Problem 1

(A) True

(B) False

(C) True

(D) False

(E) True

Problem 2

(A) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$

(B) $\sum_{k=1}^{\infty} k!\, z^k$

(C) $f(x) = \begin{cases} e^{-x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(D) Does not exist

Problem 3

Suppose $\{f_n(z)\}$ is a sequence of analytic functions on a domain D converging to $f(z)$ uniformly. To prove $f(z)$ is analytic, it suffices by Morera's Theorem to show $\int_{\partial R} f(z)\,dz = 0$ for all bounded rectangles $R \subseteq D$.

By Cauchy's Theorem $\int_{\partial R} f_n(z)\,dz = 0 \ \forall n$. Moreover, since the sequence converges uniformly, the limit can be interchanged with the integral. Thus

$$\int_{\partial R} f(z)\,dz = \int_{\partial R} \lim_n f_n(z)\,dz = \lim_n \int_{\partial R} f_n(z)\,dz = \lim_n 0 = 0.$$

$\square$

Problem 4 :

Choose $M > 0$ with $\left|\frac{f(z)}{z^n}\right| < M \quad \forall \ z > 2021$

By Cauchy's Integral Formula

$$f^{(k)}(0) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(w)}{w^{k+1}} \, dw$$

If $R > 2021$ and $k > n$:

$$\left|f^{(k)}(0)\right| \leq \frac{1}{2\pi} \oint_{|z|=R} \left|\frac{f(w)}{w^n}\right| \frac{1}{|w^{k+1-n}|} \cdot |dw|$$

$$\leq \frac{1}{2\pi} \oint_{|z|=R} M \frac{1}{|w^{k+1-n}|} \cdot |dw| = \frac{M}{R^{k-n}}$$

Taking $R \to 0$, we see $f^{(k)}(0) = 0$ for $k > n$.

Thus using the power series expansion at $z = 0$:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

so in particular $f(z)$ is a poly of deg $\leq n$.

Problem 5

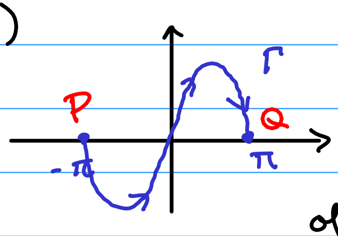
(A) By Green's Theorem

$$\oint_{\partial R} xy \, dy = \iint_R \frac{\partial (xy)}{\partial x} \, dA = \int_0^1 \int_0^1 y \, dx \, dy = \frac{1}{2}$$

(B) $$\oint_{|z|=1} \frac{1}{z^3} \, dz = \int_0^{2\pi} \frac{1}{(e^{i\theta})^3} i e^{i\theta} \, d\theta = \int_0^{2\pi} i e^{-2i\theta} \, d\theta = \frac{-1}{2} e^{-2i\theta} \Big|_0^{2\pi} = 0$$

$z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta$

(C)

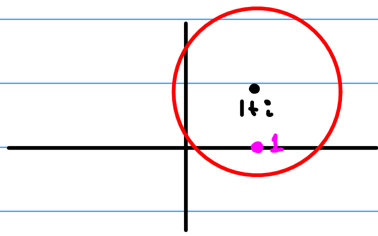


Note that $\frac{d}{dz}\sin(z) = \cos(z)$

so by the Fundamental Theorem of complex line integrals:

$$\int_\Gamma \cos(z)\,dz = \sin(Q) - \sin(P) = \sin(\pi) - \sin(-\pi) = 0.$$

(D)



By Cauchy's Integral formula:

$$\int_{|z-1-i|=\frac{5}{4}} \frac{\mathrm{Log}(z)}{(z-1)^2}\,dz = 2\pi i \left.\frac{d}{dz}\mathrm{Log}(z)\right|_{z=1} = 2\pi i \left.\frac{1}{z}\right|_{z=1} = 2\pi i$$

Problem 6:

$$f(z_0) = \frac{1}{2\pi i}\oint_{|w-z_0|=r} \frac{f(w)}{w-z_0}\,dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$\left.\begin{array}{l} w = z_0 + re^{i\theta}, \\ dw = ire^{i\theta}\,d\theta \end{array}\right\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi r}\int_{|w-z_0|=r} f(w)\,|dz| = f_{avg}$$

$\square$