__Last Time:__ free abelian groups!

- a free abelian group = abelian group w/ a basis!

__Ex:__ $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \ldots$

__Ex:__ $z = e^{i\pi\sqrt{2}}$ $\qquad G = \{z^n \mid n \in \mathbb{Z}\}$.

- because $\sqrt{2}$ is irrational $z^n = 1 \Leftrightarrow n = 0$.

Claim: G is a free abelian group w/ basis $\{z\}$.
To prove this NTS if $w \in G$, then $\exists!\ n \in \mathbb{Z}$ $\ni$
$w = \underline{z^n}$

(additive $a = n_1x_1 + \ldots + n_rx_r$)
(multiplicative $a = x_1^{n_1}x_2^{n_2}\ldots x_r^{n_r}$)

Clearly $w = z^n$ for some $n \in \mathbb{Z}$ by def. of G
if $w = z^m$, then $z^n = z^m \Rightarrow z^{n-m} = 1 \Rightarrow n = m$.

__Theorem:__ If G is a finitely generated free abelian group and $H < G$, then H is also a finitely generated free abelian group. Moreover G has a "compatible basis" $\{x_1, \ldots, x_r\}$ satisfying the property that $\{a_1x_1, \ldots a_sx_s\}$ is a basis for H for some $s \leq r$ and $a_1, \ldots, a_s \in \mathbb{Z}$.

__Ex:__ $G = \mathbb{Z}^3$, $\quad H = \langle (2,4,4), (-6,6,12), (10,-4,-16) \rangle$

$\rightsquigarrow$ __Quest:__ What is G/H?

__Idea:__ Find this "compatible basis" for G.

Find $\{x_1, x_2, x_3\} \subseteq G$ a basis of G so that $\{a_1x_1, a_2x_2, a_3x_3\}$ is a basis for H.

Smith normal form:

Def: A matrix with integer entries is in Smith normal form if it is of the form

$$\begin{bmatrix} a_1 & & & & & & & \\ & a_2 & & & & \Large 0 & & \\ & & a_3 & & & & & \\ & & & \ddots & & & & \\ & \Large 0 & & & a_r & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}$$

where $a_1 | a_2$, $a_2 | a_3$, $\ldots$ $a_{r-1} | a_r$
and $a_j > 0$ $\forall j$.

Ex: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ not in Smith normal form

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ <u>is</u> in Smith normal form

$\begin{bmatrix} 17 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ <u>is</u> in Smith normal form

<u>Theorem</u>: <u>Any</u> integer matrix can be put in Smith normal form via elementary row and column operations

- add a multiple of a row (or column) to another
- swap two rows (or columns)
- multiply a row (or column) by ± 1

Ex:

$$\begin{pmatrix} 2 & 4 & 4 \\ -6 & 6 & 12 \\ 10 & -4 & -16 \end{pmatrix} \xrightarrow{R_1+R_3} \begin{pmatrix} 12 & 0 & -12 \\ -6 & 6 & 12 \\ 10 & -4 & -16 \end{pmatrix} \xrightarrow{C_3+C_1} \begin{pmatrix} 12 & 0 & 0 \\ -6 & 6 & 6 \\ 10 & -4 & -6 \end{pmatrix}$$

$$\xrightarrow{C_1+C_2} \begin{pmatrix} 12 & 0 & 0 \\ 0 & 6 & 6 \\ 6 & -4 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Stuck ...} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 4 \\ -6 & 6 & 12 \\ 10 & -4 & -16 \end{pmatrix} \xrightarrow{C_2-2C_1} \begin{pmatrix} 2 & 0 & 4 \\ -6 & 18 & 12 \\ 10 & -24 & -16 \end{pmatrix} \xrightarrow{C_3-2C_1} \begin{pmatrix} 2 & 0 & 0 \\ -6 & 18 & 24 \\ 10 & -24 & -36 \end{pmatrix}$$

$$\xrightarrow{R_2+3R_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 18 & 24 \\ 10 & -24 & -36 \end{pmatrix} \xrightarrow{R_3-5R_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 18 & 24 \\ 0 & -24 & -36 \end{pmatrix}$$

$\gcd(18,24) = 6$

Find m,n so that

$18m + 24n = 6$

$24 - 18 = 6$ ✓

$$\xrightarrow{C_3-C_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 18 & 6 \\ 0 & -24 & -12 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 18 \\ 0 & -12 & -24 \end{pmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 18 \\ 0 & 0 & 12 \end{pmatrix} \xrightarrow{C_3-3C_2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

Find a compatible basis for the subgroup

$H = \langle \underset{R_1}{(2,4,4)}, \underset{R_2}{(-6,6,12)}, \underset{R_3}{(10,-4,-16)} \rangle$ of $\mathbb{Z}^3$

Ans: look @ row operations we did!

$R_2 + 3R_1$, $R_3 - 5R_1$, $R_3 + 2R_2$.

$\langle (2,4,4), (0,18,24), (10,-4,-16) \rangle$

$\langle (2,4,4), (0,18,24), (0,-24,-36) \rangle$

compatible basis map $\langle \underbrace{(2,4,4)}_{2}, \underbrace{(0,18,24)}_{6}, \underbrace{(0,12,12)}_{12} \rangle$

$\langle (1,2,2), (0,3,4), (0,1,1) \rangle$. $\mathbb{Z}^3$

$$G/H \cong \frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{2\mathbb{Z} \oplus 6\mathbb{Z} \oplus 12\mathbb{Z}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{12}.$$

$(1,2,2) \longleftarrow (1,0,0)$

$(0,3,4) \longleftarrow (0,1,0)$

$(0,1,1) \longleftarrow (0,0,1)$

Free group:

The free group w/ two generators x, y, is $F(\{x,y\})$ and consists of words

x, xy, xyx^2, y^2xyx^2y

x^{-1}, xy^{-1}, yxy^{-1}, $y^{-2}xyx^3y^{-3}$

Group operation is concatenation

$$xyx^2y^3 \cdot x^3y^4x = xyx^2y^3x^3y^4x$$

$$x \cdot x^{-1} = xx^{-1} = e \quad \text{(empty word)}$$

$$xyx^2 \cdot x^{-1}yx^3y = xyx^2x^{-1}yx^3y = xyxyx^3y$$

Theorem: If G is any finitely generated group then

$$G \cong F/H \quad \text{where } F \text{ free group}$$
$$H \trianglelefteq F.$$

What's so cool about compatible bases?

$$\begin{array}{ll} G & \{x_1, \dots, x_r\} \text{ basis for } G \\ | & x_1, \dots, x_r \in G \\ H & \{a_1x_1, \dots, a_sx_s\} \text{ basis for } H \end{array}$$

$s \leq r$ $\qquad a_1, \dots, a_s \in \mathbb{Z}$

Ex:

$G = \mathbb{Z}$, $H = 3\mathbb{Z}$

Compatible basis?

$\{1\}$ is a basis for $\mathbb{Z}$

$\{3\}$ is not a basis for $3\mathbb{Z}$

$\mathbb{R}^2$
$\{(1,0),(0,1)\}$
$\{(1,1),(0,1)\}$

Ex: $G = \mathbb{Z} \times \mathbb{Z}$, $H = \langle (2,2) \rangle$

Basis for G? $\{\overset{x_1}{(1,0)}, \overset{x_2}{(0,1)}\}$

NOT a compatible basis!

Basis for G $\{(1,1), (0,1)\}$ is compatible

$\{2(1,1)\}$ is a basis for H.