Econ 204 – Problem Set 5[1]

Due Friday August 14, 2020

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n \in \mathbb{N}$ with $\left|f_n'(x)\right| \leq 1$ for all n and x. Assume,
$$\lim_{n\to\infty} f_n(x) = g(x) \tag{1}$$
for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous.

   Let $a < b$ be real numbers and $\varepsilon > 0$. Take n large enough such that:
$$\left|f_n(a) - g(a)\right| < \varepsilon \quad \left|f_n(b) - g(b)\right| < \varepsilon \tag{2}$$
   Since f_n is differentiable on $\mathbb{R}$, by the mean value theorem there exists $c \in (a, b)$ such that
$$f_n(b) - f_n(a) = f_n'(c)(b - a). \tag{3}$$
   Thus,
$$|f_n(b) - f_n(a)| = |f_n'(c)|\,|b - a| \leq |b - a|\,. \tag{4}$$
   Then, using the triangle inequality:
$$\begin{aligned} \left|g(a) - g(b)\right| &\leq \left|g(a) - f_n(a)\right| + \left|f_n(a) - f_n(b)\right| + \left|f_n(b) - g(b)\right| \\ &< 2\varepsilon + \left|f_n(a) - f_n(b)\right| \\ &\leq 2\varepsilon + |b - a|\,. \end{aligned} \tag{5}$$
   This result implies that $\left|g(a) - g(b)\right| < 2\varepsilon + |b - a|$ for all $\varepsilon > 0$, hence $\left|g(a) - g(b)\right| \leq |b - a|$; concluding the Lipschitz continuity of g.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 (twice continuously differentiable) function. The function and its second derivative are bounded, namely there exist $M, N > 0$ such that $\sup_{x\in\mathbb{R}} \left|f(x)\right| \leq M$ and $\sup_{x\in\mathbb{R}} \left|f''(x)\right| \leq N$. Show that $\sup_{x\in\mathbb{R}} \left|f'(x)\right| \leq 2\sqrt{MN}$.

   Fix an arbitrary $x \in \mathbb{R}$. Then, using Taylor's theorem for every $y \in \mathbb{R}$ there exists ξ between x and y such that
$$\begin{aligned} f(y) &= f(x) + f'(x)(y - x) + \frac{1}{2} f''(\xi)(y - x)^2 \\ &\leq f(x) + f'(x)(y - x) + \frac{1}{2} N(y - x)^2. \end{aligned} \tag{6}$$
   Since $f(x) - f(y) \leq 2M$, then $\frac{1}{2}N(y - x)^2 + f'(x)(y - x) + 2M \geq 0$ for every $y \in \mathbb{R}$. Therefore, the quadratic polynomial
$$g(t) = \frac{1}{2} N t^2 + f'(x) t + 2M \tag{7}$$

[1] In case of any problems with the solution to the exercises please email farzad@berkeley.edu

is nonnegative for all $t \in \mathbb{R}$. Consequently, its $\Delta = \big(f'(x)\big)^2 - 4MN$ must be less than or equal to zero, that implies $\big|f'(x)\big| \leq 2\sqrt{MN}$. Since x was arbitrarily chosen this bound holds for every $x \in \mathbb{R}$ that concludes the proof.

3. The oscillation of an arbitrary function $f : [a, b] \to \mathbb{R}$ at $x \in [a, b]$ is [2]

$$\mathrm{osc}_x f := \lim_{r \downarrow 0} \mathrm{diam}\Big(f\left([x-r, x+r]\right)\Big), \tag{8}$$

where for every $x_1, x_2 \in [a, b]$, $f([x_1, x_2]) := \{y : y = f(x) \text{ for some } x \in [x_1, x_2]\}$. For $k > 0$, let D_k be the set of points with oscillation greater than or equal to k, i.e $D_k := \{x \in [a, b] : \mathrm{osc}_x f \geq k\}$. Prove that D_k is closed.[3]

We show that every convergent sequence in D_k converges to a point in D_k, thus we conclude D_k is closed. Take an arbitrary sequence $\{x_n\} \subset D_k$ such that $x_n \to x$. For every $\varepsilon > 0$, one can find $N \in \mathbb{N}$ such that $\forall n \geq N$: $|x_n - x| \leq \varepsilon/2$. This implies that $[x_n - \varepsilon/2, x_n + \varepsilon/2] \subset [x - \varepsilon, x + \varepsilon]$, because for every $y \in [x_n - \varepsilon/2, x_n + \varepsilon/2]$:

$$|y - x| \leq |y - x_n| + |x_n - x| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{9}$$

As a result of this inclusion, we deduce that

$$\mathrm{diam}\Big(f\left([x_n - \varepsilon/2, x_n + \varepsilon/2]\right)\Big) \leq \mathrm{diam}\Big(f\left([x - \varepsilon, x + \varepsilon]\right)\Big). \tag{10}$$

Since $x_n \in D_k$, the *lhs* to the above inequality is greater than or equal to k, therefore, $\mathrm{diam}\Big(f\left([x - \varepsilon, x + \varepsilon]\right)\Big) \geq k$. Since this result holds for every ε, then

$$\lim_{\varepsilon \downarrow 0} \mathrm{diam}\Big(f\left([x - \varepsilon, x + \varepsilon]\right)\Big) \geq k, \tag{11}$$

proving $x \in D_k$.

4. The goal of this exercise is to verify the **Banach-Steinhaus** theorem. Let $\{T_n\}$ be a sequence of bounded linear functions $T_n : X \to Y$ from a Banach (complete normed vector) space X into a normed vector space Y, such that $\{T_n(x)\}$ is bounded for every $x \in X$, that is for all $x \in X$ there exists $c_x \in \mathbb{R}_+$ such that:

$$\big\|T_n(x)\big\| \leq c_x \quad \forall n \in \mathbb{N} \tag{12}$$

Then, we want to show that the sequence of norms $\big\{\|T_n\|\big\}$ is bounded, that is there exists $c > 0$ such that $\|T_n\| \leq c$ for all $n \in \mathbb{N}$.

(a) For every $k \in \mathbb{N}$ let $A_k \subseteq X$ be the set of all $x \in X$ such that $\big\|T_n(x)\big\| \leq k$ for all

[2]The symbol '$\downarrow$' means that r *decreases* to 0 along the limit.

[3]This question is part of the exercise 19 in chapter 3 of the second edition of *Real Mathematical Analysis*, Charles Chapman Pugh.

n. Show that A_k is closed under the X-norm.

Suppose $\{x_j\}$ is a sequence in A_k converging to some point $x \in X$. Then:

$$\left\|T_n(x) - T_n(x_j)\right\| = \left\|T_n(x - x_j)\right\| \leq \|T_n\| \left\|x - x_j\right\| \tag{13}$$

The last inequality holds because $T_n \in B(X, Y)$, namely is a bounded linear function. Now one can send $j \to \infty$ and because $\left\|x - x_j\right\| \to 0$, then $\lim_{j\to\infty} T_n(x_j) = T_n(x)$. We have seen in exercise 1 of ps.3 that the metric and in particular the norm operator is a continuous mapping, hence $\lim_{j\to\infty} \left\|T_n(x_j)\right\| = \left\|T_n(x)\right\|$, thereby $T_n(x) \leq c_x$ and $x \in A_k$, which verifies that A_k is closed.

(b) Use equation (12) to show that $X = \bigcup_{k\in\mathbb{N}} A_k$.

Note that for each k, $A_k \subseteq X$ hence $\bigcup_{k\in\mathbb{N}} A_k \subseteq X$. Further, for every $x \in X$ the sequence $\{T_n(x)\}$ is bounded by c_x, hence there has to be some $k \in \mathbb{N}$ such that $k \geq c_x$ and $x \in A_k$, which implies $\bigcup_{k\in\mathbb{N}} A_k \supseteq X$.

(c) The **Baire's** theorem states that in this case since X is complete, there exists some A_{k_0} that contains an open ball, say $B_\varepsilon(x_0) \subseteq A_{k_0}$. Take this result as given, and prove there exists some constant $c > 0$ such that

$$\|T_n\| \leq c \quad \forall n \in \mathbb{N}. \tag{14}$$

Hint: For every nonzero $x \in X$ there exists $\gamma > 0$ such that $x = \frac{1}{\gamma}(z - x_0)$, where $x_0, z \in B_\varepsilon(x_0)$ and $\gamma > 0$.

Let $x \neq 0$ be arbitrary and set $z = x_0 + \gamma x$, where $\gamma = \varepsilon/2\|x\|$. Therefore, $\|z - x_0\| < \varepsilon$, hence $z \in_\varepsilon (x_0) \subseteq A_{k_0}$. This implies that for all $n \in \mathbb{N}$: $\left\|T_n(x_0)\right\| \leq k_0$ and $\left\|T_n(z)\right\| \leq k_0$. Therefore,

$$\begin{aligned}\left\|T_n(x)\right\| &= \left\|T_n\left(\frac{1}{\gamma}(z - x_0)\right)\right\| = \frac{1}{\gamma}\left\|T_n(z) - T_n(x_0)\right\| \\ &\leq \frac{1}{\gamma}\left(\left\|T_n(z)\right\| + \left\|T_n(x_0)\right\|\right) \leq \frac{2k_0}{\gamma} = \frac{4k_0\|x\|}{\varepsilon},\end{aligned} \tag{15}$$

which holds for all $x \neq 0$. This implies that $\|T_n\| \leq 4k_0/\varepsilon$ for all n (why?).

5. Suppose $\Psi : X \to 2^X$ is a non-empty and compact-valued upper-hemicontinuous correspondence. The metric space X is compact. Show that there exists a non-empty compact set $C \subset X$ such that $\Psi(C) = C$ (you can use the exercises that are proved in the sections).

First recall that we have shown in section 7 that the image of every compact subset under such a correspondence is compact. Therefore, $\Psi(X)$ is compact and $\Psi(X) \subset X$. Hence, $\Psi^2(X) := \Psi(\Psi(X)) \subset \Psi(X)$ is also compact. Consequently, we can construct a decreasing sequence of compact subsets $\{\Psi^n(X)\}$ such that $\Psi^{n+1}(X) := \Psi(\Psi^n(X))$

and $\Psi^n(X) \supset \Psi^{n+1}(X) \supset \ldots$. Let $C = \bigcap_{n\in\mathbb{N}} \Psi^n(X)$, which is non-empty because of Cantor theorem (section notes 6), and is closed because it is the intersection of closed subsets. So, C is compact as it is a closed subset of a compact set X. Since $C \subset \Psi^n(X)$ for every n, then $\Psi(C) \subset \Psi(\Psi^n(X)) = \Psi^{n+1}(X)$, and hence $\Psi(C) \subset \bigcap_{n\in\mathbb{N}} \Psi^n(X) = C$. Thus it is enough to show $C \subset \Psi(C)$. For this we offer two proofs; the first one is based on the sequential characterization of uhc and the second one uses the open set definition.

First proof: Let $y \in C$. By definition $y \in \Psi^n(X)$ for every n, so for every n there exists $z_n \in \Psi^{n-1}(X)$ such that $y \in \Psi(z_n)$. Then $\{z_n\} \subseteq X$ and X is compact, so there is a convergent subsequence $z_{n_k} \to z \in X$. Since $y \in \Psi(z_{n_k})$ for each n_k and Ψ has closed graph (because Ψ is uhc and closed-valued), must have $y \in \Psi(z)$ as well. Now claim $z \in C$. If not, then there exists N such that for all $n \geq N$, $z \notin \Psi^n(X)$. In particular, $z \notin \Psi^N(X)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \cap \Psi^N(X) = \emptyset$. But this is a contradiction, as $z_{n_k} \in \Psi^N(X)$ for all $n_k > N$, and $z_{n_k} \to z$. Therefore $z \in C$. Thus $y \in \Psi(C)$. So $C \subset \Psi(C)$.

Second proof: Assume to the contrary, $\exists z \in C \setminus \Psi(C)$. Therefore, $\{z\}$ and $\Psi(C)$ are disjoint. Since $\Psi(C)$ is closed one can find an open ball $B(z, \varepsilon_z)$ around z such that $B(z, \varepsilon_z) \cap \Psi(C) = \emptyset$. Let $\bar{B}(z, \varepsilon_z/2) = \big\{y \in X : d(y, z) \leq \varepsilon/2\big\}$ be the closed ball with radius $\varepsilon_z/2$ around z, then $\bar{B}(z, \varepsilon_z/2) \subset B(z, \varepsilon_z)$. Now for every point $x \in \Psi(C)$ one can find an open ball $B(x, \varepsilon_x)$ such that $B(x, \varepsilon_x) \cap \bar{B}(z, \varepsilon_z/2) = \emptyset$. Let G be the union of all these balls, i.e $G = \bigcup_{x\in\Psi(C)} B(x, \varepsilon_x)$, then G is an open set containing $\Psi(C)$ that is disjoint from $B(z, \varepsilon_z/2)$ containing z. Because of uhc the upper-inverse $\Psi^u(G)$ is open and covers C. There must be some $N \in \mathbb{N}$ such that for all $n \geq N$, $\Psi^n(X) \subset \Psi^u(G)$, because otherwise one could employ an elementary compactness argument to reach a contradiction. This implies $\Psi^{N+1}(X) = \Psi(\Psi^N(X)) \subset G$, so $C \subset G$, that violates the disjointness of $z \in C$ from G. Therefore, $C \subset \Psi(C)$.