Financial Econometrics Econ 40357
Vector Autoregressions (VARs)
Local Projections

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Unrestricted VAR

- Consider two zero-mean (or in deviations from the mean) covariance-stationary time series, $y_{1,t}$ and $y_{2,t}$
    - Example: $y_{1,t}$ GDP growth, $y_{2,t}$ the market excess return

$$y_t = \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix}, \; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; \epsilon_t = \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \overset{iid}{\sim} N\left(\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{0}, \underbrace{\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}}_{\Sigma} \right)$$

The VAR(1) is

$$y_t = Ay_{t-1} + \epsilon_t$$

$$\underbrace{\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}}_{y_t} = \underbrace{\begin{pmatrix} a_{11,1} & a_{12,1} \\ a_{21,1} & a_{22,1} \end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix}}_{y_{t-1}} + \underbrace{\begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}}_{\epsilon_t}$$

$$\begin{aligned} y_{1,t} &= a_{11,1} y_{1,t-1} + a_{12,1} y_{2,t-1} + \epsilon_{1,t} \\ y_{2,t} &= a_{21,1} y_{1,t-1} + a_{22,1} y_{2,t-1} + \epsilon_{2,t} \end{aligned}$$

- Same explanatory variables in each equation.
- Why are we doing this? We want to see how the market return responds to a shock to GDP (and vice-ersa).
- This is called the reduced form model. More explanation below.

Estimation

- Estimate each equation separately by least squares.
- Estimate error-covariance matrix Σ with sample counterparts from the regression residuals.
- Select lag length with information criteria (AIC, BIC, etc).
- k is total number of regression coefficients (the $a_{ij,r}$ coefficients in system. In bivariate VAR(1) $k = 6$ including constants.
- For VAR(p),

$$\text{AIC} = 2 \ln |\hat{\Sigma}_p| + \frac{2k}{T}.$$

$$\text{BIC} = 2 \ln |\hat{\Sigma}_p| + \frac{k \ln T}{T}.$$

- $|\Sigma|$ is the determinant of the covariance matrix.

Granger causality, econometric exogeneity

- y_{1t} **does not Granger cause** y_{2t} if lagged y_{1t} do not appear in the equation for y_{2t}.

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} a_{11,1} & a_{12,1} \\ 0 & a_{22,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}$$

  Conditional upon lagged y_{2t}, lagged y_{1t} do not help to predict y_{2t}.

- To test the null hypothesis that y_{1t} does not Granger cause y_{2t}, regress y_{2t} on lagged y_{1t} and lagged y_{2t}, do t-test for the significance of the coefficients on lagged y_{1t}.
- If y_{1t} **does not Granger cause** y_{2t}, then y_{2t} is **econometrically exogenous** with respect to y_{1t}.

Impulse Response Analysis

Remember MA(∞) representation of AR(1) and impulse response?

$$\begin{aligned} y_t &= \rho y_{t-1} + \epsilon_t \\ &= \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots \end{aligned}$$

Impulse responses

$$\begin{aligned} y_1 &= \epsilon_1 = 1 \\ y_2 &= \rho \\ y_3 &= \rho^3 \end{aligned}$$

Impulse Response Analysis

Do the same repeated substitution for VAR(1) to get the VMA(∞) (vector moving average)

$$\begin{aligned} y_t &= Ay_{t-1} + \epsilon_t \\ &= \epsilon_t + A\epsilon_{t-1} + A^2\epsilon_{t-2} + A^3\epsilon_{t-3} + \cdots \end{aligned}$$

$A^2 = AA$, $A^3 = AAA$ etc.

- We have moving-average representation. Next, employ impulse response analysis to evaluate the dynamic effect of shocks in each variable on (y_{1t}, y_{2t}).
- Two new issues. We want to simulate dynamic response of y_{1t} and y_{2t} to a shock to ϵ_{1t}
  1. How big should the shock be? This is an issue because you want to compare the response of y_{1t} across different shocks. We must **normalize** the size of the shocks. Usually, people set size of shock to be one standard deviation in size.

     Divide each shock by its standard deviation. (Eviews does this automatically)
  2. Need shocks that are unambiguously attributed to y_{1t} and to y_{2t}. If ϵ_{1t} and ϵ_{2t} are correlated, you can't just shock ϵ_{1t} and hold ϵ_{2t} constant. Need to make the shocks uncorrelated. (Orthogonalizing the shocks).

Orthogonalizing Correlated Variables

- Here is the idea behind orthogonalizing (decorrelating) correlated variables. Not covering the actual way VARs are orthogonalized, just the concepts.
  - Show how to build up correlated random variables from independent random variables.
  - Run the process in reverse to orthogonalize

Creating Bi-variate Normal Random Variables

- Let z_1 and z_2 be independent standard normal random variables. Build the random variables ϵ_1 and ϵ_2 as linear combinations of z_1 and z_2.

$$\begin{aligned}
\epsilon_1 &= \sigma_1 z_1 + \mu_1 \\
\epsilon_2 &= \sigma_2 \left(\rho z_1 + \sqrt{(1-\rho^2)} \right) z_2 + \mu_2
\end{aligned}$$

- ϵ_1 and ϵ_2 are normally distributed. That's because they are linear combinations of normals.
- See the overlap of z_1 in both ϵ_1 and ϵ_2? That means they are correlated.

$$\begin{aligned}
E(\epsilon_1) &= \mu_1, \; E(\epsilon_2) = \mu_2 \\
\text{Var}(\epsilon_1) &= \sigma_1^2, \; \text{Var}(\epsilon_2) = \sigma_2^2 \left(\rho^2 + \left(1 - \rho^2 \right) \right) = \sigma_2^2 \\
\text{Cov}(\epsilon_1, \epsilon_2) &= \text{E}\left(\sigma_1 z_1 \left(\sigma_2 \left(\rho z_1 + \sqrt{(1-\rho^2)} \right) z_2 \right) \right) = \sigma_1 \sigma_2 \rho \\
\text{Corr}(\epsilon_1, \epsilon_2) &= \rho
\end{aligned}$$

- We built the $\epsilon' s$ from the $z' s$, so given the $\epsilon' s$, we should be able to unpack the $z' s$.
- The ϵ_1 and ϵ_2 are like the reduced form errors in the VAR. The $z' s$ are like what we call structural shocks.

Reverse Engineer. Recover the $z's$

$$
\begin{aligned}
z_1 &= \frac{1}{\sigma_1}\left(\epsilon_1 - \mu_1\right) \\
z_2 &= \frac{\sigma_1\left(\epsilon_2 - \mu_2\right) - \rho\sigma_2\left(\epsilon_1 - \mu_1\right)}{\sigma_1\sigma_2\sqrt{1-\rho^2}}
\end{aligned}
$$

VAR method uses something called the **Choleski (**or Choleski) decomposition of the error covariance matrix, Σ

Orthogonalized Shocks

Now write in matrix form

$$\underbrace{\begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}}_{\epsilon_t} = \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} z_{1t} \\ z_{2t} \end{pmatrix}}_{z_t}$$

$$\epsilon_t = \Lambda z_t$$

substitute into vector MA(∞) representation.

$$\begin{aligned} y_t &= \epsilon_t + A\epsilon_{t-1} + A^2\epsilon_{t-2} + \cdots \\ &\Lambda z_t + A\Lambda z_{t-1} + A^2 \Lambda z_{t-2} + \cdots \end{aligned}$$

Now we can shock z_2 without disturbing z_1.

Example: Climate and the Real Exchange Rate

- The real exchange rate is price of foreign goods in terms of US goods. If S is USD per foreign currency, P the US price level, P^* the foreign price level, then

$$Q = \frac{SP^*}{P}$$

  is the real exchange rate.
- $\uparrow Q$ means USD loses in real terms. $\downarrow Q$ means USD gains in real terms.
- Exchange rate is a national asset. The relative price of two currencies (in real terms), which are claims on all the stuff in those countries.
- Asset prices (exchange rates) are forward looking. Discounted present values of future economic fundamentals.
- Appreciating USD ($\downarrow Q$) means US fundamentals look better than foreign country's, and vice versa.

Climate and the Real Exchange Rate

- We ask, what is the exposure of the US relative to a foreign country, to climate change?
- Climate variable is the cross-sectional average of temperature readings all around the world, sampled monthly.
  - This is the first principal component, the first factor of temperature. We do principal components in future class.
  - Deseasonalize and detrend the climate variable. (show picture)
- Workfile: vars_lps_climateexra.wf1

Running VAR in Eviews

1. Quick $\rightarrow$ Estimate Var.... Fill in the variables and lag choice.
2. After estimation, click on impulse tab.