1. Announcement: Last week of class devoted to paper presentations.

2. Interpretation of the factor model for the SDF $m = 1 - b(f - \mu)$. In economics, f should be consumption growth. Why? Suppose $u(c) = \ln(c)$. Then $m_{t+1} = \left(\frac{1}{1+\rho}\right) u'(c_{t+1})/u'(c_t) = \left(\frac{1}{1+\rho}\right)(c_t/c_{t+1})$, is beta times 1 divided by gross rate of consumption growth. Here, the subjective discount factor is $1/(1+\rho)$, where ρ is the subjective discount rate. Now the 'beta' in the beta-risk formulation would be the slope in this regression

$$r_{t,i}^e = \alpha_i + \beta_i \left(\frac{c_t}{c_{t-1}}\right) + \epsilon_{t,i}$$

   Consumption growth is the common factor. It has been identified in the theory. This is known as the 'consumption beta' model.

   Now what's this stuff about the CAPM, and Fama-French's 3 factor model and such? These models have asset excess returns as factors. Finance guys say consumption is poorly measured–infrequently observed in official data, and we can't see everything people consume. Their solution is to proxy the stochastic discount factor with the market's excess return. This is the CAPM

$$m_t = 1 - b\left(r_{t,m}^e - \mu_m\right)$$

   So using portfolio returns as factors partially solves the measurement problem.

3. Risk in the beta-risk framework is covariance of return with something that you care about. β_i is exposure to that risk. Positive correlation with something that is dear to you makes the security risky.

4. Beta-risk model says, for asset returns $i = 1, ..., n$

$$\bar{r}_i^e = \lambda \beta_i$$

   In word, the average excess return on security i, is proportional to it's risk exposure, β_i. Suppose I impose an additional unit of risk on this asset (increase β_i),

$$\frac{\partial \bar{r}_i^e}{\partial \beta_i} = \lambda$$

   λ is known as the 'price' of risk.

5. What about multi-factor models?

$$m_t = 1 - b_1(f_{t,1} - \mu_1) - b_2(f_{t,2} - \mu_2)$$

Posit that stochastic discount factor has a two-factor representation. Revisit the Euler equation

$$
\begin{aligned}
0 &= E\left(m_t r_{t,i}^e\right) \\
&= E\left(1 - b_1\left(f_{t,1} - \mu_1\right) - b_2\left(f_{t,2} - \mu_2\right)\right) r_{t,i}^e \\
&= E\left(r_{t,i}^e\right) - b_1 Cov\left(f_{t,1}, r_{t,1}^e\right) - b_2 Cov\left(f_{t,2}, r_{t,i}^e\right) \\
E\left(r_{t,i}^e\right) &= \underbrace{b_1 Var\left(f_{t,1}\right)}_{\lambda_1} \underbrace{\frac{Cov\left(f_{t,1}, r_{t,i}^e\right)}{Var\left(f_{t,1}\right)}}_{\beta_{i,1}} + \underbrace{b_2 Var\left(f_{t,2}\right)}_{\lambda_2} \underbrace{\frac{Cov\left(f_{t,2} r_{t,i}^e\right)}{Var\left(f_{t,2}\right)}}_{\beta_{i,2}} \\
\bar{r}_i^e &= \lambda_1 \beta_{i,1} + \lambda_2 \beta_{i,2}
\end{aligned}
$$

This is example of a two-factor model (two-beta model).

6. There are two ways to estimate these models. The time-series method, and the Fama-MacBeth method. The time-series method works if the factor or factors are asset returns. Illustrate with the one-factor model (the CAPM).

$$f_t = r_{t,m}^e$$

$$
\begin{aligned}
r_{t,i}^e &= \alpha_i + \beta_i r_{t,m}^e + \epsilon_{t,i} \\
E\left(r_{t,i}^e\right) &= \lambda \beta_i
\end{aligned}
$$

Take the expectation of the return equation. $E\left(r_{t,i}^e\right) = \alpha_i + \beta_i E\left(r_{t,m}^e\right)$. Hence, the theory implies $\alpha_i = 0$, and $\lambda = E\left(r_{t,m}^e\right)$. As a first step, let's estimate the price of risk, λ, by regressing $r_{t,m}^e$ on a constant and asking for Newey-West t-ratios.

$$r_{t,m}^e = \lambda + u_{t,m}$$

Next, estimate the α_i and β_i. This involves running $i = 1, .., n$ time-series regressions. Save the α_i and β_i. Test the hypothesis that all the $\alpha_i = 0$. Option 1. Do individual t-tests on all the $\alpha's$. But what if you want to do a joint test? Option 2 is to cheat and assume the estimate α_i are independent. In this case, each $\alpha_i's$ t-ratio has the standard normal distribution. Let's call it t_i.If these are independent standard normals, then

$$t_1^2 + t_2^2 + ... + t_n^2 \sim \chi_n^2$$

Option 3: Actually account for the dependence across the α_i. Estimate the return regressions (the α_i and β_i) jointly in a system. (consult slide on A correct test...)