Math 407 Section 1
Fall 2022
Exam I
February 22, 2022
Time Limit: 75 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam. However, you may use a single, handwritten, one-sided notesheet and a *basic* calculator.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- If the problem asks for a proof, be sure to carefully justify your work, including any theorems from class.

  Note: you may NOT use a theorem or result from class to prove something when it makes the problem entirely trivial. If you are unsure whether a particular theorem or result is allowed, just ask!

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) The group $\mathbb{Q}$ with binary operation $+$ is a cyclic group.

   (b) Every group of order 7 is cyclic.

   (c) If G is non-Abelian and $H \leq G$ then H is also non-Abelian

   (d) Every nontrivial group has a nontrivial proper subgroup.

   (e) If G is a group and $x \in G$ with $x^2 = x$ then x is the identity

2. (10 points)

   (a) State Cayley's Theorem

   (b) Write the definition of a homomorphism from a group G to a group H.

   (c) Give an example of a group of order 6 which is not cyclic.

3. (10 points)

   (a) Write down (up to isomorphism) all Abelian groups of order 64.

   (b) Write down $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6$ in prime divisor form.

   (c) Write down $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6$ in invariant factor form.

4. (10 points) $G = \{(a,b) : a, b \text{ integers}, \ 0 \leq a < 7, \ 0 \leq b < 3\}$
with the associative binary operation

$$(a,b) * (c,d) = (a +_7 (2^b c), b +_3 d).$$

For example

$$(3,2) * (5,1) = (3 +_7 (2^2 5), 2 +_3 1) = (3 +_7 20, 0) = (2,0).$$

(a) Show that G has an identity element

(b) Show each element of G has an inverse

(c) Show that G is not Abelian

(d) Find the order of the element $(1,1)$ in G. Show your work.

5. (10 points) Let G and G' be groups with identities e and e', respectively. Also let $\varphi : G \to G'$ be a group homomorphism. Prove each of the following statements

   (a) $\varphi(e) = e'$

   (b) $\varphi(a^{-1}) = (\varphi(a))^{-1}$

   (c) the kernel of φ

   $$\ker(\varphi) = \{a \in G : \varphi(a) = e'\}$$

   is a subgroup of G

6. (10 points)

   Suppose that G is a finite group (possibly non-Abelian). Show that if φ is a homomorphism

$$\varphi : G \to \mathbb{Z}$$

   then $\varphi(x) = 0$ for all $x \in G$. In other words, the only homomorphism from G to $\mathbb{Z}$ is the trivial one.