Prime and Maximal Ideals

R ring, $A \subseteq R$ subset.

Def: The ideal of R generated by A is the smallest ideal of R containing A.

Notation: $\langle A \rangle$ and if $A = \{a_1, \dots, a_r\}$ we write $\langle a_1, \dots, a_r \rangle$

An ideal of the form $\langle a \rangle$ is called principal.

Prop: Let R be a ring and $a \in R$. Then

$$\langle a \rangle = \{ r_1 a s_1 + \dots + r_n a s_n \mid r_j, s_j \in R \}$$

and if R is commutative,

$$\langle a \rangle = \{ ar \mid r \in R \}$$

Ex: $R = \mathbb{Z}$, $\langle 3 \rangle = \{ 3k \mid k \in \mathbb{Z} \} = 3\mathbb{Z}$

Ex: $R = M_2(\mathbb{C})$, $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$\langle a \rangle = M_2(\mathbb{C}).$$

Ex: $R = \mathbb{Z}[x]$, $\langle 2, x \rangle = \{ 2f(x) + x\,g(x) \mid f(x), g(x) \in R \}$

Big idea: properties of the ideal translate to properties of the quotient ring!

Def: An ideal $I \subseteq R$ is maximal if $I \neq R$ and the only ideals of I containing I are I and R

Ex: The maximal ideals of $\mathbb{C}[x]$ are $\langle x-\lambda \rangle$ for $\lambda \in \mathbb{C}$.

Ex: $\langle x \rangle$ is not a maximal ideal of $\mathbb{Z}[x]$

because it is contained in the maximal ideal $\langle x, 2 \rangle$

Let R be a commutative ring.

Theorem: $I \subseteq R$ is maximal $\Leftrightarrow$ R/I is a field

Proof: Assume I is maximal, and let $a \notin I$. Then

$$J = I + \langle a \rangle = \{x + ay \mid x \in I, y \in R\}$$

is an ideal of R containing I and $I \neq J$ so $J = R$. It follows $1 \in J$ and therefore

$$1 = x + ay \text{ for some } x \in I \text{ and } y \in R.$$

Hence

$$(a+I)(y+I) = ay + I = ay + x + I = 1 + I$$

Thus every nonzero element of R/I is a unit, so R/I is a field.

Conversely, suppose R/I is a field. Then if J is an ideal of R containing I properly, then $\exists\, a \in J \setminus I$. However then $\exists\, b \in R$ with $(a+I)(b+I) = 1+I$ so $ab-1 \in I$. Thus $1 \in J$ so $J = R$

$\square$

Ex: $\mathbb{Z}[x]/\langle x, 2\rangle \cong \mathbb{Z}_2$ field

Ex: $\mathbb{Q}[x]/\langle x^2-1\rangle \cong \mathbb{Q}\times\mathbb{Q}$ not a field

Ex: $\mathbb{Q}[x]/\langle x^2+1\rangle \cong \mathbb{Q}[i]$ field

Ex: $\mathbb{Q}[x]/\langle x^2+x+1\rangle \cong \mathbb{F}_4$ field

Def: Let R be a commutative ring and $I \subseteq R$ an ideal. Then I is prime if

$$ab \in I \Rightarrow a \in I \text{ or } b \in I \quad \forall\, a, b \in R$$

Ex: Every maximal ideal is prime

Ex: $\langle x \rangle \subseteq \mathbb{Z}[x]$ is prime but not maximal.

Theorem: $I \subseteq R$ is prime $\Leftrightarrow$ R/I is an integral domain

Ex: $R = \mathbb{Z}[x]$, $I = \langle x \rangle$, $R/I \cong \mathbb{Z}$

Ideal correspondence:

$$\begin{array}{ccc}
\left(\text{ideals}\right) & & \left(\text{quotient rings}\right) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\left(\text{prime ideals}\right) & \longleftrightarrow & \left(\text{quotient domains}\right) \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
\left(\text{maximal ideals}\right) & & \left(\text{quotient fields}\right)
\end{array}$$