PROBLEM SOLVING SEMINAR FALL 2024 POLYNOMIALS

PROF. T.W. MURPHY

(1) (Putnam 85) $p(x)$ is a polynomial of degree 5 with 5 distinct integral roots. What is the smallest number of non-zero coefficients it can have? (Hint: which polynomials have 0 as a repeated root?)

(2) (sixth Ir. M.O.) Let $a_0, a_1, \ldots a_{n-1}$ be real numbers, $n \geq 1$, and let the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

satisfy $|f(0)| = f(1)$ and also have the property that each root α of f is real with $0 < \alpha < 1$. Prove the product of the roots does not exceed $\frac{1}{2^n}$. (hint: factor f as a product of its roots. Then plug in 0 and 1 for x. This leads us to consider $g(x) = x(1-x)$. Where is this maximized?

(3) (4th Ir.M.O.) Find all polynomials

$$f(x) = x^n + a_1x^{n-1} + \ldots + a_n$$

with the property that (a) all the coefficients $a_1, \ldots, a_n$ belong to the set $\{-1, 1\}$ and (b) all roots of the equation $f(x) = 0$ are real.

(4) (4th Ir. M.O.) Find all polynomials satisfying $f(x^2) = (f(x))^2$.
(hint: equate coefficients)

(5) (seventh Ir. M.O.) Determine with proof all real polynomials f satisfying the equation

$$f(x^2) = f(x)f(x-1).$$

(hint: if α is a root, show that α^2 is. Then show that α^4 is a root. Then show $(1+\alpha)$ is also a root of unity.