Algebraic Extensions

Simple Extensions:

- α algebraic of degree d

$$F(\alpha) = \{ a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{d-1}\alpha^{d-1} \mid a_0, \dots, a_{d-1} \in F \}$$

- α transcendental

$$F(\alpha) = \{ f(\alpha)/g(\alpha) \mid f(x), g(x) \in F[x], \ g \neq 0 \}$$

Observation ~ if α is algebraic, then $F(\alpha)$ is finite dimensional as an F-vector space

__Def__: The __degree__ of an extension field E of F is

$$[E:F] = \dim_F(E)$$

__Theorem__: If $[E:F]$ is finite, then every element $\alpha \in E$ is algebraic over F.

__Def__: An extension field is __algebraic__ if every element $\alpha \in E$ is algebraic over F.
The algebraic closure of F in E is

$$\overline{F}_E = \{ \alpha \in E \mid \alpha \text{ is algebraic over } F \}$$

Theorem: F_E is a field.

There is a largest algebraic extension of a field F, called the algebraic closure.

Def: A field E is called <u>algebraically closed</u> if every polynomial $f(x) \in E[x]$ has a root in E. An algebraically closed algebraic extension E of F is called an <u>algebraic closure</u> of F.

Notation: $\overline{F}$ is the algebraic closure of F.

Theorem: $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic}\}$ is a proper subset of $\mathbb{C}$.

Proof:

$\overline{\mathbb{Q}}$ is countable, but $\mathbb{C}$ is uncountable. $\square$

Theorem: If $\alpha \in E$ is algebraic over F and $\deg \operatorname{irr}(\alpha, F) = d$, then a basis for $F(\alpha)$ as an F-vector space is

$$\{1, \alpha, \alpha^2, \dots, \alpha^{d-1}\}.$$

In particular,

$$[F(\alpha) : F] = \deg \operatorname{irr}(\alpha, F)$$

Ex: If p is prime, $\zeta = e^{2\pi i/p}$

$$\operatorname{irr}(\zeta, \mathbb{Q}) = 1 + x + x^2 + \dots + x^{p-1}$$

and

$$\mathbb{Q}(\zeta) = \{ a_0 + a_1\zeta + a_2\zeta^2 + \dots + a_{p-1}\zeta^{p-1} \mid a_0, \dots, a_{p-1} \in \mathbb{Q} \}$$
$$= \operatorname{span}_{\mathbb{Q}} \{ 1, \zeta, \zeta^2, \dots, \zeta^{p-1} \}$$

Ex: .

$$\mathbb{Q}[\sqrt{\sqrt{3}-1}] = \operatorname{span}_{\mathbb{Q}} \{ 1, \sqrt{\sqrt{3}-1}, \sqrt{3}-1, (\sqrt{3}-1)^{3/2} \}$$

Theorem:

$$[K:F] = [K:E][E:F]$$

Reminds us of subgroups!