MATH 350-2 Advanced Calculus

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Outline

1. Real Analysis Lecture 2
   - Types of real numbers
   - Integers
   - Upper Bound and Supremum

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Real numbers

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- Intuitively, it represents the fact that the real line has no holes or gaps.

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Examples:

$$\mathbb{R}, \quad \mathbb{Q}, \quad (0, \infty), \quad \mathbb{Z}, \quad \mathbb{N}, \dots$$

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- rationals are

$$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z},\ b \neq 0\}.$$

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Proof:

Let

$$S = \{n \in \mathbb{Z}_+ : 2|n(n+1)\}$$

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. We see that $1 \in S$ because 2 divides $1(1+1)$.

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Now suppose $x \in S$. (This is our usual inductive assumption).

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This means $(x+1)(x+2) = x(x+1) + 2(x+1) = 2(k+x+1)$.

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Thus 2 divides $(x+1)(x+2)$, showing that $x+1 \in S$.

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In other words $n \in S$ for every positive integer n.

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By the Principle of Induction, this means $\mathbb{Z}_+ \subseteq S$.
In other words $n \in S$ for every positive integer n.
Hence 2 divides $n(n+1)$ for every positive integers n.

Prime numbers

A positive integer p is **prime** if its only positive divisors are 1 and p.

Theorem (Apostol Theorem 1.5)

Every integer is prime or a product of primes

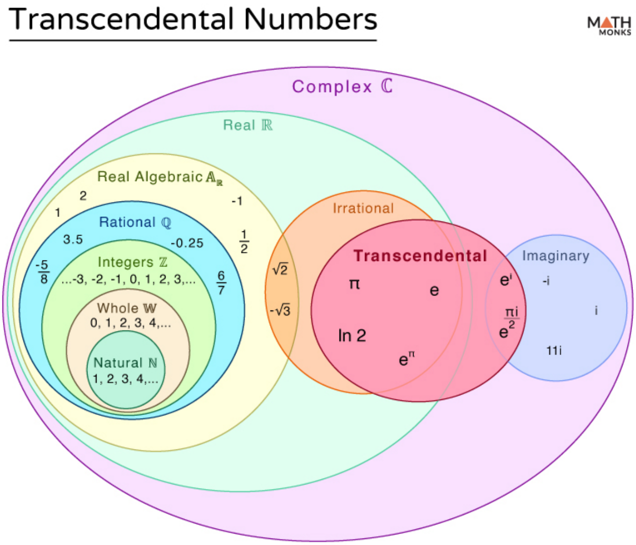
Theorem (Apostol Theorem 1.8)

If p is prime and p divides ab, then p divides a or p divides b.

Theorem (Fundamental Theorem of Arithmetic (Apostol Theorem 1.9))

Every integer $n > 1$ has a unique factorization as a product of primes, up to reordering.

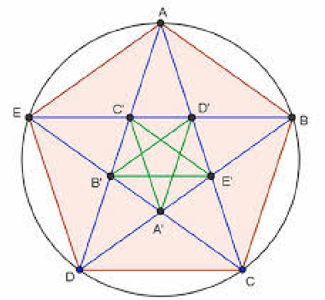
Types of numbers



Irrational numbers

Numbers which are not of the form a/b with $a, b \in \mathbb{Z}$ are called **irrational.**

- Discovered by Hippasus, a pythagorean (a student of Pythagoras)



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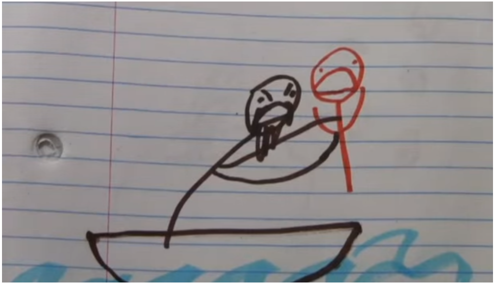

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3. transcendentals are mysterious ... but most real numbers are transcendental!

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- $[3, 7)$ has an upper bound but no maximal element

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Suppose that b_1 and b_2 are both maximal elements of S.
Then $x \leq b_1$ and $x \leq b_2$ for all $x \in S$.
Moreover, $b_1 \in S$ and $b_2 \in S$.

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Since $b_1 \in S$ and b_2 is an upper bound of S, $b_1 \leq b_2$.

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By the trichotomy, we find $b_1 = b_2$.

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Now we can say *the* maximum, $\max(S)$

Supremum

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In other words

a supremum is a **least upper bound**

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Suppose that b_1 and b_2 are both suprema of S.
Then b_1 and b_2 are both upper bounds of S.

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Suppose that b_1 and b_2 are both suprema of S.
Then b_1 and b_2 are both upper bounds of S.
Since b_1 is a least upper bound, $b_1 \leq b_2$.

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Since b_2 is also a least upper bound, $b_2 \leq b_1$.
By the trichotomy, we find $b_1 = b_2$.

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Now we can say *the* supremum, $\sup(S)$

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Let $b = \max(S)$.
Then b is an upper bound of S and $b \in S$.

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Then b is an upper bound of S and $b \in S$.
If b' is another upper bound of S, then by definition $b \leq b'$.
Thus b is the least upper bound of S.

Completeness Axiom

Now we have the machinery in place to state the Completeness Axiom.

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- 3 is the supremum of $(0, 3)$

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- 1 is the supremum of

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- π is the supremum of

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \ldots\}.$$