You have approximately 1 hour and 20 minutes to answer the questions in the midterm. Adriana and Suresh will collect the exams at 11.00 sharp. Show your work, and good luck!

**Problem 1. Quasi-linear utility for leisure.** (49 points) Martha is deciding how many hours of leisure $l$ she should take on a typical day, and how much she should consume. Martha's utility function is $u(c, l) = \alpha c + \phi(l)$, where $\phi(l)$ is a function satisfying $\phi'(l) > 0$ and $\phi''(l) < 0$. Assume $\alpha > 0$. The consumption good $c$ has price 1.

1. Compute the marginal utility of consumption $\partial u(c, l) / \partial c$ and of leisure $\partial u(c, l) / \partial l$. What is the special feature of this utility function? (3 points)

2. Martha maximizes $u(c, l)$ subject to a budget constraint. Martha starts the period with income $M$, and earns $w > 0$ for each hour $h$ worked. Every hour of work is subtracted from leisure, so $l = 24 - h$. Write down the budget constraint as a function of $c$ and $l$. Justify the steps. (4 points)

3. Write down the maximization problem of Martha that wants to achieve the highest utility subject to the budget constraint. Martha maximizes with respect to $c$ and $l$. Write down the boundary constraints for $c$ and $l$, and neglect them from now on. (3 points)

4. Assuming that the budget constraint holds with equality, write down the Lagrangean and derive the first order conditions with respect to $c$, $l$, and $\lambda$. (2 points)

5. Solve for $\lambda^*$ as a function of the parameters, $\alpha, M, w$. What does $\lambda^*$ depend on? (3 points)

6. Use the envelope function for constrained maximization to show that that $\lambda^*$ equals $\partial v(\alpha, M, w) / \partial M$, that is, $\lambda$ represents the marginal utility of wealth. Remember, $v(\alpha, M, w)$ is the indirect utility function, that is, $v(\alpha, M, w) = u(c^*(\alpha, M, w), l^*(\alpha, M, w))$. [Note: You will not need to explicitly solve for $c^*$ and $l^*$ to do this] (6 points)

7. Combine 5 and 6 to solve for $\partial v(\alpha, M, w) / \partial M$. Whay is this the case? Relate this to your answer in point 1, providing as much intuition as you can. (4 points)

8. Going back to the maximization problem, plug the value of $\lambda^*$ into the first order condition for $l$. Use the condition you obtain to derive the comparative statics of leisure with respect to income $M$ $(\partial l^*(\alpha, M, w) / \partial M)$ and wage $(\partial l^*(\alpha, M, w) / \partial w)$. What is the sign of these derivatives? [You will need the implicit function theorem for at least one of these] (5 points)

9. Given that $M$ and $w$ are both sources of earnings, why is the effect of changes on $M$ and $w$ on $l^*$ so different? Provide as detailed an answer as you can. (7 points)

10. Now provide an equation that denotes the solution for $c^*(\alpha, M, w)$. [It may be helpful to define $\omega(.) = (\phi')^{-1}(.)$ as the inverse function of $\phi'(.)$]. Could you run into problems with the non-negativity constraints for $c^*$ and $l^*$? Can you give an example of values of $\alpha, M$, and $w$ such that you do violate non-negativity? (5 points)

11. We forgot the second order conditions! Compute the Bordered Hessian and check that the sufficient conditions for an optimum are satisfied. (7 points)

**Solution to Problem 1.**

1. The marginal utility of consumption $\partial u(c, l) / \partial c$ is $\partial(\alpha c + \phi(l)) / \partial c = \alpha$ and the marginal utility of leisure $\partial u(c, l) / \partial l$ is $\partial(\alpha c + \phi(l)) / \partial l = \phi'(l) > 0$. The special feature of these preferences is that the individual has a constant marginal utility from consumption, and a positive but decreasing marginal utility of leisure.

2. Martha cannot spend more than she earns:

$$c \leq M + wh$$

or, rewriting in terms of $c$ and $l$:

$$c \leq M + w(24 - l)$$

or

$$c + wl \leq M + 24w$$

3. Martha maximizes

$$\max_{x_1,x_2} \ \alpha c + \phi(l)$$
$$\text{s.t. } c + wl \leq M + 24w \text{ and}$$
$$c \geq 0, 0 \leq l \leq 24.$$

4. The Lagrangean is $L(c, l, \lambda) = \alpha c + \phi(l) - \lambda[c + wl - M - 24w]$. This leads to the first order conditions

$$\frac{\partial L}{\partial c} = \alpha - \lambda = 0$$
$$\frac{\partial L}{\partial l} = \phi'(l) - \lambda w = 0$$
$$\frac{\partial L}{\partial \lambda} = -(c + wl - M - 24w) = 0$$

5. Using the first equation, we get $\lambda^* = \alpha$. This is a special case, since in general we expect $\lambda$ to depend on all the parameters, incuding income $M$.

6. The envelope theorem tells me that to obtain $dv(\alpha, M, w)/dM$ I can just use the partial derivative of the Lagrangean

$$\partial L(c, l, \lambda)/\partial M = \frac{\partial(\alpha c + \phi(l) - \lambda[c + wl - M - 24w])}{\partial M} = \lambda^*.$$

7. Putting together the answers from points 5 and 6, we know $dv(\alpha, M, w)/dM = \alpha$. In this particular case, therefore, the marginal utility of wealth is constant, and equal to $\lambda$. This means that the marginal effect of an increase in income on the indirect utility is constant. This depends critically on the fact that the marginal utility of consumption is constant, which is the case in a quasi-linear utility function. As we see below, every additional unit of income $M$ is spent on good $c$, yielding marginal utility $\alpha$.

8. Plugging $\lambda^*$ into the f.o.c., we get $\phi'(l^*) - \alpha w = 0$, or $l^* = \phi'^{-1}(\alpha w)$. From this, we see clearly that $l^*$ does not depend on income $M$, hence $\partial l^*(\alpha, M, w)/\partial M = 0$. To compute $\partial l^*(\alpha, M, w)/\partial w$, we use the univariate implicit function theorem:

$$\frac{dl^*(\alpha, M, w)}{dw} = -\frac{\partial[\phi'(l^*) - \alpha w]/\partial w}{\partial[\phi'(l^*) - \alpha w]/\partial l} = -\frac{-\alpha}{\phi''(l^*)} < 0.$$

As the wage increases, the amount of leisure consumed increases.

9. While $M$ and $w$ are both sources of earnings, an increase in $M$ is just a lump-sum increase in income, while an increase in $w$ also alters the trade-off between leisure and consumption, since it alters the opportunity cost of taking leisure. In a quasi-linear utility function, there is no income effect on the good that enters utility non-linearly, in this case $l$. Since there is no income effect, when $M$ increases $l$ does not increase. Even if there is no income effect, an increase in $w$ alters $l$ due to the substitution effect. As the opportunity cost of leisure goes up, leisure goes down.

10. To obtain a solution for $c^*$, we need to use the budget constraint which gives us $c^* = M + w(24 - l^*)$. We can then plug in $l^* = \phi'^{-1}(\alpha w)$, to get $c^* = M + w(24 - \phi'^{-1}(\alpha w))$ or $c^* = M + w(24 - \omega(\alpha w))$ using the notation $\omega(.) = (\phi')^{-1}(.)$. Assuming $0 \leq \phi'^{-1}(\alpha w) \leq 24$ (otherwise the constraint is already violated), the constraint $c^* < 0$ is never violated because $M + w(24 - \omega(\alpha w)) \geq M \geq 0$.. If we did not have the constraint $\phi'^{-1}(\alpha w) \leq 24$, it could be violated.

11. For the second order conditions, we have to check that the determinant of the bordered Hessian is positive. We compute the bordered Hessian as

$$H = \begin{pmatrix} 0 & -1 & -w \\ -1 & 0 & 0 \\ -w & 0 & \phi''(l^*) \end{pmatrix}.$$

The determinant is $0 * 0 - (-1) * (-\phi''(l^*)) - w * (0) = -\phi''(l^*)$, which is positive given the condition given above on $\phi()$. Notice how much easier this was relative to computing the Bordered Hessian for a Cobb-Douglas utlity function. This is a big advantage of assuming quasi-linearity of the utility function.

**Problem 2.** (19 points)

1. Consider a preference relation $\succsim$ with the properties of completeness and transitivity. Define what we mean by completeness and transitivity of a relation. (3 points)

2. As we discussed in class, a preference relation $\succsim$ defines the indifference relation $\sim$ as follows: $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. Here comes the question: If $\succsim$ is complete and transitive, does this imply that the relation $\sim$ is complete? Provide a proof or, if the statement is false, an example to the contrary (6 points)

3. If $\succsim$ is complete and transitive, does it imply that $\sim$ is transitive? Provide a proof or, if the statement is false, an example to the contrary (3 points)

4. Andrew is religious and believe in meditation. He is detouched from material things, but values prayer and meditation highly. As he states his preferencess,"I like meditation, I would always rather always do more of it, and am completely indifferent as to the consumption of material goods". Denote by $m$ the number of hours of mediations, and by $c$ the quantity of material good consumed. We can translate these preferences as follows. When comparing two bundles $x$ and $y$, with $x = (m_x, c_x)$ and $y = (m_y, c_y)$, Andrew's preferences are such that $x \succsim y$ if and only if $m_x \geq m_y$. Provide the intuition for why this is the case and plot indifference curves in the two-dimensional space $(m, c)$. (3 points)

5. Are these preferences monotonic? Are they strictly monotonic? Argue the answer, and define the terms used. (4 points)

6. [Are these preferences convex? Are they continuous? Argue the answer, and define the terms used. (6 points)]

**Solution to Problem 2.**

1. A preference relation $\succsim$ is complete if for all $x$ and $y$ in $X$, either $x \succsim y$, or $y \succsim x$, or both. A preference relation $\succsim$ is transitive if for all $x$ and $y$ in $X$ such that $x \succsim y$ and $y \succsim z$, $x \succsim z$ follows.

2. If a preference relation $\succsim$ is complete and transitive, this does not imply that the indifference relation $\sim$ is complete. Assume for example that $X$ is formed by $a$ and $b$, and the preference relation is defined by $\{a \succsim a, a \succsim b, b \succsim b\}$. This preference relation is transitive (trivially) and is complete, as is easy to check. However, the indifference relation $\sim$ is not complete. Take $a$ and $b$, it is *not* the case that $a \sim b$ or $b \sim a$, or both. The relation $a \sim b$ does not hold, since it is not true that $b \succsim a$ Since $a \sim b$ and $b \sim a$ indicate the same relation, it is also clear that $b \sim a$ does not hold. Intuitively, it is clear that the indifference relation should not be complete, or else all objects would be indifferent to each other.

3. We want to show that if $a \sim b$ and $b \sim c$, then $a \sim c$. As we showed in class, $a \sim b$ and $b \sim c$ in particular imply $a \succsim b$ and $b \succsim c$, and hence by transitivity $a \succsim c$. But $a \sim b$ and $b \sim c$ also imply $b \succsim a$ and $c \succsim b$, and hence by transitivity $c \succsim a$. Combining the two parts, we get $a \sim c$ as desired.

4. Since Andrew is indifferent with respect to the consumption of good $c$, in the space $(m, c)$ his indifference curves are vertical.

5. Monotonic preferences are such that if $x \geq y$, then $x \succsim y$. For $x \geq y$ to apply, it must be the case that $m_x \geq m_y$ and $c_x \geq c_y$. The first inequality implies $x \succsim y$, as desired. These preferences are not, however, strongly monotonic. Strictly monotonic preferences are such that if $x \geq y$ and $x_j > y_j$ for at least some $j$, then $x \succ y$. For example, $(4, 6) \geq (4, 2)$ and clearly $6 > 2$, but it is not the case that $(4, 6) \succ (4, 2)$, since Andrew is indifferent between these two commodities.

6. These preferences are convex. As per the definition, preferences are convex is for every $x$ and $y$ such that $x \succsim z$ and $y \succsim z$, then $tx + (1-t) y \succsim z$ for every $t \in [0, 1]$. In this case, by the definition of these preferences, $x \succsim z$ and $y \succsim z$ imply that $m_x \geq m_z$ and $m_y \geq m_z$. Given that $m_x \geq m_z$ and $m_y \geq m_z$, also $tm_x + (1-t) m_y \geq m_z$ holds for all $t \in [0, 1]$, and hence $tx + (1-t) y \succsim z$. The continuity part is tricky. For continuity, we need to show that for every sequence such that $\{x_n\}_{n \geq 1} \succsim y$ for all $n$, then if $\{x_n\}$ converges to a limit $x$, then also $x \succsim y$ holds. Given the preferences, $\{x_n\}_{n \geq 1} \succsim y$ translates into $m_{x_n} \geq m_y$ for each $n$. By continuity of the limit in $R$, then, $m_x \geq m_y$ also holds, where $m_x$ is the limit of the sequence of $m_{x_n}$. This proves that $x \succsim y$, as desired. Sorry that this was so involved. My point here was to show that Andrew's preferences, although they look similar to lexicographic preferences, are continuous, unlike lexicographic preferences that are not. This is because, conditional on having a choice between two bundles with the same level of $m$, Andrew does not prefer a higher level of $c$, unlike what happens for lexicographic preferences.