Quotient Groups

Let G, H be groups with $H < G$.

Notation: $G/H = \{aH \mid a \in G\}$ is the set of left cosets

In many situations, G/H also has the structure of a group!

binary operation: $(aH) * (bH) = abH$

CAUTION: this may not be well defined!

For this operation to be well-defined, since $eH = hH$ for all $h \in H$

$aH = eH * aH = hH * aH = haH$ for all $a \in G, h \in H$.

Now $aH = haH \Leftrightarrow H = a^{-1}haH \Leftrightarrow a^{-1}ha \in H$.

__Def__: A subgroup $H < G$ is called __normal__ if $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$.

Notation: $H \trianglelefteq G$ means H is a normal subgroup of G.

__Prop__: The following are equivalent.

(A) $H \trianglelefteq G$

(B) $gH = Hg$ for all $g \in G$.

(C) $g^{-1}Hg = H$ for all $g \in G$

(D) $g^{-1}Hg \leq H$ for all $g \in G$

__Example__: If G is Abelian, every subgroup of G is normal.

__Example__: $H = \{(123), (132), e\}$ is a normal subgroup of S_3

Theorem: Let $H \trianglelefteq G$. Then G/H is a group with

- binary operation $(aH)(bH) = abH$
- identity $eH = H$
- inverse $(aH)^{-1} = a^{-1}H$

Def: Let $H \trianglelefteq G$. The group G/H is called a <u>quotient group</u> or the quotient of the group G by H.
The function $G \to G/H$, $g \mapsto gH$ is a group homomorphism called the <u>quotient map</u>.

Ex: $G = \mathbb{Z}$, $H = \langle n \rangle = n\mathbb{Z}$.

$$G/H = \{ n\mathbb{Z}, 1 + n\mathbb{Z}, 2 + n\mathbb{Z}, \dots, (n-1) + n\mathbb{Z} \}$$

$$(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a+b) + n\mathbb{Z} = (a +_n b) + n\mathbb{Z}$$

In fact $G/H = \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

Ex: $G = GL_2(\mathbb{C})$, $H = \{ aI \mid a \in \mathbb{C} \}$. $H \trianglelefteq G$.

G/H is called $PGL_n(\mathbb{C})$.

Normal subgroups are kernels!

Theorem: $H \trianglelefteq G \iff H = \ker(\varphi)$ for some hom. $\varphi : G \to \tilde{G}$

Proof:

Let $H \trianglelefteq G$. Then $H = \ker(\varphi)$ for $\varphi : G \to G/H$ the quotient map. Conversely, if $H = \ker(\varphi)$ for some $\varphi : G \to \tilde{G}$, then for all $g \in G$ and $h \in H$

$$\varphi(g^{-1}hg) = \varphi(g)^{-1}\varphi(h)\varphi(g) = \varphi(g)^{-1}\tilde{e}\varphi(g) = \tilde{e}.$$

Therefore H is normal

$\square$

<u>Ex</u>: $G = \mathbb{Z} \oplus \mathbb{Z}$, $H = \langle (0,3) \rangle = 0\mathbb{Z} \oplus 3\mathbb{Z}$

$$G/H = \{(a,b)+H \mid a,b \in \mathbb{Z}\}$$
$$= \{(a,b)+H \mid a,b \in \mathbb{Z},\ 0 \leq b < 3\}$$

$$G/H = (\mathbb{Z} \oplus \mathbb{Z})/(0\mathbb{Z} \oplus 3\mathbb{Z})$$
$$\cong (\mathbb{Z}/0\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$$
$$\cong \mathbb{Z} \oplus \mathbb{Z}_3.$$

<u>First Isomorphism Theorem</u>: Let $\varphi : G \to \tilde{G}$ be a group homomorphism and $H = \ker(\varphi)$. Then

$$\mathrm{img}(\varphi) \cong G/H$$

<u>Proof</u>:

Define $\psi : G/H \to \mathrm{img}(\varphi)$ by $\psi(aH) = \varphi(a)$.

If $aH = bH$, then $a^{-1}bH = H$ so $a^{-1}b \in H$.

This means $\varphi(a^{-1}b) = \tilde{e}$ and therefore $\varphi(a)^{-1}\varphi(b) = \tilde{e}$

so that $\psi(aH) = \psi(bH)$. Thus the function is well-defined.

Also $\psi((aH)(bH)) = \psi(abH) = \varphi(ab) = \varphi(a)\varphi(b) = \psi(aH)\psi(bH)$

Thus ψ is a homomorphism

If $\psi(aH) = \tilde{e}$, then $\varphi(a) = \tilde{e}$ so $a \in H$ and $aH = H$.

Thus ψ is injective. Clearly ψ is surjective, so ψ is an isomorphism.

$\square$

Ex: $(\mathbb{Z}\oplus\mathbb{Z})/\langle(2,0)\rangle \cong ?$

$$\varphi : \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}_2\oplus\mathbb{Z}$$
$$(a,b) \longmapsto (a,b)$$

$\ker(\varphi) = \langle(2,0)\rangle$ and $\operatorname{img}(\varphi) = \mathbb{Z}_2\oplus\mathbb{Z}$

and therefore 1st Iso. Thm. $\Rightarrow (\mathbb{Z}+\mathbb{Z})/\langle(2,0)\rangle \cong \mathbb{Z}_2\oplus\mathbb{Z}$

Ex: $(\mathbb{Z}\oplus\mathbb{Z})/\langle(2,10)\rangle \cong ?$

$$\begin{tikzcd}
\langle(2,10)\rangle \subseteq \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "{\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}}"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z} \\
 & \langle(2,0)\rangle \arrow[u, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

$$\varphi : \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}_2$$
$$(a,b) \longmapsto (a, b-5a)$$

is surjective with kernel $\langle(2,10)\rangle$.

Ex: $(\mathbb{Z}_4\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8)/\langle(1,2,4)\rangle \cong ?$

$$\begin{tikzcd}
\mathbb{Z}_4\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8 \arrow[r, "{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}"] & \mathbb{Z}_4\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8 \arrow[r] & \mathbb{Z}_4\oplus\mathbb{Z}_8 \\
\langle(1,2,4)\rangle \arrow[u, phantom, sloped, "\subseteq"] & \langle(1,0,0)\rangle \arrow[u, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

$$\varphi : \mathbb{Z}_4\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8 \to \mathbb{Z}_4\oplus\mathbb{Z}_8$$
$$(a,b,c) \longmapsto (b-2a, c-4a)$$

Ex: $(\mathbb{Z}_2\oplus\mathbb{Z}_3)/\langle(1,1)\rangle \cong ?$

WRONG:

$$\begin{tikzcd}
\mathbb{Z}_2\oplus\mathbb{Z}_3 \arrow[r, "{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}"] & \mathbb{Z}_2\oplus\mathbb{Z}_3 \arrow[r] & \mathbb{Z}_3. \\
\langle(1,1)\rangle \arrow[u, phantom, sloped, "\subseteq"] & \langle(1,0)\rangle \arrow[u, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

because $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ not a hom!

Instead $\langle(1,1)\rangle$ has 6 elements... $\Big\}$ $\dfrac{\mathbb{Z}_2\oplus\mathbb{Z}_3}{\langle(1,1)\rangle} \cong \{e\}$.

$\mathbb{Z}_2\oplus\mathbb{Z}_3$ has 6 elements.

<u>Ex</u>: $(\mathbb{Z}_4\oplus\mathbb{Z}_8)/\langle(1,2)\rangle \cong ?$

$$\mathbb{Z}_4\oplus\mathbb{Z}_8 \longrightarrow \mathbb{Z}_8.$$
$$(a,b) \longmapsto b-2a$$ is surjective w/ kernel $\langle(1,2)\rangle$.

<u>Def</u>: A group with no nontrivial proper normal subgroups is called simple.

<u>Thm</u>: A_n is simple for $n \geq 5$.

<u>Thm</u>: $\mathbb{Z}_p$ is simple for p prime

<u>Big Idea</u>: Break groups down into simple groups...

- Choose $H \trianglelefteq G$ to be as big as possible
  Then G/H will be simple.
- Now choose $H_1 \trianglelefteq H$ as big as possible
  Then H/H_1 will be simple.

  $H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq H \trianglelefteq G$ is a <u>composition series for G</u>.

Special normal subgroups:

- $Z(G) = \{a \in G \mid ab = ba \ \forall\, b \in G\}$ <u>center of G</u>
- $[G,G] = \{aba^{-1}b^{-1} \mid a,b \in G\}$ <u>commutator of G</u>