Sylow Theorems

Goal: Prove 1^{st}, 2^{nd}, 3^{rd} Sylow theorems

Recall that if X is a G-set.

$$\mathrm{orb}(x) = \{g \cdot x \mid g \in G\}$$
$$\mathrm{stab}(x) = G_x = \{g \in G \mid g \cdot x = x\}$$

Orbit-Stabilizer Theorem: $|\mathrm{orb}(x)| = [G : G_x]$

Since orbits form a partition of X

$$|X| = \sum_{[x] \in X/G} |\mathrm{orb}(x)|$$

$X/G = \{\text{distinct orbits}\}$

$$|X| = \sum_{[x] \in X/G} [G : G_x]$$

$$\boxed{|X| = |X_G| + \sum_{\substack{[x] \in X/G \\ |\mathrm{orb}(x)| > 1}} [G : G_x]}$$

where here
$$X_G = \{x \in X \mid g \cdot x = x \ \forall g \in G\}$$

Def: Let $H \leq G$. The normalizer $N(H)$ of H is the set

$$N(H) = \{g \in G \mid gHg^{-1} = H\}$$

$$\begin{array}{c} G \\ \not\trianglelefteq \; | \\ N(H) \\ \trianglelefteq \; | \\ H \end{array}$$

If $H \trianglelefteq G$ then $N(H) = G$.

<u>Lemma</u>: Let H be a proper p-subgroup of a group G. Then the normalizer $N(H)$ of H in G satisfies $N(H) \neq H$. If $p \mid [G:H]$, then $p \mid [N(H):H]$.

<u>Proof</u>:
Let $X = G/H$ and let H act on X by
$$h \cdot gH = hgH.$$

$$\underbrace{|X|}_{[G:H]} = |X_H| + \sum_{\substack{[x] \in X/H \\ |\mathrm{orb}(x)| > 1}} \underbrace{[H : H_x]}_{\text{powers of } p \text{ !!!}}$$

Thus if $p \mid [G:H]$ then $p \mid |X_H|$.

Take $aH \in X_H$. Then $haH = aH \quad \forall h \in H$
$\iff a^{-1}haH = H$
$\iff a^{-1}ha \in H \iff a^{-1} \in N(H)$
$\iff a \in N(H)$

$\therefore aH \in N(H)/H$.

$X_H = N(H)/H$ and $p \mid |X_H|$

we get $p \mid |N(H)/H| = [N(H):H]$ $\square$

<u>First Sylow Theorem</u>: Let G be a group of order $p^r m$ where $p \nmid m$. Then

(A) G has a p-subgroup of order p^j for all $1 \le j \le r$

(B) If H is a p-subgroup of order p^j and $j < r$, then there is a p-subgroup $\tilde{H}$ of order p^{j+1} w/ $H \trianglelefteq \tilde{H}$.

Proof: (B) S'pose $H \leq G$ and $|H| = p^j$ w/ $j < r$.
Then by the previous lemma $p \mid [N(H):H]$.

$N(H)/H$ is a group! and $p \mid |N(H)/H|$.
Cauchy's Theorem says there exists $y \in N(H)/H$ of order p.

$$\pi : N(H) \longrightarrow N(H)/H$$
$$x \longmapsto y$$

The group generated by x and H ($\langle x \rangle \vee H$) is a subgroup of G of order p^{j+1}.

(A) Cauchy says $\exists\, x \in G$ w/ $|\langle x \rangle| = p$.

- $\langle x \rangle \to H_2$ w/ $|H_2| = p^2$
- $H_2 \to H_3$ w/ $|H_3| = p^3$

$$\vdots$$
$$p^r$$

$\square$

Second Sylow Theorem:

$|G| = p^r m$

If P, P' are Sylow p-subgroups of G, then $P' = gPg^{-1}$ for some $g \in G$.

Proof:
Let P, P' be Sylow p-subgroups of G.

$X = G/P'$ P act on X by
$$a \cdot gP' = agP'.$$

Then
$$|X| = |X_P| + \sum_{\substack{[x] \in X/P \\ |\mathrm{orb}(x)| > 1}} \underbrace{[P : P_x]}_{\text{divide } |P| = p^r}$$

$m = [G:P']$

must not be multiple of P

multiples of P

$X = G/P'$ $|X_P| \neq 0 \mod p \quad \therefore \quad |X_P| \neq 0.$

$X_P = \{x \in X \mid a \cdot x = x \ \forall a \in P\}$

Choose $gP' \in X_P.$ $\quad agP' = gP' \ \forall a \in P.$

$g^{-1}agP' = P' \ \forall a \in P$

$g^{-1}ag \in P' \ \forall a \in P.$

$$\begin{array}{ccc} p^r & & p^r \\ \downarrow & & \downarrow \\ P & \longrightarrow & P' \\ a & \longmapsto & g^{-1}ag \end{array}$$

bijection! $\quad P' = g^{-1}Pg$!!!

$\square$

<u>Third Sylow Theorem</u>: $X = \{\text{Sylow } p\text{-subgroups of } G\}$

$|X| \mid |G|$ <u>and</u> $|X| = 1 \mod p$

<u>Proof</u>:

Sara's idea G act on X by

$$g \cdot P = gPg^{-1}$$

Fix $P \in X$.

$\mathrm{orb}(P) = ???$ $\quad \mathrm{orb}(P) = X$!!!

2nd Sylow theorem!

orbit-stabilizer

$$|X| = |\mathrm{orb}(P)| = [G : G_P] \mid |G|$$

Lagrange

$$|X| \mid |G|$$

To prove $|X| = 1 \mod p$. Fix $P \in X$ and consider the action of P on X by $a \cdot P' = aP'a^{-1}$.

Think about $X_P = \{P' \mid aP'a^{-1} = P' \ \forall a \in P'\}$

If $P' \in X_P$ then $P \leq N(P')$

Since $P' \leq N(P')$ 2^{nd} Sylow theorem says P', P will be conjugates in $N(P')$

$$\underline{P} = gP'g^{-1}, \quad g \in N(P')$$
$$= P'$$

$X_P = \{P\}$

powers of P.

$$|X| = \underbrace{|X_P|}^{1} + \sum_{\substack{[x] \in P/G \\ |orb(x)| > 1}} [P : P_x]$$

$$\boxed{|X| = 1 \text{ modulo } p}$$

$\square$