The Roots of Problems

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Problem Solving Seminar

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Roots of polynomials

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Warm-up problem

Problem

Let r_1, r_2, r_3, and r_4 be the roots of the polynomial

$$f(x) = 5x^4 - 7x^3 + x^2 - x + 9.$$

Calculate

$$r_1 + r_2 + r_3 + r_4$$

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$$f(x) = 5(x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

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$$f(x) = 5(x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

$$\begin{aligned} f(x) = 5x^4 &- 5x^3(r_1 + r_2 + r_3 + r_4) \\ &+ 5x^2(r_1 r_2 + r_1 r_3 + r_2 r_4 + r_2 r_3 + r_2 r_4 +_3 r_4) \\ &- 5x(r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4) + 5 r_1 r_2 r_3 r_4. \end{aligned}$$

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$$r_1 + r_2 + r_3 + r_4 = \frac{7}{5}.$$

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$$r_1 r_2 r_3 r_4 = \frac{9}{5}.$$

Warm-up problem

Problem

Let r_1, r_2, r_3, and r_4 be the roots of the polynomial

$$f(x) = 5x^4 - 7x^3 + x^2 - x + 9.$$

Calculate

$$\frac{1}{r_1+2}\frac{1}{r_2+2}\frac{1}{r_3+2}\frac{1}{r_4+2}$$

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Warm-up problem

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has roots $r_1 + 2$, $r_2 + 2$, $r_3 + 2$, and $r_4 + 2$.

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$$h(x) = x^4 g(x^{-1})$$

$$h(x) = 151x^4 - 249x^3 + 163x^2 - 47x + 5$$

has roots $\frac{1}{r_1+2}$, $\frac{1}{r_2+2}$, $\frac{1}{r_3+2}$, *and* $\frac{1}{r_4+2}$.

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$$h(x) = 151\left(x - \frac{1}{r_1+2}\right)\left(x - \frac{1}{r_2+2}\right)\left(x - \frac{1}{r_3+2}\right)\left(x - \frac{1}{r_4+2}\right).$$

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$$5 = 151\frac{1}{r_1+2}\frac{1}{r_2+2}\frac{1}{r_3+2}\frac{1}{r_4+2}.$$

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$$\frac{1}{r_1+2}\frac{1}{r_2+2}\frac{1}{r_3+2}\frac{1}{r_4+2} = \frac{5}{151}.$$

Putnam 1951 B6

Problem

Suppose that $f(x) = x^3 + ax^2 + bx + c$ *has three real roots* r_1, r_2, r_3 *with* $r_1 \leq r_2 \leq r_3$*. Show that*

$$\sqrt{a^2 - 3b} \leq r_3 - r_1 \leq \frac{2}{\sqrt{3}}\sqrt{a^2 - 3b}$$

Solution

Solution

$$\begin{aligned} f(x) &= x^3 + ax^2 + bx + c \\ &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3. \end{aligned}$$

Solution

$$\begin{aligned} f(x) &= x^3 + ax^2 + bx + c \\ &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3. \end{aligned}$$

From this we see

$$\begin{aligned} a^2 - 3b &= r_1^2 + r_2^2 + r_3^2 - (r_1 r_2 + r_1 r_3 + r_2 r_3) \\ &= \frac{1}{2}(r_2 - r_1)^2 + \frac{1}{2}(r_3 - r_2)^2 + \frac{1}{2}(r_3 - r_1)^2 \\ &= (r_3 - r_1)^2 - (r_3 - r_2)(r_2 - r_1) \end{aligned}$$

Solution

The ***AM-GM inequality*** *says:*

$$\sqrt{AB} \leq \frac{A+B}{2}.$$

Solution

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Therefore

$$0 \leq (r_3 - r_2)(r_2 - r_1) \leq \left(\frac{(r_3 - r_2) + (r_2 - r_1)}{2}\right)^2 = \left(\frac{r_3 - r_1}{2}\right)^2$$

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$$-\left(\frac{r_3 - r_1}{2}\right)^2 \leq -(r_3 - r_2)(r_2 - r_1) \leq 0$$

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$$-\left(\frac{r_3 - r_1}{2}\right)^2 \le -(r_3 - r_2)(r_2 - r_1) \le 0$$

$$\frac{3}{4}(r_3 - r_1)^2 \le (r_3 - r_1)^2 - (r_3 - r_2)(r_2 - r_1) \le (r_3 - r_1)^2$$

Solution

Since

$$a^2 - 3b = (r_3 - r_1)^2 - (r_3 - r_2)(r_2 - r_1)$$

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Therefore

$$\sqrt{a^2 - 3b} \leq r_3 - r_1,$$

and also

$$r_3 - r_1 \leq \frac{2}{\sqrt{3}}\sqrt{a^2 - 3b}.$$

Warm-up problem

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Show that the polynomial

$$f(x) = x^3 + 4x + 1$$

has at least one real root.

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Solution

$f(1) = 6$ *and* $f(-1) = -4$*, so the* ***Intermediate Value Theorem*** *says that there is a value* x *between* 6 *and* -1 *with* $f(x) = 0$.

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Suppose $f(x)$ *has at least two roots at* $x = r_1$ *and* $x = r_2$.

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Suppose $f(x)$ *has at least two roots at* $x = r_1$ *and* $x = r_2$.
If $r_1 = r_2$, *then* $f'(r_1) = 0$, *which we know is impossible.*

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If $r_1 = r_2$, *then* $f'(r_1) = 0$, *which we know is impossible.*
If $r_1 \neq r_2$, *then* ***Rolle's Theorem*** *or the* ***Mean Value Theorem*** *implies that there is a point* x *between* r_1 *and* r_2 *with* $f'(x) = 0$, *which is also impossible.*

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If $r_1 \neq r_2$, *then* ***Rolle's Theorem*** *or the* ***Mean Value Theorem*** *implies that there is a point* x *between* r_1 *and* r_2 *with* $f'(x) = 0$, *which is also impossible.*
Therefore $f(x)$ *has no more than one root.*

Putnam 2015 B1

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Hint: consider $e^{x/2} f(x)$

Putnam 2015 B1

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Putnam 2015 B1

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$$\begin{aligned} g'''(x) &= e^{x/2}\left(\frac{1}{8}f(x) + \frac{6}{8}f'(x) + \frac{6}{4}f''(x) + f'''(x)\right) \\ &= \frac{1}{8}e^{x/2}(f(x) + 6f'(x) + 12f''(x) + 8f'''(x)) \end{aligned}$$

Solution

Let $g(x) = e^{x/2} f(x)$.
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By Rolle's Theorem, $g'(x)$ *has at least four distinct zeros.*
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$$
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g'''(x) &= e^{x/2}\left(\frac{1}{8}f(x) + \frac{6}{8}f'(x) + \frac{6}{4}f''(x) + f'''(x)\right) \\
&= \frac{1}{8}e^{x/2}\left(f(x) + 6f'(x) + 12f''(x) + 8f'''(x)\right)
\end{aligned}
$$

$e^{x/2}$ *is never zero, so the term in parentheses has two zeros!*

Putnam 1985 B1

Problem

Suppose that

$$f(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

for some distinct integers $m_1, \ldots, m_5$. If we choose the values of these integers carefully, how many of the coefficients of $f(x)$ can we force to be zero?

Putnam 1985 B1

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$$\begin{aligned} f(x) = x^5 &- (m_1 + m_2 + m_3 + m_4 + m_5)x^4 \\ &+ (m_1 m_2 + m_1 m_3 + m_1 m_4 + m_1 m_5 + m_2 m_3 \\ &\quad + m_2 m_4 + m_2 m_5 + m_3 m_4 + m_3 m_5 + m_4 m_5)x^3 \\ &- (m_1 m_2 m_3 + m_1 m_2 m_4 + m_1 m_2 m_5 + m_1 m_3 m_4 + m_1 m_3 m_5 \\ &\quad + m_1 m_4 m_5 + m_2 m_3 m_4 + m_2 m_3 m_5 + m_2 m_4 m_5 + m_3 m_4 m_5)x^2 \\ &+ (m_1 m_2 m_3 m_4 + m_1 m_2 m_3 m_5 + m_1 m_2 m_4 m_5 \\ &\quad + m_1 m_3 m_4 m_5 + m_2 m_3 m_4 m_5)x - m_1 m_2 m_3 m_4 m_5 \end{aligned}$$

Putnam 1985 B1

Solution

Take $m_5 = 0$*:*

$$\begin{aligned} f(x) = x^5 &- (m_1 + m_2 + m_3 + m_4)x^4 \\ &+ (m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4)x^3 \\ &- (m_1 m_2 m_3 + m_1 m_2 m_4 + m_1 m_3 m_4 + m_2 m_3 m_4)x^2 \\ &+ m_1 m_2 m_3 m_4 x \end{aligned}$$

Putnam 1985 B1

Solution

Take $m_4 = -(m_1 + m_2 + m_3)$*:*

$$\begin{aligned} f(x) = x^5 &+ (m_1 m_2 + m_1 m_3 + m_2 m_3 - (m_1 + m_2 + m_3)^2)x^3 \\ &- (m_1 m_2 m_3 - (m_1 m_2 + m_1 m_3 + m_2 m_3)(m_1 + m_2 + m_3))x^2 \\ &- m_1 m_2 m_3 (m_1 + m_2 + m_3)x \end{aligned}$$

Solution

Take $m_4 = -(m_1 + m_2 + m_3)$*:*

$$\begin{aligned} f(x) = x^5 &+ (m_1 m_2 + m_1 m_3 + m_2 m_3 - (m_1 + m_2 + m_3)^2)x^3 \\ &- (m_1 m_2 m_3 - (m_1 m_2 + m_1 m_3 + m_2 m_3)(m_1 + m_2 + m_3))x^2 \\ &- m_1 m_2 m_3 (m_1 + m_2 + m_3)x \end{aligned}$$

$$\begin{aligned} f(x) = x^5 &- (m_1 m_2 + m_1 m_3 + m_2 m_3 + m_1^2 + m_2^2 + m_3^2)x^3 \\ &+ (m_1 m_2 m_3 + m_1^2 m_2 + m_1 m_2^2 + m_1^2 m_3 + m_1 m_3^2 + m_2^2 m_3 + m_2 m_3^2)x^2 \\ &- m_1 m_2 m_3 (m_1 + m_2 + m_3)x \end{aligned}$$

Solution

Take $m_4 = -(m_1 + m_2 + m_3)$*:*

$$\begin{aligned} f(x) = x^5 &+ (m_1 m_2 + m_1 m_3 + m_2 m_3 - (m_1 + m_2 + m_3)^2)x^3 \\ &- (m_1 m_2 m_3 - (m_1 m_2 + m_1 m_3 + m_2 m_3)(m_1 + m_2 + m_3))x^2 \\ &- m_1 m_2 m_3 (m_1 + m_2 + m_3)x \end{aligned}$$

$$\begin{aligned} f(x) = x^5 &- (m_1 m_2 + m_1 m_3 + m_2 m_3 + m_1^2 + m_2^2 + m_3^2)x^3 \\ &+ (m_1 m_2 m_3 + m_1^2 m_2 + m_1 m_2^2 + m_1^2 m_3 + m_1 m_3^2 + m_2^2 m_3 + m_2 m_3^2)x^2 \\ &- m_1 m_2 m_3 (m_1 + m_2 + m_3)x \end{aligned}$$

$$\begin{aligned} f(x) = x^5 &- (m_1 m_2 + m_1 m_3 + m_2 m_3 + m_1^2 + m_2^2 + m_3^2)x^3 \\ &+ (m_1 + m_2)(m_1 + m_3)(m_2 + m_3)x^2 \\ &- m_1 m_2 m_3 (m_1 + m_2 + m_3)x \end{aligned}$$

Solution

Take $m_3 = -m_2$:

$$f(x) = x^5 - (m_2^2 + m_1^2)x^3 + m_1^2 m_2^2 x$$

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$$m_3 = -m_2, \quad m_4 = -m_1, \quad m_5 = 0.$$

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Suspicion: *the answer is we can two coefficients be zero.*
Question: *can we have even more be zero???*

Solution

Solution

- *Just one coefficient?*

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$$f(x) = x^5 \quad \textit{has repeated roots...impossible!}$$

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$$f(x) = x^5 \quad \textit{has repeated roots...impossible!}$$

- *Just two coefficients?*

$$f(x) = x^5 + ax \quad \textit{has complex roots...impossible!}$$

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Solution

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