $M$.

# Uniqueness of limits

**Theorem (Uniqueness of limits)**

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

**Proof.**

Suppose that $\{x_n\}$ converges to $L$ and $M$.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.

# Uniqueness of limits

## Theorem (Uniqueness of limits)

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

## Proof.

Suppose that $\{x_n\}$ converges to $L$ and $M$.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists $N_1$ such that $d(x_n, L) < \epsilon$ for $n > N_1$.

# Uniqueness of limits

## Theorem (Uniqueness of limits)

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

## Proof.

Suppose that $\{x_n\}$ converges to $L$ and $M$.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists $N_1$ such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists $N_2$ such that $d(x_n, L) < \epsilon$ for $n > N_2$.

# Uniqueness of limits

## Theorem (Uniqueness of limits)

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

## Proof.

Suppose that $\{x_n\}$ converges to $L$ and $M$.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists $N_1$ such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists $N_2$ such that $d(x_n, L) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

# Uniqueness of limits

## Theorem (Uniqueness of limits)

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

## Proof.

Suppose that $\{x_n\}$ converges to $L$ and $M$.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists $N_1$ such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists $N_2$ such that $d(x_n, L) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

$$d(L, M) \leq d(L, x_n) + d(x_n, M) < 2\epsilon = d(L, M).$$

# Uniqueness of limits

## Theorem (Uniqueness of limits)

*Let $(M, d)$ be a metric space.*
*If a sequence $\{x_n\}$ in $M$ converges to $L$ and converges to $M$, then $L = M$.*

## Proof.

Suppose that $\{x_n\}$ converges to $L$ and $M$.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists $N_1$ such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists $N_2$ such that $d(x_n, L) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

$$d(L, M) \leq d(L, x_n) + d(x_n, M) < 2\epsilon = d(L, M).$$

This is a contradiction. $\square$

# Existence of limits

**Theorem**

*If a sequence $\{x_n\}$ in a metric space $(M, d)$ converges to a value $L \in M$, then the range*

$$X = \{x_1, x_2, \ldots\} \subseteq M$$

*is a bounded set and $L$ is an adherent point of $X$.*

# Existence of limits

**Theorem**

*If a sequence $\{x_n\}$ in a metric space $(M, d)$ converges to a value $L \in M$, then the range*

$$X = \{x_1, x_2, \dots\} \subseteq M$$

*is a bounded set and $L$ is an adherent point of $X$.*

When the range of a sequence is bounded, we call the sequence $\{x_n\}$ bounded.

# Challenge!

Try to prove the previous theorem.

# Existence of limits

Sort of a converse of the previous theorem.

# Existence of limits

Sort of a converse of the previous theorem.

## Theorem

*If $S$ is a subset of a metric space $(M, d)$ and $L \in M$ is an adherent point of $S$, then there exists a sequence $\{x_n\}$ of elements of $S$ which converges to $L$.*

# Existence of limits

Recall a **subsequence** of a sequence $\{x_n\}$ is a sequence of the form $\{x_{k(n)}\}$, for $k(1) < k(2) < \dots$ positive integers.

# Existence of limits

Recall a **subsequence** of a sequence $\{x_n\}$ is a sequence of the form $\{x_{k(n)}\}$, for $k(1) < k(2) < \dots$ positive integers.

**Theorem**

*A sequence $\{x_n\}$ in a metric space $(M, d)$ converges to a value $L \in M$ if and only if every subsequence $\{x_{k(n)}\}$ also converges to $L$.*

# Challenge!

Try to prove the previous theorem.

# Monotone sequences

## Definition

# Monotone sequences

## Definition

A sequence $\{x_n\}$ of real numbers is called **increasing** or **monotone increasing** if $x_n \leq x_{n+1}$ for all $n$.

# Monotone sequences

## Definition

A sequence $\{x_n\}$ of real numbers is called **increasing** or **monotone increasing** if $x_n \leq x_{n+1}$ for all $n$.
It is called **strictly increasing** if $x_n < x_{n+1}$ for all $n$.

# Monotone sequences

## Definition

A sequence $\{x_n\}$ of real numbers is called **increasing** or **monotone increasing** if $x_n \leq x_{n+1}$ for all $n$.
It is called **strictly increasing** if $x_n < x_{n+1}$ for all $n$.
A sequence $\{x_n\}$ of real numbers is called **decreasing** or **monotone decreasing** if $x_n \geq x_{n+1}$ for all $n$.

# Monotone sequences

## Definition

A sequence $\{x_n\}$ of real numbers is called **increasing** or **monotone increasing** if $x_n \leq x_{n+1}$ for all $n$.
It is called **strictly increasing** if $x_n < x_{n+1}$ for all $n$.
A sequence $\{x_n\}$ of real numbers is called **decreasing** or **monotone decreasing** if $x_n \geq x_{n+1}$ for all $n$.
It is called **strictly decreasing** if $x_n > x_{n+1}$ for all $n$.

# Monotone sequences

**Definition**

A sequence $\{x_n\}$ of real numbers is called **increasing** or **monotone increasing** if $x_n \leq x_{n+1}$ for all $n$.
It is called **strictly increasing** if $x_n < x_{n+1}$ for all $n$.
A sequence $\{x_n\}$ of real numbers is called **decreasing** or **monotone decreasing** if $x_n \geq x_{n+1}$ for all $n$.
It is called **strictly decreasing** if $x_n > x_{n+1}$ for all $n$.
In any of these cases, we call the sequence **monotone**.

# Monotone sequences

Bounded monotone sequences always converge (with Euclidean metric)!

# Monotone sequences

Bounded monotone sequences always converge (with Euclidean metric)!

**Theorem (Monotone convergence theorem)**

# Monotone sequences

Bounded monotone sequences always converge (with Euclidean metric)!

**Theorem (Monotone convergence theorem)**

*Consider the metric space formed by $\mathbb{R}$ with Euclidean distance.*

# Monotone sequences

Bounded monotone sequences always converge (with Euclidean metric)!

**Theorem (Monotone convergence theorem)**

*Consider the metric space formed by $\mathbb{R}$ with Euclidean distance.*
*A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \ldots\}$.*

# Monotone sequences

Bounded monotone sequences always converge (with Euclidean metric)!

**Theorem (Monotone convergence theorem)**

*Consider the metric space formed by $\mathbb{R}$ with Euclidean distance.*
*A monotone increasing sequence $\{x_n\}$ of real numbers which is bounded above converges to $L = \sup\{x_1, x_2, \ldots\}$.*
*A monotone decreasing sequence $\{x_n\}$ which is bounded below converges to $L = \inf\{x_1, x_2, \ldots\}$.*

# Monotone sequences

**Proof.**

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$.

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$. Then by the Completeness Axiom $L = \sup X$ exists.

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$. Then by the Completeness Axiom $L = \sup X$ exists.

Let $\epsilon > 0$.

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$. Then by the Completeness Axiom $L = \sup X$ exists.
Let $\epsilon > 0$.
Then $L - \epsilon$ is not a supremum of $X$, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$. Then by the Completeness Axiom $L = \sup X$ exists.

Let $\epsilon > 0$.

Then $L - \epsilon$ is not a supremum of $X$, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.

Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \le x_n < L$.

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$. Then by the Completeness Axiom $L = \sup X$ exists.

Let $\epsilon > 0$.

Then $L - \epsilon$ is not a supremum of $X$, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.

Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \le x_n < L$.

It follows that

$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

# Monotone sequences

**Proof.**

Suppose that $\{x_n\}$ is a monotone increasing sequence which is bounded above and let $X = \{x_1, x_2, \dots\}$. Then by the Completeness Axiom $L = \sup X$ exists.
Let $\epsilon > 0$.
Then $L - \epsilon$ is not a supremum of $X$, so there exists $x_N \in X$ such that $x_N > L - \epsilon$.
Since the sequence is increasing, for any $n > N$, we have that $L - \epsilon < x_N \leq x_n < L$.
It follows that

$$d(x_n, L) = |x_n - L| = L - x_n < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves $x_n$ converges to $L$. $\square$