Riemann sums and integrals

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Recall the problem

Problem

Compute the sum

$$\sum_{n=1}^{2024} \sum_{k=1}^{n} \left\lfloor \frac{1}{\sqrt{nk}} \right\rfloor .$$

The key was to show this inequality:

$$1 \leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{3n}} + \cdots + \frac{1}{\sqrt{n^2}} < 2.$$

We showed this by induction. Is there another way?

The Riemann Integral is the limit of the Riemann sum

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\frac{b-a}{n}.$$

where $x_k = \begin{cases} a + k(\Delta x) & \text{for right hand rule} \\ a + (k-1)(\Delta x) & \text{for left hand rule.} \end{cases}$,

with $\Delta x = \frac{b-a}{n}$.

If we write out the Riemann Sum with right hand rule corresponding to $\int_0^1 \frac{1}{\sqrt{x}} dx$, we get

$$\sum_{k=1}^{n} \frac{1}{\sqrt{\frac{k}{n}}} \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{\sqrt{kn}}.$$

Look familiar?

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Look familiar? Can we show for $n \in \mathbb{N}$,

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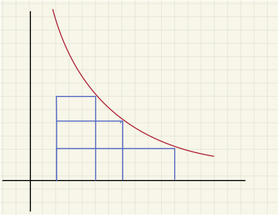
Will this help?

$$\int_0^1 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_0^1 = 2$$

Note that the function $f(x) = \frac{1}{\sqrt{x}}$ is monotonic!

The right hand rule will always be a lower bound estimate. Note that the rectangle on the right most endpoint will always be in the partitions hence $n = 1$ estimate will be the lowest. Hence $1 \leq \sum_{k=1}^{n} \frac{1}{\sqrt{kn}} < 2$.

We have an inequality principle with Riemann Sums of non-negative functions:

- For monotonically decreasing functions:
  - Right hand rule will give a lower bound.
  - Left hand rule will give an upper bound.
- Vice-versa for increasing functions:
- By carefully selecting partition points, for example, dyadically, we can obtain a monotonic sequence of estimates.

Practice: Show

$$\frac{n^{p+1}}{p+1} \leq \sum_{k=1}^{n} k^p < n^{p+1}$$

for all $n \in \mathbb{N}$ and $p > 0$, here $p \in \mathbb{R}$. What about for $-1 < p < 0$?

Aside: Faulhaber's formula is $\displaystyle\sum_{k=1}^{n} \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} B_k n^{p-k+1}$ where B_k are the Bernoulli numbers.

If we want to get a monotonic sequence, we can do something like this: Let f be a continuous increasing function on $[0, 1]$. Define

$$a_m = \frac{1}{2^m} \sum_{k=1}^{2^m} f\left(\frac{k}{2^m}\right).$$

Then $a_m \geq a_{m+1}$ and $a_m \to \int_0^1 f(x)dx$.

Show that

$$\sum_{k=1}^{2^m} \frac{k}{4^m}$$

is increasing with respect to m.

Help Wanted

Recall $\int_0^1 \log(x)dx = -1$. (Note, $\log(x) = \ln(x)$. If we apply the same principle, we have the following forms we can play with

$$
\begin{aligned}
a_m &= \frac{1}{2^m}\sum_{k=1}^{2^m} \log\left(\frac{k}{2^m}\right) \\
&= \frac{1}{2^m}\sum_{k=1}^{2^m} \log(k) - m\log(2) \\
&= \frac{1}{2^m} \log((2^m)!) - m\log(2) \\
&= \frac{1}{2^m}\sum_{k=1}^{2^m} \int_{2^m}^{k} \frac{1}{x} dx
\end{aligned}
$$

$a_m \geq a_{m+1} \geq -1$. Any problems to cook up from this?

Integrate

$$\int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \ln\left(\cos\left(\frac{x}{2^n}\right)\right) dx$$

A classic

$$\int_0^{\frac{\pi}{2}} \ln(\sin(x))dx = -\frac{\pi}{2}\ln(2).$$

Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin(x))dx$. Use substitution $x = \frac{\pi}{2} - t$. Then
$I = \int_0^{\frac{\pi}{2}} \ln(\cos(t))dx$.
So $2I = \int_0^{\frac{\pi}{2}} \ln(\sin(x)\cos(x))dx$. Use $\sin(2x) = 2\sin(x)\cos(x)$. So

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin(2x))dx - \ln(2)\frac{\pi}{2}.$$

Another substitution $2x = u$ gives us

$$2I = \frac{1}{2}\int_0^{\pi} \ln(\sin(u))du - \ln(2)\frac{\pi}{2}.$$

Finally, use $u = x + \frac{\pi}{2}$ to get $2I = I - \frac{\pi}{2}\ln(2)$.

Start with $\sin(2x) = 2\sin(x)\cos(x)$. Then

$$\begin{aligned}\sin(2^3x) &= 2\sin(2^2x)\cos(2^2x)\\ &= 2^2\sin(2x)\cos(2x)\cos(2^2x)\\ &= 2^3\sin(x)\cos(x)\cos(2x)\cos(2^2x).\end{aligned}$$

The general pattern is

$$\sin(2^nx) = 2^n\sin(x)\Pi_{k=0}^{n-1}\cos(2^kx).$$

Let $x = \frac{t}{2^n}$ and we get

$$\sin(t) = 2^n\sin\left(\frac{t}{2^n}\right)\Pi_{k=1}^{n}\cos\left(\frac{t}{2^k}\right).$$

The left side does not depend on n! We can take the limit and it will hold true:

$$\sin(t) = t\Pi_{k=1}^{\infty}\cos\left(\frac{t}{2^k}\right).$$

$$\ln(\sin(t)) = \ln(t) + \sum_{k=1}^{\infty} \ln\cos\left(\frac{t}{2^k}\right).$$

or $\ln\left(\frac{\sin(t)}{t}\right) = \sum_{k=1}^{\infty} \ln\cos\left(\frac{t}{2^k}\right)$

Related to the Dirichlet kernel, this identity holds:

$$\sum_{k=1}^{n} \cos((2k-1)x) = \frac{\sin(2nx)}{2\sin(x)}.$$

College Math Journal Problem 1247:
For every $n \in \mathbb{N}$, set

$$u_n = \sum_{k=1}^{n} \frac{1}{n + \sqrt{nk}}.$$

Find $\lim_{n\to\infty} \left(\frac{u_n}{2 - \ln(4)} \right)^n$.

Let $u = 2 - \ln(4)$. Observing that u_n is a Riemann sum, we have

$$\lim_{n\to\infty} u_n = \int_0^1 \frac{1}{1+\sqrt{x}}dx = 2 - \ln(4) = u.$$

Now

$$\left(\frac{u_n}{u}\right)^n = e^{n(\ln(u_n)-\ln(u))}.$$

By mean value theorem, we have

$$\ln(u_n) - \ln(u) = \frac{1}{c_n}(u_n - u)$$

for some c_n between u_n and u. Note that $\lim c_n = 2 - \ln(4)$.

Lemma

Suppose f is continuously differentiable on $[0,1]$. *Then*

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n} f\left(\tfrac{k}{n}\right)\tfrac{1}{n} - \int_0^1 f(x)dx\right) = \frac{f(1)-f(0)}{2}.$$

Proof of Lemma: Fix $n \in \mathbb{N}$. Rewriting $\int_{\frac{k-1}{n}}^{\frac{k}{n}} dx = \frac{1}{n}$, we have

$$\begin{aligned}\sum_{k=1}^{n} f\left(\tfrac{k}{n}\right)\tfrac{1}{n} - \int_0^1 f(x)dx &= \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\tfrac{k}{n})dx - \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx \\ &= \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\tfrac{k}{n}) - f(x)dx.\end{aligned}$$

Define $f'(a_k) = \inf\{f'(x) \mid x \in [\frac{k-1}{n}, \frac{k}{n}]\}$ and $f'(b_k) = \sup\{f'(x) \mid x \in [\frac{k-1}{n}, \frac{k}{n}]\}$. By mean value inequality, we have for each $1 \le k \le n$ and each $x \in [\frac{k-1}{n}, \frac{k}{n}]$,

$$f'(a_k)\left(\tfrac{k}{n} - x\right) \le f(\tfrac{k}{n}) - f(x) \le f'(b_k)\left(\tfrac{k}{n} - x\right).$$

Integrating both sides from $\frac{k-1}{n}$ to $\frac{k}{n}$ yields

$$\frac{f'(a_k)}{2n^2} \leq \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\tfrac{k}{n}) - f(x)dx \leq \frac{f'(b_k)}{2n^2}.$$

Now multiplying by n and summing, we have

$$\frac{1}{2}\sum_{k=1}^{n} f'(a_k)\frac{1}{n} \leq n\left(\sum_{k=1}^{n} f\left(\tfrac{k}{n}\right)\tfrac{1}{n} - \int_0^1 f(x)dx\right) \leq \frac{1}{2}\sum_{k=1}^{n} f'(b_k)\frac{1}{n}.$$

We notice that both sides are Riemann sums for the integral $\frac{1}{2}\int_0^1 f'(x)dx = \frac{f(1)-f(0)}{2}$, hence taking the limit as n goes to ∞ yields our identity.

Since $f(x) = \frac{1}{1+\sqrt{x}}$ is continuously differentiable, applying the lemma gives us the value

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{u_n}{u}\right)^n &= \lim_{n\to\infty} \exp\left(\frac{1}{c_n} n(u_n - u)\right) \\ &= \exp\left(-\frac{1}{4(2-\ln(4))}\right).\end{aligned}$$

American Math Monthly Problem 12470

Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh(2^{n-1})}\right).$$

First re-write

$$\ln\left(\frac{\tanh(2^n)}{\tanh(2^{n-1})}\right) = \ln(\tanh(2^n)) - \ln(\tanh(2^{n-1}))$$
$$= 2\int_{2^{n-1}}^{2^n} \operatorname{csch}(2x)dx$$
$$= 2^n \int_1^2 \operatorname{csch}(2^n t)dt$$

where we used a standard identity $\int \operatorname{csch}(ax)dx = a^{-1}\ln|\tanh(\frac{ax}{2})| + C$ and the change of variable $x = 2^{n-1}t$. Substituting into the original series, we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh(2^{n-1})}\right) = \sum_{n=1}^{\infty} \int_1^2 \operatorname{csch}(2^n x)dx.$$

Since $\operatorname{csch}(x) = \frac{2}{e^x - e^{-x}} \le \frac{4}{e^x}$ for $x > \frac{1}{2}\ln(2)$, by the Weierstrass M-test, the series converges uniformly on $[1, 2]$, we can interchange the integral and the sum.

Using the identity $\coth(x) = \coth(2x) + \operatorname{csch}(2x)$, we inductively apply so that

$$\coth(x) = \coth(2^k x) + \sum_{n=1}^{k} \operatorname{csch}(2^n x).$$

Since $\lim_{k\to\infty} \coth(2^k) = 1$, we have

$$\sum_{n=1}^{\infty} \operatorname{csch}(2^n x) = \coth(x) - 1.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ln\left(\frac{\tanh(2^n)}{\tanh(2^{n-1})}\right) = \int_1^2 \coth(x) - 1 dx = \ln\left(\frac{\sinh(2)}{\sinh(1)}\right) - 1.$$

Problem currently open for submission: AMM12385: Let $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$. For $0 < |t| < 1$, prove

$$(\operatorname{erf}(1))^2 \sqrt{1 - t^2} < \operatorname{erf}(\sqrt{1 - t})(\operatorname{erf}(\sqrt{1 + t}))$$