Problem 1

(A) $(13579)(2468)$

(B) $(19)(45)(67)$

Problem 2:

(A) $(8+2i)(3-i) = 26-2i$

(B)
$$(1+i)^{16} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{16} = (\sqrt{2})^{16}\, e^{i4\pi}$$
$$= 2^8\left(\cos(4\pi) + i\sin(4\pi)\right) = 256$$

Problem 3:

(A) Suppose $z, w \in U_n$. Then $z^n = 1$ and $w^n = 1$ so $(zw)^n = z^n w^n = 1$. Thus $zw \in U_n$.

- Also multiplication is associative for complex numbers.
- $1 \in U_n$ is an identity element because $1z = z1 = z$ for all $z \in U_n$.
- Lastly, if $z \in U_n$, then $\left(\frac{1}{z}\right)^n = \frac{1}{z^n} = \frac{1}{1} = 1$ so $\frac{1}{z} \in U_n$. Since $\frac{1}{z}z = z\frac{1}{z} = 1$, we have inverses!

Thus U_n is a group.

(B) First note $j +_n k = \begin{cases} j+k, & j+k < n \\ j+k-n, & j+k \geq n \end{cases}$

Therefore
$$f(j +_n k) = \begin{cases} e^{2\pi i (j+k)/n}, & j+k < n \\ e^{2\pi i (j+k-n)/n}, & j+k \geq n \end{cases}$$

Since $e^{2\pi i (j+k-n)/n} = e^{2\pi i (j+k)/n}\, e^{-2\pi i} = e^{2\pi i (j+k)/n}$

it follows
$$f(j +_n k) = e^{2\pi i (j+k)/n} = e^{2\pi i j/n}\, e^{2\pi i k/n} = f(j)f(k)$$

Lastly, we know $U_n = \{ e^{2\pi i k/n} \mid k = 0, 1, \dots, n-1 \}$ so clearly f is surjective. Since $|U_n| = |\mathbb{Z}_n|$ this implies f is bijective. Hence it is an isomorphism

$\square$

Problem 4:

(A) First of all, an automorphism will have to send a vertex to another vertex with the same number of edges coming out of it. Thus an automorphism must send $1 \mapsto 1$ and $6 \mapsto 6$ or $1 \mapsto 6$ and $6 \mapsto 1$.

Case I ($1 \mapsto 1$ and $6 \mapsto 6$): Since 1 and 2 have to be connected, this forces $2 \mapsto 2$. Likewise $5 \mapsto 5$. So the only automorphisms we get are

$$id: \begin{array}{ll} 1 \mapsto 1 & 4 \mapsto 4 \\ 2 \mapsto 2 & 5 \mapsto 5 \\ 3 \mapsto 3 & 6 \mapsto 6 \end{array} \quad \text{OR} \quad f: \begin{array}{ll} 1 \mapsto 1 & 4 \mapsto 3 \\ 2 \mapsto 2 & 5 \mapsto 5 \\ 3 \mapsto 4 & 6 \mapsto 6 \end{array}$$

Case II ($1 \mapsto 6$ and $6 \mapsto 1$): Since 1 and 2 have to be connected, this forces $2 \mapsto 5$. Likewise $5 \mapsto 2$. So the only automorphisms we get are

$$g: \begin{array}{ll} 1 \mapsto 1 & 4 \mapsto 4 \\ 2 \mapsto 5 & 5 \mapsto 2 \\ 3 \mapsto 3 & 6 \mapsto 6 \end{array} \quad \text{OR} \quad h: \begin{array}{ll} 1 \mapsto 1 & 4 \mapsto 3 \\ 2 \mapsto 5 & 5 \mapsto 2 \\ 3 \mapsto 4 & 6 \mapsto 6 \end{array}$$

Thus we have just four automorphisms total. The multiplication table is

$\circ$	id	f	g	h
id	id	f	g	h
f	f	id	h	g
g	g	h	id	f
h	h	g	f	id

Composition is assoc. and we have an identity id. Each map is its own inverse, so we have inverses. Thus this is a group.