Econ 204 – Problem Set 4

Due Tuesday, August 12

1. Similarly as it's defined in class, let $C([0,1])$ be the set of all continuous functions whose domain is the unit interval $[0,1]$ and range is $\mathbb{R}$. Let Φ be the subset consisting of all real polynomials (whose domain is restricted to the unit interval) of degree at most two:

$$\Phi \equiv \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$$

Note that the set $C([0,1])$ is a vector space over the field of real numbers and the subset Φ is a proper subspace.

(a) Are the vectors $\{ x, (x^2-1), (x^2+2x+1) \}$ linearly independent over $\mathbb{R}$?

Solution Apply the usual test for independence of vectors. Solve for A, B, and C such that

$$Ax + B(x^2 - 1) + C(x^2 + 2x + 1) = 0$$

Equating like powers of x we obtain the following system in the three unknowns:

$$B + C = 0$$

$$A + 2C = 0$$

$$C - B = 0$$

$\Rightarrow C = B = -B \Rightarrow C = B = 0 \Rightarrow A = 0.$
Thus, the three vectors are linearly independent over $\mathbb{R}$.

(b) Find a Hamel basis for the subspace Φ.

Solution Clearly, $\{1,\ x,\ x^2\}$ is linearly independent and spans Φ, so it is a Hamel basis and $\dim \Phi = 3$. Also, since the set of vectors in (a) is linearly independent and contains three elements, it is a basis.

(c) What is the dimension of Φ ? Show that $C([0,1])$ is not finite dimensional!

Solution The dimensions are three and ∞, respectively. To see that the dimension of Θ is infinite note that the set of vectors $\{1,\ x,\ x^2,\ x^3,\ ...\}$ form a linearly independent set over $\mathbb{R}$. Since Θ contains an infinite linearly indpendent set of vectors and the number of linearly independent elements of a vector space cannot exceed the dimension, the dimension of Θ must be infinite.

2. Let V have finite dimension greater than 1. Prove whether or not the set of non-invertible operators is a subspace of $L(V, V)$.

   Solution Nope, not a subspace. Fix $\dim(V) = n$, let $v = (v_1, ..., v_n)$ be in V, and define T and S by $Tv = (v_1, ..., v_{n-1}, 0)$, $Sv = (0, ..., 0, v_n)$. Then both T and S are non-invertible but $T + S$ has $(T + S)v = Tv + Sv = (v_1, ..., v_{n-1}, 0) + (0, ..., 0, v_n) = (v_1, ..., v_n) = v$. Thus $T + S$ is the identity mapping, which is invertible, and hence the set in question is not closed under addition.

3. Suppose that V is finite dimensional and $T, S \in L(V, V)$. Prove that TS is invertible if and only if both T and S are invertible.

   Solution Assume that TS is invertible. We will first check that S is invertible. Note that by the Rank-Nullity Theorem it suffices to check that $\text{Ker}(S) = 0$. If $\exists w \in V, w \neq 0$, then we find: $TS(w) = T(0) = 0$. Hence, TS has non-zero kernel, a contradiction. Thus, S is invertible. If T is not invertible put $v \in V$ with $Tv = 0$ $(v \neq 0)$. Since S is invertible, it is surjective. Thus, we can find a $w \in V$ such that $S(w) = v$, which implies that $TS(w) = T(v) = 0$. This, again, contradicts the invertibility of TS. We now prove the converse. Assuming that T, S are both invertible we wish to check that TS is invertible. We again check the sufficient condition that $\text{Ker}(TS) = 0$. To see that this is the case, we note that if $w \in \text{Ker}(TS) \Rightarrow S(w) = 0$ or, putting $v = S(w) \neq 0$, $T(v) = 0$. Thus, if both $\text{Ker}(T)$ and $\text{Ker}(S) = 0$, then $\text{Ker}(TS) = 0$.

4. $T : M_{2\times 2} \to M_{2\times 3}$ is defined by:

$$T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} & a_{11} + 3a_{22} & 0 \\ a_{11} - a_{12} & a_{12} + a_{21} & 0 \end{pmatrix}$$

   Determine $\ker T, \dim(\ker T)$, and rank T. Is T one-to-one, onto, both or neither?

   Solution: $\ker T$ is the set of all 2×2 matrices of the form

$$\begin{pmatrix} x & x \\ -x & -x/3 \end{pmatrix}$$

   which is a one-dimensional space. Rank $T = 3$ because $\dim(\ker T) + \text{rank} T = \dim M_{2\times 2} = 4$. T is not one-to-one by Theorem 2.13 on p. 126 in de la Fuente. T is not onto either since the rank of T is 3 and not 4. Alternatively, you can directly verify that T fails to be both one-to-one and onto.

5. (a) Prove that the eigenvalues of any upper or lower triangular matrix A are the diagonal entries of A;

   (b) Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i (see de la Fuente, p. 147 for a definition) is a vector space;

(c) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$;

(d) Show that if λ is an eigenvalue of the invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}.

Solution:

(a) Let us denote the diagonal elements of A by $\{t_{11}, t_{22}, t_{33}, \ldots, t_{nn}\}$. Using induction on the size of the matrix, it is easy to show by directly computing the determinant through Laplace expansion that the determinant of any triangular (or diagonal) matrix is the product of its diagonal elements. Thus the characteristic polynomial for A is:

$$\det(A - \lambda I) = (t_{11} - \lambda)(t_{22} - \lambda) \cdots (t_{nn} - \lambda),$$

so the eigenvalues are the t_{ii}'s.

(b) To show that the eigenspace is a vector space, we only need to check the existence of additive identity and inverse elements. Since the eigenspace contains the $\mathbf{0}$ vector by definition, we only need to verify the existence of inverse elements. Denote the eigenspace by E_i. Let $v \in E_i$. Then $Av = \lambda_i v$. Multiplying both sides by -1 gives us $A(-v) = \lambda_i(-v) \Rightarrow (-v) \in E_i$.

(c) We use induction to show not only that λ^k is an eigenvalue of A^k, but also that any eigenvector v corresponding to the eigenvalue λ for A also corresponds to λ^k for A^k. The base step ($k = 1$) is trivial. For the induction step, assume $Av = \lambda v$ and $A^k v = \lambda^k v$. Now consider $A^{k+1}v$:

$$A^{k+1}v = A^k(Av) = A^k(\lambda v) = \lambda(A^k v) = \lambda(\lambda^k v) = \lambda^{k+1}v$$

(d) Let $Tv = \lambda v$. Premultiply both sides by T^{-1}:

$$T^{-1}Tv = T^{-1}\lambda v \Rightarrow v = \lambda T^{-1}v \Rightarrow T^{-1}v = (1/\lambda)v$$