the following rule: A_1 is preferred to A_2 if either $q_1^{(1)} > q_1^{(2)}$ or $q_1^{(1)} = q_1^{(2)}$ and $q_2^{(1)} > q_2^{(2)}$. In this situation the preference ordering is said to be *lexicographic* and no utility function exists.

The lexicographic ordering violates the third of the above assumptions. Consider the combination $A = (q_1^0, q_2^0)$, and let Δq_1 and Δq_2 denote positive increments from A. The combination $(q_1^0 + \Delta q_1, q_2^0 - \Delta q_2)$ is preferred to A by virtue of the lexicographic ordering. Select particular positive values for the increments, and consider the infinite sequence of commodity combinations the ith member of which is

$$A_i = (q_1^0 + (\tfrac{1}{2})^i \Delta q_1, q_2^0 - \Delta q_2)$$

A_i is clearly preferred to A for any i, but the limit of the sequence

$$\lim_{i \to \infty} A_i = (q_1^0, q_2^0 - \Delta q_2)$$

is inferior to A in violation of the third assumption.

2-2 THE MAXIMIZATION OF UTILITY

The rational consumer desires to purchase a combination of Q_1 and Q_2 from which he derives the highest level of satisfaction. His problem is one of maximization. However, his income is limited, and he is not able to purchase unlimited amounts of the commodities. The consumer's budget constraint can be written as

$$y^0 = p_1 q_1 + p_2 q_2 \tag{2-7}$$

where y^0 is his (fixed) income and p_1 and p_2 are the prices of Q_1 and Q_2 respectively. The amount he spends on the first commodity $(p_1 q_1)$ plus the amount he spends on the second $(p_2 q_2)$ equals his income (y^0).

The First- and Second-Order Conditions

The consumer desires to maximize (2-1) subject to (2-7). Form the Lagrange function

$$V = f(q_1, q_2) + \lambda(y^0 - p_1 q_1 - p_2 q_2) \tag{2-8}$$

where λ is an as yet undetermined multiplier (see Sec. A-3). The first-order conditions are obtained by setting the first partial derivatives of (2-8) with respect to q_1, q_2, and λ equal to zero:

$$\begin{aligned} \frac{\partial V}{\partial q_1} &= f_1 - \lambda p_1 = 0 \\ \frac{\partial V}{\partial q_2} &= f_2 - \lambda p_2 = 0 \\ \frac{\partial V}{\partial \lambda} &= y^0 - p_1 q_1 - p_2 q_2 = 0 \end{aligned} \tag{2-9}$$

Transposing the second terms in the first two equations of (2-9) to the right and dividing the first by the second yields

$$\frac{f_1}{f_2} = \frac{p_1}{p_2} \tag{2-10}$$

The ratio of the marginal utilities must equal the ratio of prices for a maximum. Since f_1/f_2 is the RCS, the first-order condition for a maximum is expressed by the equality of the RCS and the price ratio.

The first two equations of (2-9) may also be written as

$$\frac{f_1}{p_1} = \frac{f_2}{p_2} = \lambda \tag{2-11}$$

Marginal utility divided by price must be the same for all commodities. This ratio gives the rate at which satisfaction would increase if an additional dollar were spent on a particular commodity. If more satisfaction could be gained by spending an additional dollar on Q_1 rather than Q_2, the consumer would not be maximizing utility. He could increase his satisfaction by shifting some of his expenditure from Q_2 to Q_1.

The Lagrange multiplier λ can be interpreted as the marginal utility of income. Since the marginal utilities of commodities are assumed to be positive, the marginal utility of income is positive.

The second-order condition as well as the first-order condition must be satisfied to ensure that a maximum is actually reached. Denoting the second direct partial derivatives of the utility function by f_{11} and f_{22} and the second cross partial derivatives by f_{12} and f_{21}, the second-order condition for a constrained maximum requires that the relevant bordered Hessian determinant be positive:

$$\begin{vmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} > 0 \tag{2-12}$$

Expanding (2-12)

$$2f_{12}p_1p_2 - f_{11}p_2^2 - f_{22}p_1^2 > 0 \tag{2-13}$$

Substituting $p_1 = f_1/\lambda$ and $p_2 = f_2/\lambda$ from (2-9) and multiplying through by $\lambda^2 > 0$

$$2f_{12}f_1f_2 - f_{11}f_2^2 - f_{22}f_1^2 > 0 \tag{2-14}$$

Inequality (2-14), which is the same as (2-5), is satisfied by the assumption of regular strict quasi-concavity. This assumption ensures that the second-order condition is satisfied at any point at which the first-order condition is satisfied. Inequality (2-14) is also the condition for the global univalence of solutions for Eqs. (2-9) (see Sec. A-2). Thus, regular strict quasi-concavity also ensures that constrained-utility-maximization solutions are unique.

prices and income will normally alter the consumer's expenditure pattern, but the new quantities (and prices and income) will always satisfy the first-order conditions (2-9). In order to find the magnitude of the effect of price and income changes on the consumer's purchases, allow all variables to vary simultaneously. This is accomplished by total differentiation of Eqs. (2-9):

$$\begin{aligned} f_{11}\,dq_1 + f_{12}\,dq_2 - p_1\,d\lambda &= \lambda\,dp_1 \\ f_{21}\,dq_1 + f_{22}\,dq_2 - p_2\,d\lambda &= \lambda\,dp_2 \\ -p_1\,dq_1 - p_2\,dq_2 \qquad &= -dy + q_1\,dp_1 + q_2\,dp_2 \end{aligned} \tag{2-27}$$

In order to solve this system of three equations for the three unknowns, dq_1, dq_2, and $d\lambda$, the terms on the right must be regarded as constants. The array of coefficients formed by (2-27) is the same as the bordered Hessian determinant (2-12). Denoting this determinant by $\mathscr{D}$ and the cofactor of the element in the first row and the first column by $\mathscr{D}_{11}$, the cofactor of the element in the first row and second column by $\mathscr{D}_{12}$, etc., the solution of (2-27) by Cramer's rule (see Sec. A-1) is

$$dq_1 = \frac{\lambda\mathscr{D}_{11}\,dp_1 + \lambda\mathscr{D}_{21}\,dp_2 + \mathscr{D}_{31}(-dy + q_1\,dp_1 + q_2\,dp_2)}{\mathscr{D}} \tag{2-28}$$

$$dq_2 = \frac{\lambda\mathscr{D}_{12}\,dp_1 + \lambda\mathscr{D}_{22}\,dp_2 + \mathscr{D}_{32}(-dy + q_1\,dp_1 + q_2\,dp_2)}{\mathscr{D}} \tag{2-29}$$

Dividing both sides of (2-28) by dp_1 and assuming that p_2 and y do not change ($dp_2 = dy = 0$),

$$\frac{\partial q_1}{\partial p_1} = \frac{\mathscr{D}_{11}\lambda}{\mathscr{D}} + q_1\frac{\mathscr{D}_{31}}{\mathscr{D}} \tag{2-30}$$

The partial derivative on the left-hand side of (2-30) is the rate of change of the consumer's purchases of Q_1 with respect to changes in p_1, all other things being equal. *Ceteris paribus*, the rate of change with respect to income is

$$\frac{\partial q_1}{\partial y} = -\frac{\mathscr{D}_{31}}{\mathscr{D}} \tag{2-31}$$

Changes in commodity prices change the consumer's level of satisfaction, since a new equilibrium is established which lies on a different indifference curve.

Consider a price change that is compensated by an income change that leaves the consumer on his initial indifference curve. An increase in the price of a commodity is accompanied by a corresponding increase in his income such that $dU = 0$ and $f_1\,dq_1 + f_2\,dq_2 = 0$ by (2-3). Since $f_1/f_2 = p_1/p_2$, it is also true that $p_1\,dq_1 + p_2\,dq_2 = 0$. Hence, from the last equation of (2-27), $-dy + q_1\,dp_1 + q_2\,dp_2 = 0$, and

$$\left(\frac{\partial q_1}{\partial p_1}\right)_{U=\text{const}} = \frac{\mathscr{D}_{11}\lambda}{\mathscr{D}} \tag{2-32}$$

Equation (2-30) can now be rewritten as

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial p_1}\right)_{U=\text{const}} - q_1\left(\frac{\partial q_1}{\partial y}\right)_{\text{prices}=\text{const}} \tag{2-33}$$

Equation (2-33) is known as the *Slutsky equation*. The quantity $\partial q_1/\partial p_1$ is the slope of the ordinary demand curve for Q_1, and the first term on the right is the slope of the compensated demand curve for Q_1.

An alternative compensation criterion is that the consumer is provided enough income to purchase his former consumption bundle so that $dy = q_1\,dp_1 + q_2\,dp_2$. This is the equation that led to (2-32). Here

$$\left(\frac{\partial q_1}{\partial p_1}\right)_{q_1, q_2=\text{const}} = \frac{\mathscr{D}_{11}\lambda}{\mathscr{D}}$$

which can be substituted for the first term on the right of (2-33). At first glance it might appear remarkable that two rather different compensation schemes led to the same result. However, they only define the same derivative, and may lead to quite different results for any finite move. A consumer can be induced to stay on the same indifference curve in the finite case, but he cannot be induced to purchase the same bundle if relative prices change. All subsequent analysis here is based upon (2-33).

The Slutsky equation may be expressed in terms of the price and income elasticities described in Sec. 2-3. Multiplying (2-33) through by p_1/q_1 and multiplying the last term on the right by y/y,

$$\varepsilon_{11} = \xi_{11} - \alpha_1\eta_1 \tag{2-34}$$

The price elasticity of the ordinary demand curve equals the price elasticity of the compensated demand curve less the corresponding income elasticity multiplied by the proportion of total expenditures spent on Q_1. Hence, the ordinary demand curve will have a greater demand elasticity than the compensated demand curve; that is, ε_{11} will be more negative than ξ_{11} if the income elasticity of demand is positive.

Direct Effects

The first term on the right-hand side of (2-33) is the *substitution effect*, or the rate at which the consumer substitutes Q_1 for other commodities when the price of Q_1 changes and he moves along a given indifference curve.[1] The second term on the right is the *income effect*, which states the rate at which the consumer's purchases of Q_1 would change with changes in his income, prices remaining constant. The sum of the two rates gives the total rate of change for Q_1 as p_1 changes.

In the present case the multiplier λ is the derivative of utility with respect to income with prices constant and quantities variable. From the utility

[1] Slutsky called this the *residual variability* of the commodity in question.

function (2-1) it follows that $\partial U/\partial y = f_1(\partial q_1/\partial y) + f_2(\partial q_2/\partial y)$. Substituting $f_1 = \lambda p_1$ and $f_2 = \lambda p_2$,

$$\frac{\partial U}{\partial y} = \lambda \left(p_1 \frac{\partial q_1}{\partial y} + p_2 \frac{\partial q_2}{\partial y} \right) = \lambda$$

which follows from the partial derivative of the budget constraint (2-7) with respect to $y: 1 = p_1(\partial q_1/\partial y) + p_2(\partial q_2/\partial y)$. This confirms the result inferred from (2-11) at an earlier stage.

Solving (2-27) for $d\lambda$,

$$d\lambda = \frac{\lambda \mathscr{D}_{13}\, dp_1 + \lambda \mathscr{D}_{23}\, dp_2 + \mathscr{D}_{33}(-dy + q_1\, dp_1 + q_2\, dp_2)}{\mathscr{D}} \tag{2-35}$$

Assume now that only income changes, i.e., that $dp_1 = dp_2 = 0$. Then (2-35) becomes

$$\frac{\partial \lambda}{\partial y} = -\frac{\mathscr{D}_{33}}{\mathscr{D}} = -\frac{f_{11}f_{22} - f_{12}^2}{\mathscr{D}} \tag{2-36}$$

Since $\mathscr{D}$ is positive, the rate of change of the marginal utility of income will have the same sign as $-(f_{11}f_{22} - f_{12}^2)$. This would be negative if the utility function were strictly concave. However, for ordinal utility functions only strict quasi-concavity is assumed, and the theory does not predict whether the marginal utility of income is increasing or decreasing with income.

By (2-32) the substitution effect is $\mathscr{D}_{11}\lambda/\mathscr{D}$. The determinant $\mathscr{D}$, which is the same as (2-12), is positive. Expanding $\mathscr{D}_{11}$,

$$\mathscr{D}_{11} = -p_2^2$$

which is clearly negative. This proves that the sign of the substitution effect is always negative and that the compensated demand curve is always downward sloping.

A change in real income may cause a reallocation of the consumer's resources even if prices do not change or if they change in the same proportion. The income effect is $-q_1(\partial q_1/\partial y)_{\text{prices=const}}$ and may be of either sign. The final effect of a price change on the purchases of the commodity is thus unknown. However, an important conclusion can still be derived: The smaller the quantity of Q_1, the less significant is the income effect. A commodity Q_1 is called an *inferior good* if the consumer's purchases decrease as income rises and increase as income falls; i.e., if $\partial q_1/\partial y$ is negative, which makes the income effect positive. A *Giffen good* is an inferior good with an income effect large enough to offset the negative substitution effect and make $\partial q_1/\partial p_1$ positive. This means that as the price of Q_1 falls, the consumer's purchases of Q_1 will also fall. This may occur if a consumer is sufficiently poor so that a considerable portion of his income is spent on a commodity such as potatoes which he needs for his subsistence. Assume now that the price of potatoes falls. The consumer who is not very fond of potatoes may suddenly discover that his real income has increased as a result of the price

fall. He will then buy fewer potatoes and purchase a more palatable diet with the remainder of his income.

The Slutsky equation can be derived for the specific utility function assumed in the previous examples. State the budget constraint in the general implicit form $y - p_1q_1 - p_2q_2 = 0$, and form the function

$$V = q_1q_2 + \lambda(y - p_1q_1 - p_2q_2)$$

Setting the partial derivatives equal to zero,

$$q_2 - \lambda p_1 = 0$$
$$q_1 - \lambda p_2 = 0$$
$$y - p_1q_1 - p_2q_2 = 0$$

The total differentials of these equations are

$$dq_2 - p_1\, d\lambda = \lambda\, dp_1$$
$$dq_1 - p_2\, d\lambda = \lambda\, dp_2$$
$$-p_1\, dq_1 - p_2\, dq_2 = -dy + q_1\, dp_1 + q_2\, dp_2$$

Denote the determinant of the coefficients of these equations by $\mathscr{D}$ and the cofactor of the element in the ith row and jth column by $\mathscr{D}_{ij}$. Simple calculations show that

$$\mathscr{D} = 2p_1p_2$$
$$\mathscr{D}_{11} = -p_2^2$$
$$\mathscr{D}_{21} = p_1p_2$$
$$\mathscr{D}_{31} = -p_2$$

Solving for dq_1 by Cramer's rule gives

$$dq_1 = \frac{-p_2^2\lambda\, dp_1 + p_1p_2\lambda\, dp_2 - p_2(-dy + q_1\, dp_1 + q_2\, dp_2)}{2p_1p_2}$$

Assuming that only the price of the first commodity varies,

$$\frac{\partial q_1}{\partial p_1} = -\frac{p_2\lambda}{2p_1} - \frac{q_1}{2p_1}$$

The value of λ is obtained by substituting the values of q_1 and q_2 from the first two equations of the first-order conditions into the third and solving for λ in terms of the parameters p_1, p_2, and y. Thus $\lambda = y/2p_1p_2$. Substituting this value into the above equation and then introducing into it the values of the parameters ($y = 100, p_1 = 2, p_2 = 5$) and also the equilibrium value of q_1 (25), a numerical answer is obtained:

$$\frac{\partial q_1}{\partial p_1} = -12.5$$