Problem 1.

Let A be an $n \times n$ matrix.

(a) Show that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k for $k \in \mathbb{N}$.

(b) Show that if λ is an eigenvalue of the matrix A and A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1}.

(c) Find an expression for $\det(A)$ in terms of the eigenvalues of A.

(d) The *eigenspace* of an eigenvalue λ_i is the kernel of $A - \lambda_i I$. Show that the eigenspace of any matrix A belonging to an eigenvalue λ_i is a vector space.

Solution

(a) We use induction to show not only that λ^k is an eigenvalue of A^k, but also that any eigenvector v corresponding to the eigenvalue λ for A also corresponds to λ^k for A^k. The base step ($k = 1$) is trivial. For the induction step, assume $Av = \lambda v$ and $A^k v = \lambda^k v$. Now consider $A^{k+1}v$:

$$A^{k+1}v = A^k(Av) = A^k(\lambda v) = \lambda(A^k v) = \lambda(\lambda^k v) = \lambda^{k+1}v$$

(b) Let $Tv = \lambda v$. Premultiply both sides by T^{-1}:

$$T^{-1}Tv = T^{-1}\lambda v \implies v = \lambda T^{-1}v \implies T^{-1}v = (1/\lambda)v$$

(c) The characteristic polynomial of A is given by $c(\lambda) = \det(A - \lambda I)$. The eigenvalues are the roots of this function; this is,

$$c(\lambda) = \det(A - \lambda I) = (-1)^n \prod_i (\lambda - \lambda_i)$$

Hence setting $\lambda = 0$ we get

$$\det(A) = \prod_i \lambda_i$$

so the determinant of a matrix is the product of its eigenvalues.

(d) To show that the eigenspace is a vector space, we only need to check the existence of additive identity and inverse elements. Since the eigenspace contains the $\mathbf{0}$ vector by definition, we only need to verify the existence of inverse elements. Denote the eigenspace by E_i. Let $v \in E_i$. Then $Av = \lambda_i v$. Multiplying both sides by -1 gives us $A(-v) = \lambda_i(-v) \implies (-v) \in E_i$.

Problem 2.

Let V be an n-dimensional vector space. Call a linear operator $T : V \to V$ *idempotent* if $T \circ T = T$. Prove that all such operators are diagonalizable (that is, any matrix representation $A = Mtx_U(T)$ is diagonalizable). What are the eigenvalues?

Solution

Let $\operatorname{Rank} T = r \leq n$. So we can find r linearly independent vectors $\{u_1, \ldots, u_r\} \in \operatorname{Im} T$. Then for u_i,

$$\begin{aligned} &\exists v_i : u_i = T v_i \\ \implies &\; T u_i = T^2 v_i = T v_i = u_i \end{aligned}$$

hence u_i is an eigenvector of T (with corresponding eigenvalue $\lambda_i = 1$).

If T is full rank ($r = n$), this set of eigenvectors forms a basis of V. If not, then by the Rank-Nullity Theorem, $\dim \ker T = r - n > 0$ so we can find another set of linearly independent vectors $\{w_{r+1}, \ldots, w_n\} \in \ker T$. That is, $T w_i = 0$ so w_i is an eigenvector of T (with corresponding eigenvalue $\lambda_i = 0$). Then the eigenvectors $\{u_1, \ldots, u_r, w_{r+1}, \ldots, w_n\}$ forms a basis of V. Thus any any matrix representation $A = Mtx_U(T)$ is diagonalizable.

Note for any idempotent linear operator, the eigenvalues are all either 1 or 0. Further, any idempotent operator with only one eigenvalue $\lambda = 1$ is the identity.

Problem 3.

Let V be a finite-dimensional vector space and $W \subset V$ be a vector subspace. Prove that W has a complement in V, i.e., there exists a vector subspace $W' \subset V$ such that $W \cap W' = \{0\}$ and $W + W' = V$.

Solution

Let $n = \dim V$. Let $\{v_1, \ldots, v_k\}$ be a basis for W. Extend this to a basis $\{v_1, \ldots, v_k\} \cup \{v_{k+1}, \ldots, v_n\}$ for V. Let $W' = \operatorname{span}\{v_{k+1}, \ldots, v_n\}$. Then W' is a vector subspace of V and $W + W' = V$. For the second claim, note that W and W' are both vector subspaces of V and V is a vector space, so $W + W' \subset V$. Then let $v \in V$. Since $\{v_1, \ldots, v_k\} \cup \{v_{k+1}, \ldots, v_n\}$ is a basis for V, there exist $\alpha_1, \ldots, \alpha_n$ such that

$$v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{k} \alpha_i v_i + \sum_{i=k+1}^{n} \alpha_i v_i = w + w'$$

where $w = \sum_{i=1}^{k} \alpha_i v_i \in W$ and $w' = \sum_{i=k+1}^{n} \alpha_i v_i \in W'$. Thus $V \subseteq W + W'$.

Now we have to show that $W \cap W' = \{0\}$. If $v \in W \cap W'$, then $v = \alpha_1 v_1 + ... + \alpha_k v_k$, for suitable scalars $\alpha_1, ..., \alpha_k$, and $v = \alpha_{k+1} v_{k+1} + ... + \alpha_n v_n$, for suitable scalars $\alpha_{k+1}, ..., \alpha_n$. Then, $\alpha_1 v_1 + ... + \alpha_k v_k = \alpha_{k+1} v_{k+1} + ... + \alpha_n v_n$, so

$$\alpha_1 v_1 + ... + \alpha_k v_k - \alpha_{k+1} v_{k+1} - ... - \alpha_n v_n = 0.$$

Since $\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ is a basis, is linearly independent, then $\alpha_1 = ... = \alpha_n = 0$. Thus, $v = 0$, i.e., $W \cap W' = \{0\}$.

In case you are interested, this result is also true when V is arbitrary (i.e., when V is not necessarily finite-dimensional). The argument for the proof is similar, but an additional Lemma would be needed to argue that a basis for W can be extended to a basis for V.

Problem 4.

Let U and V be vector spaces. Suppose $T : U \to V$ is a linear transformation and $v \in V$. Prove that, if the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$, then $T^{-1}(v) = \{u + z | z \in \ker T\} = u + \ker T$.

Solution

Let $M = \{u + z | z \in \ker T\}$. Suppose that $w \in M$, so $w = u + z$, where $z \in \ker T$. Then $T(w) = T(u + z) = T(u) + T(z) = v + 0 = v$, so $w \in T^{-1}(v)$. We proved that $M \subset T^{-1}(v)$.

Now, suppose $x \in T^{-1}(v)$. Then $T(x - u) = T(x) - T(u) = v - v = 0$, so $x - u \in \ker T$. This means there is a vector $z \in \ker T$ such that $x - u = z$. Rearranging this equation gives $x = u + z$, so $x \in M$. We proved that $T^{-1}(v) \subset M$.

Since $M \subset T^{-1}(v)$ and $T^{-1}(v) \subset M$, we have that $T^{-1}(v) = M$.

Problem 5.

Let V be a finite dimensional vector space and $T, S \in L(V, V)$. Prove that TS is invertible if and only if T and S are invertible.

Solution

($\Rightarrow$) Assume TS is invertible. To prove that S is invertible, we can use the Rank-Nullity theorem and check whether $\ker S = 0$. Let $w \in \ker S$, so $S(w) = 0$. Then, $TS(w) = T(0) = 0$. If $w \neq 0$, then TS has a non-zero kernel. This is a contradiction with TS begin invertible. Then, $w = 0$ and S is invertible. Towards a contradiction, assume T is not invertible. Then, $\exists v \neq 0$ such that $T(v) = 0$. Since S is invertible, it is surjective. Then $\exists w$ such that $S(w) = v$, and $v \neq 0$ implies $w \neq 0$. Then, $TS(w) = 0$, which contradicts TS being invertible. Then, T is invertible.

($\Leftarrow$) Let $v \in \ker TS$. Then, $TS(v) = 0$. Note that this implies that either (i) $S(v) = 0$, in which case $v = 0$ since S is invertible, or (ii) $w = S(v)$ and $T(w) = 0$, in which case $w = 0$ since T is invertible. Then, TS is invertible.

Problem 6.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (4x - 2y, x + y)$. Let V be the standard basis and $W = \{(5, 3), (1, 1)\}$ be another basis of $\mathbb{R}^2$.

(a) Find $Mtx_V(T)$.

(b) Find $Mtx_W(T)$.

(c) Compute $T(4, 3)$ using the matrix representation of W.

Solution

(a)

$$Mtx_V(T) = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

(b) First,

$$Mtx_{V,W}(id) = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}$$

and $Mtx_{W,V}(id) = Mtx_{V,W}(id)^{-1}$, then

$$Mtx_{W,V}(id) = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -3 & 5 \end{pmatrix}$$

Finally

$$Mtx_W(T) = Mtx_{W,V}(id) \cdot Mtx_V(T) \cdot Mtx_{V,W}(id) = \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}$$

(c) We know that $Mtx_W(T) \cdot crd_W(4, 3) = crd_W(T(4, 3))$. Solving $(4, 3) = \alpha(5, 3) + \beta(1, 1)$ yields $\alpha = 1/2$ and $\beta = 3/2$. Then

$$\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1\backslash 2 \\ 3\backslash 2 \end{pmatrix} = \begin{pmatrix} 3\backslash 2 \\ 5\backslash 2 \end{pmatrix}$$

And $\frac{3}{2} \cdot (5, 3) + \frac{5}{2}(1, 1) = (10, 7) = T(4, 3)$.