Problem 1A: $\frac{1}{z^2-z}$ is holomorphic on $0<|z|<1$ and $|z|>1$.

$$\frac{1}{z^2-z} = \frac{1}{z}\,\frac{1}{z-1} = -\frac{1}{z}\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} -z^{k-1}, \quad 0<|z|<1$$

$$\frac{1}{z^2-z} = \frac{1}{z}\,\frac{1}{z-1} = \frac{1}{z^2}\,\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\sum_{k=0}^{\infty} z^{-k} = \sum_{k=0}^{\infty} z^{-k-2}, \quad |z|>1$$

Problem 1B: $\frac{z-1}{z+1}$ is holomorphic on $|z|<1$ and $|z|>1$

$$\frac{z-1}{z+1} = \frac{z+1-2}{z+1} = 1-\frac{2}{z+1} = 1-\frac{2}{1-(-z)} = 1-2\sum_{k=0}^{\infty}(-z)^k$$

$$= 1-2\sum_{k=0}^{\infty}(-1)^k z^k, \quad |z|<1$$

$$\frac{z-1}{z+1} = 1-\frac{2}{z+1} = 1-\frac{1}{z}\,\frac{2}{1-(-1/z)} = 1-2\sum_{k=0}^{\infty}(-1)^k z^{-k-1}, \quad |z|>1$$

Problem 1C: $\frac{1}{(z^2-1)(z^2-4)}$ is holomorphic on $|z|<1$, $1<|z|<2$, $|z|>2$

$$\frac{1}{(z^2-1)(z^2-4)} = \frac{1/3}{1-z^2} - \frac{1/3}{4-z^2} = \frac{1/3}{1-z^2} - \frac{1/12}{1-z^2/4}$$

$$= \frac{1}{3}\sum_{k=0}^{\infty} z^{2k} - \frac{1}{12}\sum_{k=0}^{\infty} 4^{-k} z^{2k}$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{3}-\frac{1}{12}4^{-k}\right) z^{2k}, \quad |z|<1$$

$$\frac{1}{(z^2-1)(z^2-4)} = \frac{1/3}{1-z^2} - \frac{1/12}{1-z^2/4} = \frac{1}{z^2}\,\frac{-1/3}{1-1/z^2} - \frac{1/12}{1-z^2/4}$$

$$= -\frac{1}{3}z^{-2}\sum_{k=0}^{\infty} z^{-2k} - \frac{1}{12}\sum_{k=0}^{\infty} 4^{-k} z^{2k}$$

$$1<|z|<2$$

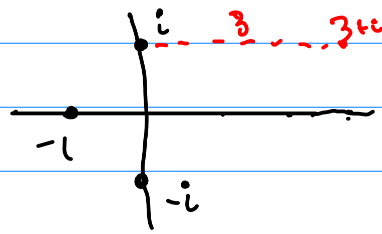
$$\frac{1}{(z^2-1)(z^2-4)} = \frac{1/3}{1-z^2} - \frac{1/12}{1-z^2/4} = \frac{1}{z^2}\,\frac{-1/3}{1-1/z^2} - \frac{1}{z^2}\,\frac{-1/3}{1-4/z^2}$$

$$= -\frac{1}{3}\sum_{k=0}^{\infty} z^{-2k} + \frac{1}{3}\sum_{k=0}^{\infty} 4^k z^{-2k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{3}(4^k - 1)\, z^{-2k}, \qquad |z| > 2$$

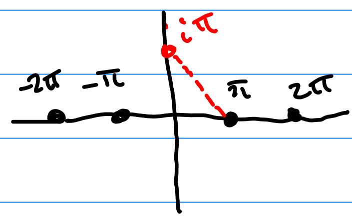
Problem 2 The radius of convergence is the distance to the nearest non-removable isolated singularity.

(A) non-removable singularities $-1, i, -i$



Closest singularity is i so radius of convergence is 3.

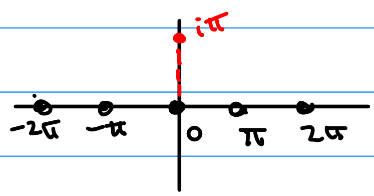
(B) non-removable singularities $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$



Closest singularity is π so radius of convergence is $\sqrt{2}\,\pi$

(C) non-removable singularities $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Closest singularity is 0 so radius of convergence is π



Problem 3 Let m be the order of the pole of $f(z)$ at z_0.

We can write $g(z) = (z - z_0)^m f(z)$, for $g(z)$ analytic on $|z| < R + 0.001$.

Using the series expansions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

we know that the radius of convergence of each series is R and at least $R+0.001$, respectively. Consequently $\{b_k R^k\}$ is bounded but $\{a_k R^k\}$ is not so

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} \frac{b_k R^k}{a_k R^k} = 0.$$

Now if $L = \lim_{k\to\infty} \frac{a_k}{a_{k+1}}$, then

$$\lim_{k\to\infty} \frac{a_k}{a_{k+m}} = \lim_{k\to\infty} \frac{a_k}{a_{k+1}} \frac{a_{k+1}}{a_{k+2}} \cdots \frac{a_{k+m-1}}{a_{k+m}} = L^m.$$

Moreover, using the binomial theorem

$$\sum_{k=0}^{\infty} b_k z^k = \sum_{j=0}^{\infty} \sum_{i=0}^{m} \binom{m}{i} a_j (-z_0)^i z^{j+m-i}$$

Thus $b_k = \sum_{i=0}^{m} \binom{m}{i} (-z_0)^i a_{k+i-m}$, $k \geq m$.

Hence

$$0 = \lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} \sum_{i=0}^{m} \binom{m}{i} (-z_0)^i \frac{a_{k+i-m}}{a_k}$$

$$= \sum_{i=0}^{m} \binom{m}{i} (-z_0)^i L^{m-i} = (L-z_0)^m.$$

Consequently, $L = z_0$.

$\square$

Problem 4

(A) $\frac{1}{\sin(z)}$ has a simple pole at $z = n\pi$, $n \in \mathbb{Z}$, so its principal part is $\frac{(-1)^n}{z-n\pi}$.

(B) $\frac{z^2}{z-1} = \frac{z^2-1+1}{z-1} = \frac{z^2-1}{z-1} + \frac{1}{z-1} = z+1 + \frac{1}{z-1}$

so the principal part at 1 is $\frac{1}{z-1}$.

Problem 5:

(A) $\frac{1}{z^2-z} = \frac{1}{z-1} - \frac{1}{z}$

(B) $\frac{1}{(z+1)(z^2+2z+2)} = \frac{1}{z+1} - \frac{1/2}{z+1+i} - \frac{1/2}{z+1-i}$

Problem 6:

(A) $\operatorname{Res}\left[\frac{1}{z^2+4}, 2i\right] = \frac{1}{4i} = -\frac{i}{4}$

(B) $\operatorname{Res}\left[\frac{\cos(z)}{z^2}, 0\right] = 0$

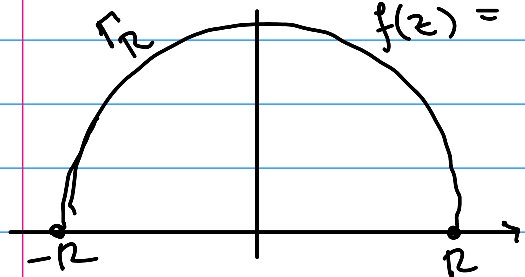
(C) $\operatorname{Res}\left[\frac{e^z}{z^5}, 0\right] = \frac{1}{4!}$

(D) $\operatorname{Res}\left[\frac{1}{z^5-1}, 1\right] = \left.\frac{1}{z^4+z^3+z^2+z+1}\right|_{z=1} = \frac{1}{5}$

Problem 7:

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{1/2 - 1/2\cos(2x)}{1+x^2}\,dx$$

$$= \operatorname{Re} \int_{-\infty}^{\infty} \frac{1/2 - 1/2\, e^{2ix}}{1+x^2}\,dx$$



$$f(z) = \frac{\frac{1}{2} - \frac{1}{2}e^{2iz}}{1+z^2}$$

$$\int_{[-R,R]\cup\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}[f(z), i]$$

$$= \left(\frac{1}{2} - \frac{1}{2}e^{-2}\right)$$

Furthermore,

$$\left| \int_{\Gamma_R} f(z)\,dz \right| \leq \int_{\Gamma_R} |f(z)| \cdot |dz| \leq \int_{\Gamma_R} \frac{1}{R^2 - 1} |dz|$$

$$< \frac{\pi R}{R^2 - 1} \to 0$$

as $R \to \infty$

Taking the limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{1/2 - 1/2\, e^{2ix}}{1+x^2}\,dx = \frac{1}{2} - \frac{1}{2}e^{-2}$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2(x)}{1+x^2}\,dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{1/2 - 1/2\, e^{2ix}}{1+x^2}\,dx = \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$