Econ 204 – Problem Set 6

Due Tuesday, August 20; before exam

1. Calculate the second and third order Taylor expansion of $(1+2x-3y)^2$ around the point $(0,0)$. Calculate the difference between the value of the function and the expansions.

   Solution We want to expand $f(x,y) = (1+2x-3y)^2$ at the point $(0,0)$. Generally, a second order expansion of $f(x,y)$ at $(0,0)$ is the following:

   $$f(x,y) = f(0,0) + Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2!}(x,y)D^2 f(0,0)\begin{pmatrix} x \\ y \end{pmatrix}$$

   Now we have to calculate some derivatives. We have:

   $$\begin{array}{ll} f_x(x,y) = 4(1+2x-3y) & f_x(0,0) = 4 \\ f_y(x,y) = -6(1+2x-3y) & f_y(0,0) = -6 \\ f_{xx}(x,y) = 8 & f_{xx}(0,0) = 8 \\ f_{xy}(x,y) = -12 & f_{xy}(0,0) = -12 \\ f_{yy}(x,y) = 18 & f_{yy}(0,0) = 18 \end{array}$$

   We also have $f(0,0) = 1$, and that all third and higher order derivatives are zero. Thus, our second order Taylor expansion should exactly equal $f(x,y)$. Plugging our derivatives in to our Taylor expansion yields:

   $$f(x,y) = 1 + (4,-6)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{bmatrix} 8 & -12 \\ -12 & 18 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

   To verify that this Taylor expansion exactly equals $f(x,y)$, we will need to do some algebra. Carrying out the matrix multiplications above, we have:

   $$f(x,y) = 1 + 4x - 6y + 4x^2 - 12xy + 9y^2$$

   To complete the check, we take the square of $(1+2x-3y)$ to find that:

   $$f(x,y) = (1+2x-3y)^2 = 1 + 4x - 6y + 4x^2 - 12xy + 9y^2$$

   Which is exactly the same as the second order Taylor expansion.

   Since the second order expansion exactly equal the function then it equals the third order expansions as well.

2. Consider the following equations:

   $$u = \frac{x}{x^2+y^2}, \quad v = \frac{y}{x^2+y^2}, \quad x^2+y^2 > 0.$$

   (a) For $(u,v) = (1/2, 1/2)$, find a pair of values (x_0, y_0) that satisfy the equations.

   Solution The point $(x_0, y_0) = (1,1)$ satisfies the equations.

(b) Describe either verbally or graphically what this transformation does. Bonus given for colorful metaphors.

Solution Points near the origin are mapped to points very far away from the origin. Points very far away are mapped to points very close to the origin. This transformation takes $\mathbf{R}^2$, rips it open at the navel $(0,0)$ and turns it inside out. Zero becomes infinity and infinity becomes zero. Each point on a given ray starting at the origin gets mapped to another point on the same ray. Points on the unit circle remain the same.

(c) Show that the above transformation implicitly defines a function in the neighborhood of (x_0, y_0) (in the sense that for every pair of values (u, v) near $(1/2, 1/2)$, there is just one corresponding pair of (x, y) values.

Solution Define $F : \mathbf{R}^4 \to \mathbf{R}^2$ by

$$F((x,y),(u,v)) = \left(u - \frac{x}{x^2+y^2},\ v - \frac{y}{x^2+y^2} \right)$$

From our answer to part (a), we see that when $\mathbf{a} = (1,1)$ and $\mathbf{b} = (1/2, 1/2)$, $F(\mathbf{a}, \mathbf{b}) = (0,0)$. We compute the Jacobian (restricted to x and y) and evaluate at $(\mathbf{a}, \mathbf{b})$ to check whether we can apply the Implicit Function Theorem. If the Jacobian is invertible, then we can. We find

$$DF_{xy}(\mathbf{a},\mathbf{b}) = \begin{bmatrix} \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{bmatrix}_{|(\mathbf{a},\mathbf{b})} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} = -\frac{1}{4}$$

Since the determinant of this matrix is non-zero, the Implicit Function Theorem guarantees the existence of a neighborhood W around $(1/2, 1/2)$ and a U containing $(\mathbf{a}, \mathbf{b})$ such that each $(u, v) \in W$ corresponds to a unique (x, y) with $((x, y),(u, v)) \in U$ and $f((x,y),(u,v)) = 0$.

(d) Compute the Jacobian of the implicit function.

Solution

$$\begin{pmatrix} \frac{\partial g}{\partial u} \\ \frac{\partial g}{\partial v} \end{pmatrix} = -\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

3. Prove that there exist functions $u, v : \mathbb{R}^4 \longrightarrow \mathbb{R}$, continuously differentiable on some open neighborhood around the point $(x, y, z, w) = (2, 1, -1, 2)$

such that $u(2,1,-1,2) = 4$ and $v(2,1,-1,2) = 3$ and the equations

$$u^2 + v^2 + w^2 = 29 \text{ and } \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17$$

both hold for all (x, y, z, w) in that neighborhood.

Solution First, we need to check that our two equations hold at $(x, y, z, w, u, v) = (2, 1, -1, 2, 4, 3)$. Plugging the appropriate values into each equation, we see that $4^2 + 3^2 + 2^2 = 29$, and $\frac{4^2}{2^2} + \frac{3^2}{1^2} + \frac{2^2}{(-1)^2} = 17$. So the equations hold at $(x, y, z, w) = (2, 1, -1, 2)$, a point which we will henceforth call s^* We shall also label $u^* = 4$ and $v^* = 3$.

To determine whether our functions u, v exist on some neighborhood around s^*, we must set up our system of equations in a way that is useful for the implicit function theroem. Define $F : \mathbb{R}^6 \longrightarrow \mathbb{R}^2$ so that:

$$F(x, y, z, w, u, v) = \begin{pmatrix} u^2 + v^2 + w^2 - 29 \\ \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} - 17 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is something to which we can apply the implicit function theorem. Note that the dimension of the range of F, which is 2, is equal to the number of endogenous variables (also 2, u and v). So, to verify whether or not we can express the endogenous variables as functions of the exogenous variables at s^*, we must verify that the determinant of the Jacobian derivative matrix of F with respect to the endogenous variables u and v is non-singular. That is, we must verify:

$$\begin{aligned} |D_{u,v}F(s^*, u^*, v^*)| &\neq 0 \\ \Leftrightarrow \left| \begin{bmatrix} \frac{\partial f_1(s^*,u^*,v^*)}{\partial u} & \frac{\partial f_1(s^*,u^*,v^*)}{\partial v} \\ \frac{\partial f_2(s^*,u^*,v^*)}{\partial u} & \frac{\partial f_2(s^*,u^*,v^*)}{\partial v} \end{bmatrix} \right| &\neq 0 \\ \Leftrightarrow \left| \begin{bmatrix} 2u^* & 2v^* \\ \frac{2u^*}{(x^*)^2} & \frac{2v^*}{(y^*)^2} \end{bmatrix} \right| &\neq 0 \\ \Leftrightarrow \left| \begin{bmatrix} 8 & 6 \\ 2 & 6 \end{bmatrix} \right| &\neq 0 \\ \Leftrightarrow 36 &\neq 0 \end{aligned}$$

So the Jacobian matrix is non-singular, and we can define our fuvnctions u and v on a neighborhood of s^*.

4. Let $E = \{(x, y) : 0 < y < x\}$ and set $f(x, y) = (x + y, xy)$ for $(x, y) \in E$.

   (a) Prove f is one-to-one from E onto $\{(s, t) : s > 2\sqrt{t},\ t > 0\}$ and find a formula for $f^{-1}(s, t)$.

   Solution First, to prove that f is one-to-one from its entire domain of E, we must verify that the conditions of the inverse function theorem hold for all of E; that is, the determinant of the Jacobian matrix of partial derivatives of f must be non-singular at all points in E. We have:

$$
\begin{aligned}
|Df(x,y)| = \begin{vmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{vmatrix} &= \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix} \\
&= x - y \\
&\neq 0 \ \forall x, y \in E \text{ since } x > y
\end{aligned}
$$

So f is one-to-one from E onto $\{(s,t) : s > 2\sqrt{t},\ t > 0\}$. To find the formula of the inverse function, we must do some algebra to express x and y as functions of s and t. Letting $y = s - x$, we have that $t = xy = x(s-x)$. Solving this for x using the quadratic formula yields:

$$x = \frac{s \pm \sqrt{s^2 - 4t}}{2}$$

Which will always be real and positive given that $s > 2\sqrt{t}$, and $t > 0$. Using $y = s - x$ to solve for y yields:

$$y = \frac{s \mp \sqrt{s^2 - 4t}}{2}$$

This initally appears to violate the just-proven fact that f is one-to-one. However, noting that we must have $0 < y < x$, it must be that the formula for x takes the positve sign, while the formula for y takes the negative sign. That is:

$$x = \frac{s + \sqrt{s^2 - 4t}}{2},\ y = \frac{s - \sqrt{s^2 - 4t}}{2}$$

Which defines $f^{-1}(s,t)$.

(b) Use the inverse function theorem to compute $D(f^{-1})(f(x,y))$ for $x \neq y$.

Solution The IFT tells us that $D(f^{-1})(f(x,y)) = (Df(x,y))^{-1}$. Therefore:

$$
\begin{aligned}
D(f^{-1})(f(x,y)) &= \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}^{-1} \\
&= \frac{1}{x-y} \begin{bmatrix} x & -1 \\ -y & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{x}{x-y} & \frac{-1}{x-y} \\ \frac{-y}{x-y} & \frac{1}{x-y} \end{bmatrix}
\end{aligned}
$$

(c) Compare the two expressions for $D(f^{-1})(f(x,y))$ that you derived directly of using the Implicit Function Theorem

Solution To compute $D(f^{-1})(f(x,y))$ directly, we use our formula from part (i): $x = \frac{s+\sqrt{s^2-4t}}{2}$, $y = \frac{s-\sqrt{s^2-4t}}{2}$. Taking derivatives, we have that:

$$\begin{aligned} D(f^{-1})(f(x,y)) = D(f^{-1})(s,t) &= \frac{1}{2}\begin{bmatrix} 1+\frac{s}{\sqrt{s^2-4t}} & \frac{-2}{\sqrt{s^2-4t}} \\ 1-\frac{s}{\sqrt{s^2-4t}} & \frac{2}{\sqrt{s^2-4t}} \end{bmatrix} \\ &= \frac{1}{2}\frac{1}{\sqrt{s^2-4t}}\begin{bmatrix} \sqrt{s^2-4t}+s & -2 \\ \sqrt{s^2-4t}-s & 2 \end{bmatrix} \end{aligned}$$

To put this into terms of x and y, we use our inital $s = x+y$ and $t = xy$. This gives us that $\sqrt{s^2-4t} = \sqrt{(x+y)^2-4xy} = \sqrt{x^2+2xy+y^2-4xy} = \sqrt{(x-y)^2} = x-y$. Substituting everything into the above yields:

$$\begin{aligned} D(f^{-1})(f(x,y)) &= \frac{1}{2(x-y)}\begin{bmatrix} x-y+x+y & -2 \\ x-y-x-y & 2 \end{bmatrix} \\ &= \frac{1}{2(x-y)}\begin{bmatrix} 2x & -2 \\ -2y & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{x-y} & \frac{-1}{x-y} \\ \frac{-y}{x-y} & \frac{1}{x-y} \end{bmatrix} \end{aligned}$$

Which is exactly what we had in part (ii). Once again, isn't it nice when math works out?

5. Consider the second order linear differential equation given by $y'' = 4y + 3y'$.

Note that this equation can be rewritten as the following *first* order linear differential equation of two variables:

$$\bar{x}'(t) = A\bar{x}(t),$$

where $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(a) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

Solution: The matrix A has two distinct eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -1$. Both of them are real, they have opposite sign what gives a saddle-point solution.

(b) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $x_1'(t) = 0$ and $x_2'(t) = 0$.

Solution: The $x_1'(t) = 0$ gives the equation $x_2 = -\frac{3}{4}x_1$ and the $x_2'(t) = 0$ locust is given by the $x_1 = 0$ equation.

(c) Give the solution of the system when $x_1(t_0) = 0$ and $x_2(t_0) = 1$.

Solution: Rewriting the system to one with a diagonal matrix as a coefficient gives

$$f' = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} f$$

where $f = Q^{-1}x$ and Q is the matrix that diagonalize A. This is a system of two independent first order linear equations that has the solution of

$$f = \begin{pmatrix} e^{4(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix} f_0$$

and the initial values are

$$f_0 = Q^{-1}x_0 = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

then rewriting the solution in terms of original variables is just

$$\begin{aligned} x &= Qf = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{4(t-t_0)} & 0 \\ 0 & e^{-(t-t_0)} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \\ &= \frac{1}{5}\begin{pmatrix} 4e^{4(t-t_0)} - 4e^{-(t-t_0)} \\ e^{4(t-t_0)} + 4e^{-(t-t_0)} \end{pmatrix} \end{aligned}$$