Problem 1

(A) $1+\sqrt{3}\,i = 2e^{i\pi/3}$

(B) $(1+i)^{2020} = \left(\sqrt{2}e^{i\pi/4}\right)^{2020} = 2^{1010} e^{i\pi 505} = 2^{1010} e^{i\pi}$

(C)
$$\sqrt{\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} = \sqrt{e^{i\pi/4}} = e^{i\pi/8}$$

(D)
$$2^{1+i} = e^{\log(2)(1+i)} = \left\{ e^{(\mathrm{Log}(2)+2\pi i k)(1+i)} \,\middle|\, k\in\mathbb{Z}\right\}$$
$$= \left\{ e^{\mathrm{Log}(2)+i\mathrm{Log}(2)-2\pi k} \,\middle|\, k\in\mathbb{Z}\right\}$$
$$= \left\{ 2e^{-2\pi k} e^{i\mathrm{Log}(2)} \,\middle|\, k\in\mathbb{Z}\right\}$$

Problem 2

If $f(x+iy) = u(x,y)+iv(x,y)$ is analytic, then u and v satisfy the Cauchy-Riemann equations:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Differentiating the first eqn. wrt y and using Clairaut's Theorem
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.$$

Thus $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\square$

Problem 3:

$f(z) = e^z \qquad f(x+iy) = e^{x+iy} = e^x e^{iy} = e^x\cos(y) + ie^x\sin(y)$

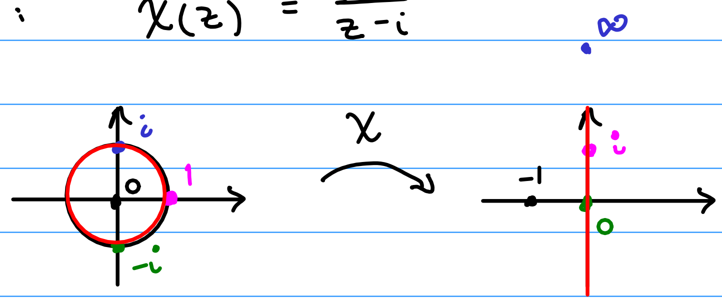
$f(x+iy) = u(x,y) + iv(x,y)$, $\quad u(x,y) = e^x\cos(y)$, $\quad v(x,y) = e^x\sin(y)$

Now
$$\frac{\partial u}{\partial x} = e^x\cos y, \quad \frac{\partial v}{\partial y} = e^x \cos y \quad \text{so} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \checkmark$$

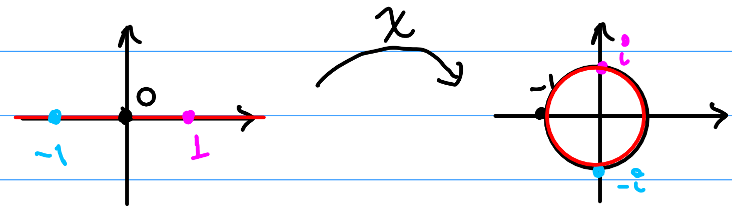
Likewise
$$\frac{\partial v}{\partial x} = e^x\sin y, \quad \frac{\partial u}{\partial y} = -e^x\sin(y) \quad \text{so} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \checkmark$$

Problem 4: $\chi(z) = \dfrac{z+i}{z-i}$

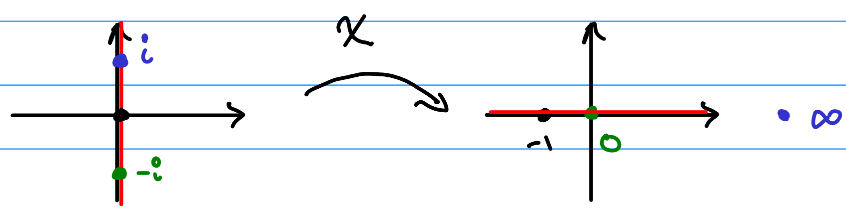
(A)



(B)



(C)



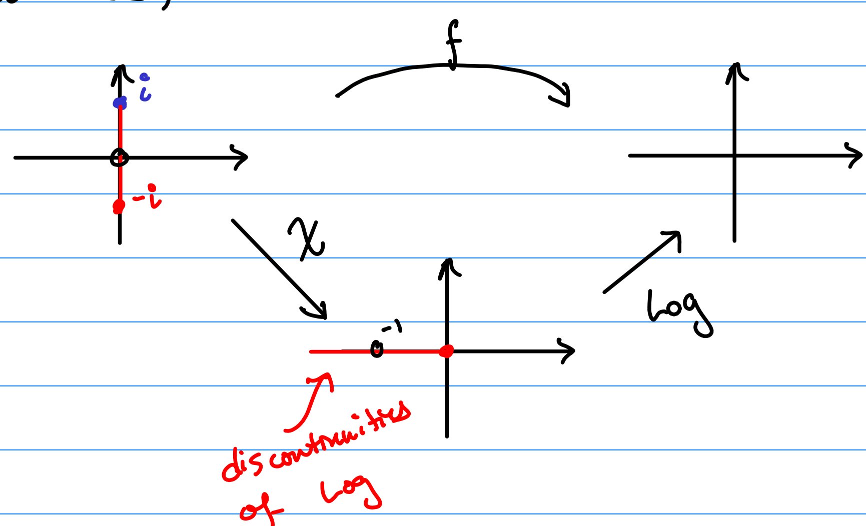
(D) $f(z) = \mathrm{Log}\left(\dfrac{z+i}{z-i}\right)$ is defined when $\dfrac{z+i}{z-i} \neq 0, \infty$

so everywhere but $-i, i$

$\mathrm{Log}(z)$ is discontinuous when $z \in (-\infty, 0]$,

so $f(z)$ is discontinuous when $\dfrac{z+i}{z-i} \in (-\infty, 0]$

Using part (C):



we see $f(z)$ is continuous on $\mathbb{C} - i[-1,1]$

Problem 5:

(A) Just show $u_{xx} + u_{yy} = 0$, which is a straightforward computation.

(B) $$v(x,y) = \int \frac{\partial v}{\partial y}\partial y = \int \frac{\partial u}{\partial x}\partial y$$

$$= \int \frac{-2xy}{(x^2+y^2)^2}\partial y = \frac{x}{x^2+y^2} + \psi(x)$$

Now, $\dfrac{\partial u}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ $\qquad \dfrac{\partial v}{\partial x} = \dfrac{y^2-x^2}{x^2+y^2} + \psi'(x)$

By CR: $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ so $\dfrac{y^2-x^2}{x^2+y^2} + \psi'(x) = -\dfrac{x^2-y^2}{x^2+y^2}$

Thus $\psi'(x) = 0 \Rightarrow \psi(x) = $ Constant C.

$$v(x,y) = \frac{x}{x^2+y^2} + C$$

Double-check:
for $z = x+iy$

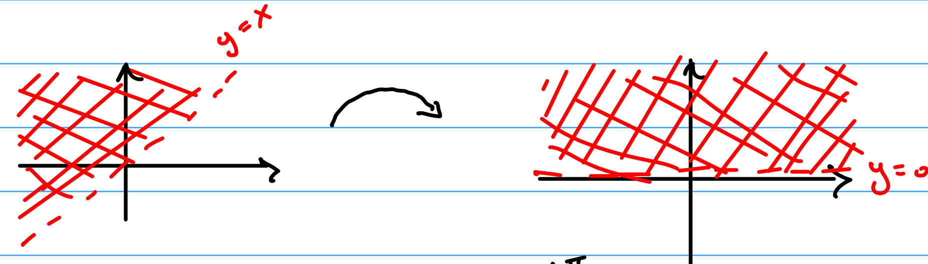
$$\frac{y}{x^2+y^2} + \frac{i x}{x^2+y^2}$$

$$= \frac{i(x-iy)}{x^2+y^2} = \frac{i\bar{z}}{z\bar{z}} = \frac{i}{z}$$

analytic!

Problem 6:

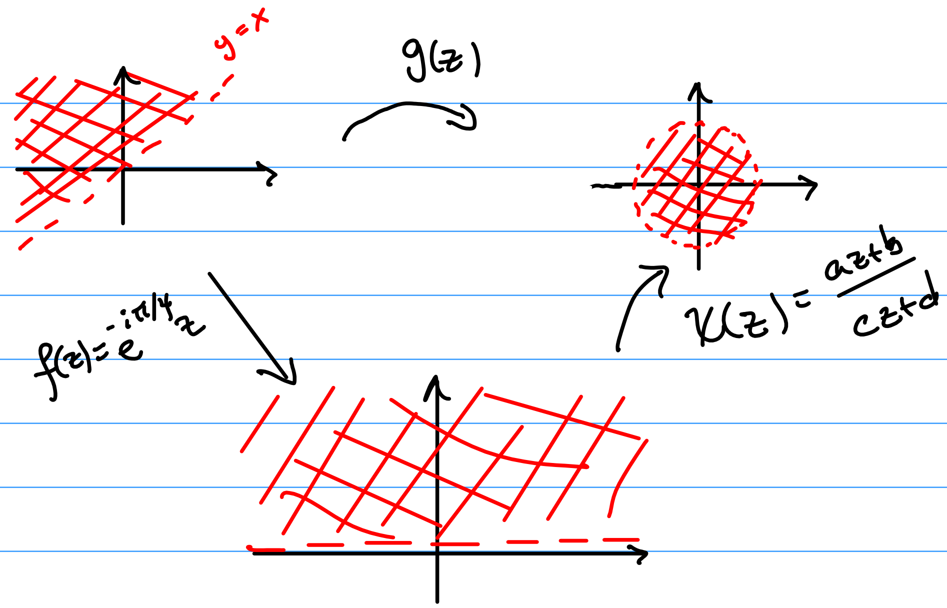
(A)



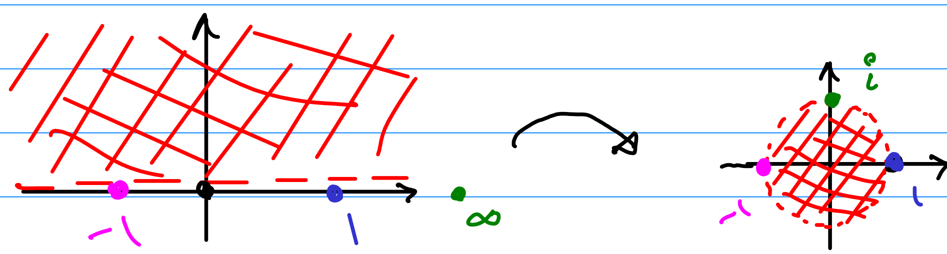
Just rotate clockwise $\frac{\pi}{4}$: $\quad f(z) = e^{-i\frac{\pi}{4}} z$

Idea: use (A) + Möbius!

(B)



To determine Möbius, choose 3 points



- WLOG: $c=1$. $\quad \chi(\infty)=i \Rightarrow a=i$
- $\chi(1)=1 \Rightarrow \frac{i+b}{1+d}=1 \rightsquigarrow d-b=i-1$
- $\chi(-1)=-1 \Rightarrow \frac{-i+b}{-1+d}=-1 \rightsquigarrow d+b=i+1$

$d=i,\ b=1$

$$\chi(z)=\frac{iz+1}{z+i}$$

So $$\boxed{g(z)=\chi(f(z))=\frac{ie^{-i\pi/4}z+1}{e^{-i\pi/4}z+i}}$$