Econ 204 – Problem Set 5

Due Friday, August 14

1. Let be X the set of at most second degree polynomials and $T : X \to X$ a linear transformation defined by $T(f(x)) = 2f(x) + xf'(x)$. Compute the matrix representation of T with respect to the basis $V = \{1, x, x^2\}$, compute $\ker T$, characterize $X/\ker T$, compute the eigenvalues and the corresponding eigenvectors of T. Is T diagonalizable?

2. For the following functions, determine at what points the derivative exists, and if the derivative function is continuous (you may use that the derivative of $\sin x$ is $\cos x$):

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad g(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, the image of the derivative function, is an interval (possibly a singleton).

4. If $a_0 + \frac{1}{2}a_1 + \cdots + \frac{1}{n}a_{n-1} + \frac{1}{n+1}a_n = 0$, where $a_0, \ldots, a_n$ are real constants, prove that the equation

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = 0$$

has at least one real root between 0 and 1.

5. Compute the second-order Taylor expansion of $f(x) = \sin^2 x + \cos x \sin x$ around the point $x_0 = \frac{\pi}{2}$