Quotient Group Examples

A first example: $G = \mathbb{Z} \oplus \mathbb{Z}$, $H = \langle (0,2) \rangle$

What are the cosets?

$(0,0) + H = \{ (0,2k) : k \in \mathbb{Z} \}$

$(0,1) + H = \{ (0,2k+1) : k \in \mathbb{Z} \}$

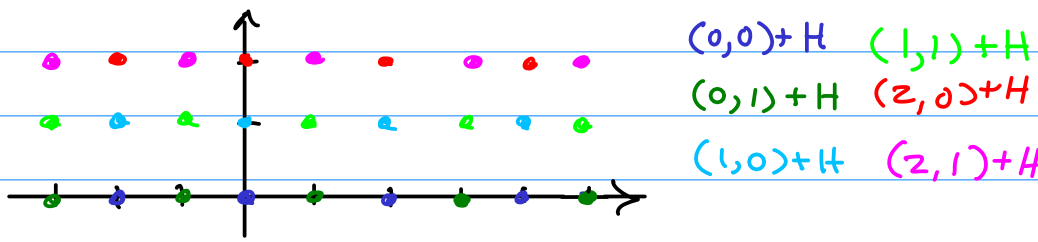
$(0,2) + H = \{ (0,2k+2) : k \in \mathbb{Z} \} = (0,0) + H$ REPEAT!

$(1,0) + H = \{ (1,2k) : k \in \mathbb{Z} \}$

$(1,1) + H = \{ (1,2k+1) : k \in \mathbb{Z} \}$

$\vdots$

$$G/H = \{ (a,b) + H \mid a \in \mathbb{Z},\ b \in \{0,1\} \}.$$



$$\big((a,b)+H\big) + \big((c,d)+H\big) = (a+c, b+d) + H$$
$$= (a+c,\ b+d \bmod 2) + H$$

So we can guess $G/H \cong \mathbb{Z} \oplus \mathbb{Z}_2$

How can we prove it? <u>First Isomorphism Theorem!</u>

Goal: Find an epimorphism $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2$ whose kernel is H.

Define $\varphi(a,b) = (a,\ b \bmod 2)$.

Clearly a surjective hom.

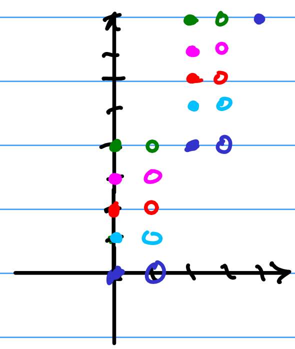
Also, $\varphi(a,b) = 0 \iff a = 0,\ b = 0 \bmod 2$

$\iff (a,b) \in \{ (0,2k) \mid k \in \mathbb{Z} \} = H$.

Thus $G/H \cong \operatorname{Img}(\varphi) = \mathbb{Z} \oplus \mathbb{Z}_2$.

Another Example : $G = \mathbb{Z} \oplus \mathbb{Z}$
$H = \langle (2,4) \rangle$

Cosets:



$$(x,y) + H = (\tilde{x}, \tilde{y}) + H \iff (\tilde{x}-x, \tilde{y}-y) \in H$$
$$\iff (\tilde{x}, \tilde{y}) = (x+2k, y+4k)$$

can make $x = 0, 1$ and y anything

$$G/H = \{(x,y) \mid x \in \{0,1\},\ y \in \mathbb{Z}\}$$ COSET REPRESENTATIVES

$$\begin{cases} (0,y)+(0,z) = (0, y+z) \\ (1,y)+(0,z) = (1, y+z) \\ (0,y)+(1,z) = (1, y+z) \\ (1,y)+(1,z) = (0, y+z-4) \end{cases}$$

$$\psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$$
$$(x,y) \mapsto (x, cx+dy)$$
$$(2,4) \mapsto (0,0)$$

$(x, y-2x)$

Examples of Quotient Groups

G group, $H \leq G$.
G/H = set of left cosets $= \{aH \mid a \in G\}$

$H \trianglelefteq G \Rightarrow G/H$ is a group with $(aH)*(bH) = abH$.

Additive version: $(a+H)+(b+H) = (a+b)+H$.

$\boxed{\text{Basic example:}}$ $G = \mathbb{Z}$, $H = \langle n \rangle = n\mathbb{Z}$

$$G/H = \{ a+H \mid 0 \leq a \leq n-1 \}$$
$$(a+H)+(b+H) = (a+b)+H = (a+b \bmod n) + H$$

As a consequence, $G/H \cong \mathbb{Z}_n$.

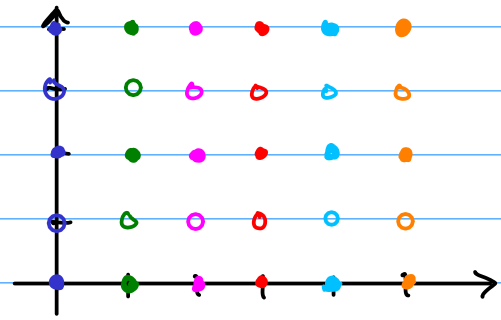
<u>Def</u>: A <u>coset representative</u> of a coset aH is a specific element of aH.

The cosets of H in G can be specified by choosing a list of coset representatives.

Coset representatives of $\mathbb{Z}/n\mathbb{Z}$ are $0, 1, 2, \ldots, n-1$.

<u>Ex</u>: $G = \mathbb{Z} \oplus \mathbb{Z}$, $H = \langle (0,2) \rangle$

Cosets? $(j,0)+H = \{(j,2k) \mid k \in \mathbb{Z}\}$
$(j,1)+H = \{(j,2k+1) \mid k \in \mathbb{Z}\}$
$(j,2)+H = \{(j,2k+2) \mid k \in \mathbb{Z}\} = (j,0)+H$



• $(0,0)+H$ ∘ $(1,0)+H$
• $(0,1)+H$ ∘ $(1,1)+H$
• $(0,2)+H$ ∘ $(1,2)+H$
• $(0,3)+H$ ∘ $(1,3)+H$
• $(0,4)+H$ ∘ $(1,4)+H$
• $(0,5)+H$ ∘ $(1,5)+H$

Coset representatives:

$(0,0)$, $(\pm 1,0)$, $(\pm 2,0)$, …
$(0,1)$, $(\pm 1,1)$, $(\pm 2,1)$, …

How does addition work on cosets?

$$((a,b)+H)+((c,d)+H) = (a+c, b+d)+H$$
$$= (a+c, b+d \bmod 2)+H$$

Suspect: $G/H \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Quest: How can we prove it?
Main tool: first isomorphism theorem!

Goal: Find an epimorphism $\varphi: \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}_2$ whose kernel is H.

Easy! $\varphi: \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}_2$
$(a,b) \mapsto (a, b \bmod 2)$

• $$\varphi((a,b)+(c,d)) = \varphi(a+c, b+d)$$
$$= (a+c, b+d \bmod 2)$$
$$= (a, b \bmod 2) + (c, d \bmod 2)$$
$$= \varphi(a,b) + \varphi(c,d)$$

- φ is obviously surjective!

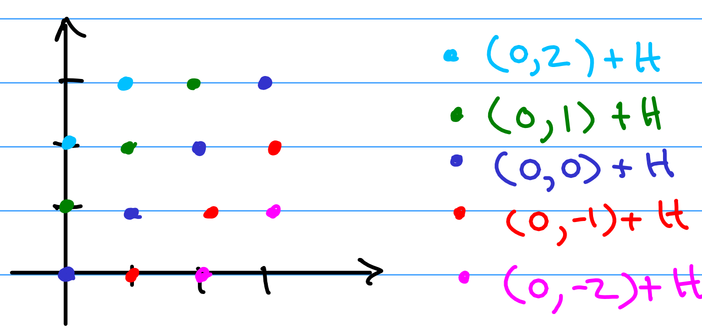
- $\varphi(a,b) = (0,0) \iff a=0 \;\&\; b=0 \mod 2$
$\iff a=0 \;\&\; b=2k,\quad k\in\mathbb{Z}$
$\iff (a,b)\in\{(0,2k) \mid k\in\mathbb{Z}\} = H.$

Therefore by the first isomorphism theorem

$$G/H \cong \operatorname{img}(\varphi) = \mathbb{Z}\oplus\mathbb{Z}_2.$$

<u>Ex</u>: $G=\mathbb{Z}\oplus\mathbb{Z}$, $H=\langle(1,1)\rangle$

Cosets? $(0,-1)+H$
$(0,0)+H$ $\quad (1,0)+H = (0,-1)+H$
$(0,1)+H$ $\quad (1,1)+H = (0,0)+H$
$(0,2)+H$ $\quad (1,2)+H = (0,1)+H$
$(0,3)+H$ $\quad (1,3)+H = (0,2)+H$



Coset representatives:
$\ldots, (0,-2), (0,-1), (0,0), (0,1), (0,2), \ldots$

Coset addition?
$$((0,a)+H)+((0,b)+H) = (0,a+b)+H.$$

Suspect $G/H \cong \mathbb{Z}$.

Goal: Find an epimorphism $\varphi: \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$
with $\ker(\varphi) = H$.

$$\varphi: \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$$
$$(x,y) \mapsto ax+by$$
$$(1,1) \mapsto 0 \quad \Rightarrow \quad a+b=0 \Rightarrow b=-a$$
$$(x,y) \mapsto ax-ay = a(x-y)$$
Surjective $\Rightarrow$ $a=1$ so $\varphi(x,y) = x-y$.

- φ is a homomorphism
- φ is surjective
- $\ker \varphi = H$

Thus by 1st Iso. Thm $G/H \cong \mathbb{Z}$.

More difficult example:

$G = \mathbb{Z}\oplus\mathbb{Z}$

$H = \langle (2,4) \rangle$

Q: What is G/H up to iso?

$$(x,y)+H = (\tilde{x},\tilde{y})+H \Leftrightarrow (\tilde{x}-x, \tilde{y}-y) \in H$$
$$\Leftrightarrow (\tilde{x},\tilde{y}) = (x+2k, y+4k), \ k\in\mathbb{Z}$$

Coset representatives: can choose $x = 0,1$.

$(0,0), (0,\pm 1), (0,\pm 2), (0,\pm 3), \ldots$

$(1,0), (1,\pm 1), (1,\pm 2), (1,\pm 3), \ldots$

How does addition work?

$$((0,a)+H) + ((0,b)+H) = (0,a+b)+H$$
$$((0,a)+H) + ((1,b)+H) = (1,a+b)+H$$
$$((1,a)+H) + ((0,b)+H) = (1,a+b)+H$$
$$((1,a)+H) + ((1,b)+H) = (2,a+b)+H$$
$$= (0,a+b-4)+H$$

Suspect $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}$

$$\varphi : \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$$
$$(a,b) \mapsto (a \bmod 2,\ b-2a)$$

- φ is a homomorphism
- φ is surjective
- $\ker\varphi = \{(2k,4k) \mid k\in\mathbb{Z}\} = H$

Thus $G/H \cong \operatorname{img}(\varphi) = \mathbb{Z}_2\oplus\mathbb{Z}$.

Q: Where did φ come from?

$$\left.\begin{array}{l} \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \\ (x,y) \mapsto (ax+by,\ cx+dy) \end{array}\right\} \text{ iso} \Leftrightarrow \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$$

!!

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}_2\oplus\mathbb{Z}$$
$$(2,4) \mapsto (2,0)$$
$$(x,y) \longmapsto (x \bmod 2,\ y-2x)$$

Ex: $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, $H = \langle (3,3,3) \rangle$

$G/H \cong ?$

$$\begin{array}{ccccc} & \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} & & & \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \rightarrow & \mathbb{Z}_3\oplus\mathbb{Z}\oplus\mathbb{Z} \\ (3,3,3) & \longmapsto & (3,0,0) & & \\ (x,y,z) & & \longmapsto & & (x, y-x, z-x) \end{array}$$

For finite groups the picture is more complicated...

Ex: $G = \mathbb{Z}_2 \oplus \mathbb{Z}_3$, $H = \langle (1,1) \rangle$

$$\begin{array}{ccccc} & \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} & & & \\ \mathbb{Z}_2\oplus\mathbb{Z}_3 & \longrightarrow & \mathbb{Z}_2\oplus\mathbb{Z}_3 & \rightarrow & \mathbb{Z}_3 \\ (1,1) & \longmapsto & (1,0) & & \\ (x,y) & & \longmapsto & & y-x \end{array}$$

So $G/H \cong \mathbb{Z}_3$ it appears...
However $H = \langle (1,1) \rangle = G$!! So $G/H \cong \{e\}$.

Q: What went wrong?
It's not a group homomorphism!

Ex: $G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$, $H = \langle (2,2) \rangle$

Caution: $G/H \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_3$

In fact $|G/H| = |G| / |H| = 24/6 = 4$

What is G/H isomorphic to?

Cosets?

$(0,0)+H = \{(0,0),(2,2),(0,4),(2,0),(0,2),(2,4)\}$
$(1,0)+H = \{(1,0),(3,2),(1,4),(3,0),(1,2),(3,4)\}$
$(0,1)+H = \{(0,1),(2,3),(0,5),(2,1),(0,3),(2,5)\}$
$(1,1)+H = \{(1,1),(3,3),(1,5),(3,1),(1,3),(3,5)\}$

$((1,0)+H) + ((0,1)+H) = (1,1)+H$
$((1,0)+H) + ((1,0)+H) = (2,0)+H = (0,0)+H$
$((0,1)+H) + ((0,1)+H) = (0,2)+H = (0,0)+H$
$((1,1)+H) + ((1,1)+H) = (2,2)+H = (0,0)+H$

All elements have order 2! $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$