Math 407 Section 1
Fall 2022
Exam II
March 24, 2022
Time Limit: 75 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- If the problem asks for a proof, be sure to carefully justify your work, including any theorems from class.

  Note: you may NOT use a theorem or result from class to prove something when it makes the problem entirely trivial. If you are unsure whether a particular theorem or result is allowed, just ask!

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) If H is a normal subgroup of a group G, then H must be Abelian.

   (b) Any group of order 4 is Abelian.

   (c) Every subgroup of a cyclic group is normal.

   (d) If P is a Sylow p-subgroup of G, then so is gPg^{-1} for all $g \in G$

   (e) The kernel of a ring homomorphism is an ideal.

2. (10 points)

   (a) State the second isomorphism theorem for groups.

   (b) State the third Sylow theorem.

   (c) Give an example of a normal subgroup of S_3.

3. (10 points)

   Let $G = \mathbb{Z} \times \mathbb{Z}$ and let H be the subgroup of G generated by the elements $(1, 1)$ and $(2, 0)$.

   (a) Write down explicitly the distinct cosets of H in G.

   (b) Determine what finite abelian group G/H is isomorphic to in invariant factor form.

4. (10 points)

Recall that the dihedral group D_n is

$$D_n = \{(a, b) : 0 \leq a < n - 1,\ b = \pm 1\}$$

with the binary operation $(a, b)(c, d) = (a + bc, bd)$.

(a) Determine the size of a Sylow 2-subgroup of D_{10}

(b) Determine the number of Sylow 7-subgroups of D_7

(c) Show that D_n always has more than one Sylow 2-subgroup for odd $n \geq 3$

5. (10 points)

Let X be a set and let R be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. So for example $g(x) = x$ and $h(x) = \sin(x)$ are elements of R. Define binary operations on R by

$$(f \cdot g)(x) = f(x)g(x) \quad \text{and} \quad (f + g)(x) = f(x) + g(x).$$

(a) Show that R is a ring

(b) Show that the set

$$I = \{f(x) \in R : f(3) = 0\}$$

is an ideal of R.

6. (10 points)

   A ring R is called **simple** if the only ideals in R are $\{0\}$ and R.

   (a) Show that if $f : R \to S$ is a ring homomorphism and R is simple, then f must be injective.

   (b) Prove that if R is a division ring (ie. a skew-field), then R is simple.