MATH 350-2 Advanced Calculus

W.R. Casper

Department of Mathematics
California State University Fullerton

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Outline

1. Real Analysis Lecture 1
   - Origin of Real Numbers
   - Properties of real numbers

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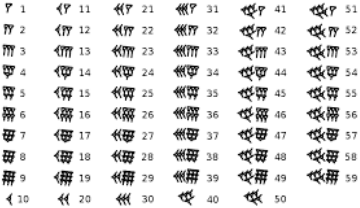
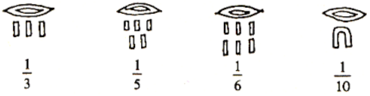
Prehistoric numbers

20,000 BC tallies on Ishango bone



3400 BC Sumerian system



1000 BC Egyptian fractions

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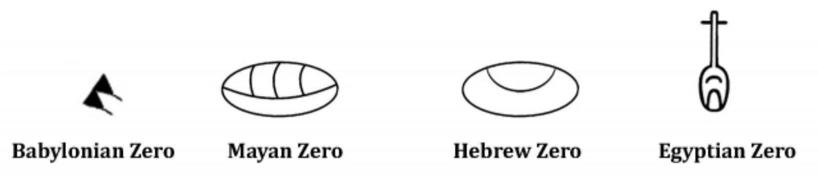
- 36 BC - Mayan/Olmec heiroglyph for zero
- 150 AD - Ptolemy using it in astronomy
- medieval scholars - debating the existence of 0



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- up to 18th century - rejected by western sources, referred to as "absurd numbers"

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- 300 BC -appear in Euclid's elements

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- 17th century - European mathematicians distinguish between transcendentals and algebraic numbers

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- the integers, rationals, etc. will be defined in terms of the reals

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Special notation: $0 := x - x$ and $1 := x/x$, and $-x := 0 - x$

Challenge!

Problem

use the axioms to show that $(x+y)z = xz + yz$.

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Solution

$$(x + y)z \stackrel{A1}{=} z(x + y) \stackrel{A3}{=} zx + zy \stackrel{A1}{=} xz + yz.$$

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Suppose $z_1 = x - x$ and $z_2 = y - y$.
Then $x = x + z_1$ and $y = y + z_2$.

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Suppose $z_1 = x - x$ and $z_2 = y - y$.
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Therefore we have

$$(x+y)+z_1 \overset{A2}{=} x+(y+z_1) \overset{A1}{=} x+(z_1+y) \overset{A2}{=} (x+z_1)+y = x+y,$$

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so that

$$z_1 = (x+y)-(x+y) = z_2.$$

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- **field of Boolean numbers**

$$\mathbb{F}_2 = \{0, 1\}$$

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0$$

$$0 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1$$

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There aren't multiplicative inverses! For example, $1/2$ doesn't make sense because there isn't an integer z with $1 = 2z$.

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Special notation:

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- $x \leq y$ means $x < y$ or $x = y$
- $x \geq y$ means $y \leq x$

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but this means both $0 < 1$ and $1 < 0$, which violates the trichotomy.

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This says $a < a$, which contradicts the trichotomy. $\square$