

Quotient Rings

Recall that the kernel of a ring homomorphism $\varphi: R \rightarrow S$ is an ideal of R . Just like the case of normal subgroups, which are characterized by being kernels, ideals are also always kernels of some homomorphism.

Def: Let $I \subseteq R$ be an ideal. The quotient ring of R by I is the set of cosets
$$R/I = \{r+I \mid r \in R\}$$

We already know this is a group with addition coming from $+$ on R . Something stronger is true:

Theorem: The Abelian group R/I is a ring with product given by
$$(r+I) \cdot (s+I) = rs + I.$$

The quotient ring comes equipped with a natural homomorphism, called the quotient map

$$\varphi: R \rightarrow R/I, \quad \varphi(r) = r+I$$

Theorem: Let I be an ideal of R and $\varphi: R \rightarrow R/I$ the quotient map. Then φ is a homomorphism of rings and $\ker(\varphi) = I$.

Ex: Let $R = \mathbb{Z}$ and $I = n\mathbb{Z} = \{kn \mid k \in \mathbb{Z}\}$
Then I is an ideal of R and the

quotient ring $R/I = \mathcal{R}/\mathcal{N}\mathcal{R}$ is isomorphic to \mathcal{R}_n .

Ex: $R = \mathbb{Q}[x]$, $I = \{f(x) \in R \mid f(\sqrt{2}) = 0\}$.
 $= \{g(x)(x^2 - 2) \mid g(x) \in \mathbb{Q}[x]\}$

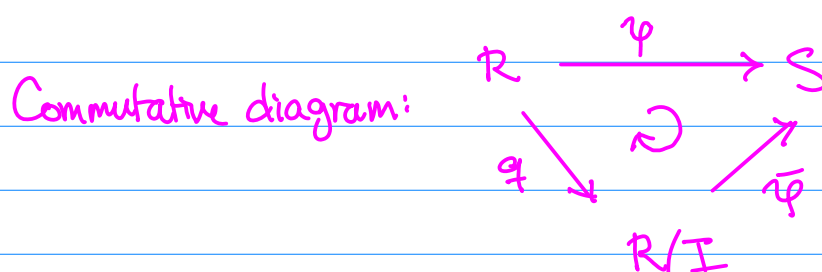
Then I is an ideal and

$$R/I = \{(a+bx) + I \mid a, b \in \mathbb{Q}\}$$

$$\begin{aligned} [(a+bx) + I] \cdot [(c+dx) + I] &= (ac + (ad+bc)x + bd x^2) + I \\ &= (ac + 2bd + (ad+bc)x) + I \end{aligned}$$

Fundamental Homomorphism Theorem:

Let I be an ideal of a ring R and
suppose $\varphi: R \rightarrow S$ is a ring homomorphism
If $I \subseteq \ker(\varphi)$, then φ descends to the quotient
to a homomorphism $\bar{\varphi}: R/I \rightarrow S$ satisfying
 $\bar{\varphi}(r+I) = \varphi(r)$



Prop: $\ker(\bar{\varphi}) = \{r+I \mid r \in \ker(\varphi)\} = \ker(\varphi)/I$.

In particular, if $I = \ker(\varphi)$ then $\bar{\varphi}$ is a
monomorphism

$$\bar{\varphi}: R/\ker(\varphi) \rightarrow S$$

Ex: Consider the evaluation homomorphism

$$\phi_{\sqrt{2}} : \mathbb{Q}[x] \rightarrow \mathbb{R}$$

The kernel of $\phi_{\sqrt{2}}$ is

$$\begin{aligned} I = \ker(\phi_{\sqrt{2}}) &= \{ f(x) \in \mathbb{Q}[x] \mid f(\sqrt{2}) = 0 \} \\ &= \{ (x^2 - 2)g(x) \mid g(x) \in \mathbb{Q}[x] \} \end{aligned}$$

Thus $\mathbb{Q}[x]/I \cong \text{img}(\phi_{\sqrt{2}}) = \{ a + \sqrt{2}b \mid a, b \in \mathbb{Q} \}$.
which is a field.