Econ 204 – Problem Set 1

Due Friday, July 31

1. Let A and B be sets, and I an arbitrary (possibly infinite) index set. Prove the following statements:

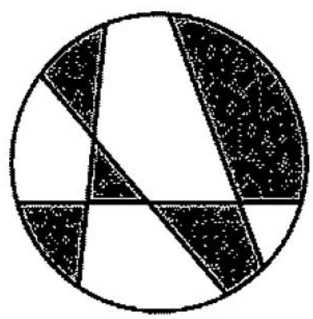
   (a) $A \bigcup \left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} \left(A \bigcup B_i\right)$

   (b) $A \bigcap \left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} \left(A \bigcap B_i\right)$

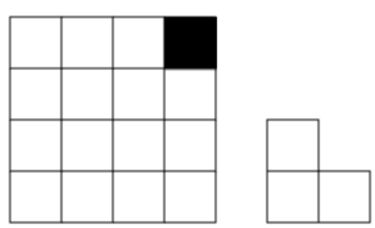
2. Use the principle of mathematical induction to prove the following statements:

   (a) A set S with n elements has 2^n subsets (note: do not forget about the empty set).

   (b) Suppose that n chords are drawn in a circle, dividing the circle into different regions. Prove that every region can be colored one of two colors such that adjacent regions are different colors. The figure below shows an example of $n = 4$.



   (c) Prove that any grid made up of $2^n \times 2^n$ tiles can be covered except for one corner tile by L-shaped triominoes (the triominoes may rotated). The figure below shows an example of a 4×4 grid (left) where all of the non-shaded tiles must be covered by a triomino (right).



3. A subset B is called a *cofinite* subset of a set A if $A - B$ is finite. In other words, B contains all but a finite number of elements of A. Prove

the following: If B and C are cofinite subsets of A, then $B \cap C$ is also a cofinite subset of A.

4. Let A and B be subsets of any uncountable set X such that their complements are countably infinite. Prove that $A \cap B \neq \varnothing$.

5. Suppose $A \subset \mathbb{R}_+$, $b \in \mathbb{R}_+$, and every finite collection of distinct elements $a_1, \ldots, a_n$ of A we have $\sum_i a_i \leq b$. Prove that A is at most countable. (Hint: consider the sets $A_n = \left\{x \in A | x \geq \frac{1}{n}\right\}$).

6. Let $f : A \to A$. Prove that there is a unique largest subset $X \subset A$ such that $f(X) = X$ (here if $Y \subset X$ we call X "larger" than Y).

7. Define the following distance function on the set of real numbers:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

   (a) Prove that $(\mathbb{R}, d)$ is a metric space.

   (b) Identify the open (and closed) balls in the topology induced by this metric.