Problem 1.

In the standard Euclidean metric space, let the set A be uncountable. Prove that there is a sequence of distinct points converging to a point in A. Is this true for every metric space?

Solution

Note that this is equivalent to saying A contains a limit point. We will show the contrapositive: if every point of a set $A \subset \mathbb{R}^n$ is isolated, then it is at most countable.

For every $a \in A$, we can find some radius $r_a > 0$ such that $B_{r_a}(x) \cap A = \{a\}$. Further, the collection of sets $\left\{ B_{\frac{r_a}{2}}(a) : a \in A \right\}$ are disjoint. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we can find a unique $q_a \in \mathbb{Q}^n$ such that $q_a \in B_{\frac{r_a}{2}}(a)$ for every $a \in A$. This defines a one-to-one mapping from A into $\mathbb{Q}^n$, which implies A is at most countable.

This is not true in general: take $[0, 1]$ with the discrete metric. This set is uncountable but the only convergent sequences are (eventually) constant.

Problem 2.

For some metric space (X, d) take any two sets such that $\operatorname{int} A = \operatorname{int} B = \varnothing$ and A is closed. Prove that $\operatorname{int}(A \cup B) = \varnothing$. What if A is not closed?

Solution

Towards contradiction, suppose $x \in \operatorname{int}(A \cup B)$. So there is some open ball $B_\varepsilon(x) \subset A \cup B$. Consider the set $E = B_\varepsilon(x) \setminus A = B_\varepsilon(x) \cap A^c$. Note that since A is closed, A^c is open, and E is the finite intersection of two open sets and hence is also open. We have two cases:

(a) $E = \varnothing$. So $B_\varepsilon(x) \subset A$ which means x is an interior point of A, which implies the interior of A is nonempty.

(b) $E \neq \varnothing$. Then for any $y \in E$, $y \in B$, so we have $E \subset B$. But since E is open, this implies B has nonempty interior.

In both cases we reach a contradiction. This proves the claim.

If A is not closed, then the statement need not hold. For example: $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ (rationals and irrationals) both have empty interior, but their union is $\mathbb{R} = \operatorname{int} \mathbb{R}$.

Problem 3.

Prove that the set of cluster points of any sequence $\{x_n\}$ is closed.

Solution

Let K be the set of all cluster points of the sequence $\{x_n\}$ and suppose z is a limit point of K. So for any $\varepsilon > 0$ we know there is some $y \in K$ such that $d(y, z) < \frac{\varepsilon}{2}$. Since y is a cluster point of $\{x_n\}$ we know that for all $N \in \mathbb{N}$ we can find an $n > N$ such that $d(x_n, y) < \frac{\varepsilon}{2}$. Then the triangle inequality gives us

$$d(x_n, z) \leq d(x_n, y) + d(y, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence z is a cluster point of $\{x_n\}$. This is, $z \in K$. Thus K contains its limit points, so K is closed.

Problem 4.

Consider ℓ^∞, the vector space defined over $\mathbb{R}$ of all bounded sequences. That is, $a \in \ell^\infty$ if $a = (a_1, a_2, \ldots)$ and $\exists M \in \mathbb{R}$ such that $|a_i| \leq M$ for every i.

a) Show that $||a||_\infty = \sup_i |a_i|$ defines a norm on this space.

b) Consider the subspace L_0 made up of sequences with only a finite number of nonzero elements. That is, $a \in L_0$ if $\exists N \in \mathbb{N}$ such that $i > N \implies a_i = 0$. Is L_0 a closed subspace of ℓ^∞?

Solution

(a) First, because $|a_i| \geq 0$ always, it follows that $\sup_i |a_i| \geq 0$. Second, $||(0, 0, \ldots)||_\infty = 0$, and if $a \neq 0$ then $\exists i$ such that $|a_i| > 0$ so $\sup_i |a_i| > 0$. Third, if $a, b \in l^\infty$ then $a + b = (a_1 + b_1, a_2 + b_2, \ldots)$ has norm given by

$$\begin{aligned}
||a + b||_\infty &= \sup_i \{|a_1 + b_1|, |a_2 + b_2|, \ldots\} \\
&\leq \sup_i \{|a_1| + |b_1|, |a_2| + |b_2|, \ldots\} \\
&= ||a||_\infty + ||b||_\infty
\end{aligned}$$

since the absolute value satisfies the triangle inequality. Finally, if $\alpha \in \mathbb{R}$ and $b \in l^\infty$, then $\alpha b = (\alpha b_1, \alpha b_2, \ldots)$ and $\sup_i |\alpha b_i| = \sup_i |\alpha||b_i| = |\alpha| \sup_i |b_i| = |\alpha|||b||_\infty$. Hence $||a||_\infty$ is a norm.

(b) The subspace L^0 is not a closed subspace of l^∞. To see this, consider the sequence given by $b_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots)$, where $n \in \mathbb{N}$. Clearly, $b_n \in L^0$ for all n. Consider $b = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \notin L^0$. Then $b_n - b = (0, 0, \ldots, \frac{1}{n+1}, \frac{1}{n+2}, \ldots)$ so that

$$\sup_i |b_n - b| = \sup \left\{ \left|\frac{1}{n+1}\right|, \left|\frac{1}{n+2}\right|, \left|\frac{1}{n+3}\right|, \ldots \right\} = \frac{1}{n+1}$$

Hence for any $\varepsilon > 0$, if we choose $N > \frac{1}{\varepsilon}$ then $n > N$ implies $||b_n - b||_\infty < \varepsilon$. This says exactly that $b_n \to b \notin L^0$, proving that the subspace is not closed.

Problem 5.

Recall the *diameter* of a set is defined $\operatorname{diam} A = \sup\{d(a,b) : a, b \in A\}$. Prove that the diameter of a set is equal to the diameter of its closure.

Solution

First let's show (similar to in section) that if we have any pair of convergent sequences $a_n \to a$, $b_n \to b$, then the sequence of real numbers $d(a_n, b_n) \to d(a,b)$. Note by repeated application of the triangle inequality, we get

$$d(a_n,b_n) \leq d(a_n,a) + d(a,b) + d(b,b_n) \implies d(a_n,b_n) - d(a,b) \leq d(a_n,a) + d(b,b_n)$$
$$d(a,b) \leq d(a,a_n) + d(a_n,b_n) + d(b_n,b) \implies d(a_n,b_n) - d(a,b) \geq -\left[d(a_n,a) + d(b,b_n)\right]$$

and putting everything together we get

$$|d(a_n,b_n) - d(a,b)| \leq d(a_n,a) + d(b_n,b)$$

Then for any $\varepsilon > 0$ we can find an N such that $n > N$ implies $d(a_n,a) < \frac{\varepsilon}{2}$ and $d(b_n,b) < \frac{\varepsilon}{2}$. So $|d(a_n,b_n) - d(a,b)| < \varepsilon$, and $d(a_n,b_n)$ converges to $d(a,b)$.

Now we turn to set diameters. First

$$A \subset \overline{A} \implies \sup\{d(a,b) : a, b \in A\} \leq \sup\{d(a,b) : a, b \in \overline{A}\}$$

so $\operatorname{diam} A \leq \operatorname{diam} \overline{A}$. Now suppose $\operatorname{diam} \overline{A} > \operatorname{diam} A$. Then we can find some $a', b' \in \overline{A}$ such that $d(a',b') > \operatorname{diam} A$. Of course this means either $a' \notin A$ or $b' \notin A$, so either a' or b' is a limit point of A. Either way, we can construct sequences $\{a_n\} \subset A$, $\{b_n\} \subset A$ (not necessarily distinct elements), such that $a_n \to a'$ and $b_n \to b'$. Now choose $\varepsilon = \frac{1}{2}\left(d(a',b') - \operatorname{diam} A\right) > 0$. Since for all n $a_n, b_n \in A$, we have $d(a_n,b_n) \leq \operatorname{diam} A < d(a',b')$. So

$$|d(a_n,b_n) - d(a',b')| = d(a',b') - d(a_n,b_n) \geq d(a',b') - \operatorname{diam} A > \varepsilon$$

which contradicts what we showed above, that $d(a_n,b_n) \to d(a',b')$. Hence $\operatorname{diam} \overline{A} \leq \operatorname{diam} A$ so $\operatorname{diam} \overline{A} = \operatorname{diam} A$.

Problem 6.

Call a metric space *discrete* if every subset is open.[1]

a) Give an example of a discrete metric space that is not complete.

b) (*Difficult!*) Show that a metric space has the property that the closure of every open set is open if and only if the metric space is discrete.

Solution

(a) Counterexample: consider the metric space $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ with the usual absolute value metric. First, for every $x = \frac{1}{n} \in X$, if $\varepsilon < \frac{1}{2n(n+1)}$ then $B_\varepsilon(x) = \{x\}$. Hence every singleton is open, and since every subset of X can be written as the union of singletons, every subset of X is open. So X is discrete. Further, the sequence $a_n = \frac{1}{n}$ is Cauchy in this metric space but is not convergent.

Remark: Using the same ambient space but the discrete metric induces the same *topology* (collection of open sets), but under the discrete metric the sequence defined above is not Cauchy!

(b) ($\Longleftarrow$) This direction is easy: since every subset of a discrete metric space is open, the closure of an open set is still open.

($\Longrightarrow$) Now we suppose for any open set V, $\overline{V}$ is also open. First note this implies that if V and W are open and disjoint, then $\overline{V}$ and $\overline{W}$ are open and disjoint. Why? If $w \in W$ then there is some neighborhood $B(w) \subset W \implies B(w) \cap V = \varnothing$ so $w \notin \overline{V}$ and we have $W \cap \overline{V} = \varnothing$. Now since $\overline{V}$ is open, for any $v \in \overline{V}$ we can find a neighborhood $B(v) \subset \overline{V}$, and from above this means $B(v) \cap W = \varnothing$. So $v \notin \overline{W}$. Hence $\overline{V} \cap \overline{W} = \varnothing$.

Now suppose there is some subset A that is not closed. A must not be finite, so construct a sequence $\{a_n\} \subset A$ (distinct) such that $a_n \to a$ where $a \notin A$. Find neighborhoods $B^{(n)}(a_n)$ that are disjoint and define the following sets:

$$U = \bigcup_{i=1}^{\infty} B^{(2i)}(a_n), \; V = \bigcup_{i=1}^{\infty} B^{(2i+1)}(a_n)$$

Then U and V are disjoint, open sets. But a is a limit point of each set, implying $a \in \overline{V} \cap \overline{W}$, a contradiction. Hence every set in the metric space is closed (and taking complements, every set is open) so the metric space is discrete.

[1] Every set equipped with the discrete metric forms a discrete metric space, but not all discrete metric spaces have the discrete metric.