Economics 204
Fall 2013
Problem Set 3 Suggested Solutions

1. Some practice with compactness:

   (a) Use the open cover definition of compactness to show that the subset $\left\{\frac{n}{n^2+1},\ n=0,1,2\ldots\right\}$ of $\mathbf{R}$ is compact.

   Solution. Denote $A=\left\{\frac{n}{n^2+1},\ n=0,1,2\ldots\right\}$. Let the collection of sets $\left\{U_\lambda\right\}_{\lambda\in\Lambda}$ be an open cover. Since $0\in A$, there exists an open set $U_{\lambda_0}\in\left\{U_\lambda\right\}$ such that $0\in U_{\lambda_0}$. Hence we can find an $\varepsilon>0$ such that $B_\varepsilon(0)\subset U_{\lambda_0}$. Note that there are only finitely many points of K not included in U_{λ_0} which are those with $\frac{n}{n^2+1}>\varepsilon$. Denote them as $a_1,...,a_n$ and for each a_j choose a U_{λ_j} from $\left\{U_\lambda\right\}_{\lambda\in\Lambda}$such that $a_j\in U_{\lambda_j}$. Then $\{U_{\lambda_0},...,U_{\lambda_n}\}$ is a finite subcover by construction. So A is compact.

   (b) Let $O_1\subset O_2\subset O_3\subset ....$ be open subsets of $\mathbf{R}$ with non-empty and bounded complement. Prove that

   $$\bigcup_{j=0}^{\infty}O_j\neq\mathbf{R}.$$

   Solution. The ${O_j}^c$ are closed and bounded, hence compact by the Heine-Borel Theorem. Note that since ${O_j}^c\supset {O_{j+1}}^c$ and each $O_j\neq\emptyset$ implies that for each n we have $\bigcap_{j=1}^{n}O_j^c\neq\emptyset$. For each n, choose an element of $\bigcap_{j=1}^{n}{O_j}^c$ and label it x_n. Note that $\{x_n\}\subseteq O_1^c$. By compactness, this sequence has a convergent subsequence with limit $l\in O_1^c$. We claim that, in fact, $l\in\bigcap_{j=1}^{\infty}O_j^c$. To see this, denote the convergent subsequence by $\{x_{n_k}\}$. From the construction of the sequence it follows that $\{x_{n_k}\}_{k=1}^{\infty}\subset O_{n_1}^c$, $\{x_{n_k}\}_{k=2}^{\infty}\subset O_{n_2}^c$, $\{x_{n_k}\}_{k=3}^{\infty}\subset O_{n_3}^c$, etc.
   Note that $l\in O_{n_k}^c$ for all k by compactness. This implies that $l\in\bigcap_{k=1}^{\infty}O_{n_k}^c=\bigcap_{j=1}^{\infty}O_j^c$. The latter equality follows because the $\{O_j\}^c$ form a decreasing sequence. Hence, given any $O_j^c\ \exists\, n_k$ such that $O_{n_k}^c\supset O_j^c$. Similarly, given any $O_{n_k}^c$, $\exists\, j$ such that $O_{n_k}^c\subset O_j^c$. Consequently, $\bigcap_{j=1}^{\infty}O_j^c\neq\emptyset\Rightarrow\bigcup_{j=1}^{\infty}O_j\neq\mathbf{R}$.

   (c) Provide an example of a decreasing sequence of closed subsets of $\mathbf{R}$ (e.g. $S_1\supset S_2\supset S_3\supset ....$) such that $\bigcap_{n=1}^{\infty}S_n=\emptyset$.

   Solution. Put $S_n=[n,\infty)$. Then, $S_n\supset X_{n+1}$ and $\bigcap_{n=1}^{\infty}S_n=\emptyset$. (note that the S_n's are closed, but not bounded).

2. Some practice with compactness and completeness:

   (a) Let (X, d) be a metric space. Suppose that for some $\varepsilon > 0$ every $\varepsilon-$ball $B_\varepsilon(x)$ in X has compact closure. Show that X is complete.

   Solution. Let $\epsilon > 0$ have a value such that every ϵ-ball in X has compact closure and let $\{x_n\}$ be a Cauchy sequence in X. We know that there exists some N such that $m, n \geq N \Rightarrow d(x_m, x_n) < \epsilon$. Consider $B_\epsilon(x_N)$ and its closure, $\overline{B_\epsilon(x_N)}$. Since $d(x_N, x_m) < \epsilon$ for all $m \geq N$ we have $x_n \in B_\epsilon(x_N) \subset \overline{B_\epsilon(x_N)}$ for all $n \geq N$. The subsequence of $\{x_n\}$ consisting of all x_n such that $n \geq N$ is itself clearly a Cauchy sequence and it is contained entirely in $\overline{B_\epsilon(x_N)}$, which, by hypothesis, is compact. Note that a sequence in a compact set must have a convergent subsequence, and Theorem 7.8 in de la Fuente establishes that a Cauchy sequence with a convergent subsequence must itself converge. Thus, the sequence contained in $\overline{B_\epsilon(x_N)}$ must converge and naturally the $\{x_N\}$ must converge as well. Therefore, X is complete.

   It is important to differentiate between closed ball and *closure* of an open ball because they are obviously not the same in general metric spaces. One prominent example of a metric space none of whose closed balls are complete is $\mathbf{Q}$ with the usual Euclidean metric. To see that, start by fixing arbitrary $q \in \mathbf{Q}$ and $\varepsilon > 0$. To differentiate between balls in $\mathbf{Q}$ and in $\mathbf{R}$, we'll use the corresponding superscript. So consider the closed ball $B_\varepsilon^{\mathbf{Q}}[q]$ in $\mathbf{Q}$. Note that $B_\varepsilon^{\mathbf{Q}}[q] = B_\varepsilon^{\mathbf{R}}[q] \cap \mathbf{Q}$ and $B_\varepsilon^{\mathbf{Q}}(q) = B_\varepsilon^{\mathbf{R}}(q) \cap \mathbf{Q}$.

   We will take advantage of the fact that both the rational numbers and the irrational numbers are dense on the real line. In particular, we will use the well-known property that every open set in $\mathbf{R}$ contains infinitely many rational and infinitely many irrational numbers. This allows us to pick some irrational $p \in B_\varepsilon^{\mathbf{R}}[q]$. Notice that, as an irrational number, p is not in $B_\varepsilon^{\mathbf{Q}}[q]$. The same property allows us to choose some rational $q_n \in B_{1/n}^{\mathbf{R}}(p) \cap B_\varepsilon^{\mathbf{R}}(q)$ for all $n \in \mathbf{N}$. More specifically, we can do that because both $B_{1/n}^{\mathbf{R}}(p)$ and $B_\varepsilon^{\mathbf{R}}(q)$ are open in $\mathbf{R}$ and so is their intersection.

   Since q_n is rational and, as noted above, $B_\varepsilon^{\mathbf{Q}}(q) = B_\varepsilon^{\mathbf{R}}(q) \cap \mathbf{Q}$, then the sequence $\{q_n\}$ is entirely contained in $B_\varepsilon^{\mathbf{Q}}(q) \subseteq B_\varepsilon^{\mathbf{Q}}[q]$. This sequence also converges in $\mathbf{R}$. To see that, observe that for any $\varepsilon > 0$ we can use the Archimedean property to find some $N \in \mathbf{N}$ such that $\varepsilon > 1/N$. But by the way we chose $\{q_n\}$, we know that $|q_n - p| < 1/n \leq 1/N < \varepsilon$ for all $n \geq N$ and hence $\{q_n\} \to p$. Since this sequence converges, it is necessarily Cauchy with respect to the Euclidean metric. Hence it is Cauchy in our metric space counterexample ($\mathbf{Q}$ with the Euclidean metric). However, it does not converge there since p - its limit in $\mathbf{R}$ - is not in $B_\varepsilon^{\mathbf{Q}}[q]$. Thus the arbitrary closed ball $B_\varepsilon^{\mathbf{Q}}[q]$ in our metric space is not complete.

(b) Continue to assume that (X, d) is a metric space. Now, suppose that for each $x \in X$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x)$ has compact closure. Will X still be complete? Prove or give counter-example.

Solution. No, and here is a counter-example. Let $X = (0, \infty)$. This set has the property that for each x there exists an $\epsilon > 0$ such that $B_\epsilon(x)$ has compact closure. For example, given some $x \in X$ we can choose $\varepsilon = x/2$. Then the closure of $B_\varepsilon(x) = \left(\frac{x}{2}, \frac{3x}{2}\right)$ is $\left[\frac{x}{2}, \frac{3x}{2}\right]$, which is a closed and bounded subset of the reals and is therefore compact. However, this space is not a complete metric space because the Cauchy sequence $x_n = 1/n$ does not converge.

3. Show that a metric space which has countably many points is connected if and only if it contains only one point. Hint: You can use (without proof) the fact that for any $a > 0$, the interval $[0, a]$ in $\mathbf{R}$ is uncountable.

Solution. Notice that necessity is trivial — one point set cannot be expressed as a union of two non-empty sets. So, lets prove sufficiency. Suppose, to contradiction, that X has countably many points and their number is greater then one. Lets enumerate points $X = \{x_1, x_2, \ldots, \}$. Pick point $x_1 \in X$, and consider $B_r(x_1)$ for all $r \in \mathbf{R}_+ +$. Note that as r increases, open ball around x_0 grow larger because the number of points in X that it can possibly contain increases. The idea of the proof is that the "size of the balls" (the number of points they contain) can't increase strictly monotonically because $\mathbf{R}_+$ is uncountable. Thus, "size of the balls" must be the constant over some interval.

More formally, consider the following set: $R = \{r_n : d(x_n, x_1), \forall n \in \mathbf{N}\}$. Observe that there must be $r > 0$ such that $r \notin R$, otherwise R is an enumeration of $\mathbf{R}_+$ (or some subset of it). Note that for such $r > 0$ we have $d(x_1, x_n) \neq r$ for all $x_n \in X$ and $d(x_1, x_n) > r$ for some $x_n \in X$ (because we assumed that X has more then one point.) Thus, we have $B_r(x_1) = B_r[x_1]$, i.e. $B_r(x_1)$ is *clopen*, open and closed at the same time. Therefore, $X \setminus B_r(x_1)$ is and open set, which immediately gives us a separation of X, $X = B_r(x_1) \cup X \setminus B_r(x_1)$ and $\overline{B_r(x_1)} \cap X \setminus B_r(x_1) = B_r(x_1) \cap \overline{X \setminus B_r(x_1)} = \emptyset$.

4. Let (X, d) and (Y, ρ) be metric spaces and let A be any subset of Y. Prove that the constant correspondence $\Phi : X \to 2^Y$ defined by $\Phi(x) = A$ for all $x \in X$ is continuous.

Solution. Pick any open set $B \subset Y$ and $x \in X$. Since $\Phi(x) = A = \Phi(x')$ for all $x, x' \in X$, we must have, firstly, that $\Phi(x) \subset B$ iff $\Phi(x') \subset B$, and $\Phi(x) \cap B \neq \emptyset$ iff $\Phi(x') \cap B \neq \emptyset$. Therefore, $\Phi(x)$ is both lhc and uhc at all points $x \in X$.

5. Let (X, d) be a compact metric space and let $\Psi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Psi(x) : X \to 2^X$ is

non-empty for every $x \in X$. Prove that there exists a compact non-empty subset K of X, such that $\Psi(K) = \cup\{\Psi(x) : x \in K\} = K$.

Solution. Consider following non-empty sequence of compact sets

$$X,\ \Psi(X),\ \Psi^2(X), \ldots$$

Observe mathematical induction implies that $X \supset \Psi(X) \supset \Psi^2(X) \supset \ldots$ because $X \supset \Psi(X)$ (base step) and $\Psi^n(X) \supset \Psi^{n+1}(X)$ yields $\Psi^{n+1}(X) \supset \Psi^{n+2}(X)$ (induction step.) Thus, the sequence $\{\Psi^n(X)\}$ is non-increasing. Moreover, using the fact that an image of compact set under compact-valued and upper hemi-continuous correspondence is compact, we have that $\Psi^n(X)$ is compact for every $n \in \mathbf{N}$. (Why? The easiest way to prove that is to use the Theorem 12 in Lecture Notes 7. If $C \subset X$ is compact, take $\{y_n\} \in \Psi(C)$ and find $\{x_n\} \in C$, such that $y_n \in \Psi(x_n)$ for all $n \in \mathbf{N}$. Then, use compactness of C to get convergence along subsequence to some $x_0 \in C$ and apply the theorem to $\{y_{n_{k_j}}\}$, the subsequence of companion subsequence. Note that you cannot really invoke Theorem 11.9 in de La Fuente as it is given without proof, or the proof is left as an exercise to interested readers.)

Lets denote by K its infinite intersection

$$K = \cap_{i=1}^{\infty} \Psi^n(X)$$

and observe that K is compact (recall, in the section you have seen the proof that an intersection of any collection of compact sets is compact). By Cantor's Intersection Theorem $K \neq \emptyset$. Now, since $K \subset \Psi^{n-1}(X)$, $\Psi(K) \subset \Psi^n(X)$ for every $n \in \mathbf{N}$, we get that $\Psi(K) \subset K$. (Can you see that? Consider $y \in \Psi(K)$, we have

$$\begin{aligned} y \in \Psi(K) &\iff \exists x \in \cap_{n=1}^{\infty} \Psi^n(X), \text{ such that } y \in \Psi(x) \\ &\iff \exists x \in \Psi^n(X) \text{ for all } n \in \mathbf{N} \text{ such that } y \in \Psi(x) \\ &\implies y \in \Psi^n(X) \text{ for all } n \in \mathbf{N} \\ &\iff y \in \cap_{n=1}^{\infty} \Psi^n(X) \end{aligned}$$

where implication follows because $\exists x \in \Psi^n(X)$ such that $y \in \Psi(x)$ means that $y \in \Psi(\Psi^n(X)) = \Psi^{n+1}(X)$).

To show that $K \subset \Psi(K)$, pick any $y \in K$. Lets construct a sequence $\{x_n\}$ such that $x_n \in \Psi^n(X)$ and $y \in \Psi(x_n)$. We use compactness of X to extract a convergent subsequence $\{x_{n_k}\}$ with a limit x. By construction of $\{x_n\}$, x must be in K. Now consider the sequence $\{x_{n_k}, y\}$ in the graph of correspondence Ψ. By Theorem 11 in Lecture Notes 7 on p. 6, we know that if $(x_{n_k}, y) \to (x, y)$ then $y \in \Psi(x)$ because Ψ has a closed graph (it is uhc and compact valued, thus, closed valued). Therefore, $y \in \Psi(K)$ which implies $K \subset \Psi(K)$.

6. Let $(X,\, d)$ be a complete metric space and $\{\, T_n\}$ be a sequence of contractive self-maps on X such that $\sup\{\beta_m \,:\, m \in \mathbf{N}\} < 1$, where β_m is a contraction modulus of T_m, $m = 1,\, 2,\, \ldots$. By the Contraction Mapping Fixed Point Theorem, T_m has a unique fixed point, say x_m. Show that if

$$\sup\{d(T_m(x), T(x)) : x \in X\} \to 0$$

for some $T : X \to X$, then T is a contraction with a unique fixed point $\lim x_m$. It is possible to weaken our assumption and just require that $d(T_m(x), T(x)) \to 0$ for every $x \in X$?

Solution. We have a sequence of self-maps on X with $d(T_m(x), T_m(x')) \le \beta_m d(x,\, x')$ for all $x,\, x'$ in X. Moreover, we have a uniform convergence of $\{T_m\}$ to T

$$T_m \rightrightarrows T \qquad \text{as } m \to \infty.$$

We need to show two things: firstly, that the limit point of $\{T_m\}$ is a contraction itself, and, secondly, that a sequence of fixed points $\{x_m\}$ converges to the fixed point of T.

Lets fix $x,\, x' \in X$ and consider the distance $d(T(x),\, T(x'))$. By repeatedly applying triangle inequality we obtain the following:

$$\begin{aligned} d(T(x),\, T(x')) &\le d(T(x),\, T_m(x)) + d(T_m(x),\, T_m(x')) + d(T_m(x'),\, T(x')) \\ &\le d(T(x),\, T_m(x)) + \beta_m d(x,\, x') + d(T_m(x'),\, T(x')) \\ &\le d(T(x),\, T_m(x)) + \beta d(x,\, x') + d(T_m(x'),\, T(x')) \end{aligned}$$

where $\beta = \sup\{\beta_m \,:\, m \in \mathbf{N}\}$. Note that in the limit the first and the third term go to zero, as $T_m \rightrightarrows T$ and we obtain $d(T_m(x),\, T(x')) \le \beta d(x,\, x')$. Thus, by our assumption of $\beta < 1$ T is a contraction. Lets observe here that in proving the first claim we do not really need a uniform convergence of $\{T_m\}$, a simple point-wise convergence $T_m \to T$ would do as x and x' are fixed by our assumption.

Now, to prove the second claim. We need to show that the distance $d(x_m,\, x) \to 0$ as $m \to \infty$. Note that

$$\begin{aligned} d(x_m,\, x) &= d(T_m(x_m),\, T(x)) \\ &\le d(T_m(x_m),\, T(x_m)) + d(T(x_m),\, T(x)) \\ &\le d(T_m(x_m),\, T(x_m)) + \beta d(x_m,\, x) \end{aligned}$$

Thus, we have $(1-\beta)d(x_m,\, x) \le d(T_m(x_m),\, T(x_m))$ which yields

$$d(x_m,\, x) = \frac{1}{1-\beta}\, d(T_m(x_m),\, T(x_m)) \to 0 \qquad \text{as } m \to \infty$$

because $d(T_m(x_m),\, T(x_m)) \le \sup\{d(T_m(x), T(x)) \,:\, x \in X\}$. Observe that this is a part were it looks like we might need uniform convergence of $\{T_m\}$ as values of x move through the sequence $\{x_m\}$ and we need to "control" (or bound) the distance between x_k and x for all $k > m$. It turns out there are no valid counterexample to this claim. In fact, point-wise convergence is enough as it was shown in the paper by Barbet and Nachi (2006) Sequences of Contractions and Convergence of Fixed Points in Monografias del Seminario Matematico Garcia de Galdeano 33, p. 51-58.

Note that the earlier paper's example that I relied on to provide you a counterexample to the claim that point-wise convergence suffices does not apply in this case. Sorry about that, it was bad and I am happy to correct it now.
(Nadler (1968) in Pacific Journal of Mathematics.).