Problem 1:

The subgroups of $\mathbb{Z}_{100}$ are all cyclic, generated by a single element $a \in \mathbb{Z}_{100}$. Moreover,

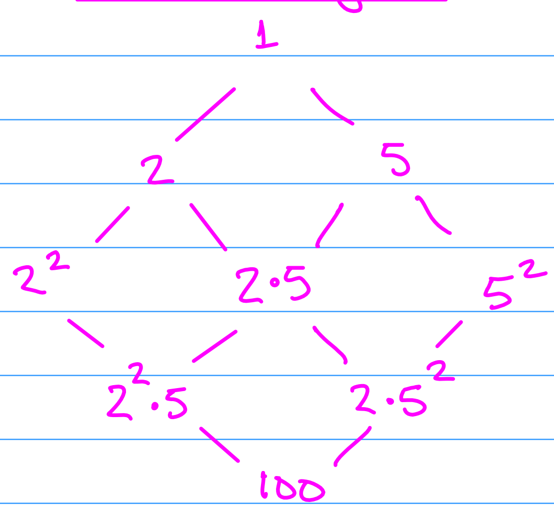
$$\langle a \rangle = \langle \gcd(a,100) \rangle$$

and therefore the subgroups of $\mathbb{Z}_{100}$ correspond precisely to the divisors of $100 = 2^2 \cdot 5^2$

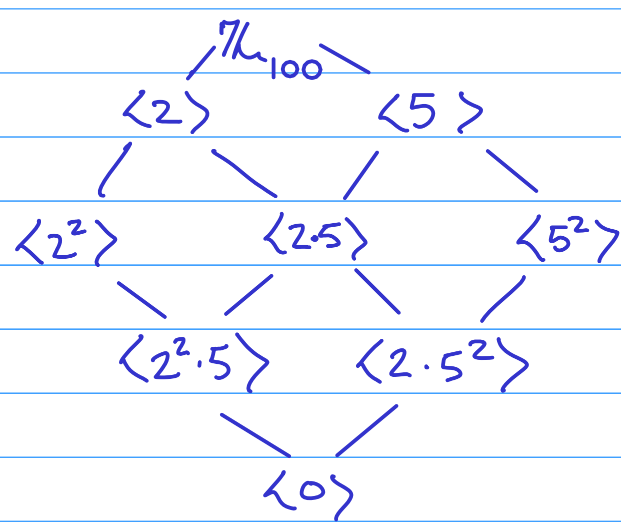
$$\langle 1 \rangle, \langle 2 \rangle, \langle 4 \rangle, \langle 5 \rangle, \langle 10 \rangle, \langle 20 \rangle, \langle 25 \rangle, \langle 50 \rangle, \langle 0 \rangle$$

The subgroup diagram corresponds to a similar divisor diagram.

Divisor Diagram



Subgroup Diagram



Problem 2:

(A) S'pose $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Therefore $ab \in H$ and $ab \in K$. Thus $ab \in H \cap K$. This shows $H \cap K$ is closed under products.

Next, since $e \in H$ and $e \in K$, we know $e \in H \cap K$.
Finally, if $a \in H \cap K$, then $a \in H$ so $a^{-1} \in H$.
Likewise $a \in K$ so $a^{-1} \in K$. Thus $a^{-1} \in H \cap K$.
This proves $a^{-1} \in H \cap K$. This shows $H \cap K$ is closed under inversion, so $H \cap K$ is a subgroup.

(B) Let $G = \mathbb{Z}$, $H = \langle 2 \rangle$, $K = \langle 3 \rangle$.
Note that $-2 \in H$ and $3 \in K$ but $(-2) + 3 = 1$ is not in $H \cup K$ so $H \cup K$ is not closed under $+$. Therefore it is not a subgroup.

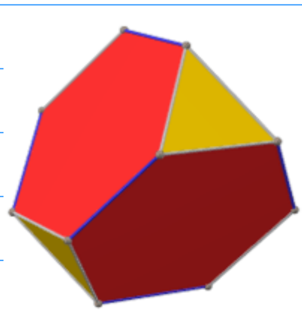
<u>Problem 3</u>:

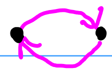
(B) Let $\sigma = (12)(34)$, $\tau = (123)$

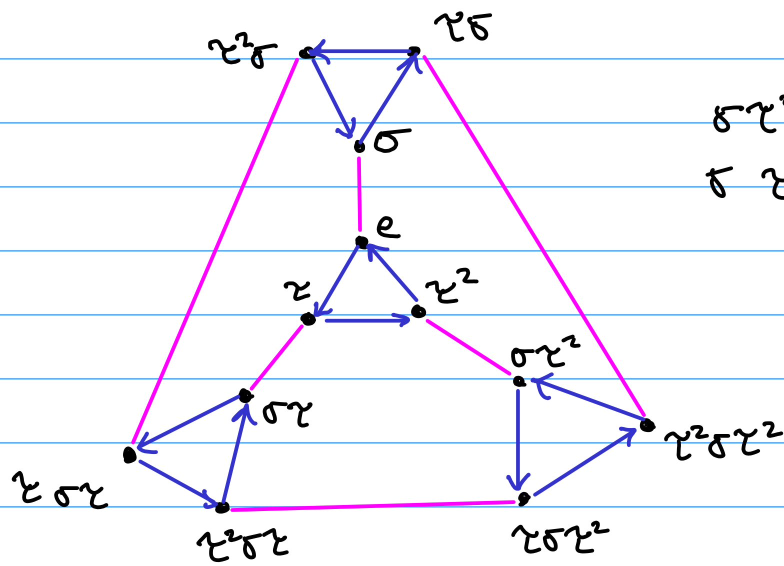
$\langle \sigma, \tau \rangle =$

$\{ e$, $\sigma = (12)(34)$, $\tau\sigma\tau^2 = (23)(14)$, $\tau^2\sigma\tau = (13)(24)$
$\tau = (123)$, $\sigma\tau = (243)$, $\tau\sigma = (134)$, $\tau^2\sigma\tau^2 = (142)$
$\tau^2 = (132)$, $\sigma\tau^2 = (143)$, $\tau\sigma\tau = (124)$, $\tau^2\sigma = (234) \}$

(C) The Cayley graph forms the vertices and edges of a <u>truncated tetrahedron</u> (a tetrahedron with the four corners cut off)



For simplicity, we will use an undirected edge 
in place of  in the graph. The graph
is then



$$\sigma\tau^2\sigma = \tau\bar{\sigma}\tau$$
$$\sigma\tau\sigma = \tau^2\bar{\sigma}\tau^2$$

Problem 4 :

(A) In a group, for fixed x and b, the equation $xy = b$ has a unique solution, Therefore every element occurs in a row exactly one time. Likewise, for fixed y and b, the equation $xy = b$ has a unique solution, so every element occurs in a column exactly one time. Thus we have a Latin square.

(B) Each row defines a permutation of the elements. Thus a Latin square defines a group if and only if the associated permutations form a subgroup of the permutation group.