Math 414 Section 1
Fall 2022
Exam III
October 26, 2021
Time Limit: 50 Minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) A topological space consisting of a single point must be connected.

   (b) The set $\mathbb{R}$ with the cofinite topology is Hausdorff.

   (c) A metric space must be T_4.

   (d) In a topological space, every open cover has a Lebesgue number.

   (e) Any set with the cofinite topology is compact.

2. (10 points)

   (a) Write down what it means for a topological space X to be disconnected and what it means for a subset $A \subseteq X$ to be disconnected.

   (b) Write down the definition of a metric space being sequentially compact.

   (c) Write down Urysohn's Lemma.

   (d) Write down the Bolzano-Weierstrass Theorem.

3. (10 points)

   (a) Give an example of an infinite connected subset of $\mathbb{R}$.

   (b) Give an example of an infinite compact subset of $\mathbb{R}$.

   (c) Give an example of a topological space which is T_0 but not T_1.

   (d) Give an example of a topological space with a compact subset which is not closed.

   (e) Give an example of a topological space with just two points which is connected.

4. (10 points) In class we proved that if $A \subseteq X$ is connected, then the closure $\overline{A}$ is also connected. Reprove that fact here.

5. (10 points) For each of the following pairs of topological spaces, explain why the two topological spaces are not homeomorphic. Assume the Euclidean topology.

   (a) the Cantor set and the set of all irrational numbers in $[0, 1]$

   (b) $\mathbb{Z}$ and the open interval $(0, 1)$

   (c) the Cantor set and the set $[0, 1]$

   (d) the closed interval $[0, 1]$ and the closed unit square $[0, 1] \times [0, 1]$

Takehome Portion!

Detach this portion of the exam and take it home with you.

Problem 1: Let X be a metric space with metric d.

(a) Prove that the function

$$\rho((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}$$

is a metric on the product set $X \times X$.

(b) Prove that the topology defined by ρ is the same as the product topology on $X \times X$.

Problem 2: Consider the unit square minus one point

$$X = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1, \quad (x, y) \neq (1/2, 1/2)\}.$$

Prove that X is connected.