Welcome to MATH 407   Abstract Algebra

- Webpage on Canvas
- Lulu's discord
- Syllabus – read it!!

Standard week: • weekly homework(s)
• quizzes

Exams – two of them
+ final exam (comprehensive)

- First homework due Friday @ midnight

Plan for this week:
– review of complex #'s
– review of sets, relations, functions
– intro to groups + binary operations

Sets, Relations, Functions

A set is (informally) a collection of objects (or elements)

Ex: the set of playing card suits is $\{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$

Ex: $\{1,2,3\} = \{1,3,2\} = \{1,2,3,3\}$

Writing $A = \{\heartsuit, 3, \text{☺}, \text{pizza}\}$ is list notation

Writing $B = \{x \mid x \text{ is a positive integer}\}$ is set builder notation

Building new sets from old ones

$A = \{1, ☺, \text{pizza}\}$, $\quad B = \{♡, ☺, ♣, 3\}$

<u>union</u>: $A \cup B = \{1, ☺, \text{pizza}, ♡, ♣, 3\}$

<u>intersection</u>: $A \cap B = \{☺\}$ ← singleton set

cartesian products

$A \times B = \{(1,♡), (1,☺), (1,♣), (1,3), (☺,♡), (☺,☺), \ldots\}$

↑ has $3 \times 4 = 12$ elements total.

<u>Notation</u>: $x \in A$ means "x is an element of A"

$A \subseteq B$ means "A is a subset of B"

<u>Def</u>: $\underbrace{A \subseteq B}$ (A is a subset of B) means that every element of A is also an element of B.

<u>Def</u>: ~~The~~ the <u>power set</u> of A is the set of <u>all</u> subsets of A.

Notation: $\mathcal{P}(A)$ or 2^A

<u>Ex</u>: $A = \{1, 2\}$, $\quad \mathcal{P}(A) = \{\{1\}, \{2\}, \{1,2\}, \emptyset\}$

$|A| = 2$ $\quad |\mathcal{P}(A)| = 2^2 = 4$

The cardinality of $\mathcal{P}(A)$ is $2^{|A|}$.

↑ ???

Use $|B|$ to denote the cardinality of B

$$|\mathcal{P}(A)| = 2^{|A|}$$

Relations:

Let A, B be sets.

Def: A relation R from A to B is a subset of $A \times B$.

Ex: $A = \{1,2,3\}$, $B = \{a,b,c\}$

Then $R_0 = \{(1,a),(3,c)\}$ is a relation from A to B.

$\hat{R}_1 = \{(1,b)\}$ is the Rana relation

Notation: we write xR_0y to mean $(x,y) \in R_0$

Previous example: $1R_0a$ is true because $(1,a) \in R_0$
$2R_0a$ is false because $(2,a) \notin R_0$

different relations = # subsets of $A \times B$
$= |\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{3 \cdot 3} = 512$

Def: A relation from A to A is called a relation on A

Properties of Relations: Let R be a relation on A.

- **reflexivity** – a relation R is reflexive if xRx for all x
- **symmetry** – a relation R is symmetric if $xRy \Leftrightarrow yRx$ for all x, y
- **transitive** – a relation R is transitive if $(xRy \text{ and } yRz) \Rightarrow xRz$

A relation which is reflexive, symmetric, and transitive is an <u>equivalence relation</u>.

<u>Ex</u>: Let $P = \{x \mid x \text{ is a living person}\}$

Define a relation R on P by

$R = \{(x,y) \mid x = y \text{ or } x \text{ is a sibling of } y\} \subseteq P \times P$

Define R by saying xRy iff $x=y$ or x is a sibling of y.

Mary Kate Olsen R Ashley Olsen is true

Ashley Olsen R Mary Kate Olsen is true

Bill Nye R Niel DeGrasse Tyson is false

Reflexive? is xRx true for all x? Yes!

Symmetric? Yes! If x is y's bro/sis then y is x's bro/sis

Transitive? Yes! If x is y's bro/sis and y is z's bro/sis then x is z's bro/sis.

This is an equivalence relation!

<u>Ex</u>: Define a relation R on $\mathbb{R}$ by

xRy iff $x^2 - xy = 0$

Lulu: $1^2 - 1 \cdot 1 = 0$ so $1R1$

David: $0^2 - 0 \cdot 4 = 0$ so $0R4$

<u>Reflexive</u>? $x^2 - xx = 0$ so xRx for all x! Yes!

Symmetric? $0R4$ because $0^2 - 0 \cdot 4 = 0$ but $4^2 - 4 \cdot 0 = 16$

No! so $4R0$ is false

David's observation: $xRy \iff x=0$ or $x=y$

S'pose xRy and yRz

$x=y$ or $x=0$ $\quad$ $y=z$ or $y=0$

Is xRz true? • If $x=0$, yes! xRz

• Otherwise $x=y$ and since yRz we get xRz

So yes! R is transitive.