Problem 1

(A) $\int_{\Gamma} 2y\,dx + (1-x)\,dy = \int_{-1}^{2} \left[2(1-t^3)\frac{dx}{dt} + (1-t)\frac{dy}{dt} \right] dt$

$x = t$, $y = 1-t^3$, $-1 \le t \le 2$

$$= \int_{-1}^{2} 2 - 3t^2 + t^3\,dt = 2t - t^3 + \tfrac{1}{4}t^4 \Big|_{-1}^{2}$$

$$= (4 - 8 + 4) - (-2 + 1 + \tfrac{1}{4})$$

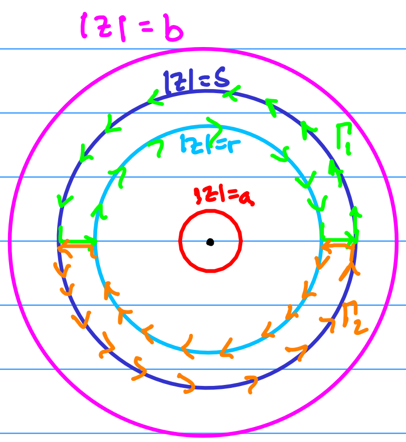
$$= \frac{3}{4}$$

(B)

$$\oint_{\Gamma} xy\,dx = \int_0^1\int_0^1 -\left(\frac{\partial xy}{\partial y}\right) dy\,dx = \int_0^1\int_0^1 -x\,dy\,dx = -\frac{1}{2}$$

Problem 2:

Let $a < r < s < b$. Then the picture is:



Green's Theorem says

$$\int_{\Gamma_2} P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$\int_{\Gamma_1} P(x,y)\,dx + Q(x,y)\,dy = 0$$

Therefore:

$$0 = \int_{\Gamma_1} P(x,y)\,dx + Q(x,y)\,dy + \int_{\Gamma_2} P(x,y)\,dx + Q(x,y)\,dy$$

$$= \oint_{|z|=s} P(x,y)\,dx + Q(x,y)\,dy - \oint_{|z|=r} P(x,y)\,dx + Q(x,y)\,dy$$

Thus $\oint_{|z|=s} P(x,y)\,dx + Q(x,y)\,dy = \oint_{|z|=r} P(x,y)\,dx + Q(x,y)\,dy$

and since r, s are arbitrary, this proves independence!

<u>Problem 3</u>: $\quad z = e^{i\theta}, \quad dz = ie^{i\theta}\, d\theta$

(A) $$\oint_{|z|=1} z^m\, dz = \int_0^{2\pi} e^{im\theta}\, ie^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} e^{i(m+1)\theta}\, d\theta = \begin{cases} 2\pi i, & m=-1 \\ 0, & m \neq -1 \end{cases}$$

(B) $$\oint_{|z|=1} \bar{z}^m\, dz = \int_0^{2\pi} e^{-im\theta}\, ie^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} e^{i(1-m)\theta}\, d\theta = \begin{cases} 2\pi i, & m = 1 \\ 0, & m \neq 1 \end{cases}$$

(C) $$\oint_{|z|=1} z^m\, |dz| = \int_0^{2\pi} e^{im\theta}\, |ie^{i\theta}|\, d\theta = \int_0^{2\pi} e^{im\theta}\, d\theta = \begin{cases} 2\pi, & m=0 \\ 0, & m \neq 0 \end{cases}$$

<u>Problem 4</u>:

$$\int_{\partial D} \bar{z}\, dz = \int_{\partial D} (x-iy)(dx+idy) = \int_{\partial D} x\,dx + y\,dy + i\int_{\partial D} -y\,dx + x\,dy$$

by Green's theorem: $$= \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) dA + i\iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial (-y)}{\partial y}\right) dA$$

$$= 0 + 2i \iint_D 1\, dA$$

$$= 2i \cdot \text{area}(A)$$