Problem 1

(A) $e^{37\pi i} = \cos(37\pi i) + i\sin(37\pi i) = -1$

(B) $(1+i)^i = \left\{ \left(e^{\frac{1}{2}\log(2) + i\pi/4 + 2\pi i k}\right)^i \,\middle|\, k \in \mathbb{Z} \right\}$

$= \left\{ e^{\frac{1}{2} i \log 2 - \pi/4 - 2\pi k} \,\middle|\, k \in \mathbb{Z} \right\}$

(C) $i^i = \left\{ \left(e^{(i\pi/4 + 2\pi i k)}\right)^i \,\middle|\, k \in \mathbb{Z} \right\} = \left\{ e^{-\pi/4 - 2\pi k} \,\middle|\, k \in \mathbb{Z} \right\}$

$i^{i^i} = i^{(i^i)} = \left\{ i^{(e^{-\pi/4 - 2\pi k})} \,\middle|\, k \in \mathbb{Z} \right\}$

$= \left\{ e^{(i\frac{\pi}{4} + 2\pi i j)(e^{-\pi/4 - 2\pi k})} \,\middle|\, j,k \in \mathbb{Z} \right\}$

$= \left\{ e^{i(\frac{\pi}{4} + 2\pi j)(e^{-\pi/4 - 2\pi k})} \,\middle|\, j,k \in \mathbb{Z} \right\}$

Problem 2:

$$\sin(z+w) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i}$$

$$= \frac{(e^{iz}+e^{-iz})(e^{iw}-e^{-iw}) + (e^{iz}-e^{-iz})(e^{iw}+e^{-iw})}{4i}$$

$$= \cos(z)\sin(w) + \sin(z)\cos(w)$$

$$\cos(z+w) = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2}$$

$$= \frac{(e^{iz}+e^{-iz})(e^{iw}+e^{-iw}) + (e^{iz}-e^{-iz})(e^{iw}-e^{-iw})}{4}$$

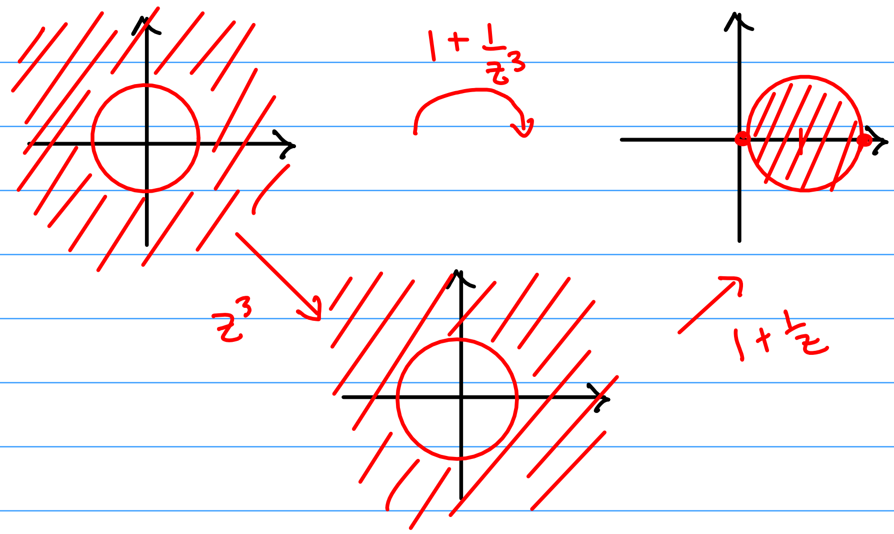
$$= \frac{(e^{iz}+e^{-iz})(e^{iw}+e^{-iw})}{4} - \frac{(e^{iz}-e^{-iz})(e^{iw}-e^{-iw})}{4i^2}$$

$$= \cos(z)\cos(w) - \sin(z)\sin(w)$$

Problem 3:

$$(f(z))^2 = z^2 + \frac{1}{z} = z^2\left(1 + \frac{1}{z^3}\right)$$

First note:



So $1 + \frac{1}{z^3}$ maps $\{z \mid |z| > 1\}$ to $\{z \mid |z-1| < 1\}$

Since the principal square root $g(z) = |z|^{1/2} e^{i\mathrm{Arg}(z)/2}$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$ and the composition of continuous functions is continuous,

$$h(z) = g\left(1 + \frac{1}{z^3}\right)$$

is continuous on $\{z \mid |z| > 1\}$.

Note $(z h(z))^2 = z^2\left(1 + \frac{1}{z^3}\right) = z^2 + \frac{1}{z}$ and so $f(z) = z h(z)$ solves the problem $\square$

Problem 4 :

$$w = \tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{\left(\frac{e^{iz}-e^{-iz}}{2i}\right)}{\left(\frac{e^{iz}+e^{-iz}}{2}\right)} = \frac{1}{i}\,\frac{e^{2iz}-1}{e^{2iz}+1}$$

$$iw(e^{2iz}+1) = e^{2iz}-1 \iff (1+iw) = e^{2iz}(1-iw)$$

$$\iff e^{2iz} = \frac{1+iw}{1-iw} \iff 2iz = \log\left(\frac{1+iw}{1-iw}\right)$$

$$\iff z = \frac{1}{2i}\log\left(\frac{1+iw}{1-iw}\right).$$

Consequently $\tan^{-1}(z) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

Problem 5 :

(A) $$\left|\frac{n}{n-1}-1\right| = \frac{1}{n-1} < \varepsilon \iff n-1 > \frac{1}{\varepsilon}$$
$$\iff n > \frac{1}{\varepsilon}+1.$$

Let $\varepsilon>0$. Choose $N > \frac{1}{\varepsilon}+1$. Then for $n \geq N$ we have $n > \frac{1}{\varepsilon}+1$ so that $\frac{1}{n-1} < \varepsilon$ and therefore $\left|\frac{n}{n-1}-1\right| < \varepsilon$.

$\square$

(B) Choose an integer $r > |z|$. Then for $n > r$:

$$\left|\frac{z^n}{n!}\right| \leq \frac{r^n}{n!} \leq \left(\prod_{j=1}^{r}\frac{r}{j}\right)\cdot\left(\prod_{j=r+1}^{n}\frac{r}{j}\right)$$

$$\leq \left(\prod_{j=1}^{r}\frac{r}{j}\right)\left(\prod_{j=r+1}^{n}\frac{r}{r+1}\right) \leq \frac{r^n}{(r+1)^{n-r}}$$

If we want $\frac{r^n}{(r+1)^{n-r}} < \varepsilon$, then

$$n\log(r) - (n-r)\log(r+1) < \log(\varepsilon)$$

$$n\log\left(\frac{r}{r+1}\right) < \log(\varepsilon) - r\log(r+1)$$

$$n > \left[\log(\varepsilon) - r\log(r+1)\right] / \log\left(\frac{r}{r+1}\right)$$

Let $\varepsilon > 0$. Choose

$$N > \left[\log(\varepsilon) - r\log(r+1)\right] / \log\left(\frac{r}{r+1}\right).$$

Then for $n \geq N$, we have $\left|\frac{z^n}{n!}\right| < \varepsilon$ and thus

$$\lim_{n\to\infty} \frac{z^n}{n!} = 0$$

$\square$

Problem 6: $\alpha = a+ib$

$$z^\alpha = e^{\mathrm{Log}(z)(a+ib)}$$

$$|z^\alpha| = e^{a\log|z| - b\,\mathrm{Arg}\,z} = |z|^a e^{-b\,\mathrm{Arg}\,z}$$

Thus if $a<0$, z^α will grow as $z\to 0$ so z^α DNE

if $a>0$, z^α will go to 0 as $z\to 0$.

If $a=0$, we have two cases:

$a=0, b\neq 0$: $z^\alpha = e^{i\log|z|\,b - b\,\mathrm{Arg}\,z}$ so as $z\to 0$ the norm of z goes to 1, but the argument of z oscillates. It ends up spinning around the unit circle

$a=0, b=0$: $z^\alpha = 1$ so the limit exists.

Problem 7 : Just plug in and check. Then

$$f(x+iy) = i\cos(x+iy)$$
$$= \sin(x)\sinh(y) + i\cos(x)\cosh(y)$$