Problem 1.

Give an example of a complete metric space which is homeomorphic to an incomplete metric space.

Solution

Define the mapping $f : \mathbb{R} \to (-1, 1)$ as

$$f(x) = \frac{x}{1 + |x|}$$

f is a continuous bijection (note $f(0) = 0$), where the inverse $f^{-1} : (-1, 1) \to \mathbb{R}$ is

$$f^{-1}(y) = \frac{y}{1 - |y|}$$

which is also continuous at all $-1 < y < 1$ (again note $f^{-1}(0) = 0$). Thus f is a homeomorphism. With the usual metric, $\mathbb{R}$ is complete but $(-1, 1)$ is incomplete.

Problem 2.

Let (E, d) be a metric space and $S \subset E$ a subset. Show that $A \subset S$ is open relative to S if and only if $A = S \cap U$ for some $U \subset E$ open.[1]

Solution

($\Rightarrow$) Suppose $A \subset S$ is open relative to S. This means that $\forall x \in A$, $\exists r_x > 0$ such that $B_{r_x}(x) \cap S \subset A$. Define $U = \bigcup_{x \in A} B_{r_x}(x)$. Since U is the union of open balls, it is open.

Note that $A \subset U$ and $A \subset S$. Then, $A \subset S \cap U$. Also, $S \cap U = \bigcup_{x \in A} B_{r_x}(x) \cap S$. And since A is open relative to S, $B_{r_x}(x) \cap S \subset A$, so is the union. Then, $S \cap U \subset A$. Then, $A = S \cap U$.

($\Leftarrow$) If $A = S \cap U$ for an open subset $U \subset E$, then $\forall x \in A$, $\exists r_x > 0$ such that $B_{r_x}(x) \subset U$. This implies that $(B_{r_x}(x) \cap S) \subset (U \cap S) = A$. Then, A is open relative to S.

[1] $A \subset S$ is open relative to S if $\forall x \in A$ $\exists r_x > 0$ such that $B_{r_x}(x) \cap S \subset A$.

Problem 3.

Let (X, d) be a metric space. Assume $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous on (X, d) and $(\mathbb{R}, |\cdot|)$, with $|\cdot|$ the absolute-value norm.

(a) Show that $f + g : X \to \mathbb{R}$ is uniformly continuous, where $(f + g)(x) = f(x) + g(x)$.

(b) Show that $\max\{f, g\} : X \to \mathbb{R}$ is uniformly continuous, where $\max\{f, g\}(x) = \max\{f(x), g(x)\}$.

(c) Give a counterexample to the following statement: $f \cdot g : X \to \mathbb{R}$ *is uniformly continuous on* (X, d) *and* $(\mathbb{R}, |\cdot|)$, where $f \cdot g = f(x) \cdot g(x)$.

Solution

(a) Since f and g are uniformly continuous, we know that $\forall \epsilon' > 0$ and $\forall x_0 \in X$, $\exists \delta' > 0$ such that if $d(x, x_0) < \delta'$, then $|f(x) - f(x_0)| < \epsilon'$ and $|g(x) - g(x_0)| < \epsilon'$. Given ϵ', let $\epsilon = 2\epsilon'$ and $\delta = \delta'$. Assume $d(x, x_0) < \delta$. Then

$$\begin{aligned}
|(f+g)(x) - (f+g)(x_0)| &= |(f(x) + g(x)) - (f(x_0) + g(x_0))|, \\
&= |(f(x) - f(x_0)) + (g(x) - g(x_0))|, \\
&\leq |f(x) - f(x_0)| + |g(x) - g(x_0)|, \\
&< \epsilon' + \epsilon', \\
&= \epsilon.
\end{aligned}$$

Then, $f + g$ is uniformly continuous.

(b) Let $x, y \in X$. Without loss of generality, suppose $\max(f(x), g(x)) \geq \max(f(y), g(y))$ (otherwise switch the roles of x and y here). Then

$$f(x) - \max(f(y), g(y)) \leq f(x) - f(y)$$

and

$$g(x) - \max(f(y), g(y)) \leq g(x) - g(y)$$

So

$$\begin{aligned}
0 &\leq \max(f(x), g(x)) - \max(f(y), g(y)) \\
&= \max(f(x) - \max(f(y), g(y)), g(x) - \max(f(y), g(y))) \\
&\leq \max(f(x) - f(y), g(x) - g(y)) \\
&\leq \max(|f(x) - f(y)|, |g(x) - g(y)|)
\end{aligned}$$

From here, the argument can be finished using the uniform continuity of f and g.

(c) Take $f(x) = g(x) = x$, so $(f \cdot g)(x) = x^2$, which is not uniformly continuous. To see this, consider the following example. Let $\varepsilon = 1$. Fix $\delta > 0$; without loss of generality take $\delta \leq 1$. Then let $x = 2/\delta$ and $y = 2/\delta - \delta$. So

$$|x - y| = \delta \text{ and } x + y = \frac{4}{\delta} - \delta = \frac{4 - \delta^2}{\delta}$$

So

$$|x^2 - y^2| = |x + y||x - y| = 4 - \delta^2 > 1 = \varepsilon$$

Problem 4.

A function $f : X \to Y$ is open if $\forall A \subset X$ such that A is open, $f(A)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic.

Solution

Towards a contradiction, assume f is not strictly monotonic. This means that for some $a < c < b \in \mathbb{R}$, either (i) $f(a) \leq f(c) \geq f(b)$ or (ii) $f(a) \geq f(c) \leq f(b)$, is true.

Consider case (i). Since $[a, b]$ is compact and f is continuous, then $f([a, b])$ is compact. Because of the extreme value theorem, $M \equiv \sup f([a, b]) \in f([a, b])$. Since $f(a) \leq f(c) \geq f(b)$, there are two options

- $f(a) = M$ or $f(b) = M$, so $f(c) = M$.
- $f(a) < M$ and $f(b) < M$.

Both cases imply that $M \in f((a, b))$. This means that $f((a, b))$ is not open, since open sets do not contain their supremum. This is a contradiction with f being open.

A similar argument can be made in case (ii) using the infimum instead of the supremum. We conclude that f is strictly monotonic.

Problem 5.

Prove that a metric space (X, d) is discrete if and only if every function on X into any other metric space (Y, ρ), where Y has at least two distinct elements, is continuous.[2]

Solution

$(\implies)$: every set in a discrete metric space is open. So for any function f and any set $A \subset Y$, $f^{-1}(A)$ is open. In particular this holds for all open subsets of Y and hence f is continuous.

$(\impliedby)$: Let $A \subseteq X$. Fix $y, y' \in Y$ with $y \neq y'$. Let $f : X \to Y$ be given by

$$f(x) = \begin{cases} y & \text{if } x \in A \\ y' & \text{if } x \in A^c \end{cases}$$

Then f is continuous, since every function from X to Y is continuous. Since $\{y\}$ is a closed set, $A = f^{-1}(\{y\})$ is closed. Similarly, $\{y'\}$ is closed, so $A^c = f^{-1}(\{y'\})$ is closed, which implies that A is also open. Since $A \subseteq X$ was arbitrary, X is discrete.

[2] A metric space (X, d) is *discrete* if every subset $A \subset X$ is open. Notice that any set equipped with the discrete metric forms a discrete metric space, but not every discrete metric space necessarily has the discrete metric.

Problem 6.

Suppose T is an operator on a complete metric space (X, d). Prove that the condition

$$d\left(T(x), T(y)\right) < d(x, y) \;\; \forall x, y \in X \;(x \neq y)$$

does not guarantee the existence of a fixed point of T.

Solution

Take $\mathbb{R}_+$ (the non-negative reals) with the usual metric. Define $T : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$T(x) = x + \frac{1}{1+x}$$

Note that $\frac{1}{1+x}$ is non-zero for any number, so clearly T has no fixed point.

To see that T satisfies the condition: take d as the usual metric. Without loss of generality, let $0 \leq x < y$. Then since T is an increasing function, we have

$$\begin{aligned}
d(x, y) - d\left(T(x), T(y)\right) &= (y - x) - \left(y + \frac{1}{1+y} - x - \frac{1}{1+x}\right) \\
&= \frac{1}{1+x} - \frac{1}{1+y} \\
&> 0
\end{aligned}$$