Last Time:

Quotient groups: $H \trianglelefteq G$

$G/H = \{xH \mid x \in G\}$ has a group structure

$(aH)(bH) = abH$. (binary operation)

H (identity)

$(aH)^{-1} = a^{-1}H$ (inverse)

Simple Groups:

Def: A simple group is a group with no nontrivial proper normal subgroups.

Ex: $\mathbb{Z}_p$ for p prime is a simple group

Ex: A_n is simple for $n \geq 5$.

(Problem in text §15 #41)

Much larger story ~ (1950-1980)

Problem: Classify all finite simple groups

Theorem: If G is a finite simple group, then up to isomorphism G is

- $\mathbb{Z}_p$ for some prime p ✓
- A_n for $n \geq 5$ ✓
- finite group of Lie type ✓
- Tits group (order 17971200)

- one of 26 <u>exceptional groups</u>, including the famous Monster group

(order $= 808017424794512875886459904961710757005754368 \cdot 10^9$)

Groups of Lie type ~ "rational pts. on Lie groups"

<u>Ex</u>: $GL_n(\mathbb{Z}_2)$ is a finite simple group

if $n=2$:

$$GL_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

<u>Def</u>: A normal subgroup $H \trianglelefteq G$ is a <u>maximal normal subgroup</u> if

- $H \neq G$
- $H \leq K \trianglelefteq G \Rightarrow H = K$ or $K = G$

<u>Theorem</u>: Let $H \trianglelefteq G$. Then G/H is simple $\Leftrightarrow$ H is a maximal normal subgroup.

<u>Ex</u>: $S_n/A_n \cong \mathbb{Z}_2$ is simple, so A_n is a maximal normal subgroup of S_n.

<u>Commutators</u>:

<u>Def</u>: Let $a, b \in G$. The <u>commutator</u> of a and b is

$$[a,b] = aba^{-1}b^{-1}.$$

The subgroup of G generated by all commutators is called the commutator subgroup $[G,G]$ of G.

$$[G,G] = \langle \{ [a,b] \mid a,b \in G \} \rangle$$

Prop: $[G,G]$ is a normal subgroup of G.

Proof: Let $H = [G,G]$ so $H < G$.
NTS

$$aHa^{-1} \subseteq H \quad \forall \quad a \in G.$$

Recall H is generated by $[x,y]$ for $x,y \in G$.

$$\begin{aligned} a[x,y]a^{-1} &= axyx^{-1}y^{-1}a^{-1} \\ &= axa^{-1}aya^{-1}ax^{-1}a^{-1}ay^{-1}a^{-1} \\ &= (axa^{-1})(aya^{-1})(axa^{-1})^{-1}(aya^{-1})^{-1} \\ &= [axa^{-1}, aya^{-1}] \end{aligned}$$

$\therefore \quad a[x,y]a^{-1} \in H \qquad \forall \ a,x,y \in G.$
Since H is generated by commutators, it follows $aHa^{-1} \subseteq H$

$\square$

Theorem: $G/[G,G]$ is abelian. and moreover
G/H is abelian $\Leftrightarrow$ $[G,G] \leq H$.

Def: The group $G/[G,G]$ is called the abelianization of G.

Proof: Let $H \trianglelefteq G$ with $[G,G] \leq H$.

$$aHbH = abH \qquad bHaH = baH$$

NTS $abH = baH$

NTS $abh_1 = bah_2$ for some $h_1, h_2 \in H$.

$b^{-1}ab = b^{-1}bah$ for $h = h_2 h_1^{-1} \in H$.

$a^{-1}b^{-1}ab = a^{-1}ah$ for some $h \in H$

$a^{-1}b^{-1}ab = h$ for some $h \in H$

I know for any $x, y \in G$, $[x,y] \in [G,G] \subseteq H$

ie $xyx^{-1}y^{-1} \in H$ $\forall x,y \in G$.

Take $x = a^{-1}$ $y = b^{-1}$. Then $a^{-1}b^{-1}ab \in H$. ✓

$\square$

<u>Examples of Quotient Groups</u>:

<u>Prop</u>: $H \leq G$ Then $ghH = gH$ for all $g \in G$, $h \in H$.

<u>Ex</u>: $G = \mathbb{Z}_4 \times \mathbb{Z}_6$, $H = \langle (2,3) \rangle$

$H = \{(2,3), (0,0)\}$

Note $H \trianglelefteq G$. (because G is abelian)

G/H is a group (finite abelian group)

Lagrange: $|G| = |H| \cdot \underbrace{[G:H]}$ $\leftarrow$ # of left cosets $= |G/H|$

$|G/H| = |G|/|H| = 24/2 = 12 = 2 \cdot 2 \cdot 3$

<u>Brian</u>: $\mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_{12}$ which is G/H

$\uparrow$ 6 kills everything!

$\uparrow$ 6 kills evens only!

$G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$

$(1,1) + H \in G/H$

$6 \cdot \big((1,1) + H \big) = (6,6) + H = (2,0) + H$

<u>Q</u>: is $(2,0) + H$ = identity in G/H

is $(2,0) + H = H$

is $(2,0) \in H$ NO!

$\therefore \; G/H \cong \mathbb{Z}_{12}$

To get isomorphism, consider homomorphism

$$\phi : G \longrightarrow \mathbb{Z}_{12}$$
$$(a,b) \longmapsto 3a - 2b$$

ϕ is surjective w/ $\ker \phi = H$

$$\boxed{G/H \cong \mathbb{Z}_{12}}$$