Combinatorics: Fancy Counting

Matt Rathbun

September 13, 2024

Putnam (2017), A6.
Scores NA–10: (40, 4, 7, 0, 0, 0, 0, 0, 3, 0, 2)

Problem

The 30 edges of a regular icosahedron are distinguished by labeling them 1, 2, . . . , 30. How many different ways are there to paint each edge red, white, or blue, such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?

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Warm-Up 1

Problem

Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

- $X_1 \cup X_2 \cup X_3 = \{1, 2, 3, 4, 5, 6, 7, 8\}$, *and*
- $X_1 \cap X_2 \cap X_3 = \emptyset$.

Warm-Up 1

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Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

- $X_1 \cup X_2 \cup X_3 = \{1\}$, *and*
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Solution

1 must be in at least one set, and in no more than two sets!

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- $X_1 = \{1\}, X_2 = X_3 = \emptyset$
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Problem

Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

- $X_1 \cup X_2 \cup X_3 = \{1, 2\}$, *and*
- $X_1 \cap X_2 \cap X_3 = \emptyset$.

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Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

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2 must be in at least one set, and in no more that two sets!

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Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

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Solution

1 must be in at least one set, and in no more than two sets!
2 must be in at least one set, and in no more that two sets!
There are 6 ways that 1's can be placed among the sets.

Problem

Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

- $X_1 \cup X_2 \cup X_3 = \{1, 2\}$, *and*
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Solution

1 must be in at least one set, and in no more than two sets!
2 must be in at least one set, and in no more that two sets!
There are 6 ways that 1's can be placed among the sets.
There are 6 ways that 2's can be placed among the sets.

Problem

Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

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Solution

1 must be in at least one set, and in no more than two sets!
2 must be in at least one set, and in no more that two sets!
There are 6 ways that 1's can be placed among the sets.
There are 6 ways that 2's can be placed among the sets.
There are 36 ways that 1 and 2 can be placed among the sets.

The Multiplication Principle

Theorem

If there are A ways of doing one thing, and B ways of doing another (independent) thing, then there are $A \cdot B$ ways of doing both things.

Warm-Up 1

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Warm-Up 1

Problem

Find the number of ordered triples of sets X_1, X_2, X_3 satisfying

- $X_1 \cup X_2 \cup X_3 = \{1, 2, 3, 4, 5, 6, 7, 8\}$, *and*
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Solution

There are 6^8 possible ordered triples of sets X_1, X_2, X_3 satisfying the conditions.

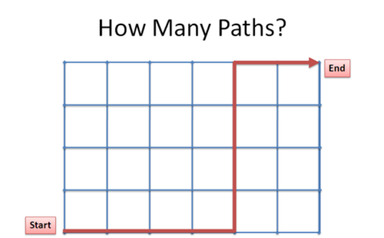
Problem-Solving Strategies

- Simplify the problem.
- Directly enumerate in a systematic/organized way.
- Multiplication Principle.

Warm-Up 2

Problem

(a) *How many routes are there from the lower-left corner to the upper-right corner of an $m \times n$ grid in which we are restricted to traveling only to the right or upward?*



(b) *Derive the formula*

$$\sum_{k=0}^{n} \frac{(k+m-1)!}{k!(m-1)!} = \frac{(m+n)!}{m!n!}.$$

Warm-Up 2

Part (a)

Solution

- *Every path consists of some number of ‘up’ moves, and some number of ‘right’ moves.*

Warm-Up 2

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- *Every path consists of some number of 'up' moves, and some number of 'right' moves.*
- *Every path has the same number of moves:* $m + n$.

Warm-Up 2

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- *Every path has the same number of moves:* $m + n$*.*

If we choose the m *positions for the 'up' moves among the* $m + n$ *positions, the rest will be 'right' moves.*

Warm-Up 2

Part (a)

Solution

- *Every path consists of some number of 'up' moves, and some number of 'right' moves.*
- *Every path has the same number of moves: $m + n$.*

If we choose the m positions for the 'up' moves among the $m + n$ positions, the rest will be 'right' moves.
So we must choose m things from among $m + n$ things:

$$_{(m+n)}C_m = \binom{m+n}{m} = \frac{(m+n)!}{m!n!}.$$

Warm-Up 2

Part (b)

- We're meant to relate this number to a *sum* of things.

Warm-Up 2
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- We're meant to relate this number to a *sum* of things.
- This should involve counting things *separately*.

The Addition Principle

Theorem

If there are A ways of doing one thing, B ways of doing another, but you can't do both, then there are $A + B$ ways of doing a thing.

Warm-Up 2

Part(b)

Solution

Let's think about when the paths first hit the top of the grid!

Warm-Up 2

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Solution

Let's think about when the paths first hit the top of the grid!
This gives a partition of the paths into n classes:

- *Every path is in at least one of the classes, and*
- *Every path is in no more than one class.*

Warm-Up 2

Part(b)

Solution

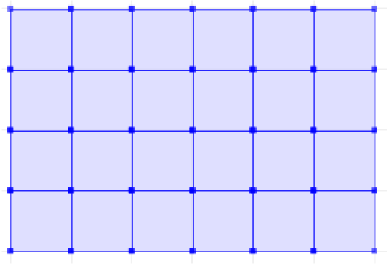
Let's think about when the paths first hit the top of the grid!
This gives a partition of the paths into n classes:

- *Every path is in at least one of the classes, and*
- *Every path is in no more than one class.*

So it suffices to count the number of paths in each class. Then we can add the number in each class to get the total.

Warm-Up 2

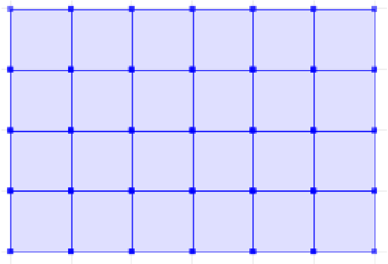
Part (b)



- How many paths first hit the top on the far left?

Warm-Up 2

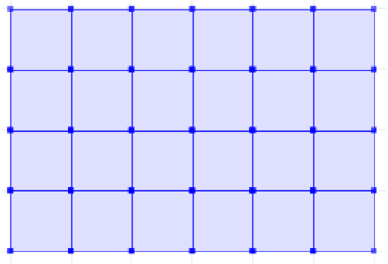
Part (b)



- How many paths first hit the top on the far left? 1.

Warm-Up 2

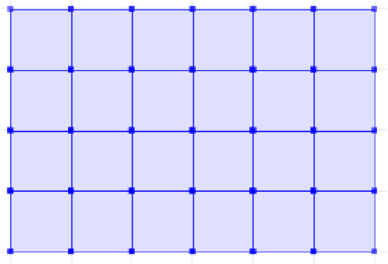
Part (b)



- How many paths first hit the top on the far left? 1.
- How many paths first hit the top one from the left?

Warm-Up 2

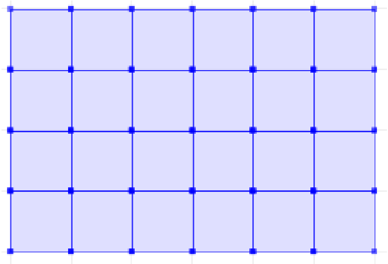
Part (b)



- How many paths first hit the top on the far left? 1.
- How many paths first hit the top one from the left? 4

Warm-Up 2

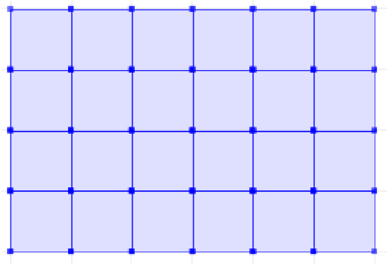
Part (b)



- How many paths first hit the top on the far left? 1.
- How many paths first hit the top one from the left? $4 = \binom{4}{1}$

Warm-Up 2

Part (b)



- How many paths first hit the top on the far left? 1.
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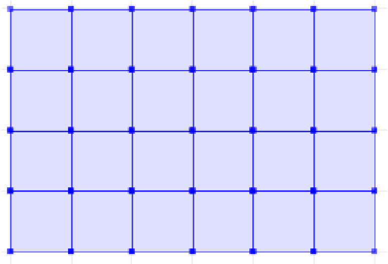
Warm-Up 2

Part (b)



- How many paths first hit the top on the far left? 1.
- How many paths first hit the top one from the left? $4 = \binom{4}{1}$
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Warm-Up 2

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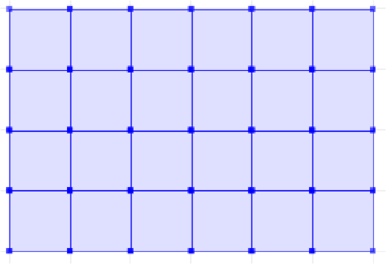
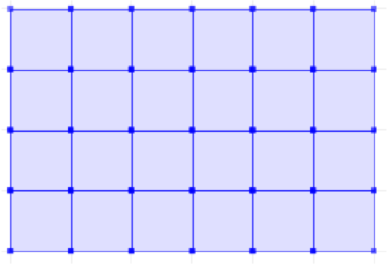


- How many paths first hit the top on the far left? 1.
- How many paths first hit the top one from the left? $4 = \binom{4}{1}$
- How many paths first hit the top two from the left? $10 = \binom{5}{2}$.

Warm-Up 2

Part (b)



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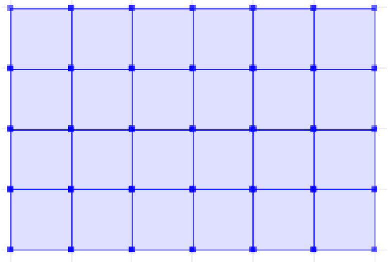
- How many paths first hit the top three from the left?

Warm-Up 2

Part (b)



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Warm-Up 2

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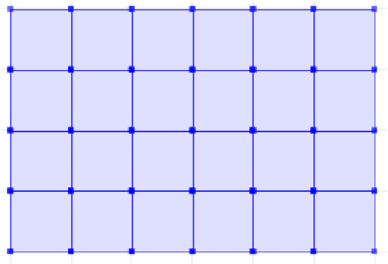

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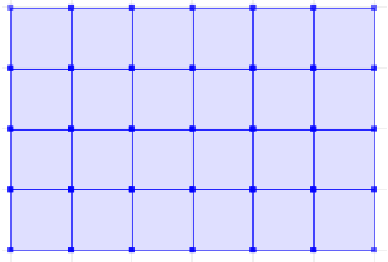
- How many paths first hit the top three from the left? $20 = \binom{6}{3}$.
- How many paths first hit the top k from the left?

Warm-Up 2

Part (b)



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- How many paths first hit the top two from the left? $10 = \binom{5}{2}$.
- How many paths first hit the top three from the left? $20 = \binom{6}{3}$.
- How many paths first hit the top k from the left? $\binom{k+4-1}{k}$.

Warm-Up 2

Part (b)

Solution

More generally, the number of paths first hitting the top of an $m \times n$ grid k from the left is

$$\binom{k+m-1}{k},$$

and the Addition Principle proves the equation in part (b).

Problem-Solving Strategies

- Multiplication Principle.
- Addition Principle.
- Break the problem into pieces.
- Analyze specific cases.
- Find patterns.

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Problem

The 30 edges of a regular icosahedron are distinguished by labeling them 1, 2, . . . , 30. How many different ways are there to paint each edge red, ~~white~~ green, or blue, such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color?

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Solution (from Bill Huang, via Art of Problem Solving user `superpi83`)

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Let

- *v and w be two antipodal vertices of the icosahedron,*

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Let

- *v and w be two antipodal vertices of the icosahedron,*
- *S_v (resp., S_w) be the set of five edges incident to v (resp., w),*

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Let

- *v and w be two antipodal vertices of the icosahedron,*
- *S_v (resp., S_w) be the set of five edges incident to v (resp., w),*
- *C_v (resp., C_w) be the set of five edges of the pentagon formed by the opposite endpoints of the five edges in S_v (resp., S_w),*

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- L *be the set of the ten remaining edges of the icosahedron.*

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- *L be the set of the ten remaining edges of the icosahedron.*
- *How many (total) possible colorings are there of L?*

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- *L be the set of the ten remaining edges of the icosahedron.*
- *How many (total) possible colorings are there of L?* 3^{10}.

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Solution

- *The edges of $C_v \cup L$ form the boundaries of five faces with no edges in common.*

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- *The edges of $C_v \cup L$ form the boundaries of five faces with no edges in common.*
- *Thus, each edge of C_v,* ***regardless of how the two corresponding edges in L are colored,***

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Solution

- *The edges of $C_v \cup L$ form the boundaries of five faces with no edges in common.*
- *Thus, each edge of C_v, **regardless of how the two corresponding edges in L are colored**, can be colored in one of two ways consistent with the given condition.*

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Solution

- *The edges of $C_v \cup L$ form the boundaries of five faces with no edges in common.*
- *Thus, each edge of C_v, **regardless of how the two corresponding edges in L are colored**, can be colored in one of two ways consistent with the given condition.*
- *Same for C_w.*
- *For $C_v \cup C_w \cup L$, how many possible colorings are there, consistent with the given condition?*

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Solution

- *The edges of $C_v \cup L$ form the boundaries of five faces with no edges in common.*
- *Thus, each edge of C_v, **regardless of how the two corresponding edges in** L **are colored**, can be colored in one of two ways consistent with the given condition.*
- *Same for C_w.*
- *For $C_v \cup C_w \cup L$, how many possible colorings are there, consistent with the given condition?* $3^{10}2^{10}$

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Solution

- *To complete the count, it suffices to check that there are exactly* 2^5 *ways to color each of* S_v *and* S_w *consistent with any given coloring of* C_v *and* C_w*, respectively.*

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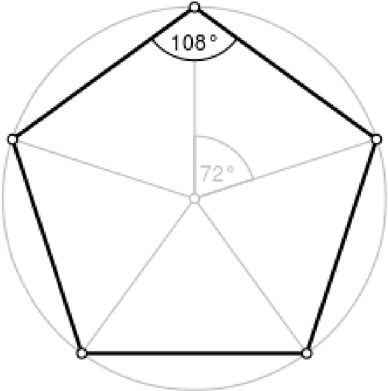
Solution

- *To complete the count, it suffices to check that there are exactly* 2^5 *ways to color each of* S_v *and* S_w *consistent with any given coloring of* C_v *and* C_w*, respectively.*
- *This makes a grand total of* $(3^{10}2^{10})2^5 2^5 = 3^{10}2^{20}$ *colorings of the edges consistent with the conditions.*

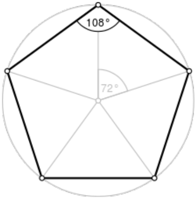
Last step



Exercise

1. *There are the same number of colorings of S_v consistent with any given coloring of C_v.*
2. *That number is $2^5 = 32$.*

Last step, Step 2

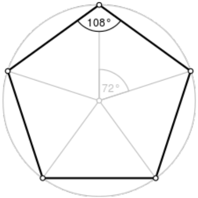


Proof.

Assume that all of the edges of C_v are colored green, say.

- How many colorings of S_v have 0 green edges?

Last step, Step 2

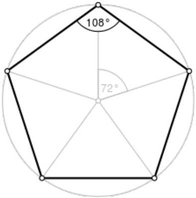


Proof.

Assume that all of the edges of C_v are colored green, say.

- How many colorings of S_v have 0 green edges? 2
- How many colorings of S_v have 1 green edge?

Last step, Step 2

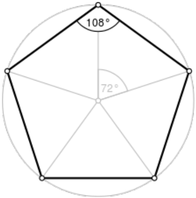


Proof.

Assume that all of the edges of C_v are colored green, say.

- How many colorings of S_v have 0 green edges? 2
- How many colorings of S_v have 1 green edge? $2 \times 5 = 10$

Last step, Step 2



Proof.

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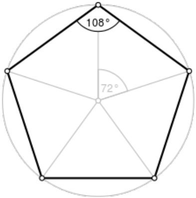
- How many colorings of S_v have 2 green edges?

Last step, Step 2

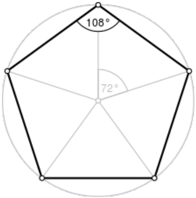


Proof.

Assume that all of the edges of C_v are colored green, say.

- How many colorings of S_v have 0 green edges? 2
- How many colorings of S_v have 1 green edge? $2 \times 5 = 10$
- How many colorings of S_v have 2 green edges? $2 \times 2 \times 5 = 20$

Last step, Step 2



Proof.

Assume that all of the edges of C_v are colored green, say.

- How many colorings of S_v have 0 green edges? 2
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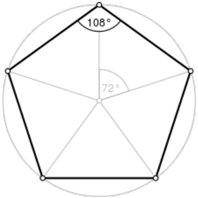
- How many colorings of S_v have > 2 green edges?

Last step, Step 2



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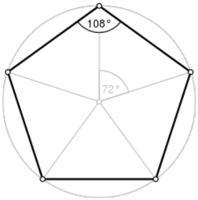
- How many colorings of S_v have >2 green edges? 0

Last step, Step 2



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Assume that all of the edges of C_v are colored green, say.

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- How many colorings of S_v have 2 green edges? $2 \times 2 \times 5 = 20$
- How many colorings of S_v have > 2 green edges? 0
- There are 32 colorings of S_v compatible with the given coloring of C_v.

□

Last step, Step 1

Proof.

Suppose C_v has **some** coloring, and there are N compatible colorings of S_v.

Last step, Step 1

Proof.

Suppose C_v has **some** coloring, and there are N compatible colorings of S_v. What happens to N if we want to change the color of one edge in S_v?

Last step, Step 1

Proof.

Suppose C_v has **some** coloring, and there are N compatible colorings of S_v. What happens to N if we want to change the color of one edge in S_v?

- Let the edge be initially colored a, and try to color it $b \neq a$.

Last step, Step 1

Proof.

Suppose C_v has **some** coloring, and there are N compatible colorings of S_v. What happens to N if we want to change the color of one edge in S_v?

- Let the edge be initially colored a, and try to color it $b \neq a$.
- The number of colorings of the adjacent triangle in L with both edges labeled $c \neq a$ and $c \neq b$ is **unchanged**

Last step, Step 1

Proof.

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- Let the edge be initially colored a, and try to color it $b \neq a$.
- The number of colorings of the adjacent triangle in L with both edges labeled $c \neq a$ and $c \neq b$ is **unchanged**
- The number of colorings of the adjacent triangle in L with a and b is **unchanged**

Last step, Step 1

Proof.

Suppose C_v has **some** coloring, and there are N compatible colorings of S_v. What happens to N if we want to change the color of one edge in S_v?

- Let the edge be initially colored a, and try to color it $b \neq a$.
- The number of colorings of the adjacent triangle in L with both edges labeled $c \neq a$ and $c \neq b$ is **unchanged**
- The number of colorings of the adjacent triangle in L with a and b is **unchanged**
- Consider the number of colorings of the adjacent triangle in L with b and b . . .

Last step, Step 1

Proof.

Suppose C_v has **some** coloring, and there are N compatible colorings of S_v. What happens to N if we want to change the color of one edge in S_v?

- Let the edge be initially colored a, and try to color it $b \neq a$.
- The number of colorings of the adjacent triangle in L with both edges labeled $c \neq a$ and $c \neq b$ is **unchanged**
- The number of colorings of the adjacent triangle in L with a and b is **unchanged**
- Consider the number of colorings of the adjacent triangle in L with b and b ...
  - Look at the triangle to the left, with labels other x and y (not both b), then third edge can be re-labeled, to keep the numbers the same.

$\square$