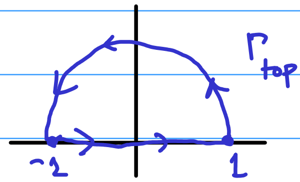
Problem 1

First suppose $c_0, \dots, c_n$ are real.



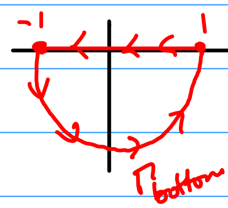
$$0 = \oint_{[-1,1]\cup\Gamma_{top}} f(z)^2\,dz = \int_{\Gamma_{top}} f(z)^2\,dz + \int_{-1}^{1} f(x)^2\,dx$$

by Cauchy's Theorem!

Since $f(x)$ is real for x real,

$$\int_{-1}^{1} |f(x)|^2\,dx = \int_{-1}^{1} f(x)^2\,dx = -\int_{\Gamma_{top}} f(z)^2\,dz \le \int_{\Gamma_{top}} |f(z)|^2 \cdot |dz|$$

Likewise,



$$0 = \oint_{[-1,1]\cup\Gamma_{bottom}} f(z)^2\,dz = \int_{\Gamma_{bottom}} f(z)^2\,dz - \int_{-1}^{1} f(x)^2\,dx$$

and therefore

$$\int_{-1}^{1} |f(x)|^2\,dx = \int_{\Gamma_{bottom}} f(z)^2\,dz \le \int_{\Gamma_{bottom}} |f(z)|^2\,dz$$

Thus

$$2\int_{-1}^{1} |f(x)|^2\,dx \le \int_{\Gamma_{top}} |f(z)|^2 \cdot |dz| + \int_{\Gamma_{bottom}} |f(z)|^2\,dz$$

$$= \int_{|z|=1} |f(z)|^2 \cdot |dz| = \int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta$$

Hence

$$\int_{-1}^{1} |f(x)|^2\,dx \le \pi \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

Notice also that $f(e^{i\theta}) = \sum_{k=0}^{n} a_k e^{ik\theta}$, $\overline{f(e^{i\theta})} = \sum_{k=0}^{n} \bar{a}_k e^{-ik\theta}$

and so $$\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{j=0}^{n}\sum_{k=0}^{n} a_j \bar{a}_k \int_0^{2\pi} e^{i(j-k)\theta} \frac{d\theta}{2\pi}$$

$$= \sum_{j=0}^{n} a_j \bar{a}_k \,\delta_{jk} = \sum_{j=0}^{\infty} |a_j|^2 .$$

Thus
$$\int_{-1}^{1} |f(x)|^2 dx \le \pi \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \pi \sum_{j=0}^{\infty} |a_j|^2$$

holds for a_j's real.

In general, if a_j's are complex:

$$f_{real}(z) = \sum_{k=0}^{n} Re(a_k) z^k \quad \text{and} \quad f_{imag}(z) = \sum_{k=0}^{n} Im(a_k) z^k$$

satisfy $f(x) = f_{real}(x) + i f_{imag}(x)$ so

$$\int_{-1}^{1} |f(x)|^2 dx = \int_{-1}^{1} |f_{real}(x)|^2 dx + \int_{-1}^{1} |f_{imag}(x)|^2 dx$$

$$\le \pi \sum_{k=0}^{n} |Re(a_k)|^2 + \pi \sum_{k=0}^{n} |Im(a_k)|^2$$

$$= \pi \sum_{k=0}^{n} |a_k|^2 = \pi \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

$\square$

Problem 2:

(A) $\displaystyle\int_{|z|=4} e^{\cos(z)}\,dz = 0$ by Cauchy's Theorem

(B) $\displaystyle\int_{|z|=1} \frac{\cosh(z)}{z^3}\,dz = 2\pi i\,\big(\cosh(z)\big)''\Big|_{z=0} = 2\pi i\cosh(0)$

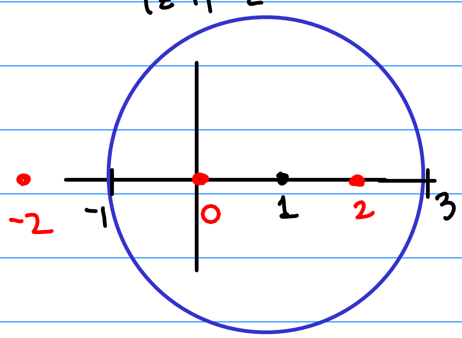
using Cauchy's Integral Formula $= 2\pi i$

(C) $\displaystyle\int_{|z|=1} \frac{1}{z^2(z^2-4)e^z}\,dz = 2\pi i\left(\frac{1}{(z^2-4)e^z}\right)'\Bigg|_{z=0}$

$\dfrac{1}{(z^2-4)e^z}$ holomorphic on $\mathbb{D}$!

$$= 2\pi i\,\frac{-[2ze^z+(z^2-4)e^z]}{(z^2-4)^2e^{2z}}\Bigg|_{z=0} = \frac{\pi i}{2}$$

(D) $\displaystyle\int_{|z-1|=2} \frac{1}{z(z^2-4)e^z}\,dz = \int_{|z-1|=2} \frac{1}{(z+2)e^z}\left(\frac{1}{z(z-2)}\right)dz$



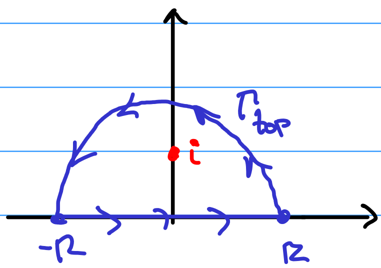
$$= \int_{|z-1|=2} \frac{1}{(z+2)e^z}\left(\frac{1/2}{z-2} - \frac{1/2}{z}\right)$$

$$= \frac{1}{2}\int_{|z-1|=2} \frac{1}{(z+2)e^z}\,\frac{1}{z-2}\,dz - \frac{1}{2}\int_{|z-1|=2} \frac{1}{(z+2)e^z}\,\frac{1}{z}\,dz$$

$$= \frac{2\pi i}{2}\left(\frac{1}{(2+2)e^2}\right) - \frac{2\pi i}{2}\left(\frac{1}{(0+2)e^0}\right)$$

$$= \boxed{\pi i\left(\frac{1}{4e^2} - \frac{1}{2}\right)}$$

Problem 3



By Cauchy's Integral Formula

$$\oint_{\Gamma_{top} \cup [-R,R]} \frac{e^{iz}}{1+z^2}\,dz = \oint_{\Gamma \cup [-R,R]} \frac{e^{iz}}{z+i}\,\frac{1}{z-i}\,dz = 2\pi i\,\frac{e^{-1}}{2i} = \frac{\pi}{e}$$

Thus $$\frac{\pi}{e} = \int_{\Gamma_{top}} \frac{e^{iz}}{1+z^2}\,dz + \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx$$

Now for $y = \operatorname{Im}(z) \geq 0$, $|e^{iz}| = |e^{ix-y}| = e^{-y} \leq 1$.

Thus for $|z| = R > 1$:

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{|1+z^2|} \leq \frac{1}{|z|^2-1} = \frac{1}{R^2-1}$$

so that

$$\left|\int_{\Gamma_{top}} \frac{e^{iz}}{1+z^2}\,dz\right| \leq \int_{\Gamma_{top}} \frac{1}{R^2-1}\,|dz| = \frac{\pi R}{R^2-1} \to 0 \quad \text{as } R \to \infty$$

Taking the limit as $R \to \infty$:

$$\frac{\pi}{e} = \lim_{R\to\infty} \int_{\Gamma_{top}} \frac{e^{iz}}{1+z^2}\,dz + \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = 0 + \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$$

Thus we have $\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$. Since $\operatorname{Re}(e^{ix}) = \cos(x)$ this gives

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \operatorname{Re}\left(\frac{\pi}{e}\right) = \frac{\pi}{e}$$

$\square$

Problem 4:

Spose $u(x,y) \le M \quad \forall (x,y) \in \mathbb{R}^2$.

Let v be a harmonic conjugate of u so

$$f(x+iy) = u(x,y) + iv(x,y) \text{ is analytic.}$$

Then

$$\left| e^{f(z)} \right| = e^{u} \le e^{M}.$$

Thus by Liouville's Theorem, $e^{f(z)}$ is constant.

$e^{f(z)} = c$. Therefore $f(z) \in \{\mathrm{Log}(c) + 2\pi i k \mid k \in \mathbb{Z}\}$ for all $z \in \mathbb{C}$. Since $f(z)$ is continuous, this implies $f(z)$ is constant. Hence u is constant.

$\square$