Quotient Rings

Recall that the kernel of a ring homomorphism $\varphi: R \to S$ is an ideal of R. Just like the case of normal subgroups, which are characterized by being kernels, ideals are also always kernels of some homomorphism.

Def: Let $I \subseteq R$ be an ideal. The quotient ring of R by I is the set of cosets

$$R/I = \{ r + I \mid r \in R \}$$

We already know this is a group with addition coming from $+$ on R. Something stronger is true:

Theorem: The Abelian group R/I is a ring with product $\cdot$ given by

$$(r+I)\cdot(s+I) = rs + I.$$

The quotient ring comes equipped with a natural homomorphism, called the quotient map

$$q: R \to R/I, \quad \varphi(r) = r + I$$

Theorem: Let I be an ideal of R and $q: R \to R/I$ the quotient map. Then q is a homomorphism of rings and $\ker(q) = I$.

Ex: Let $R = \mathbb{Z}$ and $I = n\mathbb{Z} = \{ kn \mid k \in \mathbb{Z} \}$. Then I is an ideal of R and the quotient ring $R/I = \mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$.

<u>Ex</u>: $R = \mathbb{Q}[x]$, $I = \{ f(x) \in R \mid f(\sqrt{2}) = 0 \}$.
$$= \{ g(x)(x^2-2) \mid g(x) \in \mathbb{Q}[x] \}$$

Then I is an ideal and
$$R/I = \{ (a+bx) + I \mid a, b \in \mathbb{Q} \}$$

$$[(a+bx)+I] \cdot [(c+dx)+I] = (ac + (ad+bc)x + bdx^2) + I$$
$$= (ac + 2bd + (ad+bc)x) + I$$

<u>Fundamental Homomorphism Theorem</u>:

Let I be an ideal of a ring R and suppose $\varphi : R \to S$ is a ring homomorphism. If $I \subseteq \ker(\varphi)$, then φ <u>descends to the quotient</u> to a homomorphism $\overline{\varphi} : R/I \to S$ satisfying
$$\overline{\varphi}(r+I) = \varphi(r)$$

Commutative diagram:
$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "q"'] & \arrow[d, phantom, "\circlearrowright"] & S \\ & R/I \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

<u>Prop</u>: $\ker(\overline{\varphi}) = \{ r + I \mid r \in \ker(\varphi) \} = \ker(\varphi)/I$.
In particular, if $I = \ker(\varphi)$ then $\overline{\varphi}$ is a monomorphism
$$\overline{\varphi} : R/\ker(\varphi) \to S$$

Ex: Consider the evaluation homomorphism

$$\phi_{\sqrt{2}} : \mathbb{Q}[x] \to \mathbb{R}$$

The kernel of $\phi_{\sqrt{2}}$ is

$$I = \ker(\phi_{\sqrt{2}}) = \{ f(x) \in \mathbb{Q}[x] \mid f(\sqrt{2}) = 0 \}$$
$$= \{ (x^2-2)g(x) \mid g(x) \in \mathbb{Q}[x] \}$$

Thus $\mathbb{Q}[x]/I \cong \operatorname{img}(\phi_{\sqrt{2}}) = \{ a+\sqrt{2}b \mid a,b \in \mathbb{Q} \}$.

which is a field.