Financial Econometrics Econ 40357
ARIMA
Part 2: Autoregressive Models

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Autoregressive (AR) models.

These are models with more durable, persistent dependence over time.
Let $\epsilon_t \overset{iid}{\sim} (0, \sigma_\epsilon^2)$, and $|\rho| < 1$. Then **the AR(1) model** is

$$y_t = a + \rho y_{t-1} + \epsilon_t$$

where

$$\mathrm{E}\left(y_t\right) = \mu = \frac{a}{(1-\rho)},\ \mathrm{Var}\left(y_t\right) = \frac{\sigma_\epsilon^2}{(1-\rho^2)},\ \rho\left(y_t, y_{t-k}\right) = \rho^k$$

Note: $a = \mu\left(1-\rho\right)$, which means we can also write it as

$$y_t = \mu\left(1-\rho\right) + \rho y_{t-1} + \epsilon_t$$

The MA representation of the AR(1)

The AR(1) can also be represented as an MA(∞) .

$$
\begin{aligned}
y_t &= a + \rho \underbrace{(a + \rho y_{t-2} + \epsilon_{t-1})}_{y_{t-1}} + \epsilon_t \\
&= a + \rho a + \rho^2 \underbrace{(a + \rho y_{t-3} + \epsilon_{t-2})}_{y_{t-2}} + \rho \epsilon_{t-1} + \epsilon_t \\
&= a + \rho a + \rho^2 a + \epsilon_t + \rho \epsilon_{t-1} + \rho^2 \epsilon_{t-2} + \rho^2 y_{t-3} \\
&\vdots \\
&= \underbrace{a \left(1 + \rho + \rho^2 + \rho^3 + \cdots\right)}_{a/(1-\rho)} + \epsilon_t + \rho \epsilon_{t-1} + \rho^2 \epsilon_{t-2} + \rho^3 \epsilon_{t-3} + \cdots \\
&= \frac{a}{1-\rho} + \epsilon_t + \rho \epsilon_{t-1} + \rho^2 \epsilon_{t-2} + \rho^3 \epsilon_{t-3} + \cdots
\end{aligned}
$$

What is the mean $\mathsf{E}(y_t)$?

$$\begin{aligned}
\mathsf{E}\left(y_t\right) &= \mathsf{E}\left(\frac{a}{1-\rho} + \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots\right) \\
&= \left(\frac{a}{1-\rho} + \mathsf{E}\epsilon_t + \rho\mathsf{E}\epsilon_{t-1} + \rho^2\mathsf{E}\epsilon_{t-2} + \rho^3\mathsf{E}\epsilon_{t-3} + \cdots\right) \\
&= \frac{a}{1-\rho}
\end{aligned}$$

What is the Variance $\mathrm{Var}(y_t)$?

$$
\begin{aligned}
\sigma_y^2 &= \mathrm{Var}\left(y_t\right) = \mathrm{E}\left(\epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots\right)^2 \\
&= \mathrm{E}\left(\epsilon_t^2 + \rho^2\epsilon_{t-1}^2 + \rho^4\epsilon_{t-2}^2 + \cdots + 2\rho\epsilon_t\epsilon_{t-1} + 2\rho^2\epsilon_t\epsilon_{t-2} + \cdots\right) \\
&= \left(\mathrm{E}\epsilon_t^2 + \rho^2\mathrm{E}\epsilon_{t-1}^2 + \rho^4\mathrm{E}\epsilon_{t-2}^2 + \cdots + \underbrace{2\rho\mathrm{E}\epsilon_t\epsilon_{t-1} + 2\rho^2\mathrm{E}\epsilon_t\epsilon_{t-2} + \cdots}_{0}\right) \\
&= \sigma_\epsilon^2\left(1 + \rho^2 + \rho^4 + \cdots\right) \\
&= \frac{\sigma_\epsilon^2}{1-\rho^2}
\end{aligned}
$$

What is the autocorrelation function? First, write the AR(1) in deviations from the mean form,

$$\begin{aligned} y_t &= \mu(1-\rho) + \rho y_{t-1} + \epsilon_t \\ y_t - \mu &= \rho(y_{t-1} - \mu) + \epsilon_t \end{aligned}$$

Then,

$$\begin{aligned} \gamma_1 &= \text{Cov}(y_t, y_{t-1}) = \text{E}(y_t - \mu)(y_{t-1} - \mu) \\ &= \text{E}(\rho(y_{t-1} - \mu) + \epsilon_t)(y_{t-1} - \mu) \\ &= \rho \underbrace{\text{E}(y_{t-1} - \mu)^2}_{\text{Var}(y_{t-1})} + \underbrace{\text{E}(\epsilon_t(y_{t-1} - \mu))}_{0} \\ &= \rho \text{Var}(y_t) = \rho \sigma_y \sigma_y \end{aligned}$$

Hence,

$$\rho(y_t, y_{t-1}) = \rho$$

$$
\begin{aligned}
\gamma_2 &= \mathrm{Cov}\left(y_t, y_{t-2}\right) = \mathrm{E}\left(y_t - \mu\right)\left(y_{t-2} - \mu\right) \\
&= \mathrm{E}\left(\rho\left(y_{t-1} - \mu\right) + \epsilon_t\right)\left(y_{t-2} - \mu\right) \\
&= \rho\underbrace{\mathrm{E}\left(y_{t-1} - \mu\right)\left(y_{t-2} - \mu\right)}_{\gamma_1} + \underbrace{\mathrm{E}\left(\epsilon_t\left(y_{t-2} - \mu\right)\right)}_{0} \\
&= \rho\gamma_1
\end{aligned}
$$

$$
\rho\left(y_t, y_{t-2}\right) = \frac{\rho\gamma_1}{\sigma_y\sigma_y} = \frac{\rho\rho\sigma_y\sigma_y}{\sigma_y\sigma_y} = \rho^2
$$

We can infer that

$$
\rho\left(y_t, y_{t-k}\right) = \rho^k
$$

AR(1) forecasts

$$E_t(\tilde{y}_{t+1}) = \rho\tilde{y}_t$$

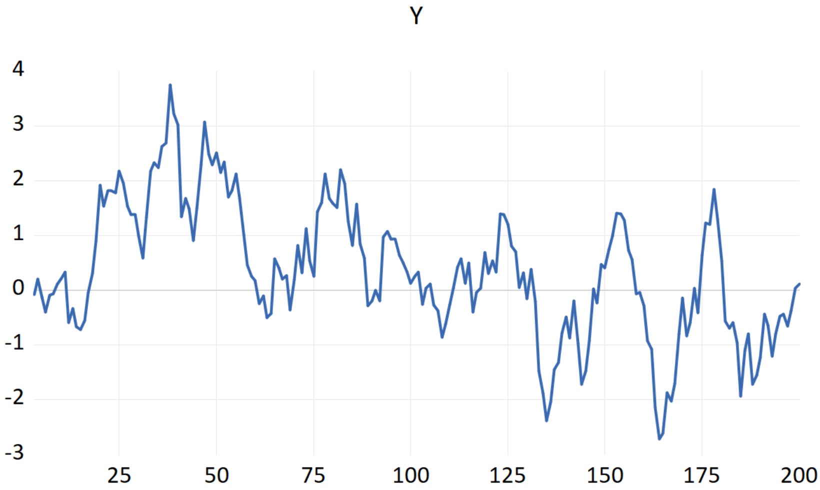
$$E_t(\tilde{y}_{t+2}) = \rho E_t(\tilde{y}_{t+1}) = \rho^2\tilde{y}_t$$

Hence,

$$E_t(\tilde{y}_{t+k}) = \rho^k\tilde{y}_t$$

Try it out on daily stock returns.

Realization of an AR(1) with $\rho = 0.96$



How to generate in Eviews

```
' Generate white noise process
series e = nrnd
'Generate persistent AR(1)
smpl @first @first
series sto = 0 ' Initial conditions
smpl @first+1 @last
series sto = .96*sto(-1)+.5*e ' Recursion
series y = sto
delete sto
```

(To get impulse response: Quick, estimate VAR)

(arima_models.wf1 and pgm)

Impulse Response Function

The impulse response function traces the effect of a one time, one-standard deviation shock today $\epsilon_t = \sigma_\epsilon$, on the current and all future values $y_t, y_{t+1}, y_{t+2}, \ldots$. Stationary processes will revert to their

mean values. Let's analyze as deviations from the mean (set $\mu = 0$).
AR(1): $y_t = \rho y_{t-1} + \epsilon_t$, $0 < \rho < 1$.

$$
\begin{aligned}
y_t &= \epsilon_t \\
y_{t+1} &= \rho y_t = \rho \epsilon_t \\
y_{t+2} &= \rho y_{t+1} = \rho^2 \epsilon_t \\
y_{t+k} &= \rho^k \epsilon_t
\end{aligned}
$$

Another representation of impulse response. MA representation (mean suppressed $\mu = 0$),

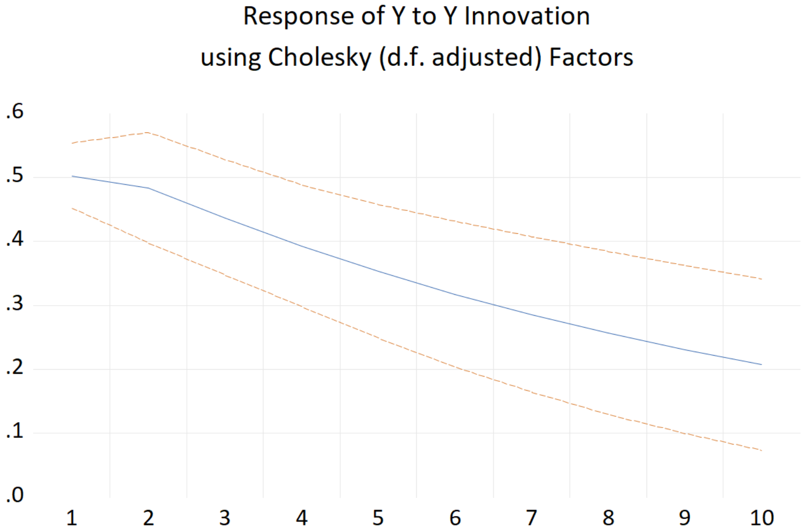
$$y_t = \epsilon_t + \rho\epsilon_{t-1} + \rho^2\epsilon_{t-2} + \rho^3\epsilon_{t-3} + \cdots$$

One time shock ϵ_t, with all other shocks shut down, $\epsilon_k = 0$, $k \neq t$

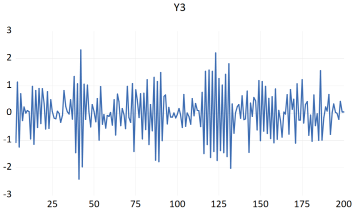
$$\begin{aligned} y_t &= \epsilon_t \\ y_{t+1} &= \rho\epsilon_t \\ y_{t+2} &= \rho^2\epsilon_t \end{aligned}$$

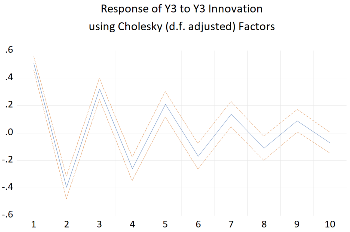
and so on.
Later, I will show you how to generate implulse responses in Eviews.

Impulse response of AR(1)



AR(1) with negative ρ

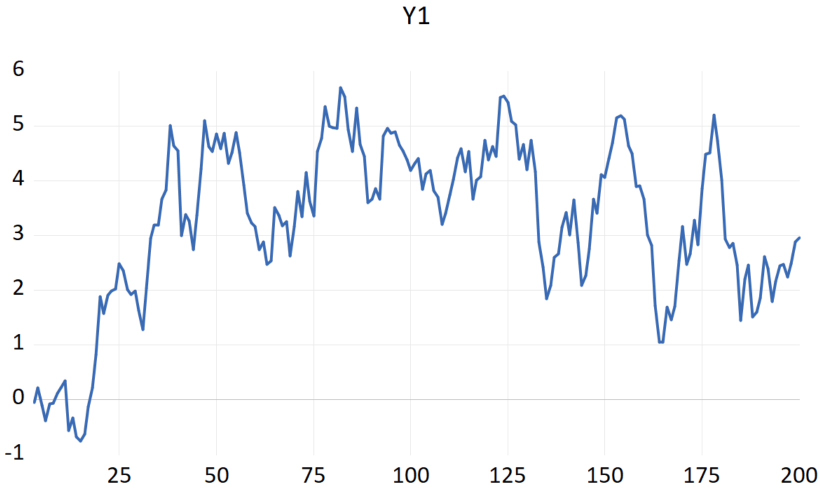




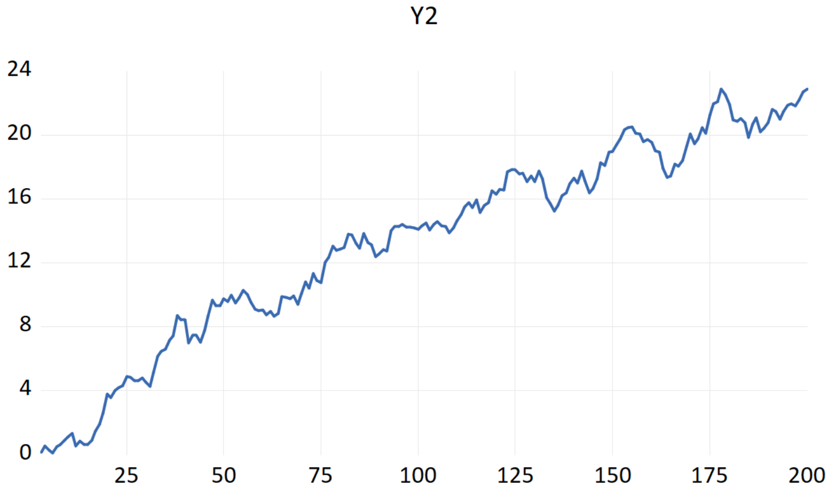
Unit Root Nonstationarity

- Why $|\rho| < 1$ is necessary for stationarity?
- It is usually the case that $0 < \rho < 1$ in economics and finance (persistence).
- What happens to the mean and the variance of y_t when $\rho = 1$?
- What happens to the impulse response function when $\rho = 1$? (permanent effect).

Realization of a driftless Random Walk



Random walk with drift



The AR(2) model. Back to Stationary Models.

Let $\epsilon_t \overset{iid}{\sim} (0, \sigma_\epsilon^2)$. The second-order autoregressive model (AR(2)) is

$$y_t = a + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t$$

and is stationary if $|\rho_1 + \rho_2| < 1$. Assume stationarity, take expectations

$$\mu_y = a + \rho_1 \mu_y + \rho_2 \mu_y$$

$$a = \frac{\mu_y}{1 - \rho_1 - \rho_2}$$

Computing variance and autocovariances by hand is too complicated. It involves taking variance and first-order covariance

$$\begin{aligned} \sigma_y^2 &= \rho_1^2 \sigma_y^2 + \rho_2^2 \sigma_y^2 + 2\rho_1 \rho_2 \gamma_1 + \sigma_\epsilon^2 \\ \gamma_1 &= \rho_1 \sigma_y^2 + \rho_2 \gamma_1 \rightarrow \gamma_1 = \frac{\rho_1 \sigma_y^2}{1 - \rho_2} \end{aligned}$$

Then you must to solve these two equations for σ_y^2 and γ_1.

AR(2) Impulse Response Function

AR(2) with $\mu = 0$ (or in deviation from mean form).

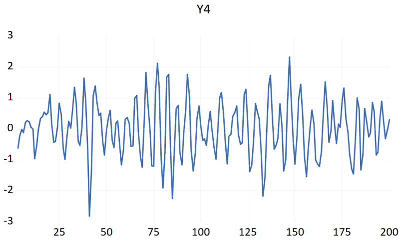
$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t$$

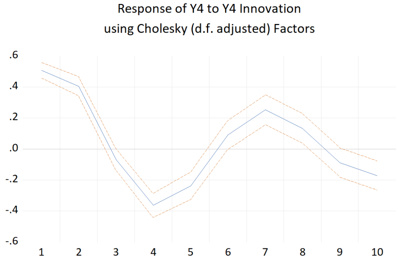
Let $y_0 = y_{-1} = 0$, One-time shock at time 1, ϵ_1, with all other shocks shut down. Trace effect recursively

$$\begin{aligned}
y_1 &= \epsilon_1 \\
y_2 &= \rho_1 y_1 = \rho_1 \epsilon_1 \\
y_3 &= \rho_1 y_2 + \rho_2 y_1 = \left(\rho_1^2 + \rho_2\right) \epsilon_t \\
y_4 &= \rho_1 y_3 + \rho_2 y_2 = \rho_1 \left(\rho_1^2 + 2\rho_2\right) \epsilon_1
\end{aligned}$$

and so on
Is possible to get **cyclical** impulse responses.

Realization and Impulse Response AR(2) $\rho_1 = 0.8, \rho_2 = -0.8$





AR(2) forecasts

Form the forecasts and input recursively.

$$\begin{aligned} E_t\left(\tilde{y}_{t+1}\right) &= \rho_1 \tilde{y}_t + \rho_2 \tilde{y}_{t-1} \\ E_t\left(\tilde{y}_{t+2}\right) &= \rho_1\left(E_t\left(\tilde{y}_{t+1}\right)\right) + \rho_2 \tilde{y}_t \\ E_t\left(\tilde{y}_{t+3}\right) &= \rho_1\left(E_t\left(\tilde{y}_{t+2}\right)\right) + \rho_2 E_t\left(\tilde{y}_{t+1}\right) \end{aligned}$$

Extensions

1. No need to stop at AR(2). Can add more and more lags.
2. In MA model, can add more and more lagged shocks.
3. Difference between MA and AR.
   AR is dependence across time of observations.
   MA is dependence across time of shocks.
4. MA memory is finite
5. AR memory is infinite (but diminishes exponentially)
6. Can combine MA and AR. Here's ARMA(1,1)

$$y_t = a + \rho y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1}$$