

Algebraic Extensions

Simple Extensions:

- α algebraic of degree d

$$F(\alpha) = \{ a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{d-1}\alpha^{d-1} \mid a_0, \dots, a_{d-1} \in F \}$$

- α transcendental

$$F(\alpha) = \{ f(\alpha)/g(\alpha) \mid f(\alpha), g(\alpha) \in F[\alpha], g \neq 0 \}$$

Observation \sim if α is algebraic, then $F(\alpha)$ is finite dimensional as an F -vector space

Def: The degree of an extension field E of F is

$$[E:F] = \dim_F(E)$$

Theorem: If $[E:F]$ is finite, then every element $\alpha \in E$ is algebraic over F .

Def: An extension field is algebraic if every element $\alpha \in E$ is algebraic over F .

The algebraic closure of F in E is

$$\bar{F}_E = \{ \alpha \in E \mid \alpha \text{ is algebraic over } F \}$$

Theorem: \overline{F}_E is a field.

There is a largest algebraic extension of a field F , called the algebraic closure.

Def: A field E is called algebraically closed if every polynomial $f(x) \in E[x]$ has a root in E .
An algebraically closed algebraic extension E of F is called an algebraic closure of F .

Notation: \overline{F} is the algebraic closure of F .

Theorem: $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic}\}$
is a proper subset of \mathbb{C} .

Proof:

$\overline{\mathbb{Q}}$ is countable, but \mathbb{C} is uncountable.
 \square

Theorem: If $\alpha \in E$ is algebraic over F
and $\deg \text{irr}(\alpha, F) = d$, then a basis
for $F(\alpha)$ as an F -vector space is

$$\{1, \alpha, \alpha^2, \dots, \alpha^{d-1}\}.$$

In particular,

$$[F(\alpha) : F] = \deg \text{irr}(\alpha, F)$$

Ex: If p is prime, $\zeta = e^{2\pi i/p}$

$$\text{irr}(\zeta, \mathbb{Q}) = 1 + x + x^2 + \dots + x^{p-1}$$

and

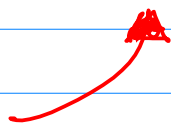
$$\begin{aligned}\mathbb{Q}(\zeta) &= \left\{ a_0 + a_1 \zeta + a_2 \zeta^2 + \dots + a_{p-1} \zeta^{p-1} \mid a_0, \dots, a_{p-1} \in \mathbb{Q} \right\} \\ &= \text{span}_{\mathbb{Q}} \{ 1, \zeta, \zeta^2, \dots, \zeta^{p-1} \}\end{aligned}$$

Ex: .

$$\mathbb{Q}[\sqrt{\sqrt{3}-1}] = \text{span}_{\mathbb{Q}} \{ 1, \sqrt{\sqrt{3}-1}, \sqrt{3}-1, (\sqrt{3}-1)^{3/2} \}$$

Theorem :

$$[K:F] = [K:E][E:F]$$



Reminds us of subgroups!

