Econ 204 – Problem Set 6

Due Wednesday, August 19; before exam

1. Calculate the first, second and third order Taylor expansion of $(1 + 2x - 3y)^2$ around the point $(0, 0)$. Calculate the difference between the value of the function and the expansions.

2. Consider the following equations:

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2}, \quad x^2 + y^2 > 0.$$

   (a) For $(u, v) = (1/2, 1/2)$, find a pair of values (x_0, y_0) that satisfy the equations.

   (b) Describe either verbally or graphically what this transformation does. Bonus given for colorful metaphors.

   (c) Show that the above transformation implicitly defines a function in the neighborhood of (x_0, y_0) (in the sense that for every pair of values (u, v) near $(1/2, 1/2)$, there is just one corresponding pair of (x, y) values.

   (d) Compute the Jacobian of the implicit function.

3. Prove that there exist functions $u, v : \mathbb{R}^4 \longrightarrow \mathbb{R}$, continuously differentiable on some open neighborhood around the point $(x, y, z, w) = (2, 1, -1, 2)$ such that $u(2, 1, -1, 2) = 4$ and $v(2, 1, -1, 2) = 3$ and the equations

$$u^2 + v^2 + w^2 = 29 \text{ and } \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17$$

   both hold for all (x, y, z, w) in that neighborhood.

4. Let $E = \{(x, y) : 0 < y < x\}$ and set $f(x, y) = (x + y, xy)$ for $(x, y) \in E$.

   (a) Prove f is one-to-one from E onto $\{(s, t) : s > 2\sqrt{t},\ t > 0\}$ and find a formula for $f^{-1}(s, t)$.

   (b) Use the inverse function theorem to compute $D(f^{-1})(f(x, y))$ for $x \neq y$.

   (c) Compare the two expressions for $D(f^{-1})(f(x, y))$ that you derived directly of using the Implicit Function Theorem

5. Consider the second order linear differential equation given by $y'' = 4y + 3y'$.

Note that this equation can be rewritten as the following *first* order linear differential equation of two variables:

$$\bar{x}'(t) = A\bar{x}(t),$$

where $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(a) Describe the solutions of the first order system (verbally) by analyzing the matrix A.

(b) In a phase diagram, show the behavior of the system using the previous analysis and by solving for $x_1'(t) = 0$ and $x_2'(t) = 0$.

(c) Give the solution of the system when $x_1(t_0) = 0$ and $x_2(t_0) = 1$.