Public Economics (ECON 131)
Section #10: Public goods

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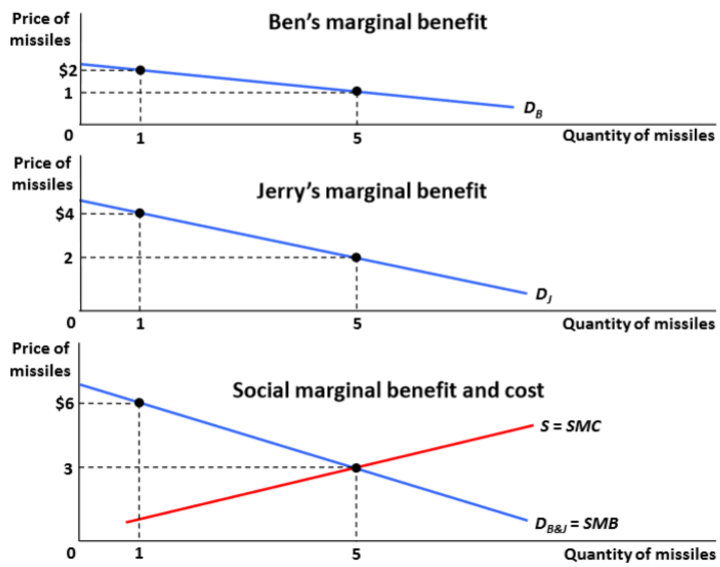
1 Key Concepts

- A pure **public good** is a good that satisfies the following two conditions:
  - Non-rivalry: The consumption of the good by one individual doesn't affect other's opportunity to consume the good.
  - Non-excludability: Individuals cannot be excluded from consuming the good.
- The **Social Demand Curve** is the aggregated demand for a good and can be found by summing the individual demand curves *vertically* for public goods (as opposed to horizontally for private goods).

  Note: Graph an example of vertical summation of individual demand curves.



- **Samuelson Rule**: Since a public good can benefit simultaneously several individuals, the optimal provision public goods is achieved when the marginal cost is set equal to the sum of the marginal rate of substitutions.

- The inability to exclude people from enjoying a public good creates the incentive fro individuals to consume it without paying or **Free Riding**. These problem generates an underprovision of public goods under the private market,

2 Practice Problems

2.1 Gruber, Ch.7, Q.13

The town of Springfield has two residents: Homer and Bart. The town currently funds its fire department solely from the individual contributions of these residents. Each of the two residents has a utility function over private goods (X_i) and total firefighters (M) of the form $U_i = 4 \cdot log(X_i) + 2 \cdot log(M)$, where $i = B, H$. The total provision of firefighters hired, M, is the sum of the number hired by each of the two persons: $M = M_H + M_B$. Homer and Bart both have income of \$100, and the price of both the private good and a firefighter is \$1.

(a) *How many firefighters are hired if the government does not intervene? How many are paid for by Homer? By Bart?*

Solution:

If each resident optimizes his own function, he will choose the number of firefighters that maximizes his own utility, taking into consideration the contribution by the other resident.

Private consumption, X_{Bart}, can be rewritten as $100 - M_{Bart}$ because all income not spent on firemen (M) can be spent on private goods.

The public good enjoyed by Bart can be rewritten as $M_{Bart} + M_{Homer}$ because public goods provided by either one are consumed by both.

Therefore, Bart's utility function can be rewritten as

$$U_{Bart} = 4 \cdot log(100 - M_{Bart}) + 2 \cdot log(M_{Bart} + M_{Homer}).$$

Set $\partial U/\partial M_{Bart}$ equal to zero:

$$-4/(100 - M_{Bart}) + 2/(M_{Bart} + M_{Homer}) = 0$$
$$4/(100 - M_{Bart}) = 2/(M_{Bart} + M_{Homer})$$

Cross-multiply, $4(M_{Bart} + M_{Homer}) = 2(100 - M_{Bart})$, and expand: $4M_{Bart} + 4M_{Homer} = 200 - 2M_{Bart}$.

Solving for M_{Bart} yields $M_{Bart} = (200 - 4M_{Homer})/6$.

The same procedure yields $M_{Homer} = (200 - 4M_{Bart})/6$.

These are response functions. They allow each resident to calculate his optimal M as a function of the contribution to M made by the other resident. Because the other resident's M carries a negative sign, the more one resident contributes, the less the other will.

Solving these response functions simultaneously is greatly eased by the fact that they are symmetric. At the solution, then, $M_{Homer} = M_{Bart}$, and hence $3M_{Bart} = 100 - 2M_{Homer} = 100 - 2M_{Bart}$. So $M_{Bart} = M_{Homer} = 20$.

(b) *What is the socially optimal number of firefighters? If your answer differs from part (a), why?*

Solution:

Method 1 - Substitution: The social planner maximizes $U_{Bart} + U_{Homer}$ subject to the budget constraints $100 = X_B + M_B$ and $100 = X_H + M_H$. Rewriting the budget constraints as $X_B = 100 - M_B$ and $X_H = 100 - M_H$ and plugging into the utility functions, we have that the social planner maximizes an aggregate of the utilities I denote "SU":

$$SU = [4 \cdot log(100 - M_{Bart}) + 2 \cdot log(M_{Bart} + M_{Homer})] + [4 \cdot log(100 - M_{Homer}) + 2 \cdot log(M_{Bart} + M_{Homer})]$$

We can find the first order conditions by taking the derivative with respect to each variable.

$$\frac{\partial SU}{\partial M_H} = -4/(100 - M_{Homer}) + 2/(M_{Bart} + M_{Homer}) + 2/(M_{Bart} + M_{Homer}) = 0$$

$$M_{Homer} = 50 - \frac{1}{2}M_{Homer}$$

And

$$\frac{\partial SU}{\partial M_B} = -4/(100 - M_{Bart}) + 2/(M_{Bart} + M_{Homer}) + 2/(M_{Bart} + M_{Homer}) = 0$$

$$M_{Bart} = 50 - \frac{1}{2}M_{Bart}$$

Now we have system of two equations with two unkowns. Plugging them in and solving, we get:

$$M_{Homer} = 33.33$$
$$M_{Bart} = 33.33$$
$$\implies M = 66.67$$

Method 2 - MRS: The socially optimal number is determined by adding each resident's marginal rate of substitution (placing the marginal utility for the public good in the numerator and for the private good in the denominator) and setting the result equal to the price ratio (1 here because both goods have the same price). Because Homer and Bart have the same utility functions, they will have the same marginal rates of substitution.

Therefore, the socially optimal number of firefighters solves $MRS_{M,X}^{Bart} + MRS_{M,X}^{Homer} = 1$.

Computing the MRS for each resident: $MU_M/MU_X = (2/M)/(4/[100 - M_i])$, where $M = M_{Homer} + M_{Bart}$, and $M_i = M_{Bart} = M_{Homer}$. The social optimum is then the solution to $(2/M)/(4/[100 - M_{Bart}]) + (2/M)/(4/[100 - M_{Homer}]) = 1$ or $[100 - M_{Bart}]/2M + [100 - M_{Homer}]/2M = 1$ or $200 - (M_{Bart} + M_{Homer}) = 2M$ or $200 - M = 2M$.

Hence, M = 200/3, and the social optimum is between 66 and 67 firefighters.

Method 3 - Lagrangian: Another way of doing this problem is to maximize the Lagrangian. The social planner maximizes $U_{Bart} + U_{Homer}$ by choosing $\{X_H, X_B, M_B, M_H\}$ subject to the budget constraints $100 = X_B + M_B$ and $100 = X_H + M_H$. This is equivalent to maximizing the following Lagrangian:

$$\begin{aligned} max_{\{X_H,X_B,M_B,M_H\}} L = [4log(X_B) + 2log(M_H + M_B)] + [4log(X_H) + 2log(M_H + M_B)] \\ + \lambda_1(100 - X_B - M_B) + \lambda_2(100 - X_H - M_H) \end{aligned}$$

Which gives first-order conditions:

(a) $\frac{4}{X_B} - \lambda_1 = 0$
(b) $\frac{4}{M_H+M_B} - \lambda_1 = 0$
(c) $\frac{4}{X_H} - \lambda_2 = 0$
(d) $\frac{4}{M_H+M_B} - \lambda_2 = 0$
(e) $100 - X_B - M_B = 0$
(f) $100 - X_H - M_H = 0$

There are a bunch of constraints, but they simplify quickly. Notice from (b) and (d) that $\lambda_1 = \lambda_2 = \frac{4}{M_H+M_B}$. Then using $\lambda_1 = \lambda_2$, we know from (a) and (b) that $\frac{4}{X_B} = \frac{4}{X_H}$ or

$X_B = X_H$ which I define as X_i. Then from (e) and (f) we know that $M_B = M_H = 100 - X_i$, and so $X_i = 100 - M_i$. Then, from (a) and things we have derived we know that $\frac{4}{M_H + M_B} = \frac{4}{X_i}$, or $2M_i = X_i = 100 - M_i$. Solving, we get $M_i = 33.3$ or $M = M_B + M_H = 33.3 + 33.3 = 66.7$.

Why are the results different? Intuitively, in the computation in part (a), we set the marginal utility of the last firefighter to each resident equal to the marginal utility of consumption for that resident. In part (b), we set the sum of the marginal utilities of the last firefighter - the social marginal utility of the firefighter - equal to the marginal utility of consumption for either resident. Since the social marginal utility of firefighters exceeds the individual marginal utilities of that firefighter, society optimally hires more than individuals would if they were acting alone.

2.2 Gruber, Ch.7, Q.15

Consider an economy with three types of individuals, differing only with respect to their preferences for monuments. Individuals of the first type get a fixed benefit of 100 from the mere existence of monuments, whatever their number. Individuals of the second and third type get benefits according to $B_{II} = 200 + 30M - 1.5M^2$ *and* $B_{III} = 150 + 90M - 4.5M^2$*, where* M *denotes the number of monuments in the city. Assume that there are 50 people of each type. Monuments cost \$3,600 each to build. How many monuments should be built?*

Solution: The important points here are to recognize that these "Benefit curves" show us total benefits, not marginal benefits. By taking the derivative with respect to M, we can arrive at the Private Marginal Benefit for each person. Second, we need to recognize the $\$3,600$ is the Social Marginal Cost. Finally, to get the social optimum M_{SO}, we need to set $SMC = SMB$, and we get SMB by aggregating the private demand curves (i.e. the private marginal benefits).

- The marginal benefit for type I individuals is 0 (for all $M > 1$).
- The marginal benefit for type II individuals is $30 - 3M$.
- The marginal benefit for type III individuals is $90 - 9M$.
- The marginal cost is \$3,600.

Aggregating marginal benefits and setting them equal to marginal cost yields:

$$50(0) + 50(30 - 3M) + 50(90 - 9M) = \$3,600 \qquad \Rightarrow \qquad M = 4$$

2.3 Gruber, Ch.7, Q.12

Andrew, Beth, and Cathy live in Lindhville. Andrew's demand for bike paths, a public good, is given by $Q = 12 - 2P$. Beth's demand is $Q = 18 - P$, and Cathy's is $Q = 8 - P/3$. The marginal cost of building a bike path is $MC = 21$. The town government decides to use the following procedure for deciding how many paths to build. It asks each resident how many paths they want, and it builds the largest number asked for by any resident. To pay for these paths, it then taxes Andrew, Beth, and Cathy the prices a, b, and c per path, respectively, where $a + b + c = MC$. (The residents know these tax rates before stating how many paths they want.)

(a) *If the taxes are set so that each resident shares the cost evenly ($a = b = c$), how many paths will get built?*

Solution: When taxes are set at $a = b = c = MC/3 = 7$, each resident faces an individual marginal cost of 7 per bike path. Rearrange the demand functions above to be in terms of price, giving you $P = 6 - \frac{1}{2}Q_A$, $P = 18 - Q_B$, and $P = 24 - 3Q_C$. These are private marginal benefits. Set each equal to the respective tax values (i.e. the effective private marginal costs to the residents): $a = 7 = 6 - \frac{1}{2}Q_A$, $b = 7 = 18 - Q_B$, and $c = 7 = 24 - 3Q_C$. At this marginal cost, Andrew wants no bike paths, Beth wants 11, and Cathy wants 5.67. The government therefore builds 11 paths.

(b) *Show that the government can achieve the social optimum by setting the correct tax prices a, b, and c. What prices should it set?*

Solution: The social optimum can be computed by reexpressing the demand curves for the three residents as $P = 6 - Q/2$, $P = 18 - Q$, and $P = 24 - 3Q$, respectively, and summing them to get marginal social benefit $MSB = 48 - 4.5Q$. Setting $MSB = MC$ and solving for Q gives $Q = 6$. We need to tax prices so that nobody will want more than 6 units (and someone will want exactly 6 units). Looking at Andrew's inverted demand curve, we see that he will want exactly 6 units at $a = 3$ (since then $a = 6 - 6/2$). Beth will want exactly 6 units at $b = 12$. And at $c = 6$, Cathy will want exactly 6 units. Since $3 + 12 + 6 = 21$, these tax rates are just enough to cover MC, and the social optimum is achieved. Note that with this tax system in place, the three residents are unanimous in the number of bike paths they desire.