Math 414 Section 1
Fall 2022
Exam I
September 26, 2021
Time Limit: 110 Minutes

Name (Print): ____________________

Student ID: ____________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit. This especially applies to limit calculations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.
- **Box Your Answer** where appropriate, in order to clearly indicate what you consider the answer to the question to be.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

Do not write in the table to the right.

1. (10 points) **TRUE or FALSE!** Write TRUE if the statement is true. Otherwise, write FALSE. Your response should be in ALL CAPS. No justification is required.

   (a) If a set A is uncountable, then A has the same cardinality as $\mathbb{R}$.

   (b) The union of infinitely many open sets is still open.

   (c) If $f : X \to Y$ is a function between metric spaces, then the preimage of any closed set must be closed.

   (d) The 3-adic distance between 5 and 14 is 1/9.

   (e) If $A \subseteq X$ is subset of the metric space X, then either A or A' must be closed.

2. (10 points) For each of the following, either prove that the function is a metric on $\mathbb{R}^2$ or explain why it is not.

   (a) $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \cdot |y_1 - y_2|$

   (b) $d((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + |x_2 - y_2|$

   (c) $d((x_1, y_1), (x_2, y_2)) = \frac{1}{1+|x_1-x_2|+|y_1-y_2|}$

   (d) $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

3. (10 points)

   (a) Let X and Y be metric spaces. Write the definition of a function $f : X \to Y$ being uniformly continuous.

   (b) Give an example of a metric space X and a sequence of continuous, real-valued functions on X which converge pointwise to a discontinuous function on X. Make sure to indicate the metric on X.

   (c) State the Cauchy-Schwarz inequality for $\mathbb{R}^n$ and describe how we used it in class.

4. (10 points)

   (a) Write the definition of two sets having the same cardinality.

   (b) State the Cantor-Schroeder-Bernstein theorem.

   (c) Show that the set
   $$\mathcal{F} = \{f : \{0,1\} \to \mathbb{N}\}.$$
   of functions from $\{0,1\}$ to $\mathbb{N}$ is countable.

5. (10 points) Let X be a metric space.

   (a) Write the definition of a limit point of a subset $A \subseteq X$.

   (b) Write the definition of a boundary point of a subset $A \subseteq X$.

   (c) Prove that the closed ball $\overline{B_r}(x)$ is a closed set.

6. (10 points) Let X be a set and suppose that d and ρ are two metrics on X satisfying the property that there exists a constant $M > 0$ with

$$\frac{1}{M}\rho(x, y) \leq d(x, y) \leq M\rho(x, y) \quad \text{for all } x, y \in X.$$

Prove that $U \subseteq X$ is open in the metric space (X, ρ) if and only if it is open in (X, d).

7. (10 points) Let $X = \mathbb{R}$ with the discrete metric and let $C \subseteq X$ be the Cantor set.

   (a) Determine the interior $\text{int}(C)$

   (b) Determine the boundary ∂C