Problem 1.

Let A and B be sets, and I an arbitrary (possibly infinite) index set. Prove the following statements:

(a) $A \bigcup \left(\bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} \left(A \bigcup B_i \right)$

(b) $A \bigcap \left(\bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} \left(A \bigcap B_i \right)$

Solution

Remember that set equality $A = B$ is defined as $A \subset B$ and $B \subset A$, which says that $x \in A \implies x \in B$ and $x \in B \implies x \in A$. Equivalently we can show $x \in A \iff x \in B$. For these examples the latter is an easier approach:

(a)

$$\begin{aligned} x \in A \bigcup \left(\bigcap_{i \in I} B_i \right) &\iff x \in A \text{ or } x \in \bigcap_{i \in I} B_i \\ &\iff x \in A \text{ or } x \in B_i \ \forall i \in I \\ &\iff \forall i \in I : \ (x \in A \text{ or } x \in B_i) \\ &\iff \forall i \in I : \ \left(x \in A \bigcup B_i \right) \\ &\iff x \in \bigcap_{i \in I} \left(A \bigcup B_i \right) \end{aligned}$$

(b)

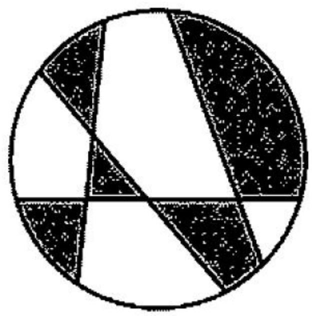
$$\begin{aligned} x \in A \bigcap \left(\bigcup_{i \in I} B_i \right) &\iff x \in A \text{ and } x \in \bigcup_{i \in I} B_i \\ &\iff x \in A \text{ and } \exists i \in I : \ x \in B_i \\ &\iff \exists i \in I : \ (x \in A \text{ and } x \in B_i) \\ &\iff \exists i \in I : \ \left(x \in A \bigcap B_i \right) \\ &\iff x \in \bigcup_{i \in I} \left(A \bigcap B_i \right) \end{aligned}$$

Problem 2.

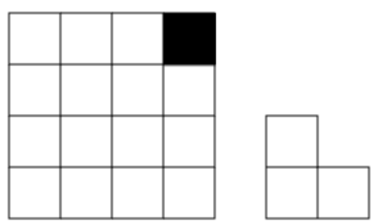
Use the principle of mathematical induction to prove the following statements:

(a) A set S with n elements has 2^n subsets. (note: do not forget about the empty set)

(b) Suppose that n chords are drawn in a circle, dividing the circle into different regions. Prove that every region can be colored one of two colors such that adjacent regions are different colors. The figure below shows an example of $n = 4$.



(c) Prove that any grid made up of $2^n \times 2^n$ tiles can be covered except for one corner tile by L-shaped triominoes (the triominoes may rotated). The figure below shows an example of a 4×4 grid (left) where all of the non-shaded tiles must be covered by a triomino (right).



Solution

(a) **Base step** ($n = 0$): The set containing 0 elements is the empty set. Since the only subset of the empty set is itself, we have $\mathcal{P}(\varnothing) = \{\varnothing\}$. Hence $|\mathcal{P}(S)| = 1 = 2^0$. Thus the claim holds for $n = 0$.

Inductive hypothesis ($n = k$): If $|S| = k$, then $|\mathcal{P}(S)| = 2^k$.

Inductive step: Take any set S such that $|S| = k + 1$. Fix some element of S (call it s_{k+1}) and consider the power set of $S \setminus \{s_{k+1}\}$, which I denote $X_1 = \mathcal{P}(S \setminus \{s_{k+1}\})$. Note that $A \subset S \setminus \{s_{k+1}\}$ if and only if $A \subset S$ and $s_{k+1} \notin A$. In other words, $X_1 = \{A : A \subset S \text{ and } s_{k+1} \notin A\}$, so $X_1 \subsetneq \mathcal{P}(S)$. By our induction hypothesis, $|X_1| = 2^k$.

Now consider the (nonempty) set $X_2 = \mathcal{P}(S) \setminus X_1$. We have $A \subset S$ and $A \notin X_1$ if and only if $A \subset S$ and $s_{k+1} \in A$, so $X_2 = \{A : A \subset S \text{ and } s_{k+1} \in A\}$. Define the following function $f : X_1 \to X_2$ as follows: for any $A \in X_1$, $f(A) = A \cup \{s_{k+1}\}$:

- f is one-to-one, since if $A \neq B$ and s_{k+1} is not an element of either A or B, then $A \cup \{s_{k+1}\} \neq B \cup \{s_{k+1}\}$.
- f is also onto, since every element of X_2 is of the form $A \cup \{s_{k+1}\}$ for some $A \in X_1$ (including the empty set!).

Hence we have defined a bijection from X_1 to X_2, so $|X_1| = |X_2|$. Then since X_1 and X_2 are finite and partition $\mathcal{P}(S)$, we have $|\mathcal{P}(S)| = |X_1| + |X_2| = 2^k + 2^k = 2^{k+1}$.

(b) **Base step** ($n = 1$): A circle with a single chord is divided into two regions so color one of the regions white and one black.

Inductive hypothesis Suppose the statement holds for any circle with n chords.

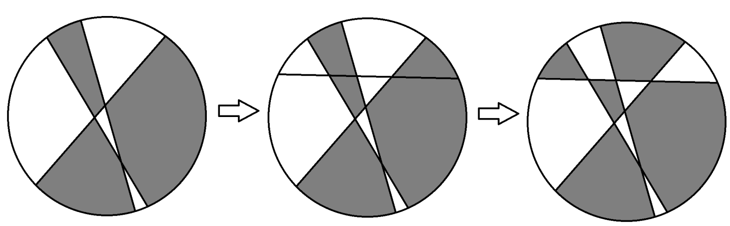
Inductive step: Take a circle with n chords and a coloring scheme that satisfies our hypothesis (from the inductive hypothesis). Let's also supposed the chords divide the circle into K regions which we call A_i.

Now add the $n + 1^{th}$ chord. If the new chord is identical to an existing chord then the current coloring satisfies our hypothesis, so let's suppose the new chord is distinct from the existing n chords. Thus the new chord intersects at least one region, creating two new adjacent regions that share the same coloring and hence the current coloring does not satisfy our hypothesis. Let's suppose the new chord leads to K' regions which we call B_j. Consider coloring the circle in the following manner:

Note the $n+1^{th}$ chord divides the circle into two regions, call them U and L. Now for all the regions in U, flip the coloring: if a region is in U and was originally white, flip it to black and vice versa. Does this new coloring satisfy our hypothesis? Let's take any two adjacent regions B_1 and B_2. There are two cases to consider:

- The common edge of B_1 and B_2 is not the $n + 1^{th}$ chord. This means that (i) we can find original regions A_i, A_j such that $B_1 = A_i$ and $B_2 = A_j$; and (ii) B_1, B_2 are both either in the set U or L. Since the original coloring scheme of the circle had no two adjacent regions with the identical coloring, this will still be the case (even if the coloring flipped).
- The common edge of B_1 and B_2 is the $n+1^{th}$ chord. This means that (i) there is some original region A_i such that B_1, B_2 partition A_i and hence with the original coloring had the same color; and (ii) exactly one of B_1 and B_2 are in U, and the other is in L. Hence exactly one of B_1, B_2 flipped colors, so the two regions are now colored differently.

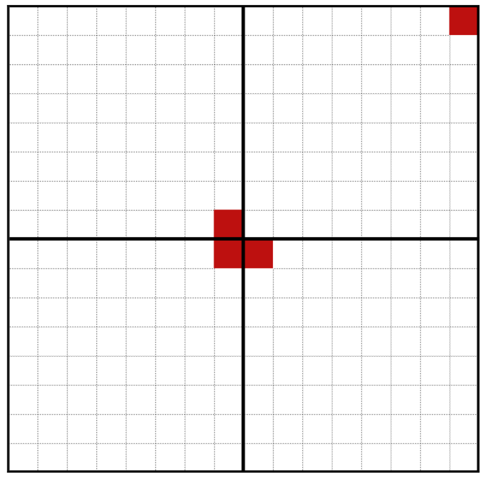
See the figure for a visual example of the coloring scheme described above for $n = 3$ to $n = 4$:



(c) **Base step** ($n = 1$): A single triomino placed on a 2×2 grid works.

Inductive hypothesis ($n = k$): Any $2^k \times 2^k$ grid can be covered by triominoes, except for one corner square.

Inductive step: Note that a $2^{k+1} \times 2^{k+1}$ is really just four different $2^k \times 2^k$ grids. By the inductive hypothesis, cover each of these grids by triominoes (except for a corner square) and organize the blocks such that three of the uncovered corner squares are in the center of the grid, while the fourth is on the corner of the larger $2^k \times 2^k$ grid. Then we can cover the three center squares by one more triomino. See figure:



Problem 3.

A subset B is called a *cofinite* subset of a set A if $A - B$ is finite.[1] In other words, B contains all but a finite number of elements of A. Prove the following: If B and C are cofinite subsets of A, then $B \cap C$ is also a cofinite subset of A.

Solution

We want to show that $A - (B \cap C)$ is finite. But first let's prove that the union of finite sets is finite, even if it seems obvious. Take any two finite sets X and Y. Since they are finite, we can find bijections $f : X \to \{1, \ldots, N_X\}$ and $g : Y \to \{1, \ldots, N_Y\}$. Now define $h : X \cup Y \to \{1, \ldots, N_X + N_Y\}$ as follows

$$h(z) = \begin{cases} f(z) & \text{if } z \in X \\ g(z) + N_X & \text{otherwise} \end{cases}$$

h is an injection (why not necessarily a bijection?), hence $|X \cup Y| \leq |\{1, \ldots, N_X + N_Y\}|$ and therefore $X \cup Y$ is finite.

Now note that:

$$\begin{aligned} A - (B \cap C) &= A \cap (B \cap C)^c \\ &= A \cap (B^c \cup C^c) \\ &= (A \cap B^c) \cup (A \cap C^c) \\ &= (A - B) \cup (A - C) \end{aligned}$$

where the second equality uses one of De Morgan's laws and the third uses distribution of intersection over union. So we have that $A - (B \cap C)$ is the union of two finite sets and therefore is itself finite.

[1] Not $B - A$.

Problem 4.

Let A and B be subsets of any uncountable set X such that their complements are countably infinite. Prove that $A \cap B \neq \varnothing$.

Solution

Remember a set S is countably infinite if there exists a bijection $f : \mathbb{N} \to S$; and if S is either finite or countably infinite, then S is countable. Otherwise we say S is uncountable.

Toward contradiction, assume that $A \cap B = \varnothing$. This is equivalent to saying $A \subseteq B^c$. But since A is a subset of a countably infinite set, A is also countable. To see why: first if A is finite then we're done. If instead A is infinite, then since B^c is countably infinite choose some bijection $f : \mathbb{N} \to B^c$. Then define the following function $g : \mathbb{N} \to A$: $g(1) = f(n_1)$ where n_1 is the smallest natural number such that $f(n_1) \in S$. Then given $n_1, \ldots, n_{k-1}$, let n_k be the smallest natural number greater than n_{k-1} such that $f(n_k) \in S$. Let $g(k) = f(n_k)$. This defines a bijection from $\mathbb{N}$ into A.

Now I'll show that $A \cup A^c$ is also countably infinite. If A is finite this is easy. So instead suppose A is also countably infinite. So we can find bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to A^c$. Consider the following function $h : \mathbb{N} \to A \cup A^c$:

$$h(n) = \begin{cases} f\left(\frac{n}{2}\right) & \text{if } n \text{ even} \\ g\left(\frac{n+1}{2}\right) & \text{if } n \text{ odd} \end{cases}$$

Then h is one-to-one: if $m \neq n$ then $f(m) \neq f(n)$, $g(m) \neq g(n)$ (since f, g are bijections), and $f(m) \neq g(n)$ (since f, g have disjoint codomains), hence $h(m) \neq h(n)$. And h is onto: for any $x \in A \cup A^c$ there exists an n such that either $f(\frac{n}{2}) = x$ or $g(\frac{n+1}{2}) = x$, so $h(n) = x$. Thus $A \cup A^c$ is countable.

But $A \cup A^c = X$ and X is uncountable, a contradiction. Thus $A \cap B \neq \varnothing$.

Remark: There was nothing special about the sets A and B^c, so we have shown that for any countable set T, if $S \subset T$ then T is countable. In fact we can extend this to showing that countable unions of countable sets are countable. Here's a somewhat informal proof:

Consider any collection of countable sets S_i for $i \in \mathbb{N}$. Since S_i is countable we can enumerate $S_i = \{s_{i,j}\}$ for $j \in \mathbb{N}$. Then the set $S = \bigcup_i S_i$ can be written in a table

$$\begin{array}{ccc} s_{1,1} & s_{1,2} & \cdots \\ s_{2,1} & s_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{array}$$

which contains all the elements of S. Then the same diagonal-style enumeration we used to show that $\mathbb{Q}$ was countable shows that S is countable as well.

Problem 5.

Suppose $A \subset \mathbb{R}_+$, $b \in \mathbb{R}_+$, and every finite collection of distinct elements $a_1, \ldots, a_n$ of A we have $\sum_i a_i \leq b$. Prove that A is at most countable. (Hint: consider the sets $A_n = \left\{x \in A | x \geq \frac{1}{n}\right\}$).

Solution

If A is finite there is nothing to show so instead assume A is infinite and define A_n as in the hint. First note that any A_n must be finite and can have at most $n \cdot b$ elements. Why? Suppose A_n had $n \cdot b + 1$ elements $a_1, \ldots, a_{n \cdot b+1}$, then we would have

$$\sum_{i=1}^{n \cdot b+1} a_i \geq \sum_{i=1}^{n \cdot b+1} \frac{1}{n} = (n \cdot b + 1)\frac{1}{n} > b$$

which contradicts our definition of the set A.

Now define $A_0 = \{0\}$ and note for any element $a \in A$, either $a = 0 \implies a \in A_0$ or $a > 0$ and hence by the Archimedean property $\exists n' \in \mathbb{N} : \frac{1}{n'} < a \implies a \in A_{n'}$. Thus

$$A \subseteq \bigcup_{n=0}^{\infty} A_n$$

Then A is a subset of the countable union of finite sets, which is countable (see the remark in the solution for Problem 4 for a sketch of a proof of the more general fact that countable unions of countable sets are countable).

Problem 6.

Let $f : A \to A$. Prove that there is a unique largest subset $X \subset A$ such that $f(X) = X$ (here if $Y \subset X$ we call X "larger" than Y).

Solution

Let's first show that the image of a function preserves unions, ie show that for any function $g : X \to Y$ and collection of sets $A_i \subset X$ indexed by I, $g\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} g(A_i)$:

$$\begin{aligned} y \in g\left(\bigcup_{i\in I} A_i\right) &\iff \exists x \in \bigcup_{i\in I} A_i : g(x) = y \\ &\iff \exists i \in I : \exists x \in A_i : g(x) = y \\ &\iff \exists i \in I : y \in g(A_i) \\ &\iff y \in \bigcup_{i\in I} g(A_i) \end{aligned}$$

Then take the collection of any set which f maps into itself (note that the image of the empty set $f(\varnothing) = \varnothing$ for any function, so there is at least one set in this collection) indexed by I and define the union of this collection:

$$\mathcal{B} = \{B_i \subset A : f(B_i) = B_i\}, B = \bigcup_{i\in I} B_i$$

Then

$$\begin{aligned} f(B) &= f\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f(B_i) \\ &= \bigcup_{i\in I} B_i = B \end{aligned}$$

So f maps B into itself, and further $B_i \subset B$ for any B_i such that $f(B_i) = B_i$.

Problem 7.

Define the following distance function on the set of real numbers:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Prove that $(\mathbb{R}, d)$ is a metric space.

(b) Identify the open (and closed) balls in the topology induced by this metric.

Solution

(a) To verify that d is a metric, you need to check that for all $x, y, z \in \mathbb{R}$, (i) $d(x, x) = 0$, (ii) $d(x, y) = d(y, x)$, and (iii) $d(x, y) + d(y, z) \geq d(x, z)$. Requirements (i) and (ii) are easily verified. To verify (iii) there are two cases to consider: $x = z$ or $x \neq z$.

Case I: If $x \neq z$, then either $x \neq y$ or $y \neq z$ (why? $x = y$ and $y = z$ implies $x = z$, so take the contrapositive). Thus either $d(x, y) = 1$ or $d(y, z) = 1$ and we have $d(x, y) + d(y, z) \geq 1 = d(x, z)$.

Case II: If $x = z$ then $d(x, z) = 0 \leq d(x, y) + d(y, z)$.

So (iii) holds and d is a metric.

(b) Given any point x consider the ball centered at x with radius ε. When $\varepsilon \leq 1$ we have $B_\varepsilon(x) = \{y \in \mathbb{R} : d(x, y) < \varepsilon\} = \{x\}$ since $d(x, y) < \varepsilon \leq 1 \implies d(x, y) = 0 \implies x = y$. Similarly, when $\varepsilon > 1$ we obtain $B_\varepsilon(x) = \{y \in \mathbb{R} : d(x, y) < \varepsilon\} = \mathbb{R}$ since $\forall y \in \mathbb{R}$ where $y \neq x$, then $d(x, y) = 1 < \varepsilon$. Therefore, the open balls in this space either look like singleton points $\{x\}$ or the entire space. For $\varepsilon < 1$, the closed ball coincides with the open ball of radius ε. For $\varepsilon \geq 1$, the closed ball is the entire space.

Remark: Note that the analysis above shows that points are *open* in this space. Since arbitrary unions of open sets are open it follows that every subset of this space is open. On the other hand, since a closed set is defined to be the complement of an open set, every subset of this space is also closed. A space with such a topology (where every point is both open and closed) is said to be *discrete*.