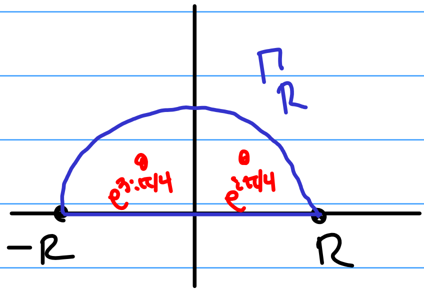
Problem 1 :



$$\oint_{\Gamma_R \cup [-R,R]} \frac{1}{z^4+1}\,dz = 2\pi i \,\mathrm{Res}\left[\frac{1}{z^4+1}, e^{i\pi/4}\right] + 2\pi i \,\mathrm{Res}\left[\frac{1}{z^4+1}, e^{3i\pi/4}\right] = \frac{\sqrt{2}}{2}\pi$$

Furthermore
$$\left|\int_{\Gamma_R} \frac{1}{z^4+1}\right| \le \int_{\Gamma_R} \left|\frac{1}{z^4+1}\right| \cdot |dz| \le \int_{\Gamma_R} \frac{1}{R^4-1}|dz| = \frac{\pi R}{R^4-1} \longrightarrow 0$$
as $R \to \infty$,

Therefore taking the limit as $R \to \infty$:
$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{2}\pi .$$

Problem 2 :

$$z = e^{i\theta} \to dz = ie^{i\theta}\,d\theta \to d\theta = \frac{1}{iz}dz$$

$$\int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta = \oint_{|z|=1} \frac{1}{a+b(z-z^{-1})/2i}\,\frac{1}{iz}\,dz$$

$$\frac{-ia \pm \sqrt{b^2-a^2}}{b}$$

$$= \oint_{|z|=1} \frac{1}{iaz+\frac{b}{2}z^2-\frac{b}{2}}\,dz$$

$$= \frac{2}{b}\oint_{|z|=1} \frac{1}{z^2+(2ia/b)z-1}\,dz$$

$$= \frac{2}{b}\oint_{|z|=1} \frac{1}{(\lambda_+-\lambda_-)}\left[\frac{1}{z-\lambda_+} - \frac{1}{z-\lambda_-}\right]dz$$

$$= \frac{2}{b}\,\frac{1}{\lambda_+-\lambda_-}\,2\pi i$$

where here $\lambda_{\pm} = \dfrac{-ia \pm \sqrt{b^2-a^2}}{b}$ so

$\lambda_+ - \lambda_- = \frac{2}{b}\sqrt{b^2-a^2}$. Hence

$$\int_0^{2\pi} \frac{1}{a+b\sin\theta}\, d\theta = \frac{2\pi i}{\sqrt{b^2-a^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Problem 3

Assume $r<1$.

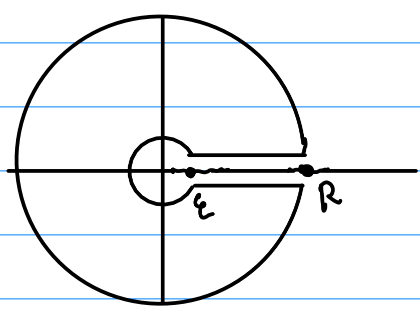
$$\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}\frac{d\theta}{2\pi} = \frac{1}{2\pi}\oint_{|z|=1} \frac{1-r^2}{1-r(z+z^{-1})+r^2}\frac{1}{iz}\,dz$$

$$= -\frac{1-r^2}{2\pi i r}\oint_{|z|=1} \frac{1}{z^2-(r+r^{-1})z+1}\,dz$$

$$= -\frac{(1-r^2)}{2\pi i r}\oint \frac{1}{r-r^{-1}}\left(\frac{1}{z-r}-\frac{1}{z-r^{-1}}\right)dz$$

$$= \frac{1}{2\pi i}\oint\left(\frac{1}{z-r}-\frac{1}{z-r^{-1}}\right)dz = 1$$

Problem 4



$$\oint_{\text{Keyhole}} \frac{z^{-a}}{(1+z)^m}\,dz = 2\pi i \operatorname{Res}\left[\frac{z^{-a}}{(1+z)^m}, -1\right]$$

where here $z^{-a} = e^{-a\widetilde{\operatorname{Log}}(z)}$

for

$$\widetilde{\operatorname{Log}}(z) = \log(r) + i\theta, \quad \theta \in [0, 2\pi)$$
$$z = re^{i\theta}.$$

Now $\quad z^{-a} = (-1)^{-a}\left(1+(-(z+1))\right)^{-a} = e^{-ia\pi}\left(1+(-(z+1))\right)^{-a}$

$$= e^{-ia\pi}\sum_{n=0}^{\infty}\binom{-a}{n}(-1)^n(z+1)^n$$

Thus $\quad \dfrac{z^{-a}}{(z+1)^m} = e^{-ia\pi}\displaystyle\sum_{n=0}^{\infty}\binom{-a}{n}(-1)^n(z+1)^{n-m}$

The residue is the coeff. of $(z+1)^{-1}$, which is

$$e^{-ia\pi}\binom{-a}{m-1}(-1)^{m-1} = e^{-ia\pi}\frac{1}{(m-1)!}(-a)(-a-1)\dots(-a-m+2)(-1)^{m}$$

$$= e^{-ia\pi}\frac{1}{(m-1)!}a(a+1)\dots(a+m-2)$$

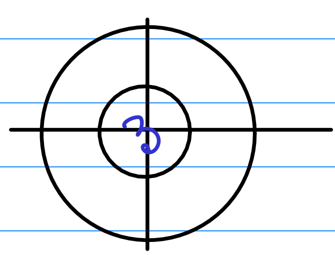
Using the usual keyhole argument, it follows

$$\int_0^\infty \frac{x^{-a}}{(1+x)^m}\,dx - e^{-2\pi i a}\int_0^\infty \frac{x^{-a}}{(1+x)^m}\,dx = 2\pi i\, e^{-ia\pi}\frac{1}{(m-1)!}a(a+1)\dots(a+m-2)$$

Hence $\quad \displaystyle\int_0^\infty \frac{x^{-a}}{(1+x)^m}\,dx = \frac{2\pi i}{1-e^{-2\pi i a}}e^{-ia\pi}\frac{1}{(m-1)!}a(a+1)\dots(a+m-2)$

$$= \frac{\pi}{\sin(\pi a)}\,\frac{a(a+1)\dots(a+m-2)}{(m-1)!}$$

<u>Problem 5</u> $\quad p(z) = z^9 + z^5 - 8z^3 + 2z + 1$



$p(z) = f(z) + h(z)$

$f(z) = -8z^3$

$h(z) = z^9 + z^5 + 2z + 1$

$|f(z)| = 8 > 4 \geq |h(z)|$ for $|z| = 1$

so Rouche $\Rightarrow$ $p(z)$ has 3 roots in $|z| < 1$

$$p(z) = f(z) + h(z)$$
$$f(z) = z^9$$
$$h(z) = z^5 - 8z^3 + 2z + 1$$

$$|h(z)| \leq |z|^5 + 8|z|^3 + 2|z| + 1 = 101 < 512 = |f(z)|$$
for $|z| = 2$.

Thus by Rouche,
$p(z)$ has same # roots as $f(z)$ in $|z| < 2$
which is 9.

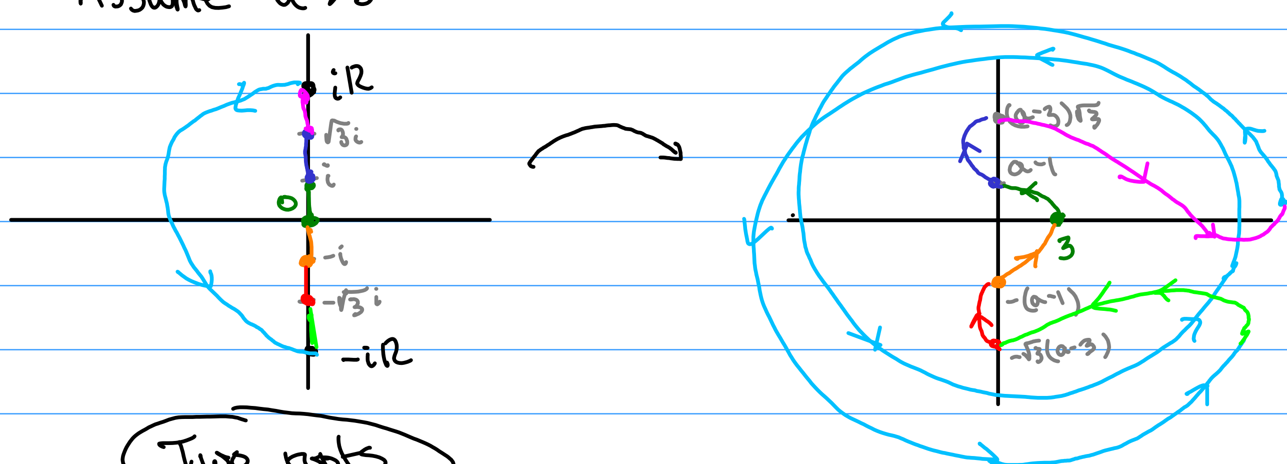
It follows there are $9 - 3 = 5$ in between

$\square$

Problem 6:

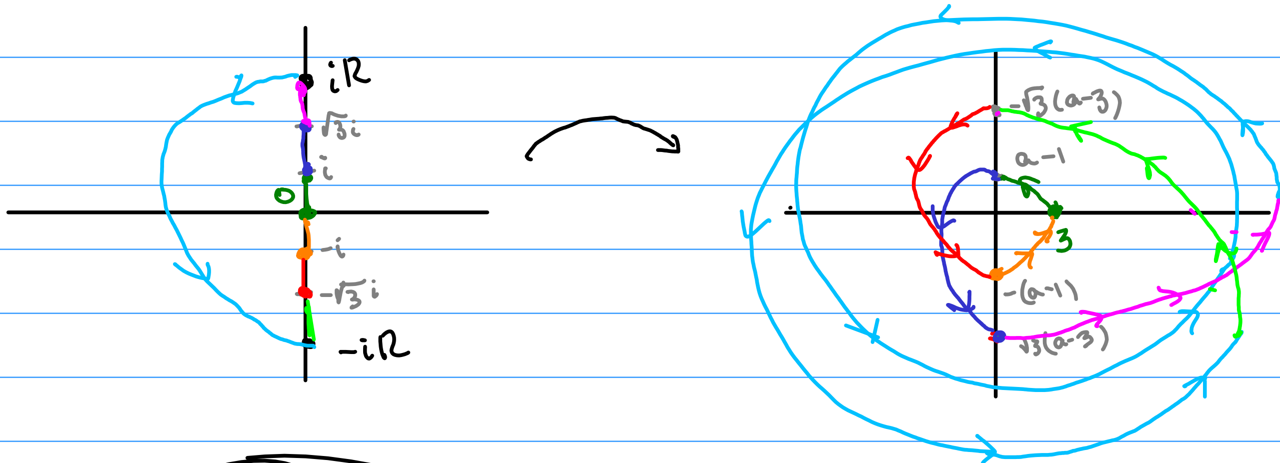
$$f(z) = z^4 + z^3 + 4z^2 + az + 3$$
$$f(iy) = (y^4 - 4y^2 + 3) + iy(a - y^2)$$
$$= (y^2 - 3)(y^2 - 1) + iy(a - y^2)$$

Assume $a > 3$



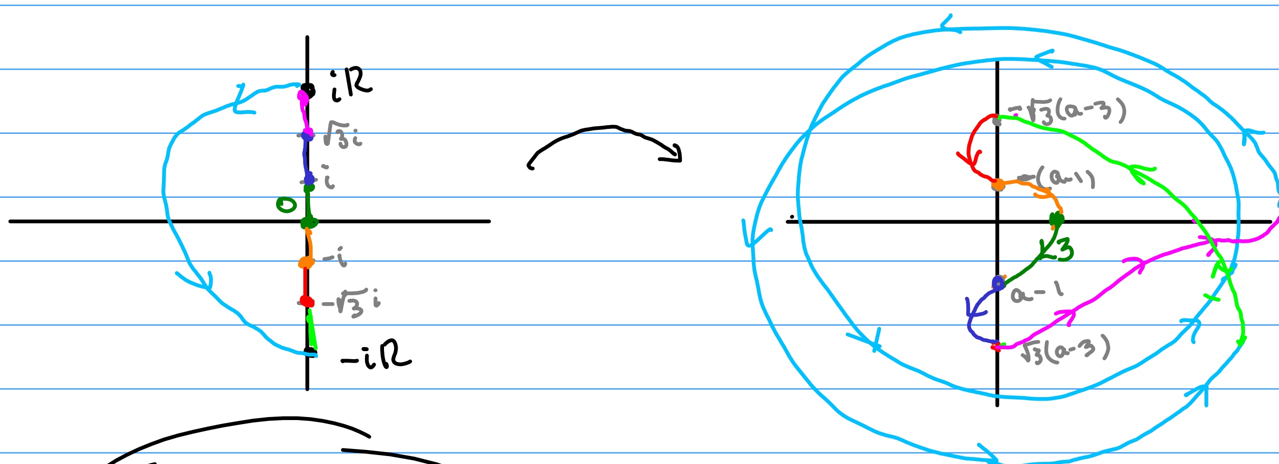
Two roots

Assume $3 > a > 1$



Four roots

Assume $1 > a$



Two roots

<u>Problem 7</u>: Let $f(z) = (z-a_1)(z-a_2)\ldots(z-a_n)$

(A) if $f(z) \neq 0$ but $f'(z) = 0$, then

$$0 = \frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_n}$$

(B) Note $\dfrac{1}{z-a_j} = \dfrac{\bar{z}-\bar{a}_j}{|z-a_j|^2}$. Therefore

$$0 = \frac{\bar{z}-\bar{a}_1}{|z-a_1|^2} + \dots + \frac{\bar{z}-\bar{a}_n}{|z-a_n|^2}$$

$$= \sum_j \frac{\bar{z}-\bar{a}_j}{|z-a_j|^2} = \sum_j \frac{\bar{z}}{|z-a_j|^2} - \sum_j \frac{\bar{a}_j}{|z-a_j|^2}$$

Thus
$$\bar{z}\left(\sum_j \frac{1}{|z-a_j|^2}\right) = \sum_j \frac{\bar{a}_j}{|z-a_j|^2}.$$

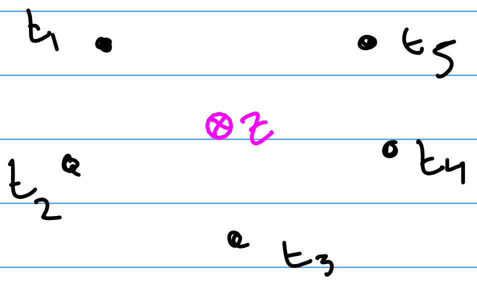
(C) Conjugate B to get

$$z\left(\sum_j \frac{1}{|z-a_j|^2}\right) = \sum_j \frac{a_j}{|z-a_j|^2}.$$

Let $t_j = \dfrac{1}{|z-a_j|^2}$. Then

$$z = \frac{a_1t_1 + a_2t_2 + \dots + a_nt_n}{t_1+t_2+\dots+t_n}$$

Think about putting an object weighing t_j grams in the complex plane at position $a_j \in \mathbb{C}$. Then z above is the center of mass!



Hence it occurs inside the convex hull

$\square$