Welcome to Math 407

- Groups
- Rings
- Fields
- Galois Theory

Not talked about here:
Modules
Representations

Algebra is multiplying, adding, subtracting, dividing, etc.

Today: Binary operations

Def: A binary operation on a set A is a function $* : A \times A \to A$. $(a_1, a_2) \mapsto a_3$

Notation: $\cancel{*(a_1, a_2)} = a_3$ we write $\boxed{a_1 * a_2}$ in place of $*(a_1, a_2)$

Ex: $A = \mathbb{R}$, $a * b = a+b$

$3 * 2 = 5$

$7 * 1 = 8$

Ex: $A = (0, \infty)$, $a * b = \sqrt{ab}$

$2 * 2 = \sqrt{2 \cdot 2} = \sqrt{4} = 2$

Ex: $A = (0, \infty)$, $a * b = \frac{a}{b}$

$4 * 2 = 4/2 = 2$

Non-example: $A = \mathbb{R}$, $a * b = \sqrt{ab}$

$1 * (-1) = \sqrt{-1}$ not real!

$A \times A \to A$

Ex: $A=\mathbb{R}$, $a*b = e^{ab}$

Ex: $A = M_n(\mathbb{R})$, $B*C = BC$

Ex: $A = \{f: X \to X\}$ is the set of all functions from X to X.

$f \in A$, $g \in A$ then $f,g : X \to X$
$f \circ g : X \to X$

Define $f * g = f \circ g$. Really important example!

$$\boxed{\text{Theorem: } (f\circ g)\circ h = f\circ(g\circ h) \quad \text{associativity!}}$$

Q: Do binary operations have nice properties like this?

Def: A binary operation $*$ on a set A is associative if
$(a*b)*c = a*(b*c)$ for all $a,b,c \in A$.

Ex: $A = \mathbb{Q}$, $a*b = a+b$.

$a,b,c \in \mathbb{Q}$

$$\begin{aligned} a*(b*c) &= a*(b+c) \\ &= a+(b+c) \\ &= (a+b)+c \\ &= (a*b)+c \\ &= (a*b)*c \quad ✓ \end{aligned}$$

This is associative!!

Ex: $A = \mathbb{Q}$, $a*b = (a+1)b$

Question: associative??

$$0*(1*2) = 0*((1+1)2)$$
$$= 0*4$$
$$= (0+1)4 = 4$$

$$(0*1)*2 = ((0+1)\cdot 1)*2$$
$$= 1*2$$
$$= (1+1)2 = 4$$

$$(a*b)*c = a*(b*c)$$

$$1*(2*3) = 1*9 = 18$$
$$(1*2)*3 = 4*3 = 15$$

$\neq$ NOT ASSOC.

Def: A binary operation $*$ on a set A is <u>commutative</u> if

$$a*b = b*a \quad \text{for all } a,b \in A.$$

Previous example: $A = \mathbb{Q}$, $a*b = (a+1)b$

$$\left.\begin{array}{l} 0*1 = (0+1)\cdot 1 = 1 \\ 1*0 = (1+1)\cdot 0 = 0 \end{array}\right\} \text{NOT COMMUTATIVE}$$

Ex: $A = (0,\infty)$, $a*b = e^{a+b}$

What properties does A have??

$$a*b = e^{a+b} = e^{b+a} = b*a$$ ✓ COMMUTATIVE!

ASSOCIATIVE?

$$\left.\begin{array}{l} (2*5)*7 = e^7*7 = e^{(e^7+7)} \\ 2*(5*7) = 2*e^{12} = e^{(2+e^{12})} \end{array}\right\} \neq$$

This is not associative!

Ex: $A = \{1,2,3\}$

$A \times A \to A$

$1*1 = 3$	$2*1 = 1$	$3*1 = 2$
$1*2 = 2$	$2*2 = 1$	$3*2 = 3$
$1*3 = 3$	$2*3 = 1$	$3*3 = 3$

Q: How many diff. binary ops. on the set $\{1,2,3\}$

$\underline{3}\ \underline{3}\ \underline{3}\ \underline{3}\ \underline{\cdots}\ \underline{3}$ $\quad 3^9$

Multiplication Table:

$*$	1	2	3
1	$1*1$	$1*2$	$1*3$
2	$2*1$	$2*2$	$2*3$
3	$3*1$	$3*2$	$3*3$

$*$	1	2	3
1	3	2	3
2	1	1	1
3	2	3	3

$*$ commutative iff symmetric

Q: How many commutative binary operations are there on $\{1,2,3\}$?

$*$	1	2	3
1	x	y	z
2	y	p	q
3	z	q	f

$\underline{x, y, z, q, p, f}$

3^6

$$x * y = \begin{cases} x+y, & x+y \le 12 \\ x+y-12, & x+y > 12 \end{cases}$$