Def: A subgroup $H \leq G$ is **normal** if $aH = Ha$ for all $a \in G$.

In this case, we write $H \trianglelefteq G$.

Ex: If G is abelian, then every subgroup of G is normal.

Ex: G = group of symmetries of the square. (eg. D_4)

$$G = \{R_0, R_{\pi/2}, R_\pi, R_{3\pi/2}, S_0, S_{\pi/2}, S_\pi, S_{3\pi/2}\}$$

$H = \{R_0, R_{\pi/2}, R_\pi, R_{3\pi/2}\}$ is normal

$K = \{R_0, S_0\}$ is not normal.

Theorem: Suppose $\varphi: G \to G'$ is a hom. Then $H = \ker(\varphi)$ is a normal subgroup of G.

Proof:

NTS $aH = Ha \quad \forall a \in G$

Choose $h \in H$

$$\varphi(ah) = \varphi(a)\varphi(h) = \varphi(a)e = \varphi(a)$$

$$\begin{aligned}\varphi(aha^{-1}) &= \varphi(a)\varphi(h)\varphi(a^{-1}) \\ &= \varphi(a)\varphi(h)\varphi(a)^{-1} \\ &= \varphi(a)\, e\, \varphi(a)^{-1} = \varphi(a)\varphi(a)^{-1} = e.\end{aligned}$$

$$\varphi(aha^{-1}) = e \implies aha^{-1} \in \ker(\varphi)$$
$$\implies aha^{-1} \in H.$$

$aha^{-1} = \tilde{h}$ for some $\tilde{h} \in H$.

$ah = \tilde{h}a \in Ha$ $\qquad$ $\underline{aH \subseteq Ha}$.

If we start with ha, then

$\varphi(a^{-1}ha) = e$ so $a^{-1}ha \in H$ so $a^{-1}ha = \tilde{h}$

so $ha = a\tilde{h}$ so $Ha \subseteq aH$

$\therefore$ $aH = Ha$

$\square$

Ex: Consider

$$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^{\times}$$
$$A \longmapsto \det(A)$$

Since $\det(AB) = \det(A)\det(B)$ so
$\det$ is a group homomorphism!

$$\ker(\det) = \{ A \in GL_n(\mathbb{R}) \mid \det(A) = 1 \} = SL_n(\mathbb{R})$$

This shows $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Theorem: Let $H \leq G$. The following are equivalent.

a) H is a normal subgroup of G
b) $gHg^{-1} = H \quad \forall g \in G$
c) $gHg^{-1} \subseteq H \quad \forall g \in G$
d) $ghg^{-1} \in H \quad \forall g \in G \ \forall h \in H$
e) $\forall g \in G \ \exists \tilde{g} \in G$ with $gH = H\tilde{g}$.

Proof: a) $\Rightarrow$ b) $\quad gH = Hg \ \forall g \in G \Rightarrow gHg^{-1} = H \ \forall g \in G$

b) $\Rightarrow$ c) $\quad gHg^{-1} = H \Rightarrow gHg^{-1} \subseteq H$.

c) $\Rightarrow$ d) $\quad$ duh.

d) $\Rightarrow$ a) $\quad$ Assume that $ghg^{-1} \in H \ \forall g \in G$.

Then $\varphi_g : H \to H$, $h \mapsto ghg^{-1}$ is well-defined.

Note φ_g and $\varphi_{g^{-1}}$ are inverse functions.

φ_g is bijective! $\therefore$ $H = gHg^{-1} \Rightarrow Hg = gH$.

a) $\Rightarrow$ e) Assume $gH = Hg$ $\forall g \in G$.

NTS $\forall g \in G$ $\exists \tilde{g} \in G$ w/ $gH = H\tilde{g}$.

Obvious, just take $\tilde{g} = g$.

e) $\Rightarrow$ a) Assume $\forall g \in G$ $\exists \tilde{g} \in G$ w/ $gH = H\tilde{g}$.

NTS $\cancel{\tilde{g} = g}$ $gH = Hg$

I know $g = ge \in gH = H\tilde{g}$ so $g = h\tilde{g}$ for some $h \in H$.

but then $\tilde{g} = h^{-1}g$, so

$b = ah^{-1}$

$$H\tilde{g} = \{a\tilde{g} \mid a \in H\} = \{ah^{-1}g \mid a \in H\} = \{bg \mid b \in H\} = Hg.$$

Thus $gH = H\tilde{g} = Hg$. $\square$

Quotient Groups

Book calls these "factor groups".

$X, Y \subseteq G$ (not nec. subgroups)

Notation $XY = \{xy \mid x \in X, y \in Y\}$.

Proposition: $H \leq G$ is normal $\Longleftrightarrow$

$$(aH)(bH) = abH \qquad \forall a, b \in G.$$

Proof: Try at home! $\square$

Consequently:

Theorem: Let $H \trianglelefteq G$. Then the set of left cosets

$$G/H = \{aH \mid a \in G\}$$

is a group w/

- binary operator $(aH)(bH) = abH$
- identity eH
- inverse of $aH = a^{-1}H$

<u>Definition</u>: The group G/H is called a <u>factor group</u> or <u>quotient group</u>, or the <u>quotient of G by H</u>.

<u>Ex</u>: $G = \mathbb{Z}$. $H = \langle m \rangle = m\mathbb{Z}$

$G/H = ???$

$$\begin{aligned}
0 + m\mathbb{Z} &= \{0 + mk \mid k \in \mathbb{Z}\} \\
1 + m\mathbb{Z} &= \{1 + mk \mid k \in \mathbb{Z}\} \\
2 + m\mathbb{Z} &= \{2 + mk \mid k \in \mathbb{Z}\} \\
&\vdots \\
(m-1) + m\mathbb{Z} &= \{m-1 + mk \mid k \in \mathbb{Z}\} \\
m + m\mathbb{Z} &= \{m(k+1) \mid k \in \mathbb{Z}\} \\
(m+1) + m\mathbb{Z} &= \{1 + m(k+1) \mid k \in \mathbb{Z}\} \\
&\vdots
\end{aligned}$$

same

same

$G/H = \{0 + m\mathbb{Z}, 1 + m\mathbb{Z}, 2 + m\mathbb{Z}, \dots, m-1 + m\mathbb{Z}\}$.

H is normal! (abelian) so G/H has a group structure!

$(aH)(bH) = abH \rightsquigarrow (a+H) + (b+H) = (a+b) + H$.

$(j + m\mathbb{Z}) + (k + m\mathbb{Z}) = (j+k) + m\mathbb{Z}$.

$2 + m\mathbb{Z} + (m-1) + m\mathbb{Z} = 1 + m\mathbb{Z}$ (addition mod m!!!)

$$\boxed{\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_m}$$

$k + m\mathbb{Z} \longleftarrow\!\shortmid k$

Ex: Let $G = V$ vector space (group w/ operation $+$)

(identity $\vec{0}$)
(inverse of $\vec{v}$ is $-\vec{v}$)

$H = W$ where W is a subspace of V.

$G/H = V/W$ ← it's also a vector space, called the quotient space of V by W.

$G = \mathbb{R}^2$, $H = \text{span}\{\begin{pmatrix}1\\1\end{pmatrix}\}$

What is in G/H?

$\begin{pmatrix}1\\0\end{pmatrix} + H = \begin{pmatrix}2\\1\end{pmatrix} + H$.

$\begin{pmatrix}2\\1\end{pmatrix} = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\substack{\in\\ H}} + \begin{pmatrix}1\\0\end{pmatrix}$

$G/H = \{\text{cosets of } H \text{ in } G\}$

$= \underline{\{\begin{pmatrix}x\\0\end{pmatrix} + H \mid x \in \mathbb{R}\}}$

$\underline{\begin{pmatrix}x\\y\end{pmatrix} + H} = \begin{pmatrix}x-y\\0\end{pmatrix} + \begin{pmatrix}y\\y\end{pmatrix} + H = \underline{\begin{pmatrix}x-y\\0\end{pmatrix} + H}$

$\mathbb{R} \cong G/H$
$x \mapsto \begin{pmatrix}x\\0\end{pmatrix} + H$.

$\underline{\text{Ex}}$:

$$\overset{G}{\overset{\|}{\mathbb{R}}} \times \overset{K}{\overset{\|}{\mathbb{R}}} \xrightarrow{\psi} \overset{H}{\overset{\|}{\mathbb{R}}} \times \overset{K}{\overset{\|}{\mathbb{R}}}$$

$$(\alpha, \beta) \overset{\cong}{\longmapsto} (\alpha - \beta, \alpha + \beta)$$

(α, β) (α-only, β-only)

$$(\alpha, \beta) \longmapsto (\beta, \alpha)$$