Econ 204 – Problem Set 2

Due Tuesday, August 4

1. In the standard Euclidean metric space, let the set A be uncountable. Prove that there is a sequence of distinct points converging to a point in A. Is this true for every metric space?

2. For some metric space (X, d) take any two sets such that $\text{int } A = \text{int } B = \varnothing$ and A is closed. Prove that $\text{int}(A \cup B) = \varnothing$. What if A is not closed?

3. Prove that the set of cluster points of any sequence $\{x_n\}$ is closed.

4. Consider ℓ^∞, the vector space defined over $\mathbb{R}$ of all bounded sequences. That is, $a \in \ell^\infty$ if $a = (a_1, a_2, \ldots)$ and $\exists M \in \mathbb{R}$ such that $|a_i| \leq M$ for every i.

   a) Show that $||a||_\infty = \sup_i |a_i|$ defines a norm on this space.

   b) Consider the subspace L_0 made up of sequences with only a finite number of nonzero elements. That is, $a \in L_0$ if $\exists N \in \mathbb{N}$ such that $i > N \implies a_i = 0$. Is L_0 a closed subspace of ℓ^∞?

5. Recall the *diameter* of a set is defined $\operatorname{diam} A = \sup\{d(a, b) : a, b \in A\}$. Prove that the diameter of a set is equal to the diameter of its closure.

6. Call a metric space *discrete* if every subset is open.[1]

   a) Give an example of a discrete metric space that is not complete.

   b) (*Difficult!*) Show that a metric space has the property that the closure of every open set is open if and only if the metric space is discrete.

[1] Every set equipped with the discrete metric forms a discrete metric space, but not all discrete metric spaces have the discrete metric.