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Free Abelian Groups
   " linear algebra with groups"
Def: Let G be an abolion group w/ brinary operation +
 We say G is a free abelian group if there is a subset X = G such that every a ∈ G has a unique
    expression of the form
   a = n_1 x_1 + n_2 x_2 + ... + n_r x_r  where n_5 \in \mathbb{N}.

We call X a basis for G.

x_j \in X
Ex: G = 1/2 Me is a free abelian group w/ basis
  X = { (1,0), (0,1) }
"Proof": QeG a=(m,n)
              \alpha = \underline{M}(1,0) + \underline{M}(0,1)
Ex: G= 76 8 M also has basis X= { (1,0), (1,1)}
          (m_1 n) = c(1,0) + d(1,1)
             m = c+d?
n = d
                         d=n, c= m-d = m-n
          (m,n) = (m-n)(1,0) + n(1,1)
           (still need to show this is unique)
Ex: G= 760 7/2. The set X= {(2,0), (0,2) }
  is not a basis!
         c(2,0) + d(0,2) = (2c,2d)
   Only getting pairs of even integers!!
 Not all groups are free abelian groups !
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Ex: G = 1/2 0 1/2 Claim G >5 not a free abelien group

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Proof of claim: Assume X is a basis for G,
(1,0) = n_1 x_1 + n_2 x_2 + ... + n_r x_r
for some n_j \in \mathbb{N} and x_j \in X.

urigueness.
 But
         3(40) = (3,0) = (1,0)
      (1,0) = 3(1,0) = 3(n,x,+n,x,+ + ...+n,xr)
                  (1,0) = 3 n, x, +3n2x2 + ... +3n+x+
          Couldn't have been a basis ...
      Thus this is a contradiction and there is no basis!
Theorem: If G is a free abelian arrows w/ basis

X (finite) | X = + , then G = 7/4
            Write X = {x1,x2,...,x1}.
  Define a map

\begin{array}{ccc}
P: \mathcal{K}^r & \longrightarrow G \\
(n_1, n_2, ..., n_r) & \longmapsto & \sum_{j=1}^r n_j x_j
\end{array}

   Clara: This is an isomorphism's
        (n_1, n_2, \dots, n_r) + (m_1, m_2, \dots, m_r) = (n_1 + m_1, n_2 + m_2, \dots, n_r + m_r)
           \sum_{j=1}^{r} n_j x_j + \sum_{j=1}^{r} m_j x_j = \sum_{j=1}^{r} (n_j m_j) x_j
                 = n; x; + m; x; =
             Thus y is a homomorphism
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:. 3(2G) = 27/ = 27/ \ 27/ \ \D ... \ 27/. 9(26) = 27/ because 29 158. G → 7/2 -> 7/2/27/2 1st isomorphism theorem 6/26 2 7/2/27/2. 7/27/ = 1/0/10 ... @ 1/2 = 1/2 @ 1/2 @ ... @ 1/2 27662760.D176 -- many copies 2 many elements 2 = 6/261 Summary: if X = 0 basis and |X|=r, thun $|G/2G| = 2^r$ If Y is another basis and IYI=s, then · 16/261 = 2 23 = 2" ⇒ S=r Definition: The rank of a free group & 13 Theorem: Let G be a free abolian group with basis X.

and H be another abolian group.

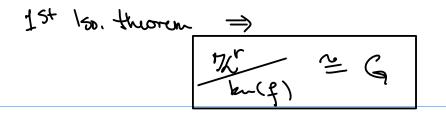
Then any function $f: X \to H$ extends uniquely

to a group homomorphism , g · C → H ie f(x)=f(x) for xex. Proof: (Sketch)

Any element ce EG must be of the form a=n,x,+n2x2+...+n,x, for some x,5-,x,eX and n, ..., n, e 7h f(a) = f(n,x,+h,2x2+ ...+n,xr) = f(n,x,)+f(n2x2)+..+g(n,x) = $\eta_1 f(x_1) + \eta_2 f(x_2) + ... + \eta_r f(x_r)$ f(a) = n, fo(x,) + n2fo(x2) + ... + nr fo(xr) Try this at home: prove that f is well-defend end a homomorphism. Quick note on well-defined: make sure if a E G Mun 7! b EH with f(a) = b. Corollary: Any finishly generated abolion group G Proof:

Let {9,,92,...,9r} to be a set of

generators of G. Defrue a homomorphism ing (g)= & f(n,,,nr) (n,,-,nr) ETX & er = (0,0,0,...,1) = { \frac{1}{2} n_1 \, \frac{1}{2} \ ! Surjective = < 29,,..,9,3 = 6



Theorem: If H < G and G is a free group

Then H is also a free group. Furthermore we

can find a basis {x₁, ..., x_r} of G such

that there are integers a₁, a₂, ..., a_s (54 r)

w/ a₁ | a₂ | a₃ | ... | a_s Sotisfying

2 a₁x₁, ..., a₅x₅x₅ is a basis of H.

EX: G=76, H=376, Bosis for G 15 {1} Basis for H is 23.1}.

