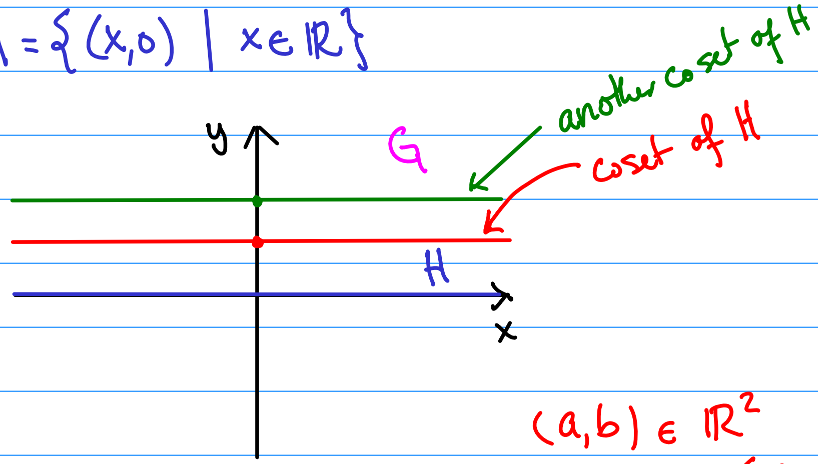
Quotient Group Examples

Ex: $G = \mathbb{R}^2$ w/ binary operation $+$.

$H = \{(x,0) \mid x \in \mathbb{R}\}$



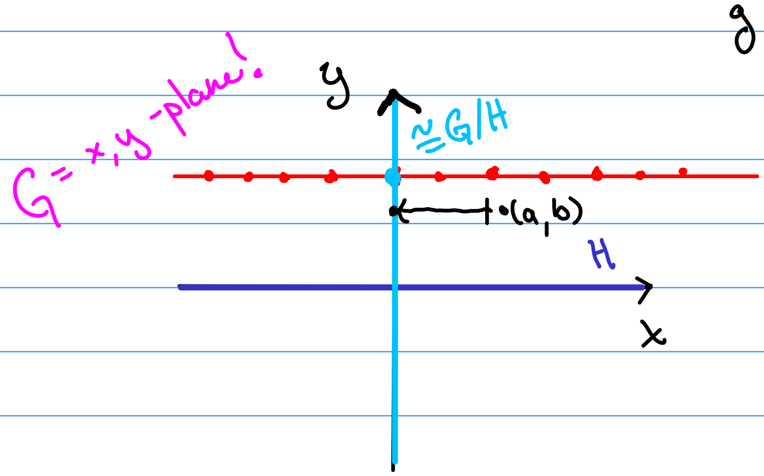
$(a,b) \in \mathbb{R}^2$

$(a,b) + H = \{(a+x, b) \mid x \in \mathbb{R}\}$

Quest: What is G/H?

$(a,b) + H = (c,d) + H \iff b = d$

Def: If H is a normal subgroup of G, the quotient map is the natural homomorphism
$$\pi : G \longrightarrow G/H$$
$$g \longmapsto gH$$



quotient map: $\pi : G \longrightarrow G/H$

$$(a,b) \longmapsto (a,b) + H = \{(a+x, b) \mid x \in \mathbb{R}\}$$
$$= \{(x,b) \mid x \in \mathbb{R}\}$$
$$= (0,b) + H$$

morally speaking: $(a,b) \longmapsto b$

Ex: $G = \mathbb{R}^2$
$H = \{(x,x) \mid x \in \mathbb{R}\}$

- $(1,0) + H = \{(1+x, x) \mid x \in \mathbb{R}\}$
- $(2,1) + H = \{(2+x, 1+x) \mid x \in \mathbb{R}\}$

same!

Prop: If $H \le G$, then $gH = \tilde{g}H \Leftrightarrow \tilde{g}^{-1}g \in H$

$(2,0) + H \ne (1,0) + H$ because $(2,0) - (1,0) \notin H$

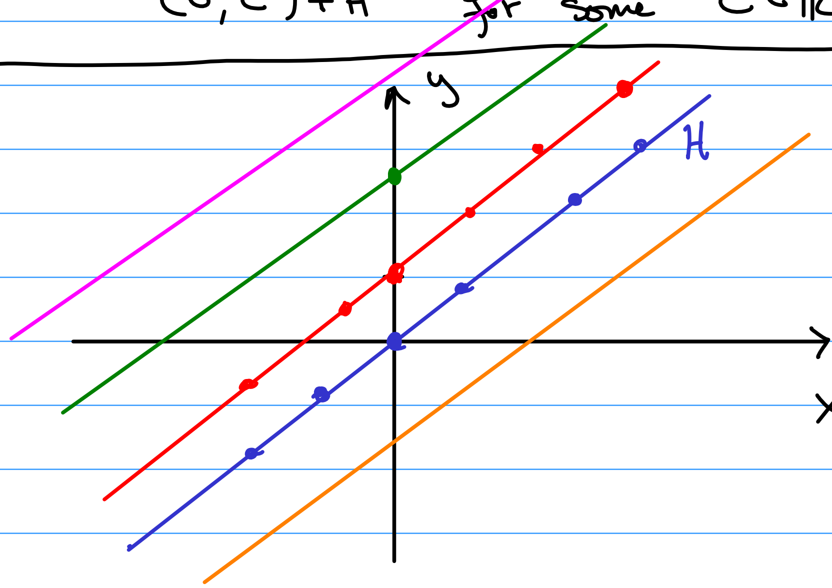
$(2,1) + H = (1,0) + H$ because $(2,1) - (1,0) \in H$

Quest: What are all the elements of G/H?

$(a,b) + H = (c,d) + H \Leftrightarrow (a-c, b-d) \in H = \{(x,x) \mid x \in \mathbb{R}\}$
$\Leftrightarrow a - c = b - d$

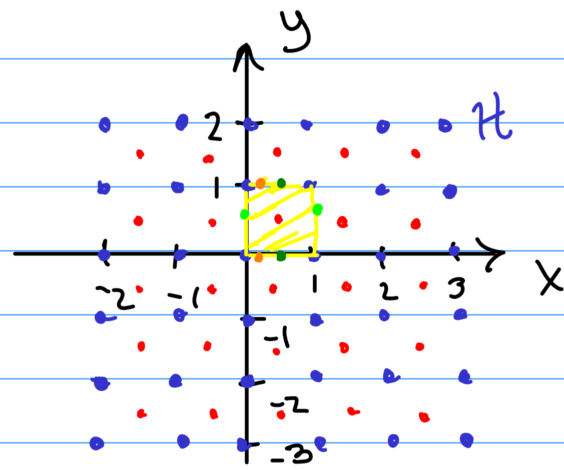
$(a,b) + H = (a,b) - (a,a) + H = (0, b-a) + H$

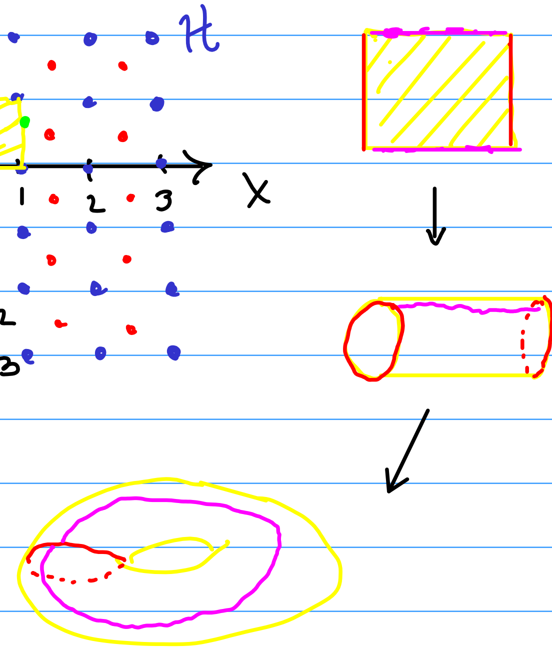
The cosets are all elements of the form $(0,c) + H$ for some $c \in \mathbb{R}$.



Ex: $G = \mathbb{R}^2$

$H = \mathbb{Z} \times \mathbb{Z}$





First, Second, and Third Isomorphism Theorems

Q: How do we create maps from a quotient group to another group?

G group, $H \trianglelefteq G$ $\qquad G/H \longrightarrow G'$???

Idea: Build a homomorphism $\varphi : G \longrightarrow G'$

Theorem: If $\varphi : G \to G'$ is a homomorphism and $H \leq \ker(\varphi)$, then there exists $\varphi' : G/H \to G'$ such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G' \\ G/H \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

ie $\varphi = \varphi' \circ \pi$.

Def: If $H \leq \ker(\varphi)$, we say φ <u>descends to the quotient</u> to the map $\varphi' : G/H \to G'$.

<u>First Isomorphism Theorem</u>: Let $\varphi: G \to G'$ be a group homomorphism and $H = \ker(\varphi)$. Then φ induces an isomorphism $\varphi': G/H \to \operatorname{img}(\varphi)$ satisfying

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & G' \\ G/H \arrow[r, "\varphi'"'] & \operatorname{img}(\varphi) \arrow[u, "\text{IU}"'] \end{tikzcd} \qquad \varphi = \varphi' \circ \pi$$

Punch line: $G/\ker(\varphi) \cong \operatorname{img}(\varphi)$

$$gH \longmapsto \varphi(g)$$

Wait a minute! Is this well-defined???
We already noticed that $gH = \tilde{g}H$ can hold even if $g \neq \tilde{g}$.

No worries mate! If $gH = \tilde{g}H$, then $\tilde{g}^{-1}g \in H = \ker(\varphi)$

so $\varphi(\tilde{g}^{-1}g) = e$

$$\varphi(\tilde{g}^{-1}g) = \varphi(\tilde{g})^{-1}\varphi(g) = e \Rightarrow \varphi(g) = \varphi(\tilde{g})$$

<u>Second Isomorphism Theorem</u>

Prop: If $N \trianglelefteq G$ and $H < G$, then $HN < G$.

Proof: You did this on the exam (hopefully).

<u>Def</u>: Let $H, K \leq G$. The <u>join</u> of H and K is the subgroup generated by HK

$$H \vee K = \langle \{hk \mid h \in H \text{ and } k \in K\} \rangle$$

<u>Prop</u>: Let $H \leq G$ and $N \trianglelefteq G$. Then $HN = H \vee N = NH$

Moreover if $H \trianglelefteq G$ then $HN \trianglelefteq G$.

<u>Proof</u>:

Idea ~ if I can show HN is a subgroup, then

$$HN = \langle HN \rangle = H \vee N.$$

If I can also show $HN = NH$, then we are done w/ first part.

Take $h_1, h_2 \in H$ and $n_1, n_2 \in N$. $h_1 n_1 \in HN$, $h_2 n_2 \in HN$.

I know N is normal.

NTS $h_1 n_1 h_2 n_2 \in HN$ $\qquad$ $h^{-1} n h \in N \;\; \forall \; h \in H, n \in N.$

$$h_1 n_1 h_2 n_2 = \underbrace{\underbrace{h_1}_{\in H}\underbrace{h_2}_{\in H}}_{\in H}\underbrace{\underbrace{h_2^{-1} n_1 h_2}_{\in N}\underbrace{h_2^{-1} n_1 h_2}_{\in N}\underbrace{n_2}_{\in N}}_{\in N} \in HN$$

NTS $(h_2 n_2)^{-1} = n_2^{-1} h_2^{-1} \in HN.$

$$n_2^{-1} h_2^{-1} = \underbrace{h_2^{-1}}_{\in H}\underbrace{h_2 \underbrace{n_2^{-1}}_{\in N} h_2^{-1}}_{\in N} \in HN.$$