MATH 350-2 Advanced Calculus

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Outline

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1. Real Analysis Lecture 13
   - Metric subspaces
   - Limits of sequences

Metric subspace

Definition

Let (M, d) be a metric space.

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If $S \subseteq M$ is a subset, then $d : M \times M \to \mathbb{R}$ restricts to a metric $d : S \times S \to \mathbb{R}$, called the **relative metric.**

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The space (S, d) with the relative metric is called a **metric subspace.**

Open balls in S:

$$B_S(x; r) = B_M(x; r) \cap S.$$

Example

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Open sets in subspaces

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This is a union of open sets, and is therefore open.

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Furthermore $U \cap S = V$. $\square$

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We need to show that $V := U \cap S$ is open.

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If $x \in V$, then $x \in U$ and there exists $r > 0$ such that $B_M(x; r) \subseteq U$.
It follows that $B_S(x, r) = B_M(x, r) \cap S \subseteq U \cap S = V$.

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Since $x \in V$ was arbitrary, this shows that V is open in S. $\square$

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We say $\{x_n\}$ **converges** to L and L is the **limit of the sequence.**

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Proof.

Suppose that it converges to a number L.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.

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Suppose that it converges to a number L.
Then for all $\epsilon > 0$, there must exists an $N \in \mathbb{Z}_+$ such that $d(x_n, L) < \epsilon$ for all $N \geq n$.
Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.

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Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.
With the discrete metric $d(x_n, L) < 1$ implies $x_n = L$.

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Take $\epsilon = 1$ and choose $N \in \mathbb{Z}_+$ such that $d(x_n, L) < 1$ for all $N \geq n$.
With the discrete metric $d(x_n, L) < 1$ implies $x_n = L$.
This means $x_n = 1/n = L$ for all $n \geq N$, which is a contradiction! □

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Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.

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Proof.

Suppose that $\{x_n\}$ converges to L and M.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.

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Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists N_2 such that $d(x_n, M) < \epsilon$ for $n > N_2$.

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Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists N_2 such that $d(x_n, M) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

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If a sequence $\{x_n\}$ in M converges to L and converges to M, then $L = M$.

Proof.

Suppose that $\{x_n\}$ converges to L and M.
Assume further that $L - M \neq 0$ and let $\epsilon = d(L, M)/2$.
Then there exists N_1 such that $d(x_n, L) < \epsilon$ for $n > N_1$.
Likewise, there exists N_2 such that $d(x_n, M) < \epsilon$ for $n > N_2$.
Then for $n > \max\{N_1, N_2\}$, the triangle inequality says

$$d(L, M) \leq d(L, x_n) + d(x_n, M) < 2\epsilon = d(L, M).$$

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This is a contradiction. $\square$