MATH 350-2 Advanced Calculus

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October 21, 2024

Outline

1. Real Analysis Lecture 14
   - Cauchy sequences
   - Limit of a function

Outline

Cauchy sequence

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- every convergent sequence is Cauchy

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- every convergent sequence is Cauchy
- is every Cauchy sequence convergent?

Challenge!

Problem

Prove that if $\{x_n\}$ is a Cauchy sequence in a metric space (M, d), then it is bounded.

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Solution

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Let $\epsilon = 2024$.
Then $\exists\ N \in \mathbb{Z}_+$ with $d(x_m, x_n) < 2024$ for all $m, n \geq N$.
Therefore for all $n \geq N$, we have $d(x_n, x_N) \leq 2024$.

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So if we define

$$R = \max\{d(x_1, x_N), d(x_2, x_N), \dots, d(x_{n-1}, x_N), 2024\},$$

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then $d(x_n, x_N) \leq 2024$ for all $n \geq 1$.
Hence $X = \{x_1, x_2, \dots\} \subseteq B_M(x_N, R)$, and is bounded.

Cauchy sequence

Every convergent sequence is Cauchy.

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Theorem

Let (M, d) be a metric space and suppose that $\{x_n\}$ is a sequence in M which converges to $L \in M$.
Then $\{x_n\}$ is Cauchy.

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Let (M, d) be a metric space and suppose that $\{x_n\}$ is a sequence in M which converges to $L \in M$.
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Proof.

Let $\epsilon > 0$.

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Let $\epsilon > 0$.
Then there exists N such that $n \geq N$ implies $d(x_n, L) < \epsilon/2$.

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Therefore for any $m, n \geq N$ we have

$$d(x_m, x_n) \leq d(x_m, L) + d(L, x_n) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves that $\{x_n\}$ is Cauchy. $\square$

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Partial converse.

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Theorem

Let $M = \mathbb{R}^n$ with the Euclean metric, and suppose that $\{x_n\}$ is a Cauchy sequence in M.
Then $\{x_n\}$ converges.

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Consider the range $X = \{x_1, x_2, \dots\}$ of $\{x_n\}$.

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We consider two cases:

- when X is finite
- when X is infinite

$\square$

Cauchy sequence

Proof.

Case I: Assume X is finite.

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Then
$$\{d(x,y) : x,y \in X,\ x \neq y\}$$
is a finite set of positive values.

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Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon$ for all $m, n \geq N$.
This implies $x_n = x_N$ for all $n \geq N$.

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This implies $x_n = x_N$ for all $n \geq N$.
Hence $\lim_{n\to\infty} x_n = x_N$. □

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Then by the Bolzano-Weierstrass Theorem, X has an accumulation point $L \in M$.

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Then by the Bolzano-Weierstrass Theorem, X has an accumulation point $L \in M$.
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Since $\{x_n\}$ is Cauchy, we can choose $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon/2$ for all $m, n \geq N$.

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Proof.

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Let $\epsilon > 0$.
Since $\{x_n\}$ is Cauchy, we can choose $N \in \mathbb{Z}_+$ with $d(x_m, x_n) < \epsilon/2$ for all $m, n \geq N$.
Moreover, the ball $B(L, \epsilon/2)$ contains infinitely many points of X, so we can choose $\ell \geq N$ with $x_\ell \in B(L, \epsilon/2)$.

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$$d(x_n, L) \leq d(x_n, x_\ell) + d(x_\ell, L) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

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Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to L. $\square$

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Consider the metric space $M = \mathbb{Q}$ with the Euclidean metric d.

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- *tries* to converge to $\sqrt{2}$
- however $\sqrt{2} \notin M$, so it doesn't converge!
- somehow M is "missing" some points

Complete spaces

Definition

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A metric space (M, d) is called **complete** if every Cauchy sequence in M converges.
A subset $S \subseteq M$ is called **complete** if the subspace (S, d) is a complete metric space.

- $\mathbb{R}^n$ with the Euclidean metric is complete
- $[0, 1]$ with the Euclidean metric is complete
- $(0, 1)$ with the Euclidean metric is not complete

Challenge!

Problem

Show that the interval $(0, 1) \subseteq \mathbb{R}$ with the Euclidean metric is not a complete metric space.

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Theorem

Let (M, d) be a metric space.

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Let (M, d) be a metric space and $S \subseteq M$ compact.

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Proof.

Let (M, d) be a metric space and $S \subseteq M$ compact.
To show that (S, d) is complete, we must show that every Cauchy sequence in S converges to a value in S.

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Suppose that $\{x_n\}$ is a Cauchy sequence in (S, d).
If the set X of all values of $\{x_n\}$ is finite, then the Cauchy condition implies $\{x_n\}$ converges.

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If the set X of all values of $\{x_n\}$ is finite, then the Cauchy condition implies $\{x_n\}$ converges.
Assume instead X is infinite.

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Suppose that $\{x_n\}$ is a Cauchy sequence in (S, d).
If the set X of all values of $\{x_n\}$ is finite, then the Cauchy condition implies $\{x_n\}$ converges.
Assume instead X is infinite.
TIME OUT! □

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Lemma

Let (M, d) be a metric space and let $S \subseteq M$ be compact.

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Let (M, d) be a metric space and let $S \subseteq M$ be compact. Then for any infinite subset $X \subseteq S$, the set S has an accumulation point of X.

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To see this, suppose otherwise.

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Proof.

To see this, suppose otherwise.
Then for all $x \in S$, there exists $r_x > 0$ such that $B_S(x, r_x) \cap X$ is empty if $x \notin X$ or equal to $\{x\}$ if $x \in X$.

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The open cover $\{U_i : i \in I\}$ with $I = S$ and $U_i = B_S(i; r_i)$ of S has no finite subcover.

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Lemma

Let (M, d) be a metric space and let $S \subseteq M$ be compact. Then for any infinite subset $X \subseteq S$, the set S has an accumulation point of X.

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To see this, suppose otherwise.
Then for all $x \in S$, there exists $r_x > 0$ such that $B_S(x, r_x) \cap X$ is empty if $x \notin X$ or equal to $\{x\}$ if $x \in X$.
The open cover $\{U_i : i \in I\}$ with $I = S$ and $U_i = B_S(i; r_i)$ of S has no finite subcover.
This is a contradiction, so S has an accumulation point of X. □

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Proof.

Let L be an accumulation point of X in S.

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Proof.

Let L be an accumulation point of X in S.
Let $\epsilon > 0$.

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Let L be an accumulation point of X in S.
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Then $\exists N > 0$ such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon/2$.

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Proof.

Let L be an accumulation point of X in S.
Let $\epsilon > 0$.
Then $\exists N > 0$ such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon/2$.
Moreover, $B_S(L, \epsilon/2)$ contains infinitely many points of X.

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Hence it contains x_m for some $m \geq N$.

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It follows that for all $n \geq N$,

$$d(x_n, L) \leq d(x_n, x_m) + d(x_m, L) < \epsilon/2 + \epsilon/2 = \epsilon.$$

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Since $\epsilon > 0$ was arbitrary, this proves $\{x_n\}$ converges to L. □

Outline

Limit of a function

Let (S, d_S) and (T, d_T) be metric spaces and $A \subseteq S$.

Limit of a function

Let (S, d_S) and (T, d_T) be metric spaces and $A \subseteq S$. Also, let $f : A \to T$ be a function.

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Definition

Then for any accumulation point p of A and element $L \in T$, we say that the **limit as x approaches p of $f(x)$ is L**

Limit of a function

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$$\lim_{x \to p} f(x) = L$$

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$$\lim_{x \to p} f(x) = L$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < d_S(x, p) < \delta \;\Rightarrow\; d_T(f(x), L) < \epsilon.$$

Limit of a function

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$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{n \to \infty} f(x_n) = L$$

for all sequences $\{x_n\}$ of values in $A \backslash \{a\}$ which converge to a.