Problem 1 :

a) False
b) True
c) False (need H or K to be normal)
d) True (Cauchy's Theorem)
e) False (ex. $\{(2,0,0), (0,2,0), (0,0,2)\}$)

Problem 2 :

a) If $K \triangleleft G$, $H \triangleleft G$ and $K < H$ then

$$(G/K)/(H/K) \cong G/H.$$

b) If P_1 and P_2 are two Sylow p-subgroups then there exists $g \in G$ with $g^{-1} P_1 g = P_2$.

c) The center of S_n is $\{e\}$ for $n > 2$.

Problem 3 : Let n_p = # Sylow p-subgroups

a)

$$n_3 = 1, \cancel{3}, \cancel{5}, \cancel{17}, 3\cdot\cancel{5}, \cancel{3\cdot17}, 5\cdot17, 3\cdot\cancel{5}\cdot17$$

must be $\equiv 1 \mod 3$.

Thus $n_3 = 1$ or $n_3 = 85$

b)

$$n_5 = 1, \cancel{3}, \cancel{5}, \cancel{17}, 3\cdot\cancel{5}, 3\cdot17, \cancel{5\cdot17}, \cancel{3\cdot5}\cdot17$$

must be $\equiv 1 \mod 5$

Thus $n_5 = 1$ or $n_5 = 51$

c)

$$n_{17} = 1, \cancel{3}, \cancel{5}, \cancel{17}, \cancel{3\cdot5}, \cancel{3\cdot17}, \cancel{5\cdot17}, \cancel{3\cdot5}\cdot17$$

must be $\equiv 1 \mod 17$

Thus $n_{17} = 1$.

Problem 4:

a) Note $H = \langle (6,9),(2,2) \rangle = \langle (0,3),(2,2) \rangle$

Thus $(a,b)+H = (a-2j, b-2j)+H$
$= (a-2j, b-2j-3k)+H$ for any j,k

Thus we see that the only cosets are

$(0,0)+H$, $(0,1)+H$, $(0,2)+H$
$(1,0)+H$, $(1,1)+H$, $(1,2)+H$

b) We're looking for an abelian group w/ 6 elements so the only possibility is $\mathbb{Z}_6$.

Alternatively, we can determine an explicit isomorphism.

The homomorphism
$$\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_6$$
$$(m,n) \mapsto m+2n$$
is surjective and has kernel H so by the first isomorphism theorem $G/H \cong \mathbb{Z}_6$.

Problem 5:

Proof: Let $x \in G$ and consider the element $g = xH$ of the quotient group G/H.
By Lagrange's theorem, $|\langle g \rangle|$ divides $|G| = n$ and consequently g^n is equal to the identity eH in G/H. Thus $eH = g^n = (xH)^n = x^n H$.
This means $x^n \in H$.

$\blacksquare$

Problem 6:

Proof: Consider the action of G on the set $X = G$ defined by $g \cdot y = gyg^{-1}$ for $g \in G$, $y \in X$.
Now for fixed $x \in G$, the orbit is
$$\text{orb}(x) = \{g \cdot x \mid g \in G\} = \{gxg^{-1} \mid g \in G\} = K$$
and the stabilizer is
$$G_x = \{g \in G \mid g \cdot x = x\} = \{g \in G \mid gxg^{-1} = x\} = H$$
By the orbit-stabilizer theorem
$$|K| = |\text{orb}(x)| = [G : G_x] = [G : H].$$
Thus $|G| = [G:H] \cdot |H| = |K| \cdot |H|$.
and since $p \mid |G|$ we must have $p \mid |K|$ or $p \mid |H|$
$\square$

Problem 7:

Proof: We use the same action as on the previous problem on $X = G$. Then the subset X_G of X which G acts on trivially is
$$X_G = \{x \in X : g \cdot x = x \ \forall g \in G\} = \{x \in X \mid gxg^{-1} = x \ \forall g \in G\}$$
This is exactly the center $Z(G)$ of G!

Now use the usual counting trick:
$$|X| = |X_G| + \sum_{\substack{\text{orb}(x) \in X/G \\ |\text{orb}(x)| > 1}} [G : G_x]$$
where $X_G = Z(G)$.

Now since $Z(G)$ is a subgroup, we know $|Z(G)|$ is $1, p, p^2$, or p^3. When $|\text{orb}(x)| > 1$, $G_x \neq G$ so $[G : G_x] \neq 1$ and must be divisible by p. Thus $|X_G| = 0 \mod p$ by our counting trick so $|Z(G)| \neq 1$.

$$\frac{p^3 - p^2}{p} = p^2 - 1$$

We just need to show $|Z(G)| \neq p^2$.

If $|Z(G)| = p^2$, then $G/Z(G)$ is a group of order p, hence it is abelian (in fact, cyclic). It follows that G is abelian, so that $Z(G) = G$ meaning $|Z(G)| = p^3$ $\Rightarrow\Leftarrow$

Thus $Z(G) = p$ or $Z(G| = p^?$. $\square$

<u>Remark:</u> We could also have just used the fact proved in class that if G is a p-group, then $Z(G) \neq \{e\}$ to do the first part, instead of the group action