Econ 204 (2015) - Final

08/19/2015

Instructions: This is a closed book exam. You have 3 hours. The weight of each question is indicated next to it. Write clearly, explain your answers, and be concise. You may use any result from class unless you are explicitly asked to prove it. Good luck!

1. (18pts) Suppose that (X, d) is a metric space and let $A \subset X$ be a subset. Let A_1 be the intersection of all closed sets that contain A. Let A_2 be the set of all $x \in X$ for which there exists a sequence $\{x_n\}$ in A such that $x_n \to x$. Show that $A_1 = A_2$, i.e., the two alternative definitions of the closure of A agree.

   Solution: Note that $A \subset A_1$, and A_1 is closed since it is the intersection of closed sets. Furthermore, since any closed set that contains A is included in that intersection, A_1 is the smallest closed set that contains A.

   "$A_2 \subset A_1$": Let $x \in A_2$, i.e., there exists a sequence $\{x_n\}$ in A such that $x_n \to x$. Since $A \subset A_1$, $\{x_n\}$ is also a sequence in the closed set A_1. By the sequential characterization of closed sets, its limit x must also be in A_1.

   "$A_1 \subset A_2$:" First note that $A \subset A_2$, because for any $x \in A$, the constant sequence x in A converges to x. We will next show that $X \setminus A_2$ is open, i.e., A_2 is closed, which will prove that $A_1 \subset A_2$ since A_1 is the smallest closed set that contains A.

   Let $x \in X \setminus A_2$. There exists $\epsilon > 0$ such that $B_\epsilon(x) \cap A = \emptyset$, otherwise we can construct a sequence $\{x_n\}$ in A such that $x_n \to x$, which would imply that $x \in A_2$, a contradiction.[1] Take any $y \in B_\epsilon(x)$, and let $\delta = \epsilon - d(x, y) > 0$. Since,

   $$z \in B_\delta(y) \implies d(z,x) \leq d(z,y) + d(y,x) < \delta + d(y,x) = \epsilon \implies z \in B_\epsilon(x),$$

   we have that $B_\delta(y) \subset B_\epsilon(x)$. Therefore we also have $B_\delta(y) \cap A \subset B_\epsilon(x) \cap A = \emptyset$. Therefore there is no sequence $\{y_n\}$ in A such that $y_n \to y$, implying that $y \in X \setminus A_2$. This shows that $B_\epsilon(x) \subset X \setminus A_2$, proving that $X \setminus A_2$ is open.

[1] We have done this construction in class when we proved the sequential characterization of closed sets. If there does not exist $\epsilon > 0$ such that $B_\epsilon(x) \cap A = \emptyset$, then for every $n \in \mathbb{N}$ we can chose $x_n \in B_{\frac{1}{n}}(x) \cap A$. For any $\epsilon' > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon'$, then for all $n > N$, $\frac{1}{n} < \frac{1}{N} < \epsilon'$, hence $x_n \in B_{\frac{1}{n}}(x) \subset B_{\epsilon'}(x)$. Therefore, $x_n \to x$.

2. (16pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = x_1^4 x_2^5$. Give the second order Taylor expansion of f around $x^* = (1, 1)$. What is the order of the error term?

Solution: The function f is infinitely differentiable since it is a polynomial. Therefore, the remainder in its second-order Taylor expansion is $O(|h|^3)$. The Jacobian and Hessian of f are

$$Df(x) = (4x_1^3 x_2^5, 5x_1^4 x_2^4) \text{ and } D^2 f(x) = \begin{pmatrix} 12x_1^2 x_2^5 & 20x_1^3 x_2^4 \\ 20x_1^3 x_2^4 & 20x_1^2 x_2^3 \end{pmatrix}$$

The second-order Taylor expansion of f around $x^* = (1, 1)$ is given by:

$$\begin{aligned} f(x^* + h) &= f(x^*) + Df(x^*)h + \frac{1}{2} h^T D^2 f(x^*) h + O(|h|^3) \\ &= 1 + (4, 5) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \frac{1}{2}(h_1, h_2) \begin{pmatrix} 12 & 20 \\ 20 & 20 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + O(|h|^3) \\ &= 1 + 4h_1 + 5h_2 + 6h_1^2 + 20h_1 h_2 + 10h_2^2 + O(|h|^3) \end{aligned}$$

3. (16pts) Define $F : \mathbb{R}^2 \to \mathbb{R}$ by:

$$F(x, a) = \sin^2(x) - a \qquad x, a \in \mathbb{R}.$$

For each of the following (x_0, a_0) values, state whether you can use the Implicit Function Theorem to conclude that there exist open sets $U, W \subset \mathbb{R}$ such that $x_0 \in U$, $a_0 \in W$, and a C^1 function $g : W \to U$ satisfying: (i) $g(a_0) = x_0$, and (ii) for every $a \in W$, $x = g(a)$ is the unique solution of $F(x, a) = 0$ for $x \in U$. If your answer is yes, find $g'(a_0)$.

(a) $(x_0, a_0) = (\pi, 0)$.
(b) $(x_0, a_0) = (\frac{\pi}{4}, \frac{1}{2})$.
(c) $(x_0, a_0) = (\frac{5\pi}{4}, -\frac{1}{2})$.

Solution: The function F is C^1 and the number of unknowns and the number of equations agree, i.e., $n = m = 1$ in the statement of the Implicit Function Theorem. Therefore, in order to apply the Theorem, we need to check the two remaining conditions: (i) $F(x_0, a_0) = 0$ and (ii) $det(D_x F(x_0, a_0)) \neq 0$, i.e., $\frac{dF}{dx}(x_0, a_0) = 2\sin(x_0)\cos(x_0) \neq 0$.

(a) $F(\pi, 0) = \sin^2(\pi) - 0 = 0 - 0 = 0$, but $\frac{dF}{dx}(\pi, 0) = 2\sin(\pi)\cos(\pi) = 2 \times 0 \times (-1) = 0$ so we can not apply the Implicit Function Theorem.

(b) $F(\frac{\pi}{4},\frac{1}{2}) = \sin^2(\frac{\pi}{4}) - \frac{1}{2} = \left(\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{2} = 0$ and $\frac{dF}{dx}(\frac{\pi}{4},\frac{1}{2}) = 2\sin(\frac{\pi}{4})\cos(\frac{\pi}{4}) = 2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 1 \neq 0$, so we can apply the Implicit Function Theorem.

(c) $F(\frac{5\pi}{4},-\frac{1}{2}) = \sin^2(\frac{\pi}{4}) + \frac{1}{2} = \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} = 1 \neq 0$, so we can not apply the Implicit Function Theorem.

4. (16pts) Find the solution $y : \mathbb{R} \to \mathbb{R}^2$ of the following initial value problem:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

You can leave the solution in the form of a product of matrices and a matrix inverse.[2] In one sentence, explain how the solution behaves qualitatively.

Solution: The eigenvalues of

$$M = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

are given by the roots of its characteristic polynomial $\lambda^2 + 1$, which are $\lambda_1 = i$ and $\lambda_2 = -i$. A pair of eigenvectors associated with these eigenvalues are given by $v_1 = (-1-i, 1)^T$ and $v_2 = (i-1, 1)^T$. Therefore, by Theorem 2 from lecture 14 notes, we know that the general solution of the linear differential equation in part (b) is given by:

$$y(t) = V \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} V^{-1} y(0),$$

where V is the matrix whose columns are the eigenvectors v_1 and v_2, and V^{-1} is its inverse. Since the real part of the eigenvalues are zero, the solution follows a closed loop around the steady state $y_s = 0$.

5. (16pts) Let X, Y, Z be vector spaces over the same field $\mathbb{F}$, and suppose that X and Y are finite dimensional. Show that for any $T \in L(X,Y)$ and $S \in L(Y,Z)$:

$$\dim(X) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T) \cap \mathrm{Ker}(S)) + \dim(\mathrm{Im}(S \circ T)).$$

Solution: Let $\hat{S} \in L(\mathrm{Im}(T), Z)$ be the restriction of the function S to $\mathrm{Im}(T) \subset Y$, i.e., $\hat{S}(y) = S(y)$ for any $y \in \mathrm{Im}(T)$. Applying the Rank-Nullity Theorem to T and $\hat{S}$, we obtain:

$$\dim(X) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) \tag{1}$$

[2] That is, you do not have to carry out the matrix multiplication and matrix inversion.

$$\dim(\mathrm{Im}(T)) = \dim(\mathrm{Ker}(\hat{S})) + \dim(\mathrm{Im}(\hat{S})) \tag{2}$$

Note that $\mathrm{Ker}(\hat{S}) = \mathrm{Im}(T) \cap \mathrm{Ker}(S)$, because for any $y \in Y$:

$y \in \mathrm{Ker}(\hat{S}) \Leftrightarrow y \in \mathrm{Im}(T)$ & $\hat{S}(y) = 0 \Leftrightarrow y \in \mathrm{Im}(T)$ & $S(y) = 0 \Leftrightarrow y \in \mathrm{Im}(T) \cap \mathrm{Ker}(S)$

Also, note that $\mathrm{Im}(\hat{S}) = \mathrm{Im}(S \circ T)$, because for any $z \in Z$:

$z \in \mathrm{Im}(\hat{S}) \Leftrightarrow \exists y \in \mathrm{Im}(T) : \ \hat{S}(y) = S(y) = z \Leftrightarrow \exists x \in X : S(T(x)) = z \Leftrightarrow z \in \mathrm{Im}(S \circ T)$

Therefore, we can rewrite the Rank-Nullity Theorem for $\hat{S}$ in Equation (2) as:

$$\dim(\mathrm{Im}(T)) = \dim(\mathrm{Im}(T) \cap \mathrm{Ker}(S)) + \dim(\mathrm{Im}(S \circ T))$$

Substituting $\dim(\mathrm{Im}(T))$ in Equation (1) by the above term gives the desired equality.

6. (18pts) Let $(X, \|\cdot\|)$ be a normed vector space and assume that $B_1[0] = \{x \in X : \|x\| \leq 1\}$ is compact. Show that the Heine-Borel theorem applies to $(X, \|\cdot\|)$, that is, a subset $A \subset X$ is compact if and only if it is closed and bounded. Hint: Show first that for any $\beta \in \mathbb{R}$ and $x_0 \in X$, the function $f : X \to X$ defined by $f(x) = \beta x + x_0$ is continuous.

   Solution: Let $\beta \in \mathbb{R}$ and $x_0 \in X$, and define the function $f : X \to X$ by $f(x) = \beta x + x_0$. To see that f is continuous, note that for any sequence $\{x_n\}$ in X such that $x_n \to x$ for some $x \in X$, we have

$$\|f(x_n) - x\| = \|(\beta x_n + x_0) - (\beta x + x_0)\| = |\beta| \|x_n - x_0\| \longrightarrow 0.$$

   We next prove that for $\beta > 0$, $B_\beta[x_0] = f(B_1[0])$.

   "$B_\beta[x_0] \subset f(B_1[0])$": Let $y \in B_\beta[x_0]$, i.e., $\|y - x_0\| \leq \beta$. Then,

$$\|\frac{1}{\beta}(y - x_0)\| \leq 1 \implies \frac{1}{\beta}(y - x_0) \in B_1[0] \implies y = f\left(\frac{1}{\beta}(y - x_0)\right) \in f(B_1[0]).$$

   "$f(B_1[0]) \subset B_\beta[x_0]$": Let $y \in f(B_1[0])$, i.e., there exists $x \in B_1[0]$ such that $y = f(x)$. Then,

$$\|y - x_0\| = \|f(x) - x_0\| = \|\beta x\| = \beta \|x\| \leq \beta.$$

We know from class that any compact set in a metric space is closed and bounded. To prove the converse, suppose that $A \subset X$ is closed and bounded. Boundedness implies that there exists $x_0 \in X$ and $\beta > 0$ such that $A \subset B_\beta[x_0]$. We know from above that $B_\beta[x_0]$ is the image of the compact set $B_1[0]$ under a continuous function, so $B_\beta[x_0]$ is also compact. Then, A is compact because it is a closed subset of a compact set.

7. (Bonus, 20pts) Let $u : [0, 1] \to \mathbb{R}$ be a continuous function, and $0 < \delta < 1$. Show that there exists a unique continuous function $V : [0, 1] \to \mathbb{R}$, that solves the equation:

$$V(y) = \max_{c\in[0,y]} u(c) + \delta V(y - c) \quad \text{for all } y \in [0, 1].$$

Solution: For any $V \in C([0, 1])$, define a new function $TV : [0, 1] \to \mathbb{R}$ by

$$TV(y) = \max_{c\in[0,y]} [u(c) + \delta V(y - c)] \quad \text{for all } y \in [0, 1]. \tag{3}$$

Note that the maximum exists because $[0, y]$ is compact and the objective function is continuous when u and V are continuous.

Step 1: $TV \in C([0, 1])$

Let $\epsilon > 0$. Since u and V are continuous on the compact set $[0, 1]$, they are uniformly continuous, i.e., there exists $\delta_u, \delta_V > 0$ such that:

$$\forall x, x' \in [0, 1] : \ |x - x'| < \delta_u \implies |u(x) - u(x')| < \epsilon/2$$

$$\forall x, x' \in [0, 1] : \ |x - x'| < \delta_V \implies |V(x) - V(x')| < \epsilon/2$$

Let $\delta = \min\{\delta_u, \delta_V\}$ and take any $y, y' \in [0, 1]$ such that $|y - y'| < \delta$.

Let $c^* \in [0, y]$ be a solution of the maximization in Equation (3), and define $\hat{c} = \min\{c^*, y'\}$. Note that $\hat{c} \in [0, y']$.

If $c^* \leq y'$, then $\hat{c} = c^*$. If $c^* > y'$, then $\hat{c} = y' < c^* \leq y$, so $|c^* - \hat{c}| \leq |y - y'| < \delta$, implying that $|u(\hat{c}) - u(c^*)| < \epsilon/2$. In either case, we have:

$$u(\hat{c}) \geq u(c^*) - \epsilon/2$$

If $c^* \leq y'$, then $\hat{c} = c^*$ and $|V(y' - \hat{c}) - V(y - c^*)| < \epsilon/2$ because $|y - c^* - (y' - \hat{c})| = |y - y'| < \delta$. If $c^* > y'$, then $\hat{c} = y' < c^* \leq y$, so $|y - c^*| < |y - y'| < \delta$, implying that $|V(y' - \hat{c}) - V(y - c^*)| < \epsilon/2$. In either case, we have:

$$V(y' - \hat{c}) \geq V(y - c^*) - \epsilon/2$$

Therefore,

$$\begin{aligned} TV(y') &= \max_{c\in[0,y']} [u(c) + \delta V(y'-c)] \\ &\geq u(\hat{c}) + \delta V(y' - \hat{c}) \\ &\geq u(c^*) - \epsilon/2 + \delta[V(y-c^*) - \epsilon/2] \\ &> u(c^*) + \delta V(y-c^*) - \epsilon = TV(y) - \epsilon, \end{aligned}$$

implying that $TV(y) - TV(y') < \epsilon$. By a symmetric argument interchanging the roles of y and y' above, we also have that $TV(y') - TV(y) < \epsilon$. Hence $|TV(y) - TV(y')| < \epsilon$, proving that $TV \in C([0,1])$.

Step 2: $T : C([0,1]) \to C([0,1])$ is a contraction of modulus δ

Let $V, W \in C([0,1])$. Let $y \in [0,1]$, and let $c^* \in [0,y]$ be a solution of the maximization in Equation (3). Then,

$$\begin{aligned} TV(y) - TW(y) &= \max_{c\in[0,y]} [u(c) + \delta V(y-c)] - \max_{c\in[0,y]} [u(c) + \delta W(y-c)] \\ &= u(c^*) + \delta V(y-c^*) - \max_{c\in[0,y]} [u(c) + \delta W(y-c)] \\ &\leq u(c^*) + \delta V(y-c^*) - [u(c^*) + \delta W(y-c^*)] \\ &\leq \delta[V(y-c^*) - W(y-c^*)] \\ &\leq \delta\|V - W\|_\infty \end{aligned}$$

By a symmetric argument interchanging the roles of V and W above, we also have that $TW(y) - TV(y) \leq \delta\|V - W\|_\infty$. Hence

$$|TW(y) - TV(y)| \leq \delta\|V - W\|_\infty$$

Taking supremum over all $y \in [0,1]$ in the left hand side of the above inequality, we conclude that

$$\|TV - TW\|_\infty \leq \delta\|V - W\|_\infty$$

Therefore, T is a contraction mapping of modulus δ.

Step 3: Application of the Contraction Mapping Theorem

We have shown above that T is a contraction mapping on the Banach space $(C([0,1]), \|\cdot\|_\infty)$. Therefore, T has a unique fixed point. The desired conclusion follows, since for any $V \in C([0,1])$, V solves the equation in the question if and only if it is a fixed point of T.