# Financial Econometrics Econ 40357
# Factors
# Single-Factor Models–The market model and the CAPM

*N.C. Mark*

University of Notre Dame and NBER

October 19, 2020

# Textbook

- Brooks pp. 586-588.

# The market model and the CAPM

- Finance people like to talk about (common) **factors**
- Factor is systematic component driving the cross-section over time.
- Factor may be observed or latent (unobserved)
- Returns driven by common and idiosyncratic factors
- Investors are paid to bear systematic risk (part driven by common factors)
- CAPM is a single-factor model. Factor is the market return.
- Later, we talk about multi-factor models.
- Finance people like to embed factor models within the **beta-risk** framework.

# The Beta-Risk Model

- **Question is**: Which assets pay high returns and which pay low returns over long periods of time, and why?

  e.g., Big versus small firms. Do small firms pay more or less? If more, what's the risk in small firms that make people afraid of them?

- **Answer is** those assets with greater **exposure** to the **risk factor.** Measure exposure with **beta.** The **big question** here, is **what is (are) risk factor(s)**?

# The Beta-Risk Model

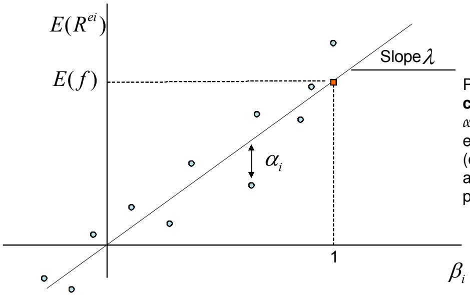
**Asset pricing model**: In finance, all models take the form

$$E\left(r^e_{t,i}\right) = \beta_i \lambda + \alpha_i$$

where

$$r^e_{t,i} = \alpha_i + \beta_i f_t + \epsilon_{t,i}$$

In the CAPM, factor $f_t = r^e_{t,m}$ is the **market excess return.**




Picture shows relation between risk and return. Risk is **covariance**. Excess returns vary proportionally to $\beta_i$. $\alpha_i$ is the deviation (Jensen's alpha). $\beta$ is the asset's exposure to the risk factor, $f$. It says, the risk-premium (expected excess return) varies in proportion to the asset's exposure to risk factor. $\lambda$ is that factor of proportionality.

# The Beta-Risk Model

- Let $f_t = r^e_{t,m}$. Each asset's return $i = 1, ..., N$, is assumed to be generated by

$$r^e_{t,i} = \alpha_i + \beta_i f_t + \epsilon_{t,i}$$

- Take **time-series** expectation

$$E\left(r^e_{t,i}\right) = E\left(f_t\right)\beta_i + \alpha_i$$

1. Jensen's alpha, $\alpha_i$, is risk-adjusted performance measure. The average return on security (portfolio) above or below that predicted by theory (e.g., CAPM).
2. $\alpha_i = 0$, portfolio manager has no value. $\alpha_i > 0$, manager has special talent.
3. Key implication from model: Excess return explained entirely by exposure to risk factor

$$\lambda = E(f_t)$$
$$\alpha_i = 0$$

# All finance models take beta-risk form (short version)

- Investor's Euler equation. $x_{t+1,j}$ is payoff from asset $j$ that costs $p_{t,j}$.

$$p_{t,j} u'(c_t) = E_t\left[\beta u'(c_{t+1})\, x_{t+1,j}\right] \tag{1}$$

  If asset is stock, $x_{t+1,j} = p_{t+1,j} + d_t$. If asset is coupon bond, replace $d_t$ with coupon. If asset is discount bond, $x_{t+1,j} = 1$.

- Express in return form,

$$1 = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{x_{t+1,j}}{p_{t,j}}\right]$$

- Change notation: $m_{t+1} = \beta u'(c_{t+1}) / u'(c_t)$ is the **stochastic discount factor**. $(1 + r_{t+1,j}) = x_{t+1,j} / p_{t,j}$ is the gross return.
- Rewrite the Euler equation one more time

$$1 = E_t(m_{t+1}(1 + r_{t+1,j})) \tag{2}$$

- Holds for all traded assets $j = 1, ..., N$. Also holds for the **risk free** asset whose return is $1 + r_t^f$.

$$1 = E_t(m_{t+1}(1 + r_t^r)) \tag{3}$$

# All finance models take beta-risk form (short version)

- Subtract (3) from (2) to get

$$0 = E_t(m_{t+1} r^e_{t+1,j})$$

Take unconditional expectations of both sides,

$$0 = E(m_{t+1} r^e_{t+1,j})$$

Now the timing $t+1$, $t$ doesn't matter.

- Assume a **one-factor** representation for the SDF. $\Leftarrow$ this is key

$$m_t = 1 - b(f_t - \mu_f) \tag{4}$$

What is factor $f_t$? Could be consumption growth, could be asset returns.

- Substitute (4) into Euler equation to get the beta-risk representation

$$\begin{aligned} 0 &= E(r^e_t(1 - b(f_t - \mu_f))) \\ &= E(r^e_t) - bCov(r^e_t, f_t) \\ &= E(r^e_t) - bVar(f_t)\frac{Cov(r^e_t, f_t)}{Var(f_t)} \end{aligned} \tag{5}$$

Hence,

$$E(r^e_t) = \lambda_f \beta$$

# Estimate and Test the CAPM with Time-Series Method

- This method works when **factor** is an **excess return**.
- Preliminary analysis
  - Estimate and test if price of risk $E(f_t) = \lambda$ is statistically significant: Run the regression

    $$f_t = c + \epsilon_t$$

    of the factor (excess return) on constant.
  - Constant is estimate of $\lambda$. Do Newey-West on the constant, test if it is greater than 0.

# Estimate and test CAPM with Time-Series Method

- Run the time-series regression

$$r^e_{t,i} = \alpha_i + \beta_i f_t + \epsilon_{t,i}$$

  for each asset $i = 1, ..., n$, using Newey-West. Do individual t-tests on the $\alpha_{i.}$

- A cheap and not entirely correct joint test: If all the $\alpha_i$ are zero, then the sum of the $\alpha_i$ is zero. If the $\alpha_i$ estimates are independent, then

$$t_1^2 + t_2^2 + \cdots t_n^2 \sim \chi_n^2$$

  where $t_i^2$ is the squared value of the Newey-West t-ratio on $\alpha_i$.

- This test is not entirely right because it ignores possible correlation across the $\alpha_i$

# A correct joint test on the $\alpha$'s

Let

$$\underline{\hat{\alpha}} = \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_N \end{pmatrix}$$

$$\Sigma_\epsilon = \begin{pmatrix} Var(\epsilon_1) & Cov(\epsilon_1, \epsilon_2) & \cdots & Cov(\epsilon_1, \epsilon_N) \\ Cov(\epsilon_2, \epsilon_1) & Var(\epsilon_2) & \cdots & Cov(\epsilon_2, \epsilon_N) \\ \vdots & & & \\ Cov(\epsilon_N, \epsilon_1) & \cdots & & Var(\epsilon_N) \end{pmatrix}$$

$$Var(\epsilon_i) = \frac{1}{T}\sum_{t=1}^{T} \epsilon_{t,i}^2, \quad Cov(\epsilon_i, \epsilon_j) = \frac{1}{T}\sum_{t=1}^{T} \epsilon_{t,i}\epsilon_{t,j}$$

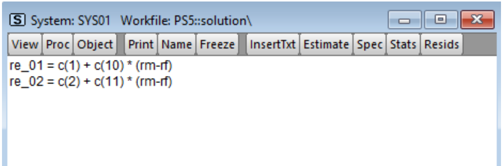
$$\bar{f} = \frac{1}{T}\sum_{t=1}^{T} f_t, \quad \Sigma_f = \frac{1}{T}\sum_{t=1}^{T} (f_t - \mu_f)(f_t - \mu_f)'$$

Then test statistic is,

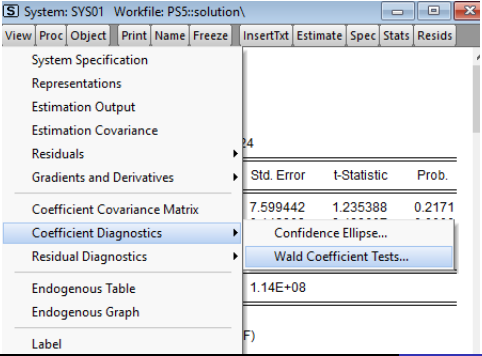
$$T\left(1 + \bar{f}'\Sigma_f^{-1}\bar{f}\right)^{-1}\left(\underline{\hat{\alpha}}'\Sigma_\epsilon^{-1}\underline{\hat{\alpha}}\right) \sim \chi_N^2$$

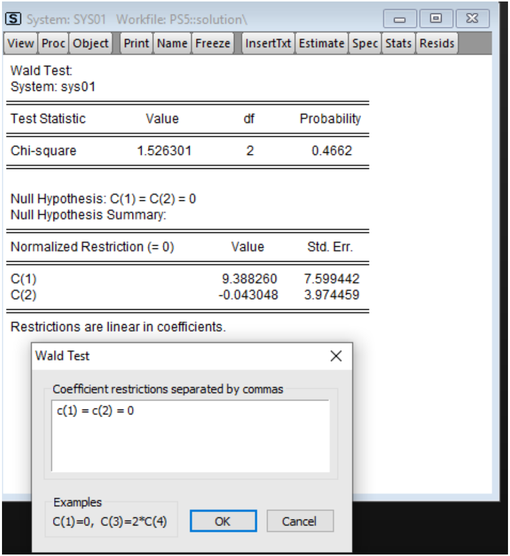
- Chances are, you didn't learn this in econometrics class. Why? Because in econometrics, you learned about constructing standard errors (and t-ratios) for a single regression. Here we are looking at the joint distribution of $\alpha_i$ and $\alpha_j$ across different regressions
- How to do this in Eviews? Estimate as system, ask for the joint test.

- Object → New Object → System
- Write down the system model




- Estimate by Ordinary Least Squares → View → Coefficient Diagnostics






Wald Test:
System: sys01

| Test Statistic | Value | df | Probability |
|---|---|---|---|
| Chi-square | 1.526301 | 2 | 0.4662 |

Null Hypothesis: C(1) = C(2) = 0
Null Hypothesis Summary:

| Normalized Restriction (= 0) | Value | Std. Err. |
|---|---|---|
| C(1) | 9.388260 | 7.599442 |
| C(2) | -0.043048 | 3.974459 |

Restrictions are linear in coefficients.

# Estimate/test CAPM with Time-Series Method

- In the time-series regression,

$$r^e_{i,t} = \alpha_i + \beta_i f_t + \epsilon_{i,t}$$

  Let us impose the restriction that mean returns are proportional to betas,

$$E(r^e_{i,t}) = \beta_i \lambda_f = \alpha_i + \beta_i E(f_t)$$

  Then the intercept should be

$$\alpha_i = \beta_i(\lambda - E(f_t))$$

- The intercept in the regression controls the mean return. If $\lambda = E(f_t)$, the intercept will be zero. In order to test this restriction, you need an estimate of $\lambda$, and this only works if the factor is a return.
- If the model is true, $\alpha_i = 0$. (why is that?)