Problem 1 :

(A) Notice $p(z) = (z^n - 1) - \prod_{k=0}^{n-1} (z - e^{2\pi i k/n})$ is a polynomial of degree $< n$. However, it has n roots so by FTA $p(z) \equiv 0$.

Hence
$$z^n - 1 = \prod_{k=0}^{n-1} (z - e^{2\pi i k/n})$$

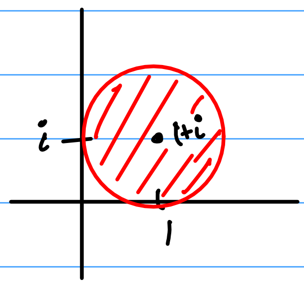
(B) Using (a), we see $0^n - 1 = \prod_{k=0}^{n-1} (0 - e^{2\pi i k/n})$

Thus $-1 = (-1)^n \prod_{k=0}^{n-1} e^{2\pi i k/n}$, so $\prod_{k=0}^{n-1} e^{2\pi i k/n} = (-1)^{n+1}$.

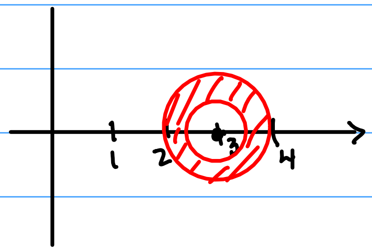
(C) Since
$$z^n - 1 = \prod_{k=0}^{n-1} (z - e^{2\pi i k/n})$$
$$= z^n - \left(\sum_{k=0}^{n-1} e^{2\pi i k/n}\right) z^{n-1} + \dots$$

Since these two polynomials are equal, it follows
$$\sum_{k=0}^{n-1} e^{2\pi i k/n} = 0 .$$

Problem 2 :

(A) 

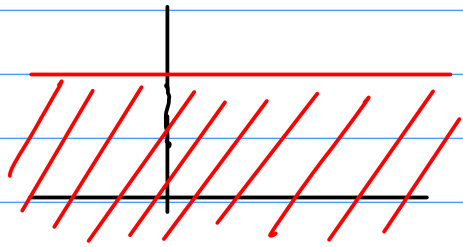
$|z - (1+i)| < 1$

(B) 

$1 < |2z - 6| < 2$

$1/2 < |z - 3| < 1$

(c)



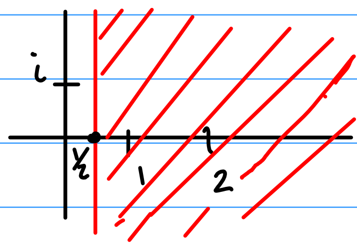
$\mathrm{Re}(iz+2) > 0$

$z = x+iy$

$\mathrm{Re}(ix-y+2) > 0$

$2 > y$

(D)



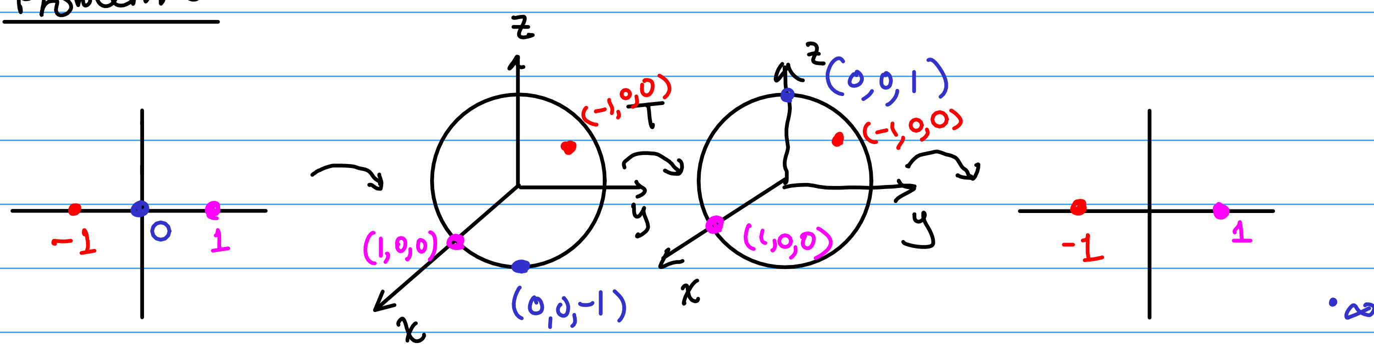
$|z-1| < |z|$

$|z-1|^2 < |z|^2$

$(x-1)^2+y^2 < x^2+y^2$

$1-2x < 0 \qquad x > 1/2$

Problem 3 :



$\chi : \; -1 \longmapsto -1$

$\quad\; 1 \longmapsto 1$

$\quad\; 0 \longmapsto \infty$

$$\chi(z) = \frac{az+b}{cz+d}$$

$\chi(0) = \infty \Rightarrow d = 0$

WLOG: $c = 1$.

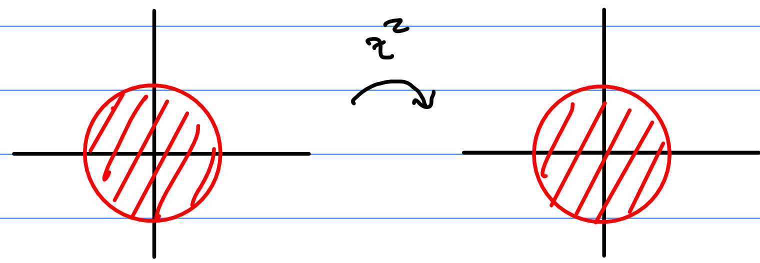
$$\chi(z) = \frac{az+b}{z}$$

$$\left.\begin{array}{l} \chi(1) = 1 \Rightarrow a+b = 1 \\ \chi(-1) = -1 \Rightarrow a-b = -1 \end{array}\right\} \begin{array}{l} b = 1 \\ a = 0 \end{array}$$

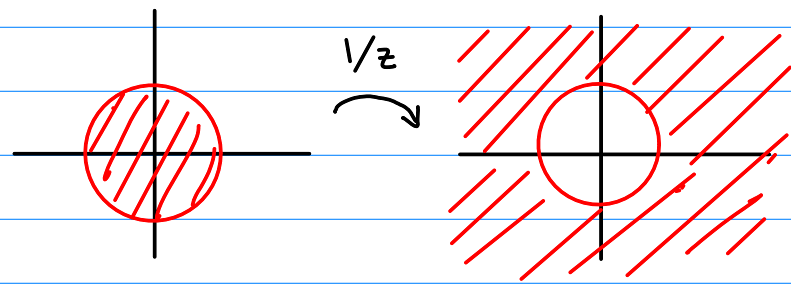
$$\chi(z) = \frac{1}{z}$$

Problem 4:

(A)



(B)



(C)