Last Time:

- Group actions
- Orbit-Stabilizer theorem
- Applications

One application:

Theorem: If $|G| = p^n$ for p prime, then G has a nontrivial center!

Corollary: Every group of order p^2 is abelian!

Ex: Up to isomorphism all groups of order 121 are $\mathbb{Z}_{11} \times \mathbb{Z}_{11}$ or $\mathbb{Z}_{121}$.

special case

Cauchy's Theorem*: If G is a group of order p^n then G has an element of order p.

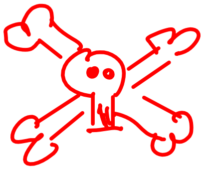
Proof:

Let H be the center of G. Then $H \neq \{e\}$ and $|H| \mid |G|$ so $|H| = p^k$. Hence $p \mid |H|$ and H is abelian so H contains an element of order p.

Cauchy's Theorem: If $p \mid |G|$ then G has an element of order p.

Why is this super cool???

For an abelian group G, if $m \mid |G|$, then G has a subgroup of order m.



• NOT TRUE FOR NON-ABELIAN GROUPS •

Ex: A_n for $n \geq 5$ $\left(A_n = \{\sigma \in S_n \mid \sigma \text{ even}\} \right)$

Remember A_n is simple for $n \geq 5$.

meaning it has no normal subgroups except A_n, $\{e\}$.

Lemma: If $H < G$ of index 2 $\left([G:H] = 2 \right)$ then H is normal.

Proof: $G/H = \boxed{\{H, gH\}}$ for $g \in G$.

H normal $\Leftrightarrow$ every left coset is also a right coset

H is both a left + right coset

NTS gH is a right coset!

NTS $gH = Hg$ $\qquad H = gHg^{-1}$

$G = H \cup gH$ $\qquad$ If $x \in gHg^{-1}$

$x = ghg^{-1}$ for some $h \in H$

$x \in H$ or $x \in gH$ $\qquad$ if $x \in gH$ $\qquad ghg^{-1} \in gH$

$\cancel{g}hg^{-1} = \cancel{g}\tilde{h}$ for some $\tilde{h} \in H$

$hg^{-1} = \tilde{h}$

$h\tilde{h}^{-1} = g$

$\in H$

$\Rightarrow gH = H$

$\Rightarrow\Leftarrow$

$gHg^{-1} = H$

$\therefore$ $gH = Hg$ so H is normal.

$\square$

Ex: A_n does not have a subgroup of order $\frac{n!}{4}$.

$\downarrow$

$\frac{n!}{2}$ $\qquad$ If $H \leq A_n$ and $|H| = \frac{n!}{4}$

Lagrange $\quad |A_n| = |H| \cdot [A_n : H]$

$$\frac{n!}{2} = \frac{n!}{4} \cdot 2$$

$\Rightarrow H \trianglelefteq A_n$

Cauchy's Theorem: at least we know G has a subgroup of order p for any prime $p \mid |G|$.

<u>Definition</u>: A <u>p-group</u> is a group whose order is p^k for some $k > 0$.

BIG PIC

of Sylow Theory!

Sylow p-subgroup

G	order $n = p^r a$ where $p \nmid a$	
H_r	order p^r	(1st Sylow theorem)
$\vdots$		
normal		
H_3	order p^3	(1st Sylow theorem)
normal		
H_2	order p^2	(1st Sylow theorem)
normal		
H_1	order p	(Cauchy says so)

2nd Sylow theorem

Def: A <u>Sylow p-subgroup</u> H of G is a p-subgroup which is not contained in any other proper p-subgroup of G.

A group G can have more than one Sylow p-subgroup!

$$\begin{array}{cccc}
 & G & & \\
 & | \quad \backslash & \backslash & \\
H_r & H_r' & H_r'' & \\
\vdots & \vdots & \vdots & \\
H_3 & H_3' & H_3'' & \cdots \\
| & | & | & \\
H_2 & H_2' & H_2'' & \\
| & | & | & \\
H_1 & H_1' & H_1'' &
\end{array}$$

<u>2nd</u> Sylow theorem:
If H_r and H_r' are both Sylow p-subgroups, then

$$H_r = gH_r'g^{-1} \text{ for some } g \in G.$$

"they are conjugates of each other"

Third Sylow theorem:
The number of Sylow p-subgroups divides $|G|$ and is equal to 1 mod p.

<u>Ex</u>: G group of order $15 = 3 \cdot 5$
It has a Sylow 3-subgroup H $(|H| = 3)$
and a Sylow 5-subgroup K $(|K| = 5)$

How many Sylow 5 subgroups are there in G?
divides $3 \cdot 5$ so $1, \cancel{3}, \cancel{5}, \cancel{15}$
$\equiv 1 \bmod 5$
just <u>one</u> such group!

gHg^{-1} is also a Sylow 3-subgroup for all $g \in G$

$\therefore$ $gHg^{-1} = H$ $\forall g \in G$. H is normal !!

Similarly, K is a normal subgroup of order 5

$$\boxed{K \trianglelefteq G, \quad H \trianglelefteq G \qquad K \cap H = \{e\} \quad HK = G}$$

$$\Rightarrow \quad G \cong H \times K \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$$

Ex: No simple groups of order 20. $= 2^2 \cdot 5$

Subgroups: one of order 5

one of order $2, 2^2$

$A_5 = 5 \cdot 4 \cdot 3$

Sylow 5-subgroups

lots of them.

m = number of Sylow 5 subgroups?

m divides 20 — $1, \cancel{2}, \cancel{4}, \cancel{5}, \cancel{10}, \cancel{20}$

m equal to 1 mod 5

just one Sylow 5 subgroup!

$H \leq G$ Sylow 5-subgroup

gHg^{-1} is also a Sylow 5-subgroup

$gHg^{-1} = H$ $\therefore$ H is normal

G can't be simple!

Ex: A_5 has order $5 \cdot 4 \cdot 3$

$H = \langle (12345) \rangle$ Sylow 5-subgroup

$\langle (21345) \rangle$ Sylow 5-subgroup

$\langle (32514) \rangle$ Sylow 5-subgroup.

Sylow 5-subgroups of $A_5 = \{ \langle (a_1 a_2 a_3 a_4 a_5) \rangle \}$

$5 \cdot 4 \cdot 3 = 5 \cdot 2^2 \cdot 3$

$1, \cancel{2}, \cancel{4}, 6, \cancel{12},$

6 Sylow 5 subgroups