Problem 1: $\quad z^4 = 2e^{i\pi/3 + 2\pi i k}, \; k \in \mathbb{Z}$

$$z = \sqrt[4]{2}\, e^{i\pi/12 + i\pi k/2}, \quad k \in \mathbb{Z}$$

This repeats every four k, giving us four solutions

$$z \in \left\{ \sqrt[4]{2}e^{i\pi/12},\ \sqrt[4]{2}e^{i\pi/12}i,\ -\sqrt[4]{2}e^{i\pi/12},\ -\sqrt[4]{2}e^{i\pi/12}i \right\}$$

Problem 2:

$$(1+\sqrt{3}i)^i = \left(e^{(\log 2 + i\pi/3 + 2\pi i k)}\right)^i, \quad k \in \mathbb{Z}$$

$$= e^{[i\log 2 - (\pi/3 + 2\pi k)]}, \quad k \in \mathbb{Z}$$

This gives infinitely many values.

Problem 3: Since $f(z)$ is holomorphic it is continuous.

Therefore

$$M = \max\{|f(z)| : 0 \leq Re(z) \leq 1,\ 0 \leq Im(z) \leq 1\} < \infty$$

Consequently, the fact that $f(z) = f(z+1)$ and $f(z) = f(z+i)$ for all z, it follows $|f(z)| \leq M \ \forall\ z \in \mathbb{C}$.

Liouville's Theorem implies $f(z)$ is constant

Problem 4:

$$\int_0^{2\pi} \frac{\cos(x)}{\cos(x)+a}\,dx = \oint_{|z|=1} \frac{\frac{z+z^{-1}}{2}}{\frac{z+z^{-1}}{2} + a} \frac{1}{iz}\,dz$$

$$= \oint_{|z|=1} \frac{z^2+1}{z^2+2az+1} \frac{1}{iz}\,dz$$

$$= \oint_{|z|=1} \frac{1}{iz} + \frac{2ai}{z^2+2az+1}\,dz$$

$$= \oint_{|z|=1} \frac{1}{iz} + \frac{ai/\sqrt{a^2-1}}{z+a-\sqrt{a^2-1}} - \frac{ai/\sqrt{a^2-1}}{z+a+\sqrt{a^2-1}} \, dz$$

$$= 2\pi i \, \mathrm{Res}[\text{" . "}, 0] + 2\pi i \, \mathrm{Res}[\text{" . "}, -a+\sqrt{a^2-1}]$$

$$= 2\pi i \frac{1}{i} + 2\pi i \frac{ai}{\sqrt{a^2-1}} = 2\pi\left(1 - \frac{a}{\sqrt{a^2-1}}\right)$$

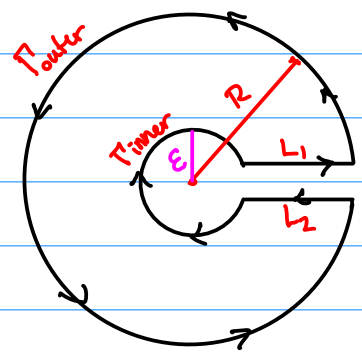
Problem 5

(A) $$\mathrm{Res}\left[\frac{1}{1+z^2}, i\right] = \frac{1}{(1+z^2)'}\Big|_{z=i} = \frac{1}{2z}\Big|_{z=i} = -\frac{1}{2}i$$

(B) $$\mathrm{Res}\left[\frac{e^{2z}}{(z-1)^2}, 1\right] = \mathrm{Res}\left[e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(z-1)^{n-2}, 1\right] = 2e^2$$

(C) $$\mathrm{Res}\left[z^3 e^{3/z}, 0\right] = \mathrm{Res}\left[\sum_{n=0}^{\infty} \frac{3^n}{n!} z^{3-n}, 0\right] = \frac{3^4}{4!}$$

Problem 6 $$\int_0^{\infty} \frac{\sqrt[n]{x}}{x^2+4}\, dx = ?$$



$$\oint_{\text{Keyhole}} \frac{\sqrt[n]{z}}{z^2+4}\, dz = 2\pi i \, \mathrm{Res}\left[\frac{\sqrt[n]{z}}{z^2+4}, 2i\right] + 2\pi i \, \mathrm{Res}\left[\frac{\sqrt[n]{z}}{z^2+4}, -2i\right]$$

$$= 2\pi i \left(\frac{(2i)^{1/n}}{4i} - \frac{(-2i)^{1/n}}{4i} \right)$$

$$= \frac{\pi}{2} 2^{1/n} \left(e^{i\pi/2n} - e^{3i\pi/2n} \right)$$

Note as $R \to \infty$

$$\left| \int_{\Gamma_{outer}} \frac{\sqrt[n]{z}}{z^2+4} \right| \le \int_{\Gamma_{outer}} \frac{R^{1/n}}{R^2-4} |dz| < \frac{2\pi R^{1+1/n}}{R^2-4} \to 0$$

as $R \to \infty$

Likewise, as $\varepsilon \to 0$

$$\left| \int_{\Gamma_{inner}} \frac{z^{1/n}}{z^2+4}\, dz \right| \le \int_{\Gamma_{inner}} \frac{\varepsilon^{1/n}}{4-\varepsilon^2} |dz| \le \frac{2\pi \varepsilon^{1+1/n}}{4-\varepsilon^2} \to 0$$

as $\varepsilon \to 0$.

Hence we see

$$\int_0^\infty \frac{x^{1/n}}{x^2+4}\, dx - e^{2\pi i/n} \int_0^\infty \frac{x^{1/n}}{x^2+4}\, dx = \frac{\pi}{2} 2^{1/n} \left(e^{i\pi/2n} - e^{3i\pi/2n} \right)$$

So that

$$\int_0^\infty \frac{x^{1/n}}{x^2+4}\, dx = \frac{\pi}{2} 2^{1/n} \frac{e^{i\pi/2n} - e^{3i\pi/2n}}{1 - e^{2\pi i/n}}$$

$$= \frac{\pi}{2} 2^{1/n} \frac{e^{-i\pi/2n} - e^{i\pi/2n}}{e^{-i\pi/n} - e^{i\pi/n}}$$

$$= \frac{\pi}{2} 2^{1/n} \frac{\sin(\pi/2n)}{\sin(\pi/n)}.$$

Problem 7:

Let $f(x+iy) = u(x,y) + iv(x,y)$.

Then $f(z)$ is analytic and $|f(z)| = u^2+v^2 > 0$ on D.

Thus $\log(f(z))$ is analytic on D. Hence its real part is harmonic

Thus $\mathrm{Re}(\log(f(z))) = \log|f(z)| = \log(u^2+v^2)$ is harmonic

Problem 8

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1/2}{z} - \frac{1}{z-1} + \frac{1/2}{z-2}$$

Therefore

on $0<|z|<1$ $\quad f(z) = \frac{1}{2}z^{-1} + \sum_{n=0}^{\infty} z^n - \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} z^n$

on $1<|z|<2$ $\quad f(z) = \frac{1}{2}z^{-1} - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} z^n$

on $|z|>2$ $\quad f(z) = \frac{1}{2}z^{-1} - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{2z}\sum_{n=0}^{\infty} 2^n z^{-n}$

Problem 9: $f(\frac{1}{n}) = \frac{1}{n^3} - \frac{1}{n^5}$ for all $n \geq 2$

Since $f(z)$ is continuous, it follows $f(0)=0$.

Therefore $f(z) - (z^3 - z^5)$ has zeros on the set $\{0\} \cup \{1/n : n \in \mathbb{Z}, n \geq 2\}$

This set has an accumulation point, so

$$f(z) - (z^3 - z^5) = 0 \quad \forall z \quad \text{by Uniqueness Theorem}$$

Thus $f(z) = z^3 - z^5$ and $f(2) = 8 - 32 = -24$

Problem 10:

(A) $u(x,y)$ is harmonic means $u(x,y)$ is twice differentiable and $u_{xx} + u_{yy} = 0$.

(B) Principal log and any point on the negative real axis

(C) $z_0 \in \mathbb{C}$ is an essential singularity of $f(z)$ if it is an isolated singularity and the Laurent series expansion of $f(z)$ at z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$ has $a_n \neq 0$ for infinitely many $n<0$

(D) If $D \subseteq \mathbb{C}$ is a simply connected, proper subset then there exists a conformal map $f(z)$ of D surjectively onto $\mathbb{D}$.

<u>Problem 11:</u> $p(z) = z^5 + 15z + 1$.

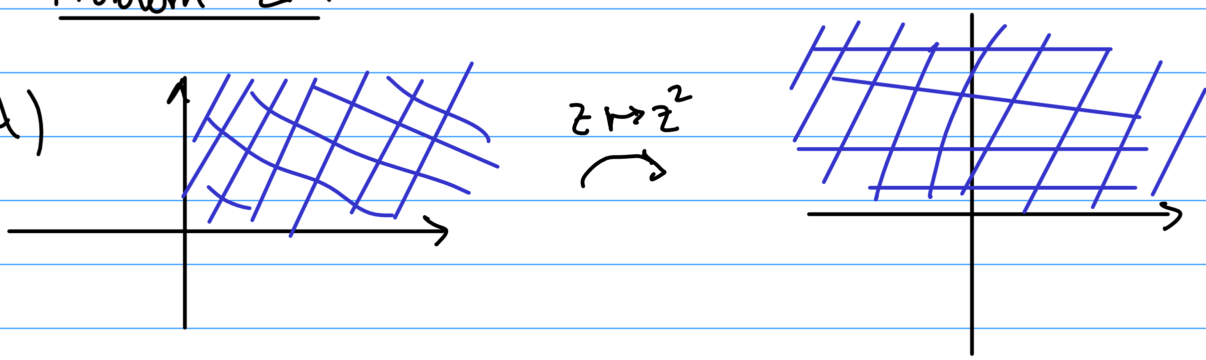
$$= f(z) + h(z) \quad \text{for} \quad f(z) = z^5$$
$$h(z) = 15z + 1$$

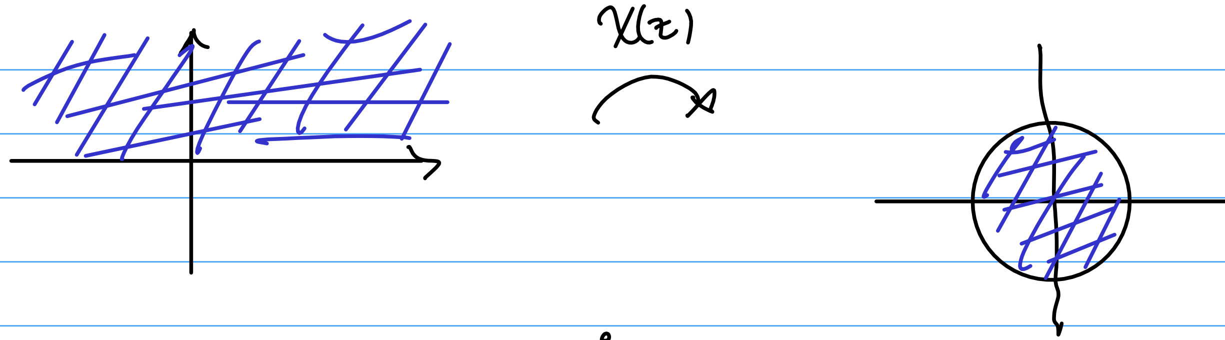
Note for $|z| = 2$,
$$|f(z)| = |z|^5 = 32$$
$$|h(z)| \leq 15|z| + 1 = 31$$
so $|f(z)| > |h(z)|$ for all $|z| = 2$.

Thus by Rouché's Theorem $p(z)$ and $f(z)$ have the same number of zeros in $\{z \mid |z| < 2\}$.
Since $f(z)$ has five (zero at 0 w/ mult. 5) we know $p(z)$ must have all five zeros in this disk.

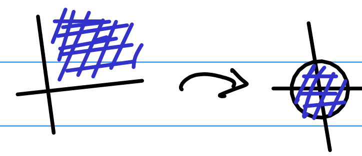
<u>Problem 12:</u>

(A)





$$\chi(z) = \frac{z-i}{z+i}$$

Thus $\quad f(z) = \chi(z^2) = \dfrac{z^2-i}{z^2+i}$ maps 

(B) Using our classification of the conformal maps of the disk to itself, all maps must be of the form

$$g(z) = e^{i\phi}\,\frac{f(z)-a}{1-\bar{a}f(z)} \quad \text{for } a \in \mathbb{D},\ 0 \le \phi < 2\pi$$

and $f(z)$ defined in (A)