Ec240a, Fall 2019

Professor Bryan Graham

Problem Set 2

Due: November 8th, 2019

Problem sets are due at 5PM in the GSIs mailbox. You may work in groups, but each student should turn in their own write-up (including a printout of a narrated/commented and executed Jupyter Notebook if applicable). Please also e-mail a copy of any Jupyter Notebook to the GSI (if applicable).

1. Let $\mathcal{H}$ be an Hilbert space and y a fixed vector within it. Show that for each $\epsilon > 0$ that there exists a $\delta > 0$ such that
$$|\langle h_1, y\rangle - \langle h_2, y\rangle| \leq \epsilon$$
for all $h_1, h_2 \in \mathcal{H}$ where $\|h_1 - h_2\| \leq \delta$ (HINT: use the Cauchy-Schwarz Inequality).

2. The linear regression of Y into X is
$$\mathbb{E}^*\left[Y|\, X\right] = X'\gamma_0,\ \ \gamma_0 = \mathbb{E}\left[XX'\right]^{-1} \times \mathbb{E}\left[XY\right].$$
Let $X = (1, W')'$, with W a $K \times 1$ vector of linearly independent random variables. Show that
$$\mathbb{E}\left[XX'\right]^{-1} = \left[\begin{array}{cc} 1 + \mathbb{E}\left[W\right]' \mathbb{V}\left(W\right)^{-1} \mathbb{E}\left[W\right] & -\mathbb{E}\left[W\right]' \mathbb{V}\left(W\right)^{-1} \\ -\mathbb{V}\left(W\right)^{-1} \mathbb{E}\left[W\right] & \mathbb{V}\left(W\right)^{-1} \end{array}\right]$$
and hence also that
$$\gamma_0 = \left(\begin{array}{c} \alpha_0 \\ \beta_0 \end{array}\right) = \left[\begin{array}{c} \mathbb{E}\left[Y\right] - \mathbb{E}\left[W\right]' \beta_0 \\ \mathbb{V}\left(W\right)^{-1} \mathbb{C}\left(W, Y\right) \end{array}\right].$$
You may assume that all the expectations and variances in the expression above are well-defined.

3. Let X be a $K \times 1$ vector of covariates with a constant as first element. Let W be a $J \times 1$ vector of additional covariates (excluding a constant). Consider the **long regression** of Y onto X and W:
$$\mathbb{E}^*\left[Y|\, X, W\right] = X'\beta_0 + W'\gamma_0.$$
Further consider the **short regression** of Y onto X alone
$$\mathbb{E}^*\left[Y|\, X\right] = X'b_0.$$
Finally consider the **auxiliary linear** (multivariate) regression of W given X
$$\mathbb{E}^*\left[W|\, X\right] = \Pi_0 X.$$
Here Π_0 is the $J \times K$ coefficient matrix $\Pi_0 = \mathbb{E}\left[WX'\right] \times \mathbb{E}\left[XX'\right]^{-1}$. Let $U = Y - \mathbb{E}^*\left[Y|\, X, W\right]$.
   (a) Use the Projection Theorem to show that $\mathbb{E}^*\left[U|\, X\right] = 0$.
   (b) Use the Projection Theorem to show that $\mathbb{E}^*\left[X|\, X\right] = X$.

(c) Use you the results from (a) and (b) above as well as linearity of the projection operator to further show that

$$\mathbb{E}^*\left[Y|\,X\right] = X'\beta_0 + \mathbb{E}^*\left[W|\,X\right]'\gamma_0$$

and hence that

$$b_0 = \beta_0 + \Pi_0'\gamma_0.$$

(d) Interpret your result as an "omitted variable bias" (OVB) formula.

(e) Further argue that you have shown the **law of iterated linear predictors:**

$$\mathbb{E}^*\left[Y|\,X\right] = \mathbb{E}^*\left[\mathbb{E}^*\left[Y|\,X,W\right]|\,X\right].$$

4. Show that

$$\mathbb{V}\left(Y\right) = \mathbb{V}\left(Y - \mathbb{E}^*\left[Y|\,X\right]\right) + \mathbb{V}\left(\mathbb{E}^*\left[Y|\,X\right]\right).$$

5. Let $\mathbf{Y}$ be an $N \times 1$ vector of outcomes and $\mathbf{X}$ and $N \times K$ vector of covariates (which includes a constant in column 1). The projection of $\mathbf{Y}$ onto the column space of $\mathbf{X}$ coincides with the least squares fit

$$\hat{\mathbf{Y}} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y}.$$

Let $\hat{\mathbf{U}} = \mathbf{Y} - \hat{\mathbf{Y}}$ be the fitted residuals. Using vector space methods show that:

(a) $\mathbf{X}'\hat{\mathbf{U}} = 0$

(b) $\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right)'\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right) = \hat{\mathbf{U}}'\hat{\mathbf{U}} + \left(\hat{\mathbf{Y}} - \frac{\mathbf{Y}'\iota}{N}\right)'\left(\hat{\mathbf{Y}} - \frac{\mathbf{Y}'\iota}{N}\right)$

(c) $0 \leq R^2 \leq 1$ for $R^2 = 1 - \frac{\hat{\mathbf{U}}'\hat{\mathbf{U}}}{\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right)'\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right)}$

6. Compute the following exercises from Hansen (2019): 2.4, 2.16, 3.2, 3.3, 3.6 (using a Projection Theorem argument), 3.8, 3.9