Ec240a, Fall 2016

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Problem Set 1

Due: October 28th, 2016

Problem sets are due at 5PM in the GSIs mailbox. You may work in groups, but each student should turn in their own write-up (including a printout of a narrated/commented and executed iPython Notebook if applicable). Please also e-mail a copy of any iPython Notebook to the GSI (if applicable).

# 1 Binomial distribution

You are conducting a survey of the presidential voting intentions of Cal undergraduates. Let $Y = 1$ if a randomly sampled Cal undergraduate plans to vote for Hillary Clinton, and zero if they plan to vote for an alternative candidate. Among the population of Cal undergrads $\theta = \Pr(Y = 1)$ is the true population frequency of individuals who intend to vote for Clinton. You take a random sample of size $N$ from the Cal student body. Let $Z_N = \sum_{i=1}^{N} Y_i$ equal the total number of sampled students who indicate their intention to vote for Obama.

1. Derive a formula that can be used to calculate the ex ante (i.e., pre-sample) probability of the event that $Z_N < z$ for any $z \in \{1, 2, \ldots, N\}$. Provide a 3 - 4 sentence written description of your reasoning.

2. Let $\bar{Y}_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$. Using your answer above, provide an expression that can be used to calculate the ex ante probability of the event $\frac{\sqrt{N}\left(\bar{Y}_N - \theta\right)}{\sqrt{\theta(1-\theta)}} < c$.

3. Using your formula plot, in an iPython Notebook, $\Pr\left(\frac{\sqrt{N}\left(\bar{Y}_N - \theta\right)}{\sqrt{\theta(1-\theta)}} < c\right)$ as a function of $c$ for $N = 5, 10, 100, 1000$ and $\theta = 1/2$. Make a single figure with 4 subplots arrayed $2 \times 2$. Title each figure and label all axes.

4. Let $X \sim \mathcal{N}(0, 1)$ . Plot $\Pr(X < c)$ as a function of c on *each* of the four plots created in the previous problem.

5. Repeat questions 3 and 4 with $\theta = 1/20$. Comment on your figures (4 - 6 sentences).

# 2 Binomial-Beta learning

Let $\theta$, as before, denote the probability than a randomly sampled Cal undergraduate intends to vote for Hillary Clinton. Assume that your beliefs about $\theta$ are summarized by a prior distribution. In particular the probability that you assign to different possible values of $\theta$ is given by a $\text{beta}\,(a, b)$ distribution (i.e., $\theta \sim \text{beta}\,(a, b)$). Let $Z_N$, also as before, equal the number of Cal students, out of a random sample of size $N$, who say they intend to vote for Clinton.

1. What is the conditional distribution of $Z_N$ given $\theta$?

2. Calculate the joint distribution of $Z_N$ and $\theta$.

3. Calculate the conditional distribution of $\theta$ given $Z_N$. What is the mean of this distribution? Why might posterior be a good name for this distribution? (5 - 6 sentences)

4. Assume that $a = b = 1/2$. Comment on this prior (2 to 3 sentences).

# 3 Multivariate normal distribution

Let $\mathbf{Y} = (Y_1, \ldots, Y_K)'$ be a $K \times 1$ random vector with density function

$$f(y_1, \ldots, y_K) = (2\pi)^{-K/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{y} - \mu)' \Sigma^{-1} (\mathbf{y} - \mu)\right),$$

for $\Sigma$ a symmetric positive definite $K \times K$ matrix and $\mu$ a $K \times 1$ vector. We say that $\mathbf{Y}$ is a multivariate normal random variable with mean $\mu$ and covariance $\Sigma$ or

$$\mathbf{Y} \sim \mathcal{N}(\mu, \Sigma).$$

The multivariate normal distribution arises frequently in econometrics and a mastery of its basic properties is essential for both applied and theoretical work in econometrics. This problem provides an opportunity for you to review and/or learn some of these properties. There are many useful references on the multivariate normal distribution, for example, T. W. Anderson's *An Introduction to Multivariate Statistical Analysis.*

1. Let $C$ be a $K \times K$ nonsingular matrix. Show that $\mathbf{Z} = C\mathbf{Y}$ is distributed according to $\mathcal{N}(C\mu, C\Sigma C')$.

2. Partition $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$ into $K_1 \times 1$ and $K_2 \times 1$ sub-vectors with $K_1 + K_2 = K$. Let $\mu = (\mu_1', \mu_2')'$ and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

be conformable partitions of $\mu$ and $\Sigma$ (note that symmetry implies $\Sigma_{12} = \Sigma_{21}'$). Show that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are independent random vectors if $\Sigma_{12} = \Sigma_{21}' = \underline{0}\underline{0}'$ (i.e., a matrix of zeros).

3. Let

$$C = \begin{pmatrix} I_{K_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{K_2} \end{pmatrix}$$

for $I_P$ a $P \times P$ identity matrix. Derive the distribution of $\mathbf{Z} = C\mathbf{Y}$. Are the first $K_1$ elements of $\mathbf{Z}$ independent from the second $K_2$? Interpret your result?

4. Let $D$ be a $P \times K$ ($P \leq K$) matrix of rank $P$. Arrange the first $P$ columns of $D$, denoted by $D_{11}$, such that they are non-singular. Denote the remaining $K - P$ columns by $D_{12}$. Find a $(K - P) \times K$ matrix $E$ such that

$$\begin{pmatrix} \mathbf{Z} \\ \mathbf{X} \end{pmatrix} = \begin{pmatrix} D \\ E \end{pmatrix} \mathbf{Y}$$

is a non-singular transformation of $\mathbf{Y}$. Finally show that $\mathbf{Z}$ is distributed according to $\mathcal{N}(D\mu, D\Sigma D')$.

5. Consider the partition of $\mathbf{Y}$ introduced in Problem 2 above. Derive the conditional distribution of $\mathbf{Y}_1$ given $\mathbf{Y}_2 = \mathbf{y}_2$.

6. Let $\{\mathbf{Y}_i\}_{i=1}^{N}$ be a random sample of size $N$ drawn from the multivariate normal population described above. Show that $\sqrt{N}\left(\overline{\mathbf{Y}}-\mu\right)$ is a $\mathcal{N}(0,\Sigma)$ random variable for $\overline{\mathbf{Y}}=\frac{1}{N}\sum_{i=1}^{N}\mathbf{Y}_i$, the sample mean (HINT: Use independence of the $i=1,\ldots,N$ draws and your result in Problem 4 above).

7. Let $\mathbf{W}=N\cdot\left(\overline{\mathbf{Y}}-\mu\right)'\Sigma^{-1}\left(\overline{\mathbf{Y}}-\mu\right)$. Show that $\mathbf{W}\sim\chi^2_K$ (i.e., $\mathbf{W}$ is a chi-square random variable with $K$ degrees of freedom).

8. Let $\chi_K^{2,1-\alpha}$ be the $(1-\alpha)^{th}$ quantile of the $\chi^2_K$ distribution (i.e., the number satisfying the equality $\Pr\left(\mathbf{W}\leq\chi_K^{2,1-\alpha}\right)=1-\alpha$ with $\mathbf{W}$ a chi-square random variable with $K$ degrees of freedom). Let $D$ be a $P\times K$ $(P\leq K)$ matrix of rank $P$ and $d$ a $P\times 1$ vector of constants. Consider the hypothesis

$$\begin{aligned}H_0 &: D\mu=d\\ H_1 &: D\mu\neq d.\end{aligned}$$

*Maintaining* $H_0$ derive the sampling distribution of $D\overline{\mathbf{Y}}$ as well as that of

$$\mathbf{W}=N\cdot\left(D\overline{\mathbf{Y}}-d\right)'\left(D\Sigma D\right)^{-1}\left(D\overline{\mathbf{Y}}-d\right).$$

You observe that, for the sample in hand, $\mathbf{W}>\chi_P^{2,1-\alpha}$ for $\alpha=0.05$. Assuming $H_0$ is true, what is the ex ante (i.e., pre-sample) probability of this event? What are you inclined to conclude after observing $\mathbf{W}$ in the sample in hand?

# 4 Normal learning

Let $\theta$ be some parameter of interest. For example the average number of hours per week a graduate student in economics spends studying. Upon arriving in graduate school you summarize your beliefs/uncertainty about $\theta$ by assuming that $\theta\sim\mathcal{N}\left(\bar{\theta},\frac{1}{\rho_\theta}\right)$.

1. If $\bar{\theta}=10$ and $\rho_\theta=1/10$, then what is the probability that you assign to the possibility that $\theta$ exceeds 40 hours a week? Is less than 10 hours per week?

2. Let $S_t=\theta+\epsilon_t$ with $\epsilon_t\sim\mathcal{N}\left(0,\frac{1}{\rho_\epsilon}\right)$. Compute the conditional distribution of $\theta$ given $S_1$.

3. You observe the additional signals $S_2,\ldots,S_T$. Compute the conditional distribution of $\theta$ given all $T$ signals.