Math 414 Section 1
Fall 2022
Exam II
October 26, 2021
Time Limit: 110 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) Given any subset A of $\mathbb{R}$, A is open in its relative topology.

   (b) The product topology on $\mathbb{R} \times \mathbb{R}$ and the Euclidean topology on $\mathbb{R} \times \mathbb{R}$ are the same.

   (c) The Cantor set is compact.

   (d) Let P be a poset. If $m \in P$ is a maximal element, then $m \geq x$ for all $x \in P$.

   (e) Let X be compact. Then for any continuous, real-valued function f on X, there exists $a \in X$ such that $f(a) \geq f(x)$ for all $x \in X$.

2. (10 points)

   (a) Let X and Y be topological spaces. Write down the definition of the topology on $X \times Y$.

   (b) Write down the Heine-Borel Theorem.

   (c) Write down Zorn's Lemma.

   (d) Write down the definition of a compact topological space.

3. (10 points)

   (a) Give an example of a topological space X with a point $x \in X$ whose singleton set $\{x\}$ is not closed.

   (b) Give an example of a topological space which is Lindelof but not second countable.

   (c) Suppose
   $$X = \{1, 2, 3\}.$$
   Give three different examples of topologies on X.

4. (10 points) You are walking in the forest at 2AM. While passing a magical pond, you spot a frog and a toad having an intense debate about which subspaces of $\mathbb{R}$ are homeomorphic. Given your history with these two mystical amphibians, you decide to help. For each of the following pairs of subspaces of $\mathbb{R}$, determine **with proof** whether the spaces are homeomorphic.

   (a) closed interval $[0, 1]$ and $\mathbb{Q}$

   (b) $\mathbb{Q}$ and $\mathbb{Z}$

5. (10 points) In class, we proved the following theorem.

   Theorem: Let X be a compact topological space and suppose $A \subseteq X$ is closed. Then A is compact.

   Reprove this theorem here.

6. (10 points) Let X be a set with the cofinite topology, where the open sets are $\varnothing$ and complements of finite sets.

   Prove that X is compact.

7. (10 points) Consider two separate topologies on $\mathbb{Z} \times \mathbb{Z}$

- the cofinite topology;
- the product topology from the product of the cofinite topology on $\mathbb{Z}$ with the cofinite topology on $\mathbb{Z}$.

Are these topologies the same? Is one topology stronger (finer) than the other? Justify your answers.