Complex Numbers

Def: A complex number is something of the form $a+ib$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$

numbers of the form bi with $b \in \mathbb{R}$ are imaginary.

Why complex numbers: looking at roots of polynomials

$z^2 + 1 = 0$ no solutions over $\mathbb{R}$

two solutions $i, -i$ in complex numbers

Fundamental Theorem of Algebra: A polynomial of degree n has exactly n roots over the complex numbers!

Complex numbers are a ring.

addition: $\boxed{(a+ib) + (x+iy) = (a+x) + i(b+y)}$

multiplication: $(a+ib)\cdot(x+iy) = ax + aiy + ibx + ibiy$

$i = \sqrt{-1}$, $i^2 = -1$

$= ax + iay + ibx - by$

$$\boxed{(a+ib)\cdot(x+iy) = (ax-by) + i(ay+bx)}$$

Anatomy of a complex # $\underbrace{x}_{\text{real part}} + i\underbrace{y}_{\text{imaginary part}}$

Modulus (aka absolute value) $|x+iy| = \sqrt{x^2+y^2}$

Examples: $(2+3i)(1-4i)$
$= 14 - 5i$

$$3(1-5i) + (4-2i) = 7 - 17i$$

The complex numbers $\mathbb{C}$ are a field.
Meaning we have inverses!

$$\frac{1}{2+3i} = \frac{1}{2+3i}\left(\frac{2-3i}{2-3i}\right) = \frac{2-3i}{(2+3i)(2-3i)}$$

$$= \frac{2-3i}{4+\cancel{6i}-\cancel{6i}+9} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$$

Ex: $\frac{1+7i}{2-i} = \frac{1+7i}{2-i}\,\frac{2+i}{2+i} = \frac{(1+7i)(2+i)}{(2-i)(2+i)}$

Brenden $-1+3i$

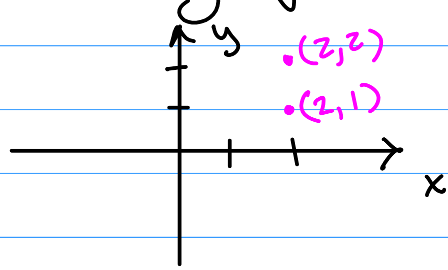
$$= \frac{2+14i+i-7}{4-2i+2i+1} = \frac{-5+15i}{5}$$

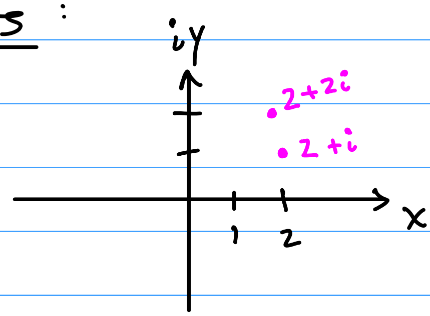
$$= -1+3i$$

Definition: The complex conjugate $\bar{z}$ of a complex number
$z = x+iy$ is $\bar{z} = x - iy$

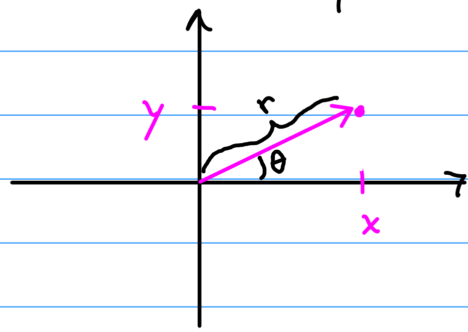
$z\bar{z} = |z|^2$ $\qquad (x+iy)(x-iy) = x^2 + y^2$

Geometry of Complex Numbers:



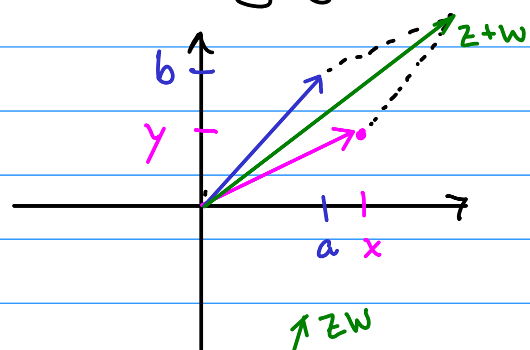


$z = x+iy$

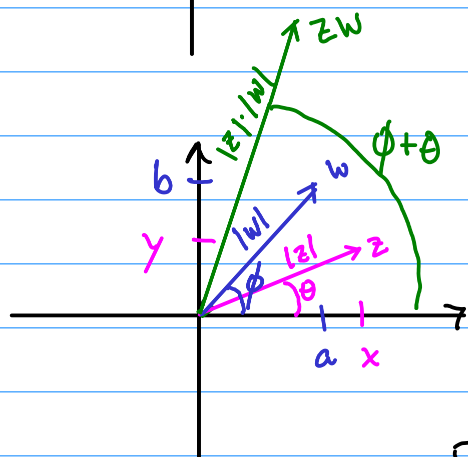


$|z| = r = \sqrt{x^2+y^2}$

Addition is really just vector addition!



$z = x+iy$
$w = a+ib$



$z = x+iy$
$w = a+ib$

Really cool property of $|\cdot|$: $\boxed{|zw| = |z|\cdot|w|}$

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

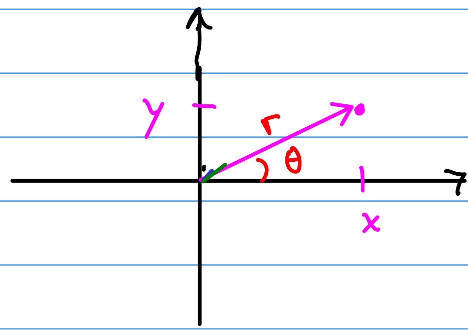
$$e^{\smile} = \sum_{n=0}^{\infty} \frac{\smile^n}{n!}$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^n}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$\boxed{e^{i\theta} = \cos(\theta) + i \sin\theta}$$

$$e^{i\pi} + 1 = 0$$



$$z = x + iy = re^{i\theta}$$

Euler's formula is built for multiplication!

$$(re^{i\theta})(se^{i\phi}) = rse^{i\theta + i\phi} = rse^{i(\theta+\phi)}$$

Ex:

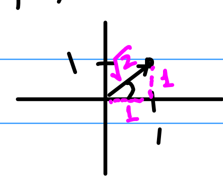
$$(1+i)^{2021}$$

$$1+i = re^{i\theta}$$

cartesian $\longrightarrow$ polar

$$(x,y) \longmapsto (r,\theta)$$

$$(1,1) \longmapsto (\cancel{\tfrac{\pi}{4}, \sqrt{2}}) \quad (\sqrt{2}, \tfrac{\pi}{4})$$



$$\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{2021}$$

$$(\sqrt{2})^{2021}(e^{i\pi/4})^{2021}$$

$$2^{1010}\sqrt{2}\, e^{i\pi 2021/4} \qquad 2^{1010}\sqrt{2}\, e^{i(\pi 505 + \pi/4)}$$

$$2^{1010}\sqrt{2}\left(\cos(505\pi + \tfrac{\pi}{4}) + i\sin(505\pi + \tfrac{\pi}{4})\right)$$

$$2^{1010}\sqrt{2}\left(\cos(5\pi/4) + i\sin(5\pi/4)\right)$$

$$= 2^{1010}\cancel{\sqrt{2}}\left(-\frac{1}{\cancel{\sqrt{2}}} + i\left(-\frac{1}{\cancel{\sqrt{2}}}\right)\right)$$

$$= 2^{1010}(-1 - i) = -2^{1010} - 2^{1010} i$$

Multiplication Tables:

Consider a set S with a binary operation $*$
Try to visualize the operation.

$S = \{0, 1, 2\}$ binary operation $+_3$

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Isomorphisms: $S = \{0, 1, \dots, 11\}$, $+_{12}$
$C = \{1, 2, \dots, 12\}$, $+_{clock}$

Definition: Let $(G, *)$ and $(H, \star)$ be two sets with binary operations. An isomorphism is a bijection $f: G \to H$ respecting the group operation

$$f(a * b) = f(a) \star f(b) \quad \text{for all } a, b \in G.$$

Ex: $f: S \to C$ $\quad f(k) = \begin{cases} k, & k \neq 0 \\ 12, & k = 0 \end{cases}$

$$f(1 +_{12} 1) = f(2) = 2 \quad \checkmark$$

$$f(1) +_{clock} f(1) = 1 +_{clock} 1 = 2$$