Practice Exam 2 Version 1

Problem 1

(a) False

(b) True

(c) False

(d) True

(e) False

Problem 2

(a) Let H and K be normal subgroups of a group G with $K < H$. Then
$$(G/K)/(H/K) \cong G/H$$

(b) If P and Q are both Sylow p-subgroups of a group G, then there exists $a \in G$ with
$$Q = aPa^{-1}.$$

(c) S_3

Problem 3

(a) $n_3 \in \{1, \cancel{3}, \cancel{5}, \cancel{15}, \cancel{17}, \cancel{51}, 85, \cancel{255}\}$ to divide 255

options eliminated because $n_3 = 1 \bmod 3$

Therefore $n_3 = 1$ or $n_3 = 85$

(b) $n_5 \in \{1, \cancel{3}, \cancel{5}, \cancel{15}, \cancel{17}, 51, \cancel{85}, \cancel{255}\}$

options eliminated because $n_5 = 1 \bmod 5$

Therefore $n_5 = 1$ or $n_5 = 51$

(c) $n_{17} \in \{1, \cancel{3}, \cancel{5}, \cancel{15}, \cancel{17}, \cancel{51}, \cancel{85}, \cancel{255}\}$

options eliminated because $n_7 = 1 \bmod 17$

Therefore $n_{17} = 1$.

<u>Problem 4</u>: To do this problem, it helps to first rewrite H.

Note $(6,9) - 3(2,2) = (0,3) \in H$ and in fact

$$H = \langle (2,2), (0,3) \rangle.$$

Then for any $(j,k) \in \mathbb{Z} \times \mathbb{Z}$,

$$(j,k) + H = (j+2a, k+2a+3b) + H$$

for all $a, b \in \mathbb{Z}$. Thus WLOG we may take $j = 0,1$ and $k = 0,1,2$ so the only distinct cosets are

$$(0,0)+H, \quad (0,1)+H, \quad (0,2)+H$$
$$(1,0)+H, \quad (1,1)+H, \quad (1,2)+H.$$

(b) To do this we can reduce the matrix $\begin{bmatrix} 2 & 2 \\ 6 & 9 \end{bmatrix} \xrightarrow[R_2 - 3R_1]{} \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \xrightarrow[C_2 - C_1]{} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Therefore $(\mathbb{Z} \oplus \mathbb{Z})/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

Finally, we put this in invariant factor form to get $\mathbb{Z}_6$.

<u>Alternative approach</u>: G/H is an Abelian group of order 6 and there is only one of those up to isomorphism, namely $\mathbb{Z}_6$.

Problem 5:

Let $x \in G$ and consider $xN \in G/N$. The order of xN must divide $[G/N] = [G:N] = n$ and therefore $(xN)^n = eN$. Since $(xN)^n = x^n N$ it follows $x^n N = N$ and therefore $x^n \in N$.

Problem 6: To show $a \in \mathbb{Z}_n$ is a unit we must show there exists $x \in \mathbb{Z}_n$ with $ax = 1$ in $\mathbb{Z}_n$

Note $\gcd(a,n) = 1 \Leftrightarrow \exists\, x, y \in \mathbb{Z}$ with

$ax + ny = 1$

$\Leftrightarrow \exists x \in \mathbb{Z}$ with $ax = 1 \mod n$

$\Leftrightarrow \exists\, x \in \mathbb{Z}_n$ with $ax = 1$ in $\mathbb{Z}_n$.

Thus $\gcd(a,n) = 1 \Leftrightarrow a$ is a unit

$\square$

Problem 7: If $a \in N(R)$ and $b \in R$ then there exists $n > 0$ with $a^n = 0$ and therefore

$$(ab)^n = a^n b^n = 0 \cdot b^n = 0 \quad \text{so } ab \in N(R)$$

Likewise, if $a, b \in N(R)$ are both nilpotent then there exists $m > 0$, $n > 0$ with $a^m = 0$ and $b^n = 0$. It follows

$$(a+b)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} a^k b^{m+n-k} = \sum 0 = 0.$$

So $a+b \in N(R)$

$\square$

Practice Exam 2 Version 2

Problem 1

(a) False
(b) False
(c) True
(d) True
(e) False

Problem 2

(a) Let $\varphi : G \to \tilde{G}$ be a group homomorphism. Then φ induces an isomorphism

$$\overline{\varphi} : G/\ker(\varphi) \to \mathrm{Img}(\varphi)$$

(b) Let G be a finite group. If p is a prime divisor of $|G|$ then G has an element of order p.

(c) $GL_2(\mathbb{R})$

Problem 3

(a) $|G| = 48 = 2^4 \cdot 3$ so the Sylow 2-subgroups must have $2^4 = 16$ elements and the Sylow 3-subgroups must have 3 elements

(b) The divisors of 48 equal to 1 mod 3 are

$1, \cancel{2}, \cancel{3}, 4, \cancel{6}, \cancel{8}, \cancel{12}, 16, \cancel{24}, \cancel{48}$

Therefore there are 1, 4, or 16 Sylow 3-subgroups. If $n_3 = 16$, then G has $n_3 \cdot (3-1) = 16 \cdot 2 = 32$ elements of order 3. Note that if $A \in G$ has order 3, then $-A$ in G has order 6. Thus $\{\pm A \mid A \text{ has order } 3\}$ is a subset of G of 64 elements. G only has 48 elements, so this is impossible. Hence $n_3 = 1$ or $n_3 = 4$.

Note $\left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Z}_3\right\}$ and $\left\{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \middle| a \in \mathbb{Z}_3\right\}$ are two different subgroups of order 3 so $n_3 \neq 1$. Hence $n_3 = 4$.

(c) The divisors of 48 equal to 1 mod 2 are

$1, \cancel{2}, 3, \cancel{4}, \cancel{6}, \cancel{8}, \cancel{12}, \cancel{16}, \cancel{24}, \cancel{48}$

Hence $n_2 = 1$ or $n_2 = 3$.

$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ has order 8

$B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ has order 2

and

$AB = BA^3$ so $P = \langle A, B \rangle$ is a subgroup of order 16 (ie a 2-Sylow subgroup). P is not normal so $n_2 \neq 1$. Hence $n_2 = 3$.

Problem 4

(a) We first rewrite the generators of H using

$$-(4,6)+2(2,10)=(0,14)$$

Therefore $H=\langle(2,10),(0,14)\rangle$. It follows that for all $j,k\in\mathbb{Z}$

$$(j,k)+H = (j+2x,\, k+10x+14y)+H$$

for all $x,y\in\mathbb{Z}$. In particular we can take $j=0,1$ and $0\le k\le 13$.

The distinct cosets are

$$(j,k)+H \quad , \quad 0\le j\le 1, \quad 0\le k\le 13.$$

(b) Via Smith normal form

$$\begin{bmatrix} 2 & 10 \\ 4 & 6 \end{bmatrix} \xrightarrow[R_2-2R_1]{} \begin{bmatrix} 2 & 10 \\ 0 & -14 \end{bmatrix} \xrightarrow[C_2-5C_1]{} \begin{bmatrix} 2 & 0 \\ 0 & -14 \end{bmatrix} \xrightarrow[-C_2]{} \begin{bmatrix} 2 & 0 \\ 0 & 14 \end{bmatrix}$$

So $(\mathbb{Z}\times\mathbb{Z})/H \cong \mathbb{Z}_2\oplus\mathbb{Z}_{14}$.

Alternative proof: G/H has order 28 so

$$G/H \cong \mathbb{Z}_2\oplus\mathbb{Z}_{14} \text{ or } \mathbb{Z}_{28}$$

However

$$\begin{aligned} 14\big((j,k)+H\big) &= (14j, 14k)+H \\ &= (0, 14k-140j)+H \\ &= H \end{aligned}$$

so every element has order ≤ 14. Hence it can't be $\mathbb{Z}_{28}$.

Problem 5

$G = \langle a \rangle$ for some $a \in G$.

Therefore

$$G/H = \{xH \mid x \in G\}$$
$$= \{a^k H \mid k \in \mathbb{Z}\} = \langle aH \rangle.$$

In particular G/H is cyclic, generated by aH.

Problem 6

Since R is a ring, mult. is associative. Also $1_R \in R^\times$ is an identity. Finally, if $u \in R^\times$, then u is a unit so $\exists\, v \in R^\times$ with $uv = vu = 1_R$. Hence u has an inverse. Thus $R^\times$ is a group.

Problem 7

First note the binomial coefficient

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k(k-1)(k-2)\cdots(2)(1)}$$

is an integer for $0 \leq k \leq p$ and when $0 < k < p$ the numerator is divisible by p but the denominator isn't. ~~for~~

Thus
$$\binom{p}{k} \equiv 0 \mod p \text{ for } 0 < k < p$$

Hence (

$$(a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^k b^{p-k}$$

$$= \binom{p}{0} a^0 b^p + \cancel{\binom{p}{1} a^1 b^{p-1}} + \dots + \binom{p}{p} a^p b^0$$

$$= \binom{p}{0} b^p + \binom{p}{p} a^p = b^p + a^p,$$

(b) $F_p(x+y) = (x+y)^p = x^p + y^p = F_p(x) + F_p(y)$

by part (a)

Also $F_p(xy) = (xy)^p = x^p y^p = F_p(x) F_p(y)$

So F_p is a ring homomorphism