Econ 204 – Problem Set 1

Due July 31[1]

1. Use induction to prove the following:

   (a) $2^{2n} - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

   (b) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$

   Solution:

   (a) The base step is for $n = 1$, which is true because $2^2 - 1 = 3$ is divisible by 3! Now assume the claim holds for $n \in \mathbb{N}$ and verify the induction step, which is to show $2^{2(n+1)} - 1$ is divisible by 3:

   $$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \times 2^{2n} - 1 = 3 \times 2^{2n} + 2^{2n} - 1 \tag{1}$$

   First term is clearly divisible by 3, so as the second one, because of the induction step assumption.

   (b) Base case ($n = 1$): $1 \leq 2 = 2 \cdot \sqrt{1}$. Induction step: assume the statement holds for $n = k$. Adding $\frac{1}{\sqrt{k+1}}$ to both sides gives

   $$1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} \tag{2}$$

   Hence if we can show $2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}$ we are done. Then:

   $$\begin{aligned} 2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} &\iff 2\sqrt{k}\sqrt{k+1} + 1 \leq 2(k+1) \\ &\iff 2\sqrt{k}\sqrt{k+1} \leq 2k+1 \\ &\iff 4k(k+1) \leq (2k+1)^2 \\ &\iff 4k^2 + 4k \leq 4k^2 + 4k + 1 \\ &\iff 0 \leq 1 \end{aligned}$$

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2. Define the infinite **cartesian product** of a set X with itself as $X^\omega := \prod_{i \in \mathbb{N}} X$. Prove by contradiction that for $X = \{0, 1\}$, X^ω is uncountable. (Hint: suppose there exists a surjective map $f : \mathbb{N} \to X^\omega$, and find an element in X^ω which is not in the image of f).

   Solution: Proof by contradiction: Suppose such a surjective function $f : \mathbb{N} \to X^\omega$ exists. Then, for every $n \in \mathbb{N}$ let

   $$f(n) := (x_{n1}, x_{n2}, \ldots) \in X^\omega \tag{3}$$

   Now take $y = (y_1, y_2, \ldots)$ such that:

   $$y_n = \begin{cases} 1 & \text{if } x_{nn} = 0 \\ 0 & \text{if } x_{nn} = 1 \end{cases} \tag{4}$$

[1] In case of any problems with the exercises please email farzad@berkeley.edu

Now it is clear that y can not be contained in the codomain set of f, i.e there does not exist any $n \in \mathbb{N}$ such that $f(n) = y$. This uncovers the contradiction and verifies that $\{0,1\}^\omega$ is uncountable.

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3. In the following examples, show that the sets A and B are numerically equivalent by finding a specific bijection between the two.

   (a) $A = [0,1],\ B = [10,20]$

   (b) $A = [0,1],\ B = [0,1)$

   (c) $A = (-1,1),\ B = \mathbb{R}$

   Solution: In each case, we want to define a valid *one-to-one* and *onto* mapping $f : A \to B$ (by valid, I mean well-defined for each element of the domain and whose image lies in the codomain).

   (a) Define
   $$f(x) = 10x + 10 \tag{5}$$
   $f : A \to B$ is well-defined since for any $x \in [0,1]$, we have $10x + 10 \in [10,20]$.
   Onto: for any $y \in B = [10,20]$, $x = \frac{y-10}{10} \in A = [0,1]$ and $f(x) = y$
   One-to-one: $x \neq x' \implies 10x + 10 \neq 10x' + 10$

   (b) Take any infinite countable subset $C \subset A$ enumerated $C = \{x_1, x_2, \ldots\}$ where $x_1 = 1$ (e.g $x_n = 1/n\ \forall n$), and let $C' = A \setminus C$. Define:
   $$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \in C \\ x & \text{if } x \in C' \end{cases} \tag{6}$$
   Note that C, C' partition A so the function is well-defined over the entire domain.
   Onto: we have $f(C') = C'$ and $f(C) = C \setminus \{1\}$. Then
   $$\begin{aligned} f(A) &= f(C' \cup C) \\ &= f(C') \cup f(C) \\ &= C' \cup C \setminus \{1\} \\ &= A \setminus \{1\} = B \end{aligned}$$
   One-to-one: take $x \neq x'$. If $x, x' \in C$ or $x, x' \in C'$ this is immediate (note the elements of C are unique). If $x \in C, x' \in C'$ or vice versa, since the images $f(C), f(C')$ are disjoint we have $f(x) \neq f(x')$.

   (c) Define $f(x) = \tan(\frac{\pi x}{2}) = \frac{\sin(\pi x/2)}{\cos(\pi x/2)}$ or $f(x) = \frac{x}{1-|x|}$, and verify it is a well-defined bijective function that maps $(-1,1)$ to the entire $\mathbb{R}$.

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4. **(Dynkins's π-λ system Theorem)**: The goal of this exercise is to prove this theorem, and review/practice some of the set theoretical results.

Suppose Ω is some arbitrary set (which need not have any topological or algebraic structure). Then, $\mathscr{F} \subset 2^{\Omega}$ as a collection of subsets of Ω is called a **σ-algebra** if it satisfies following properties:

- $\emptyset \in \mathscr{F}$.
- For every $A \subset \Omega$ where $A \in \mathscr{F}$, $A^c \in \mathscr{F}$ (A^c refers to the complement of set A, i.e $\Omega \setminus A$).
- For every *countable* sequence of subsets $\{A_n\}$ where $A_n \in \mathscr{F}$ for all n, $\bigcup_n A_n \in \mathscr{F}$.

(a) Show that $\Omega \in \mathscr{F}$.

(b) Prove that $\mathscr{F}$ is closed under countable intersection.

Two more definitions: first, $\Lambda \subset 2^{\Omega}$ is called a **λ-system** if:

- $\Omega \in \Lambda$.
- If $A, B \in \Lambda$ and $A \subset B$, then $B \setminus A \in \Lambda$.
- If $\{A_n\}$ is an increasing sequence of subsets, i.e $A_1 \subset A_2 \subset \ldots$, with each element being in Λ, then $\bigcup_n A_n \in \Lambda$.

Second, $\Pi \subset 2^{\Omega}$ is called a **π-system**, if it is closed under *finite* intersection. Now **assume** Π is a π-system such that $\Pi \subset \Lambda$, where Λ is a λ-system. The Dynkin's theorem which we want to prove states that the smallest σ-algebra containing Π (denoted by $\sigma(\Pi)$) is a subset of Λ. Try to keep on with each step below until the final result drops out:

(c) Let $\lambda(\Pi)$ be the *smallest*[2] λ-system containing Π. Explain why $\lambda(\Pi) \subset \Lambda$. (Hint: note that $\Pi \subset \Lambda$).

(d) Let $B \in \Pi$ and define $\mathcal{A}_B := \{A \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Show that $\mathcal{A}_B$ is itself a λ-system and contains $\lambda(\Pi)$, i.e $\lambda(\Pi) \subset \mathcal{A}_B$.

(e) Now let $A \in \lambda(\Pi)$, and define $\mathcal{B}_A := \{B \subset \Omega : A \cap B \in \lambda(\Pi)\}$. Again, show that $\mathcal{B}_A$ is a λ-system containing $\lambda(\Pi)$.

(f) Use the previous two steps to show $\lambda(\Pi)$ is also a π-system.

(g) Given that we have seen so far that $\lambda(\Pi)$ is both a π-system and a λ-system, deduce that it has to be a σ-algebra.

(h) Conclude that $\sigma(\Pi) \subset \Lambda$.

Solution:

(a) The second property of the σ-algebra states that it is closed under complementation. Therefore, given $\emptyset \in \mathscr{F}$, its complement Ω must also be in $\mathscr{F}$.

(b) Suppose $A_n \in \mathscr{F}$ for all n. Therefore, $A_n^c \in \mathscr{F}$ too, and because of the third property of σ-algebra $\bigcup_n A_n^c \in \mathscr{F}$. Using the the second property again:

$$\bigcap_n A_n = \left(\bigcup_n A_n^c\right)^c \in \mathscr{F} \tag{7}$$

[2]By smallest we mean: $\lambda(\Pi) = \bigcap_{\{\Lambda_\alpha \text{ is } \lambda\text{-system in } 2^{\Omega}:\Pi\subset\Lambda_\alpha\}} \Lambda_\alpha$

Therefore, the σ-algebra is closed under countable intersection.

(c) $\lambda(\Pi)$ by definition is the smallest λ-system containing Π, meaning that it is the intersection of all λ-systems on Π. Therefore, it has to be a subset of Λ, since Λ is a λ-system containing Π.

(d) To show that $\mathcal{A}_B$ is a λ-system, we have to verify all three properties of a generic λ-system. First, $\Omega \in \mathcal{A}_B$, because $\Omega \cap B = B \in \Pi$, and $\Pi \subset \lambda(\Pi)$, then $\Omega \in \lambda(\Pi)$ as well. Second, suppose $E, F \in \mathcal{A}_B$ and $E \subset F$. Both $E \cap B$ and $F \cap B$ are by construction elements of $\lambda(\Pi)$, and given that $E \cap B \subset F \cap B$, and the λ-system is closed under proper difference, hence $(F \cap B) \setminus (E \cap B) \in \lambda(\Pi)$. Then,

$$\begin{aligned}(F \cap B) \setminus (E \cap B) &= (F \cap B) \cap (E^c \cup B^c) \\ &= (F \cap B \cap E^c) \cup \underbrace{(F \cap B \cap B^c)}_{\emptyset} = (F \setminus E) \cap B\end{aligned} \tag{8}$$

implies that $F \setminus E \in \mathcal{A}_B$. Third, we have to show if $\{A_n\}$ is an increasing sequence of subsets in $\mathcal{A}_B$, then their union is also in $\mathcal{A}_B$.

$$\left(\bigcup_n A_n\right) \cap B = \bigcup_n (A_n \cap B) \tag{9}$$

Since $A_n \cap B \in \lambda(\Pi)$, and they constitute an increasing sequence of subsets, i.e $A_n \cap B \subset A_{n+1} \cap B$, their union also belongs to $\lambda(\Pi)$ (because remember that $\lambda(\Pi)$ is itself a λ-system). Therefore, $\left(\bigcup_n A_n\right) \cap B \in \lambda(\Pi)$, implying $\bigcup_n A_n \in \mathcal{A}_B$. So far, we have proved that $\mathcal{A}_B$ is a λ-system, however it remains to show that it contains $\lambda(\Pi)$. But, this immediately falls out, because $\Pi \subset \mathcal{A}_B$ (why?), therefore $\lambda(\Pi) \subset \mathcal{A}_B$.

(e) Showing that $\mathcal{B}_A$ is a λ-system follows from the exact same lines of previous part. However, we still need to show $\lambda(\Pi) \subset \mathcal{B}_A$. To see this, take some $B \in \Pi$. From the previous part for any $A \in \mathcal{A}_B$ (and in particular in $\lambda(\Pi)$) $A \cap B \in \lambda(\Pi)$, thus $B \in \mathcal{B}_A$, that in turn implies $\Pi \subset \mathcal{B}_A$. Since $\mathcal{B}_A$ is shown to be a λ-system, hence $\lambda(\Pi) \subset \mathcal{B}_A$.

(f) All needed is to prove that $\lambda(\Pi)$ is closed under finite intersection. Pick some $A, B \in \lambda(\Pi)$. Previous part implies that $B \in \mathcal{B}_A$ (because $\lambda(\Pi) \subset \mathcal{B}_A$), therefore $A \cap B \in \lambda(\Pi)$, concluding that $\lambda(\Pi)$ is also a π-system.

(g) To show the *closedness* under complementation take $A \in \lambda(\Pi)$. Given $A \subset \Omega$ and $\Omega \in \lambda(\Pi)$, then $A^c = \Omega \setminus A \in \lambda(\Pi)$. Same argument implies that $\emptyset \in \lambda(\Pi)$. It is thus left to show $\lambda(\Pi)$ is closed under countable union of not necessarily increasing sequence of subsets. For this first take $A_1, A_2 \in \lambda(\Pi)$, and see that $A_1^c, A_2^c \in \lambda(\Pi)$. Since $\lambda(\Pi)$ is also a π-system, then $A_1^c \cap A_2^c \in \lambda(\Pi)$, that leads to $A_1 \cup A_2 = (A_1^c \cap A_2^c)^c \in \lambda(\Pi)$. This implies that $\lambda(\Pi)$ is closed under finite union. Now take an arbitrary sequence of subsets $\{A_n\}$ in $\lambda(\Pi)$. Closedness under finite union implies that for each n:

$$C_n := \bigcup_{m=1}^{n} A_m \in \lambda(\Pi) \tag{10}$$

and this is an increasing sequence of subsets, i.e $C_n \subset C_{n+1}$, hence $\bigcup_n C_n \in \lambda(\Pi)$. Further, you should be able to check (by *pointwise* technique[3]) that

$$\bigcup_n C_n = \bigcup_n A_n, \tag{11}$$

which concludes that $\lambda(\Pi)$ is closed under countable union, therefore it satisfies all the properties of a σ-algebra.

(h) It falls out immediately from the fact that $\sigma(\Pi) \subset \lambda(\Pi) \subset \Lambda$. In fact, as a further exercise you can verify that $\sigma(\Pi) = \lambda(\Pi)$.

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5. Let $\mathcal{U}$ and $\mathcal{Z}$ be two sets, and $P : \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ be a bounded function. Define the *upper and lower value functions* as:

$$\begin{aligned} V_+ &= \inf_{u\in\mathcal{U}} \sup_{z\in\mathcal{Z}} P(u,z) \\ V_- &= \sup_{z\in\mathcal{Z}} \inf_{u\in\mathcal{U}} P(u,z) \end{aligned} \tag{12}$$

(a) Show that $V_+ \geq V_-$.

(b) Call any function $\beta : \mathcal{U} \to \mathcal{Z}$ a *strategy* for the maximizing side. Denote the space of all such strategies as $\mathscr{B}$. Prove the following identity, and explain why it is not in contrast with part (a).

$$V_+ = \inf_{u\in\mathcal{U}} \sup_{\beta\in\mathscr{B}} P(u,\beta(u)) = \sup_{\beta\in\mathscr{B}} \inf_{u\in\mathcal{U}} P(u,\beta(u)) \tag{13}$$

Solution:

(a) Let $(\bar{u},\bar{z})$ be an arbitrary point in $\mathcal{U}\times\mathcal{Z}$. Then because of definition of supremum we have:

$$\sup_{z\in\mathcal{Z}} P(\bar{u},z) \geq P(\bar{u},\bar{z}) \tag{14}$$

Similarly, by the definition of infimum,

$$P(\bar{u},\bar{z}) \geq \inf_{u\in\mathcal{U}} P(u,\bar{z}), \tag{15}$$

that implies,

$$\sup_{z\in\mathcal{Z}} P(\bar{u},z) \geq \inf_{u\in\mathcal{U}} P(u,\bar{z}) \quad \forall \bar{z}\in\mathcal{Z}. \tag{16}$$

Therefore, $\sup_{z\in\mathcal{Z}} P(\bar{u},z)$ is an upper bound for $\left\{\inf_{u\in\mathcal{U}} P(u,\bar{z}) : \bar{z}\in\mathcal{Z}\right\}$. Using the definition of supremum,

$$\sup_{z\in\mathcal{Z}} P(\bar{u},z) \geq \sup_{z\in\mathcal{Z}} \inf_{u\in\mathcal{U}} P(u,z) \quad \forall \bar{u}\in\mathcal{U}. \tag{17}$$

Since $\sup_{z\in\mathcal{Z}} \inf_{u\in\mathcal{U}} P(u,z)$ is a lower bound for $\left\{\sup_{z\in\mathcal{Z}} P(\bar{u},z) : \bar{u}\in\mathcal{U}\right\}$, then

$$V_+ = \inf_{u\in\mathcal{U}} \sup_{z\in\mathcal{Z}} P(u,z) \geq \sup_{z\in\mathcal{Z}} \inf_{u\in\mathcal{U}} P(u,z) = V_-. \tag{18}$$

(b) Let's first justify the first identity. For this note that for every $u\in\mathcal{U}$,

$$\left\{P(u,\beta(u)) : \beta\in\mathscr{B}\right\} = \left\{P(u,z) : z\in\mathcal{Z}\right\}, \tag{19}$$

[3] By taking $\omega \in \bigcup_n C_n$ and showing it has to belong to $\bigcup_n A_n$ as well, and vice versa.

therefore,

$$\sup_{\beta\in\mathscr{B}} P(u,\beta(u)) = \sup_{z\in\mathcal{Z}} P(u,z), \quad \forall u\in\mathcal{U}. \tag{20}$$

Then taking the infimum over $\{u\in\mathcal{U}\}$ leads to

$$V_+ = \inf_{u\in\mathcal{U}}\sup_{z\in\mathcal{Z}} P(u,z) = \inf_{u\in\mathcal{U}}\sup_{\beta\in\mathcal{B}} P(u,\beta(u)), \tag{21}$$

that justifies the first identity. Similar argument as of part (a) leads to

$$\inf_{u\in\mathcal{U}}\sup_{\beta\in\mathscr{B}} P(u,\beta(u)) \geq \sup_{\beta\in\mathscr{B}}\inf_{u\in\mathcal{U}} P(u,\beta(u)), \tag{22}$$

thus it is just left to justify the reverse inequality to prove the second identity in part (b). For this pick an arbitrary small $\varepsilon>0$. Then, for every $u\in\mathcal{U}$ one can find $\beta_u\in\mathcal{U}$ such that

$$\sup_{\beta\in\mathscr{B}} P(u,\beta(u)) \leq P(u,\beta_u(u)) + \varepsilon. \tag{23}$$

Therefore, using the infimum definition on $\mathcal{U}$ we would get:

$$\inf_{u\in\mathcal{U}}\sup_{\beta\in\mathscr{B}} P(u,\beta(u)) \leq \inf_{u\in\mathcal{U}} P(u,\beta_u(u)) + \varepsilon \tag{24}$$

Using the definition of supremum one can see that,

$$\inf_{u\in\mathcal{U}} P(u,\beta_u(u)) \leq \sup_{\beta\in\mathscr{B}}\inf_{u\in\mathcal{U}} P(u,\beta(u)). \tag{25}$$

Therefore,

$$\inf_{u\in\mathcal{U}}\sup_{\beta\in\mathscr{B}} P(u,\beta(u)) \leq \sup_{\beta\in\mathscr{B}}\inf_{u\in\mathcal{U}} P(u,\beta(u)) + \varepsilon. \tag{26}$$

Since ε was chosen arbitrarily, this concludes the reverse direction and hence the proof of part (b).

In the general case V_+ need not be equal to V_-, and this is not in contrast with the identity in (13), mainly because the maximization in the latter case is over the larger class of strategies rather than the actual action space.

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6. Let $f:[a,b]\to\mathbb{R}$. The set $P=\{x_0,x_1,\ldots,x_n\}$ is called a *partition* for $[a,b]$, if $a=x_0<x_1<\ldots<x_n=b$. Define $V(f;P) := \sum_{j=1}^{n}\left|f(x_j)-f(x_{j-1})\right|$. The *variation* of f on $[a,b]$ is defined as

$$V(f;[a,b]) := \sup\left\{V(f;P) : P \text{ is a partition for } [a,b]\right\}. \tag{27}$$

When $V(f;[a,b])$ is finite, we say that f is of *bounded variation* on $[a,b]$.

(a) Show that the class of functions of bounded variation on $[a,b]$ is closed under addition. That is if f and g have bounded variation on $[a,b]$, then $f+g$ also has bounded variation on $[a,b]$.

(b) Show that if f is of bounded variation on $[a,b]$ and $a \le c \le b$, then

$$V(f;[a,b]) = V(f;[a,c]) + V(f;[c,b]). \tag{28}$$

Solution:

(a) Suppose f and g are of bounded variation. Then, for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$

$$\begin{aligned} V(f+g;P) &= \sum_{j=1}^{n} \big|(f+g)(x_j) - (f+g)(x_{j-1})\big| \\ &\le \sum_{j=1}^{n} \big|f(x_j) - f(x_{j-1})\big| + \sum_{j=1}^{n} \big|g(x_j) - g(x_{j-1})\big| \\ &\le V(f;[a,b]) + V(g;[a,b]) \end{aligned} \tag{29}$$

Therefore, the set $\big\{V(f+g;P) : P \text{ is a partition for } [a,b]\big\}$ is bounded above by $V(f;[a,b]) + V(g;[a,b])$, and hence has a finite supremum.

(b) Given $\varepsilon > 0$ find the partitions $P_1 = \{x_0, x_1, \ldots, x_n\}$ of $[a,c]$ and $P_2 = \{y_0, y_1, \ldots, y_m\}$ of $[c,b]$ such that

$$\begin{aligned} V(f;[a,c]) - \varepsilon/2 \le V(f;P_1) \\ V(f;[c,b]) - \varepsilon/2 \le V(f;P_2), \end{aligned} \tag{30}$$

which results from the property of supremum. The union $P_1 \cup P_2$ is a partition for $[a,b]$. It is easy to see $V(f;P_1) + V(f;P_2) = V(f;P_1 \cup P_2) \le V(f;[a,b])$, therefore

$$V(f;[a,c]) + V(f;[c,b]) - \varepsilon \le V(f;[a,b]). \tag{31}$$

Above holds for every $\varepsilon > 0$, thus sending $\varepsilon \to 0$ leads to

$$V(f;[a,c]) + V(f;[c,b]) \le V(f;[a,b]). \tag{32}$$

Now we try to prove the reverse inequality. Given $\varepsilon > 0$, and using the property of supremum, find a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $V(f;[a,b]) - \varepsilon \le V(f;P)$. First assume $c \notin P$, hence there is $\{x_m, x_{m+1}\} \subset P$ such that $x_m < c < x_{m+1}$. Extend the partition by adding the point c to P, and get a refined partition

$$\widetilde{P} = \{x_0, x_1, \ldots, x_m, c, x_{m+1}, \ldots, x_n\}.$$

One can see $V(f;P) \leq V(f;\widetilde{P})$, because

$$\begin{aligned} V(f;P) &= \sum_{j=1}^{n} \left| f(x_j) - f(x_{j-1}) \right| \\ &\leq \sum_{j=1}^{m} \left| f(x_j) - f(x_{j-1}) \right| \\ &\qquad\qquad + \left| f(c) - f(x_m) \right| + \left| f(x_{m+1}) - f(c) \right| + \sum_{j=m+1}^{n} \left| f(x_j) - f(x_{j-1}) \right| \\ &= V(f;\widetilde{P}). \end{aligned} \tag{33}$$

This implies $V(f;\widetilde{P}) \geq V(f;[a,b]) - \varepsilon$. One can decompose $\widetilde{P}$ into two sub-partitions: $\widetilde{P}_1 = \{x_0, \ldots, x_m, c\}$ on $[a,c]$ and $\widetilde{P}_2 = \{c, x_{m+1}, \ldots, x_n\}$ on $[c,b]$, and see $V(f;\widetilde{P}_1) + V(f;\widetilde{P}_2) = V(f;\widetilde{P})$. Note that $V(f;\widetilde{P}_1) \leq V(f;[a,c])$ and $V(f;\widetilde{P}_2) \leq V(f;[c,b])$. Therefore,

$$V(f;[a,b]) - \varepsilon \leq V(f;P) \leq V(f;\widetilde{P}) \leq V(f;[a,c]) + V(f;[c,b]). \tag{34}$$

Since this holds for every $\varepsilon > 0$, the reverse inequality falls out.

Now assume $c \in P$. The proof of reverse inequality in this case does not need the refinement step, and otherwise is the same, hence omitted.