Relations, Functions, and Equivalence

Let A and B be sets.

Def: A relation from A to B is a subset $R \subseteq A \times B$

Notation: we write aRb to mean $(a,b) \in R$

Ex: $A = \{x \mid x \text{ is a resident of Orange County}\}$
$B = \{x \mid x \text{ is a resident of L.A. County}\}$

xRy if and only if x and y are siblings

Def: A relation on A is a relation from A to A.
(i.e. a subset of $A \times A$)

Ex: Let $A = \{1,2,3,4\}$ and define a relation R on A by
$$mRn \Leftrightarrow mn \text{ is even}$$

Alternative description:

$$\begin{aligned}A \times A &= \{(m,n) \mid m,n \in A\} \\ &= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4), \\ &\quad (3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\} \\ R &= \{(m,n) \mid m,n \in A,\ mn \text{ even}\} \\ &= \{(1,2),(1,4),(2,1),(2,2),(2,3),(2,4), \\ &\quad (3,2),(3,4),(4,1),(4,2),(4,3),(4,4)\}\end{aligned}$$

Two most important types of relations:

FUNCTIONS	EQUIVALENCE RELATIONS
Def: A function from A to B is a relation R from A to B satisfying • for every $a \in A$ there is a $b \in B$ with aRb and with b is unique Notation: $R(a) = b$ means aRb	Def: An equivalence relation on A is a relation R on A satisfying three properties • reflexive: aRa for all $a \in A$ • symmetric: $aRb \Leftrightarrow bRa$ • transitive: if aRb and bRc then aRc

Ex: $A = \mathbb{Z}$

$aRb \Leftrightarrow a-b$ is even

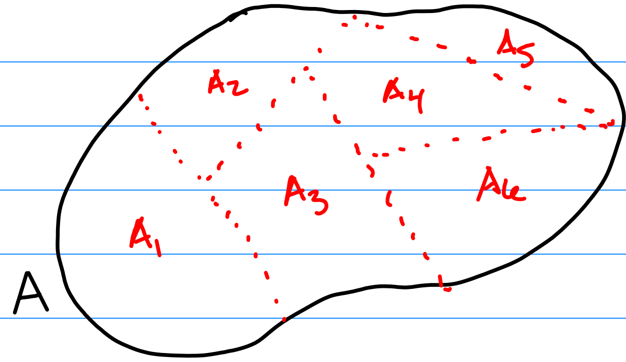
Reflexive? Let $a \in \mathbb{Z}$. $a-a = 0$ (even) $\therefore aRa$ ✓

Symmetric? Let $a, b \in \mathbb{Z}$ with aRb. Then $a-b$ is even.
This means $-(a-b)$ even, ie $b-a$ is even. $\therefore bRa$
Thus $aRb \Rightarrow bRa$ ✓

Transitive? Let $a, b, c \in \mathbb{Z}$ with aRb and bRc. Then
$a-b$ and $b-c$ are both even,
Therefore $a-c = (a-b)+(b-c)$ is even. $\therefore aRc$.
Thus aRb and $bRc \Rightarrow aRc$. ✓

Thus we are an equivalence relation!

Partitions:



Def: A partition of A is a collection of subsets $A_1, \dots, A_n$ of A satisfying

- pairwise disjoint: $A_j \cap A_k = \emptyset \quad j \neq k$
- union is A: $A_1 \cup A_2 \cup \dots \cup A_n = A$.

Equivalence relations give rise to partitions via equivalence classes.

Def: Let R be an equivalence relation on A.
The equivalence class of $a \in A$ is

$$[a] = \bar{a} = \{x \in A \mid aRx\}$$

Theorem: The set of equivalence classes of A is a partition of A.

Previous Example Continued: R equiv. relation on $\mathbb{Z}$.
$aRb \Leftrightarrow a-b$ even

$$\bar{0} = \{x \mid 0Rx\} = \{x \mid 0-x \text{ even}\} = \{x \mid x \text{ even}\}$$
$$= \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$
$$\bar{1} = \{x \mid 1Rx\} = \{x \mid 1-x \text{ even}\} = \{x \mid x \text{ odd}\}$$
$$= \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$$
$$\bar{2} = \{x \mid 2Rx\} = \{x \mid 2-x \text{ even}\} = \{x \mid x \text{ even}\} = \bar{0}$$

$$\bar{3} = \dots = \bar{1}$$

All of the equivalence classes (no repeats): $\bar{0}$, $\bar{1}$

$$\left.\begin{aligned} \bar{0} \cap \bar{1} &= \emptyset \\ \bar{0} \cup \bar{1} &= \mathbb{Z} \end{aligned}\right\} \quad \text{Partition!!!}$$