Asset Pricing
Econ 70427
Asset Pricing in Macro

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Concepts to cover

- Recursive utility (Epstein-Zin)
- Long-Run Risk
- Long-Run Risk versus Habit persistence

Recursive Utility

- Epstein and Zin (1989 JPE, 1991 Econometrica), extending work by Kreps and Porteus.
- Current utility depends on the current consumption (and labor) flow and future utility.
- Breaks the link between risk aversion and intertemporal substitution.

Upper case quantities are levels. M, C. Upper case returns are gross returns. Lower case quantities are logs. Lower case returns are rates of return.

Contrast with Time-Separable Utility

Current utility flow

$$u\left(C_t\right)$$

Lifetime expected utility

$$V_t = E_t \sum_{j=0}^{\infty} \beta^j u\left(C_{t+j}\right)$$

$$V_t = u\left(C_t\right) + \beta E_t V_{t+1}$$

CES version

$$\begin{aligned} V_t^{1-\rho} &= (1-\beta) u\left(C_t\right)^{1-\rho} + \beta E_t V_{t+1}^{1-\rho} \\ V_t &= \left((1-\beta) u\left(C_t\right)^{1-\rho} + \beta E_t V_{t+1}^{1-\rho}\right)^{\frac{1}{1-\rho}} \end{aligned}$$

Epstein-Zin (EZ)

EZ preferences. This is both current utility and lifetime utility.

$$V_t = \left[(1-\beta) C_t^{1-\rho} + \beta \left(E_t V_{t+1}^{1-\gamma} \right)^{\frac{1-\rho}{1-\gamma}} \right]^{\frac{1}{1-\rho}} \tag{1}$$

where γ is risk aversion, $\rho = 1/\psi$ where ψ is the IES.
Utility is CRRA if $\gamma = 1/\psi$.

A different perspective on the future

- Under time-separable utility, current utility depends just on current consumption. Future utility doesn't affect current happiness.

- Suppose you're offered a meal cooked by a Michelin awarded chef. Happy $u(C_t)$. Now, you are offered the same meal, but you are on death row, scheduled to be executed in 3 hours. Do you feel the same, or does future utility affect current utility?

Let's write with some short-cut notation

$$V_t = F\left(C_t, R\left(V_{t+1}\right)\right) = \left\{\left(1-\beta\right)\left(C_t\right)^{1-\rho} + \beta\left(R_t\left(V_{t+1}\right)\right)^{1-\rho}\right\}^{\frac{1}{1-\rho}}$$

where

$$R_t\left(V_{t+1}\right) = \left(E_t\left(V_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{1-\gamma}}$$

Take some derivatives

$$\begin{aligned}
\frac{\partial V_t}{\partial C_t} &= F_1\left(C_t, R\left(V_{t+1}\right)\right) = \left(1-\beta\right)\left(C_t\right)^{-\rho}\ V_t^{\rho} \\
\frac{\partial V_t}{\partial R\left(V_{t+1}\right)} &= F_2\left(C_t, R\left(V_{t+1}\right)\right) = \beta\left(\frac{V_t}{R\left(V_{t+1}\right)}\right)^{\rho} \\
\frac{\partial R\left(V_{t+1}\right)}{\partial V_{t+1}} &= \left(\frac{V_{t+1}}{R\left(V_{t+1}\right)}\right)^{-\gamma} \\
\frac{\partial V_t}{\partial C_{t+1}} &= \frac{\partial V_t}{\partial R\left(V_{t+1}\right)}\frac{\partial R\left(V_{t+1}\right)}{\partial V_{t+1}}\frac{\partial V_{t+1}}{\partial C_{t+1}} \\
&= F_2\left(C_t, R\left(V_{t+1}\right)\right)\left(\frac{\partial R\left(V_{t+1}\right)}{\partial V_{t+1}}\right) F_1\left(C_{t+1}, R\left(V_{t+2}\right)\right) \\
&= \left(1-\beta\right)\beta C_{t+1}{}^{-\rho}\ V_t^{\rho}\left(\frac{V_{t+1}}{R\left(V_{t+1}\right)}\right)^{\rho-\gamma}
\end{aligned}$$

Details on $\partial R(V)/\partial V$

Differentiation rule

$$\frac{d}{dw(x)}\left(\int w(x) f(x)\, dx\right) = \int f(x)\, dx$$

Apply to an expectation

$$\frac{d}{dV_{t+1}^{1-\gamma}}\left(E_t V_{t+1}^{1-\gamma}\right) = \frac{d}{dV_{t+1}^{1-\gamma}}\left(\int V_{t+1}(s)^{1-\gamma} f(s)\, ds\right) = \int f(s)\, ds = 1$$

where the pdf of the underlying states s is $f(s)$

$$\frac{dR(V_{t+1})}{dV_{t+1}} = \frac{d\left(E_t V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}{dV_{t+1}} = \underbrace{\frac{d\left(E_t V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}}{E_t V_{t+1}^{1-\gamma}}}_{(a)} \underbrace{\frac{dE_t V_{t+1}^{1-\gamma}}{dV_{t+1}}}_{(b)}$$

$$(a) = \frac{1}{1-\gamma} E_t \left(V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}-1}$$

$$(b) = \frac{dE_t V_{t+1}^{1-\gamma}}{dV_{t+1}^{1-\gamma}} \frac{dV_{t+1}^{1-\gamma}}{dV_{t+1}} = \underbrace{\frac{d\int V_{t+1}^{1-\gamma} f(s)\, ds}{dV_{t+1}^{1-\gamma}}}_{1} (1-\gamma) V_{t+1}^{-\gamma}$$

Stochastic Discount Factor

$$M_{t,t+1} = \frac{\frac{\partial V_t}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}} = \frac{\frac{\partial V_t}{\partial R(V_{t+1})} \frac{\partial R(V_{t+1})}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}}$$

$$\begin{aligned} M_{t+1} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{R\left(V_{t+1}\right)} \right)^{(\rho-\gamma)} \\ &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\left(E_t \left(V_{t+1}^{1-\gamma} \right) \right)^{\frac{1}{1-\gamma}}} \right)^{(\rho-\gamma)} \end{aligned}$$

Early or Late Resolution of Uncertainty

- Option 1: Can find out now if you have the Alzheimer's gene. Option 2 wait and see
- Your comps have been graded. You're going on 2 weeks vacation. Option 1: find your score now. Option 2: Wait until you come back.
- Lottery 1 either pays you 1 forever or 0 forever, probability is 1/2. Lottery 2 pays 1 or 0 each period forever with probability 1/2. Which lottery do you prefer?

In each case, if you like option 1, you have preference for early resolution of uncertainty. We get this in recursive utility if

$$\gamma > \rho = \frac{1}{\psi}$$

Connection to Asset Pricing

- In asset pricing, if $IES < 1$, people don't like growing consumption. They have a hard time having low consumption now and high consumption later. They want to borrow to eat more now. This urge to borrow drives up $P_{t,f}$ and drives down $R_{t,f}$.
- The standard case is for $\gamma > 1/\psi$. The asset pays when there is an upward revision in expected consumption growth. That makes the asset risky. When $\psi < 1$, the stock price increases when volatility increases (i.e., when returns are low). The risk premium for volatility shocks is negative.

Household Wealth

V_t is homogeneous of degree 1. Euler's theorem implies

$$V_t = \frac{\partial V_t}{\partial C_t} C_t + E_t \left(\frac{\partial V_t}{\partial R\left(V_{t+1}\right)} \frac{\partial R\left(V_{t+1}\right)}{\partial V_{t+1}} V_{t+1} \right)$$

Divide by $\partial V_t / \partial C_t$, multiply and divide second term by $\partial V_{t+1} / \partial C_{t+1}$.

$$\frac{V_t}{\frac{\partial V_t}{\partial C_t}} = C_t + E_t \left(\underbrace{\frac{\frac{\partial V_t}{\partial R(V_{t+1})} \frac{\partial R(V_{t+1})}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial C_{t+1}}}{\frac{\partial V_t}{\partial C_t}}}_{M_{t+1}} \frac{V_{t+1}}{\frac{\partial V_{t+1}}{\partial C_{t+1}}} \right)$$

Let's define the left hand side as

$$W_t \equiv \frac{V_t}{\frac{\partial V_t}{\partial C_t}}$$

then

$$W_t = C_t + E_t M_{t+1} W_{t+1}$$

What ever this thing W_t is, it's current consumption plus the expected discounted value of next period's thing. **Wow, it looks like wealth**. Let's call it wealth.

Return on Wealth

Next, we show that

$$R_{m,t+1} = \frac{1}{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}}{R\left(V_{t+1}\right)}\right)^{1-\rho} \tag{2}$$

Proof: By substitution of derivatives obtained earlier, we get,

$$W_{t+1} = \frac{V_{t+1}}{\left(\frac{\partial V_{t+1}}{\partial C_{t+1}}\right)} = \frac{C_{t+1}^{\rho} V_{t+1}^{1-\rho}}{(1-\beta)} \tag{3}$$

Hence,

$$R_{m,t+1} \equiv \frac{W_{t+1}}{W_t - C_t} = \frac{C_{t+1}^{\rho} V_{t+1}^{1-\rho}}{C_t^{\rho} V_t^{1-\rho} - (1-\beta)\, C_t} = \left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}^{1-\rho}}{V_t^{1-\rho} - (1-\beta)\, C_t^{1-\rho}}\right) \tag{4}$$

Substitute

$$V_t^{1-\rho} = (1-\beta)\, C_t^{1-\rho} + \beta R\left(V_{t+1}\right)^{1-\rho}$$

into (4),

$$
\begin{aligned}
R_{m,t+1} &= \left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}^{1-\rho}}{(1-\beta)\,C_t^{1-\rho}+\beta R\left(V_{t+1}\right)^{1-\rho}-(1-\beta)\,C_t^{1-\rho}}\right) \\
&= \left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}^{1-\rho}}{\beta R\left(V_{t+1}\right)^{1-\rho}}\right) \\
&= \frac{1}{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}}{R\left(V_{t+1}\right)}\right)^{1-\rho}
\end{aligned}
$$

Why is this useful? It helps to show that the SDF depends on consumption growth and the market return

SDF, consumption growth, market return

From (2), solve for $V_{t+1} / R\left(V_{t+1}\right)$

$$\frac{V_{t+1}}{R\left(V_{t+1}\right)} = \left(\beta R_{m,t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right)^{\frac{1}{1-\rho}}$$

Substitute this into the SDF

$$\begin{aligned} M_{t+1} &= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\beta R_{m,t+1}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right)^{\frac{\rho-\gamma}{1-\rho}} \\ &= \beta^{\frac{1-\gamma}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\left(\frac{1-\gamma}{1-\rho}\right)} R_{m,t+1}^{\frac{\rho-\gamma}{1-\rho}} \qquad (5) \end{aligned}$$

Let

$$\theta = \frac{1-\gamma}{1-\rho} = \frac{1-\gamma}{1-\frac{1}{\psi}}, \text{ and } \psi = \frac{1}{\rho}$$

(Note: under usual assumptions $\theta < 0$). CRRA is when $\theta = 1$.

Then (5) becomes

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{-\theta}{\psi}} R_{m,t+1}^{\theta-1}$$

Take logs (ignore β)

$$m_{t+1} = -\frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta)\, r_{m,t+1} \tag{6}$$

where

$$\frac{\theta}{\psi} = \frac{1-\gamma}{1-\psi} > 0$$

$$(1-\theta) = \left(\frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}} \right) < 0$$

under the usual assumptions.

Expected Returns

Return on asset i satisfies

$$1 = E_t M_{t+1} R_{i,t+1} \tag{7}$$

$$1 = E_t \left(\beta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{\frac{-\theta}{\psi}} R_{m,t+1}^{\theta-1} R_{i,t+1} \right)$$

Suppose C_{t+1}/C_t, $R_{m,t+1}$, $R_{t+1,i}$ are jointly log normal. (Note: when we take logs, I will ignore β.) Then

$$-\frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1)\, r_{m,t+1} + r_{i,t+1} \sim N\left(\mu, \sigma^2\right)$$

where

$$\mu = \frac{-\theta}{\psi}\mu_{\Delta c} + (\theta - 1)\,\mu_m + \mu_i$$

$$\sigma^2 = \left(\frac{\theta}{\psi}\right)^2 \sigma_{\Delta c}^2 + (\theta - 1)^2 \sigma_m^2 + \sigma_i^2 + 2\frac{\theta}{\psi}(1 - \theta)\, Cov\,(\Delta c, r_m)$$
$$- 2\frac{\theta}{\psi} Cov\,(\Delta c, r_i) + 2\,(\theta - 1)\, Cov\,(r_m, r_i)$$

Expected Returns

Rewrite (7) as,

$$
\begin{aligned}
1 &= e^{\mu + \frac{\sigma^2}{2}} \\
0 &= \frac{-\theta}{\psi}\mu_{\Delta c} + (\theta - 1)\,\mu_m + \mu_i + \left(\frac{\theta}{\psi}\right)^2 \frac{\sigma^2_{\Delta c}}{2} + (\theta - 1)^2 \frac{\sigma^2_m}{2} + \frac{\sigma^2_i}{2} \\
&\quad + \frac{\theta}{\psi}(1 - \theta)\, Cov\,(\Delta c, r_m) - \frac{\theta}{\psi} Cov\,(\Delta c, r_i) + (\theta - 1)\, Cov\,(r_m, r_i) \\
\mu_i &= \frac{\theta}{\psi}\mu_{\Delta c} + (1 - \theta)\,\mu_m - \left(\frac{\theta}{\psi}\right)^2 \frac{\sigma^2_{\Delta c}}{2} - (\theta - 1)^2 \frac{\sigma^2_m}{2} \\
&\quad - \frac{\sigma^2_i}{2} - \frac{\theta}{\psi}(1 - \theta)\, Cov\,(\Delta c, r_m) + \underbrace{\frac{\theta}{\psi} Cov\,(\Delta c, r_i) + (1 - \theta)\, Cov\,(r_m, r_i)}_{\text{CCAPM and CAPM}}
\end{aligned}
$$

Under CRRA

$$
\mu_i = \frac{1}{\psi}\mu_{\Delta c} - \left(\frac{1}{\psi}\right)^2 \frac{\sigma^2_{\Delta c}}{2} - \frac{\sigma^2_i}{2} + \frac{1}{\psi} Cov\,(\Delta c, r_i)
$$

Market return

$$1 = E_t M_{t+1} R_{m,t+1} \tag{8}$$

$$1 = E_t \left(\beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{-\theta}{\psi}} R_{m,t+1}^{\theta} \right) \tag{9}$$

Assume log-normality, take logs, re-arrange

$$0 = \frac{-\theta}{\psi} \mu_{\Delta c} + \theta \, \mu_m + \left(\frac{\theta}{\psi} \right)^2 \frac{\sigma_{\Delta c}^2}{2} + \theta^2 \frac{\sigma_m^2}{2} - \frac{\theta^2}{\psi} Cov\left(\Delta c, r_m\right) \tag{10}$$

$$\mu_m = \frac{1}{\psi} \mu_{\Delta c} - \frac{\theta}{\psi^2} \frac{\sigma_{\Delta c}^2}{2} - \theta \frac{\sigma_m^2}{2} + \frac{\theta}{\psi} Cov\left(\Delta c, r_m\right) \tag{11}$$

More covariance between consumption growth and market return lowers the mean market return (if $\theta < 0$), whereas Under CRRA,

$$\mu_m = \gamma \mu_{\Delta c} - \frac{\gamma^2 \sigma_{\Delta c}^2 + \sigma_m^2}{2} + \gamma Cov\left(\Delta c, r_m\right)$$

Risk-Free Rate

$$1/R_{f,t} = E_t\left(M_{t+1}\right) = E_t\left(e^{-\frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta) r_{m,t+1}}\right)$$

$$\begin{aligned} E\left(m_{t+1}\right) &= -\frac{\theta}{\psi}\mu_{\Delta c} - (1-\theta)\,\mu_m \\ Var\left(m_{t+1}\right) &= \left(\frac{\theta}{\psi}\right)^2 \sigma_{\Delta c}^2 + (1-\theta)^2\,\sigma_m^2 + 2\,(1-\theta)\,\frac{\theta}{\psi} Cov\left(\Delta c, r_m\right) \end{aligned}$$

$$r_{f,t} = \frac{\theta}{\psi}\mu_{\Delta c} + (1-\theta)\,\mu_m - \left(\frac{\theta}{\psi}\right)^2 \frac{\sigma_{\Delta c}^2}{2} - (1-\theta)^2\,\frac{\sigma_m^2}{2} - (1-\theta)\,\frac{\theta}{\psi} Cov\left(\Delta c, r_m\right)$$

- $\psi > 1$ Substitution effect dominates wealth effect
- $\psi = 1$ Substitution and wealth effects offset
- Market volatility lowers $r_{f,t}$

Under CRRA

$$r_{f,t} = \frac{1}{\psi}\mu_{\Delta c} - \left(\frac{1}{\psi}\right)^2 \frac{\sigma_{\Delta c}^2}{2}$$

Equity Premium

$$\mu_m = \frac{1}{\psi}\mu_{\Delta c} - \frac{\theta}{\psi^2}\frac{\sigma^2_{\Delta c}}{2} - \theta\frac{\sigma^2_m}{2} + \frac{\theta}{\psi} Cov\left(\Delta c, r_m\right) \tag{12}$$

$$r_{f,t} = \frac{\theta}{\psi}\mu_{\Delta c} + (1-\theta)\,\mu_m - \left(\frac{\theta}{\psi}\right)^2 \frac{\sigma^2_{\Delta c}}{2} - (1-\theta)^2\,\frac{\sigma^2_m}{2} - (1-\theta)\,\frac{\theta}{\psi} Cov\left(\Delta c, r_m\right)$$

Solve the simultaneous equation system:

$$\mu_m - r_{f,t} = (1-2\,\theta)\,\frac{\sigma^2_m}{2} + \frac{\theta}{\psi}\,Cov\left(\Delta c, r_m\right)$$

This is weird. Higher covariance between the market and consumption growth lowers the equity premium. Market volatility raises the equity premium. Under CRRA

$$\mu_m - r_{f,t} = \frac{1}{\psi}\,Cov\left(\Delta c, r_m\right) - \frac{\sigma^2_m}{2}$$

Bansal and Yaron, JF, 2004

$R_{a,t}$ Gross return on asset that pays aggregate consumption as dividend (unobserved).
$R_{m,t}$ Gross return on the market portfolio (observable?) that pays the aggregate dividend. (6) becomes

$$m_{t+1} = \theta \ln(\beta) - \frac{\theta}{\psi}\Delta c_{t+1} - (1-\theta) r_{a,t+1}$$

$$\begin{aligned} \Delta c_{t+1} &= \mu + x_t + \sigma_t \eta_{t+1} && (13) \\ \Delta d_{t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} && (14) \\ x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} && (15) \\ \sigma^2_{t+1} &= \sigma^2 + \nu_1 \left(\sigma_t^2 - \sigma^2\right) + \sigma_w w_{t+1} && (16) \end{aligned}$$

with $e_{t,} u_{t,} \eta_t, w_t \sim NID(0,1)$. (13)-(16) are the LRR dynamics.
$\phi > 1$, $\varphi_d > 1$, interepreted as leverage. ϕ is the leverage ratio on expected consumption growth. The (latent) long-run risk is x_t.
(note: sketch of solution method in Consolidated_Notes_Global_Risks.tex)