Recursion Problems

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Problem Solving Seminar

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Recursive definition

Recursive definitions involves two things:

- a base case – where the thing is defined explicitly
- a recursive step – a rule which relates the next case to previous cases

Putnam 1990 A1

Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

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Hint: compare to the first few terms of the sequence n!

$$1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880$$

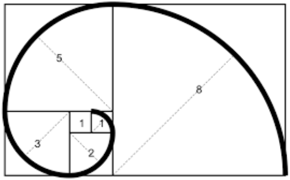
- Base case:

$$F_1 = 1, \quad F_2 = 1,$$

- Recursive step:

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 2$$

$$F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \ldots \ldots$$



A closed form

Show that

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\varphi + \varphi^{-1}},$$

where here φ is the **golden ratio**

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

Practice again

Suppose that $u_0 = 0$, $u_1 = 1$ and

$$u_{n+2} = 4u_{n+1} - 4u_n.$$

Determine the value of u_{16}.

Generalization

Suppose that u_0, u_1, and u_2 are three real numbers. Define recursively

$$u_{n+2} = \frac{u_{n+1} + u_n + u_{n-1}}{3}.$$

Determine the limit of u_n.

Suppose that $a_0 = 1$, $a_1 = 2$, and

$$a_n = 4a_{n-1} - a_{n-2}, \quad n \geq 2.$$

Determine an odd prime divisor of a_{2015}.

Nonlinear recursion

Determine the value of the sequence x_n satisfying $x_0 = 1$ and

$$x_n = 1 + 1/x_{n-1}, \quad \text{for} \quad n > 0$$

Doex x_n converge? If so, to what?

Let z_0 and z_1 be real numbers. Determine an explicit formula in terms of z_0 and z_1 for the value of

$$z_n^2 - z_{n+1}z_{n-1} = 1, \quad n \geq 1.$$