Econ 204 – Problem Set 3

Due August 7 [1]

1. Let (X, d) be a metric space:

   (a) Let $y \in X$ be given. Define the function $d_y : X \to \mathbb{R}$ by

$$d_y(x) = d(x, y) \tag{1}$$

   Show that d_y is a continuous function on X for each $y \in X$.

   Pick the sequence of elements $\{x_n\} \subset X$ such that $x_n \to x$. To verify that d_y is continuous we only need to show $d_y(x_n) \to d_y(x)$, that is $d(x_n, y) \to d(x, y)$. Because of triangle inequality:

$$\begin{aligned} d(x_n, y) \leq d(x_n, x) + d(x, y) &\Rightarrow d(x_n, y) - d(x, y) \leq d(x_n, x) \\ d(x, y) \leq d(x, x_n) + d(x_n, y) &\Rightarrow -\big(d(x_n, y) - d(x, y)\big) \leq d(x_n, x) \end{aligned} \tag{2}$$

   That in turn implies the *reverse triangle inequality*:

$$\big|d(x_n, y) - d(x, y)\big| \leq d(x_n, x) \tag{3}$$

   This verifies that $\lim_{n\to\infty} \big|d(x_n, y) - d(x, y)\big| = 0$, because $d(x_n, x) \to 0$. Hence $d(x_n, y) \to d(x, y)$.

   (b) Let A be a subset of X and $x \in X$. Recall that the distance from the point x to the set A is defined as:

$$\rho(x, A) = \inf \big\{d(x, a) : a \in A\big\} \tag{4}$$

   Show that the closure of set A is the set of all points with zero distance to A, that is:

$$\bar{A} = \big\{x \in X : \rho(x, A) = 0\big\} \tag{5}$$

   Let's denote $A' := \big\{x \in X : \rho(x, A) = 0\big\}$. Pick $x \in A'$, then $\rho(x, A) = 0$. Because of the infimum property for every $n \in \mathbb{N}$ there exists $a_n \in A$ such that $d(x, a_n) \leq 1/n$, that in turn implies the sequence $\{a_n\} \subset A$ converges to x, hence x is in the closure of set A, concluding that $A' \subset \bar{A}$.

   Conversely, take $x \in \bar{A}$, which is to say there exists a sequence $\{b_n\} \subset A$ such that $b_n \to x$. Using the continuity of function d_x proved in part (a) we deduce that $d_x(b_n) \to d_x(x) = 0$. This implies $x \in A'$ (why?) and finishes the reverse direction, i.e $\bar{A} \subset A'$.

   (c) Now let $A \subset X$ be a compact subset. Show that $\rho(x, A) = d(x, a)$ for some $a \in A$.

   We have seen that the function d_x is continuous on X, so is on A (why?). One can represent $\rho(x, A)$ as

$$\rho(x, A) = \inf\{d_x(a) : a \in A\}. \tag{6}$$

[1] In case of any problems with the exercises please email farzad@berkeley.edu

It has been proved in the lecture notes that the continuous functions defined on the compact subsets assumes their extremums. Namely, there has to be some $a \in A$ such that $\rho(x, A) = d_x(a)$.

2. Let D be the space of all functions $f : [0,1] \to \mathbb{R}$ such that f is continuous and such that for some $\varepsilon > 0$, $f : (-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ is differentiable and $f' : (-\varepsilon, 1+\varepsilon) \to \mathbb{R}$ is continuous. For each $f \in D$, let

$$\|f\|_\infty = \sup\left\{|f(t)| : t \in [0,1]\right\} \quad \text{and} \quad \|f'\|_\infty = \sup\left\{|f'(t)| : t \in [0,1]\right\}. \tag{7}$$

Define the function $\|\cdot\| : D \to \mathbb{R}_+$ by

$$\|f\| := \|f\|_\infty + \|f'\|_\infty \tag{8}$$

(a) Show that $(D, \|\cdot\|)$ is a normed vector space.

First, one needs to show that D is a vector space, that requires checking all the properties of vector space definition. That means we need to define vector space operations '+' and '.' and check for any $f, g \in D$, $\alpha \in \mathbb{R}$, $f + g \in D$ and $\alpha f \in D$. Further, we need to define the 0 element which in this case is the constant zero function, i.e $f(t) = 0$ for each $t \in [0,1]$.

Then, we need to verify that $\|\cdot\|$ induces a norm function on D. For this note that for every $f \in D$, both $\|f\|_\infty$ and $\|f'\|_\infty$ are finite (because f and f' are assumed to be continuous functions on the compact interval $[0,1]$, therefore they are bounded). Hence, the function $\|\cdot\| : D \to R_+$ is well-defined.

Now, we need to show why $\|\cdot\|_\infty$ is indeed a norm on the space $C[0,1]$, and will then use this to show the suggested $\|\cdot\|$ function induces a norm on D. Remember that we defined $\|\cdot\|_\infty$ as

$$\|f\|_\infty = \sup\left\{|f(t)| : t \in [0,1]\right\}. \tag{9}$$

Therefore, $\|f\|_\infty = 0$ *iff* $f = 0$, implying the first property of the norm. Secondly, for every $\alpha \in \mathbb{R}$

$$\begin{aligned}
\|\alpha f\|_\infty &= \sup\left\{|\alpha f(t)| : t \in [0,1]\right\} \\
&= \sup\left\{|\alpha|\,|f(t)| : t \in [0,1]\right\} \\
&= |\alpha| \sup\left\{|f(t)| : t \in [0,1]\right\} = |\alpha| \|f\|_\infty,
\end{aligned} \tag{10}$$

that verifies the second property of the norm function. Lastly, for every $f, g \in C[0,1]$ and $t \in [0,1]$:

$$\begin{aligned}
|f(t) + g(t)| &\le |f(t)| + |g(t)| \\
&\le \|f\|_\infty + \|g\|_\infty
\end{aligned} \tag{11}$$

Therefore, the set $\Big\{ \big| f(t) + g(t) \big| : t \in [0,1] \Big\}$ is upper bounded by $\|f\|_\infty + \|g\|_\infty$, hence because of the property of supremum

$$\sup \Big\{ \big| f(t) + g(t) \big| : t \in [0,1] \Big\} = \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \,, \tag{12}$$

that concludes the third property of the norm function, i.e triangle inequality. Now we know that $\|\cdot\|_\infty$ is a norm on $C[0,1]$. It is just left to verify that the prescribed function $\|\cdot\| : D \to \mathbb{R}_+$ defines a norm on D. Clearly, $\|f\| = 0$ implies $f = 0$ and vice versa. Also for every $\alpha \in \mathbb{R}_+$

$$\|\alpha f\| = \|\alpha f\|_\infty + \left\|\alpha f'\right\|_\infty = |\alpha| \left(\|f\|_\infty + \|g\|_\infty\right) = |\alpha| \|f\| \,, \tag{13}$$

where in the first identity we used the definition of $\|\cdot\|$, and in the second identity we used the norm property of $\|\cdot\|_\infty$ (which we proved in last paragraph). Finally, for every $f, g \in D$, $f + g \in D$ (because of the vector space property of D), and

$$\begin{aligned} \|f + g\| &= \|f + g\|_\infty + \left\|f' + g'\right\|_\infty \\ &\leq \|f\|_\infty + \|g\|_\infty + \left\|f'\right\|_\infty + \left\|g'\right\|_\infty \\ &= \|f\| + \|g\| \,, \end{aligned} \tag{14}$$

where in the inequality part we used the norm property of $\|\cdot\|_\infty$ and in last identity we used the definition of $\|\cdot\|$. This concludes the verification of norm properties for $\|\cdot\|$, and proves that $(D, \|\cdot\|)$ is a normed vector space.

(b) Define the function $J : D \to \mathbb{R}$ as

$$J(f) = \int_0^1 e^{-x} \frac{f(x)}{1 + f'(x)^2} \mathrm{d}x. \tag{15}$$

Prove that J is continuous.

Let f and g be two arbitrary functions in D. Then

$$\begin{aligned} \big|J(f) - J(g)\big| &= \left| \int_0^1 e^{-x} \left(\frac{f(x)}{1 + f'(x)^2} - \frac{g(x)}{1 + g'(x)^2} \right) \mathrm{d}x \right| \\ &\leq \int_0^1 e^{-x} \left| \frac{f(x)(1 + g'(x)^2) - g(x)(1 + f'(x)^2)}{(1 + f'(x)^2)(1 + g'(x)^2)} \right| \mathrm{d}x \\ &\leq \int_0^1 e^{-x} \left| f(x)(1 + g'(x)^2) - g(x)(1 + f'(x)^2) \right| \mathrm{d}x \\ &\leq \int_0^1 e^{-x} \left| f(x) - g(x) \right| \mathrm{d}x + \int_0^1 e^{-x} \left| f(x) g'(x)^2 - g(x) f'(x)^2 \right| \mathrm{d}x \end{aligned} \tag{16}$$

The first integral above is less than or equal to $\|f - g\|_\infty$. Let's denote the second integral by $\mathcal{I}_2$. We can apply the triangle inequality and obtain

$$\begin{aligned}
\mathcal{I}_2 \leq & \int_0^1 e^{-x} \left| f(x)g'(x)^2 - f(x)f'(x)^2 \right| \mathrm{d}x + \int_0^1 e^{-x} \left| f(x)f'(x)^2 - g(x)f'(x)^2 \right| \mathrm{d}x \\
\leq & \|f\|_\infty \int_0^1 e^{-x} \left| g'(x) - f'(x) \right| \left| g'(x) + f'(x) \right| \mathrm{d}x + \left\| f' \right\|_\infty^2 \int_0^1 e^{-x} \left| f(x) - g(x) \right| \mathrm{d}x \\
\leq & \|f\|_\infty \|g' - f'\|_\infty \|g' + f'\|_\infty + \left\| f' \right\|_\infty^2 \|f - g\|_\infty \\
\leq & \|f\|_\infty \|g' - f'\|_\infty \left(\|g' - f'\|_\infty + 2\|f'\|_\infty \right) + \left\| f' \right\|_\infty^2 \|f - g\|_\infty \\
\leq & \left[\|f\|_\infty \left(\|g - f\| + 2\|f'\|_\infty \right) + \left\| f' \right\|_\infty^2 \right] \|g - f\|
\end{aligned} \tag{17}$$

For every sequence $\{g_n\} \subset D$ such that $g_n \to f$, that is $\|g_n - f\| \to 0$, one can deduce from the above analysis that

$$\left| J(g_n) - J(f) \right| \leq \left[1 + \|f\|_\infty \left(\|g_n - f\| + 2\|f'\|_\infty \right) + \left\| f' \right\|_\infty^2 \right] \|g_n - f\| , \tag{18}$$

therefore $\left| J(g_n) - J(f) \right| \to 0$ as $n \to \infty$, because the term in the bracket is finite (as $f \in D$) and $\|g_n - f\| \to 0$. This concludes the proof of the continuity of J in the metric space D.

3. Let X be a connected space. Let $\mathcal{R}$ be an equivalence relation on X such that for each $x \in X$, there exists an open set $\mathcal{O}_x$ containing x such that $\mathcal{O}_x \subset [x]$. Show that $\mathcal{R}$ has only one (distinct) equivalence class.

   For the fixed point $x \in X$, and some arbitrary element $y \in [x]$, there exists an open set like $\mathcal{O}_y$ containing y that is located in $[x]$ ($\mathcal{O}_y \subset [x] = [y]$). This implies that $[x]$ is an open set for every $x \in X$. Therefore the union of all equivalence classes but one is also open, hence a single equivalence class is both open and closed. Since X is connected, this could only happen if there is only one equivalence class (recall from the section notes, that a metric space is disconnected if it contains a proper subset which is both open and closed).

4. Define the correspondence $\Gamma : [0,1] \to 2^{[0,1]}$ by:

$$\Gamma(x) = \begin{cases} [0,1] \cap \mathbb{Q} & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ [0,1] \setminus \mathbb{Q} & \text{if } x \in [0,1] \cap \mathbb{Q} \end{cases}. \tag{19}$$

   Show that Γ is not continuous, but it is lower-hemicontinuous. Is Γ upper-hemicontinuous at any rational? At any irrational? Does this correspondence have a closed graph?

   Consider the open set $V = (0,1)$ which contains $\Gamma(q) = [0,1] \setminus \mathbb{Q}$ for every $q \in [0,1] \cap \mathbb{Q}$. Then any open set containing q will also contain an irrational number $x \in [0,1] \setminus \mathbb{Q}$, and $\Gamma(x) = [0,1] \cap \mathbb{Q} \not\subset V$. Hence Γ is not upper-hemicontinuous at any rational number.

   Now fix some $y \in [0,1] \setminus \mathbb{Q}$ and consider the open set $V = [0,y) \cup (y,1]$ in $[0,1]$.

For any $x \in [0,1] \setminus \mathbb{Q}$ we have $\Gamma(x) \subset V$, but every open set containing x will also contain a rational number $q \in [0,1] \cap \mathbb{Q}$ and $\Gamma(q) = [0,1] \setminus \mathbb{Q} \not\subset V$. Thus Γ is nowhere upper-hemicontinuous and hence nowhere continuous.

Next, let V be any open set satisfying $V \cap [0,1] \neq \varnothing$. Then we have $V \cap ([0,1] \cap \mathbb{Q}) \neq \varnothing$ and $V \cap ([0,1] \setminus \mathbb{Q}) \neq \varnothing$, since every ε-ball in the reals contains both rational and irrational numbers. But then $\Gamma(x) \cap V \neq \varnothing$ for every x in the domain of Γ. This proves that Γ is lower-hemicontinuous.

The correspondence does not have a closed graph. Remember that $\mathrm{gr}(\Gamma)$ is a subset of $[0,1] \times [0,1]$. Fix some $y \in [0,1] \setminus \mathbb{Q}$ and take any sequence $\{q_n\} \subset [0,1] \cap \mathbb{Q}$ such that $q_n \to y$. Then the sequence $(q_n, y) \in \mathrm{gr}(\Gamma)$ but $(y,y) \notin \mathrm{gr}(\Gamma)$. Hence the graph is not closed.

5. Let X be a metric space, and $I : X \to \mathbb{R}_+$ be a lower semi-continuous function [2].

   (a) Prove that for every given $\varepsilon > 0$ there exists an open set U_ε containing $x \in X$ such that
$$\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon. \tag{20}$$

   Because of lower semi-continuity the set $U_\varepsilon := \{y : I(y) > I(x) - \varepsilon\}$ is open in X, that trivially contains x, and hence is non-empty. Now, it should be clear that
$$\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon. \tag{21}$$

   (b) Let $x \in X$. For each $n \in \mathbb{N}$ let
$$m_n = \inf\left\{I(y) : y \in B_{1/n}(x)\right\}. \tag{22}$$

   Show that $\{m_n\}$ is an increasing sequence and that $m_n \to I(x)$.

   The sequence $\{m_n : n \in \mathbb{N}\}$ is clearly weakly increasing, namely $m_n \leq m_{n+1} \leq m_{n+2} \leq \dots$. Further, for every n, we have $m_n \leq I(x)$. Now, for any given ε find $U_\varepsilon \ni x$ as in part (a) such that $\inf\{I(y) : y \in U_\varepsilon\} \geq I(x) - \varepsilon$. The set U_ε is an open set around x, hence there exists some $N \in \mathbb{N}$ such that for all $n \geq N$: $B_{1/n}(x) \subset U_\varepsilon$. This in turn implies that for all $n \geq N$, we have $m_n \geq I(x) - \varepsilon$. This concludes the proof of $m_n \to I(x)$.

6. Let x and y be moving objects in $\mathbb{R}$. Time is discrete, namely $t \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. In addition, $\beta > 1$ is a fixed parameter. For $a, b \in \mathbb{R}$, let $\rho(a,b) := |a - b| \wedge 1$ (as mentioned in the section, the symbol $\wedge$ is sometimes used to refer to the minimum of

[2] A function $I : X \to \mathbb{R}$ is called lower semi-continuous *iff* for every α the set $\{x : I(x) > \alpha\}$ is open in X.

two elements). Then for any $x, y \in \mathbb{R}^\omega$ [3], let

$$d(x,y) = \sum_{t\in\mathbb{Z}_+} \beta^{-t}\rho(x_t, y_t) \tag{23}$$

denotes the distance between $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$, where x_t is the position of x at time t on the real line.

(a) Show that d is a metric on $\mathbb{R}^\omega$.

The first two metric properties are easy to validate, and we only verify the triangle inequality. We have seen in the section that ρ is a bounded metric on $\mathbb{R}$, hence $\rho(x_t, z_t) \le \rho(x_t, y_t) + \rho(y_t, z_t)$. Multiply both sides by β^{-t} and sum over t; since the right hand side is bounded by $\sum_{t\in\mathbb{Z}_+} \beta^{-t}$, this sum is well-defined. Therefore

$$\sum_{t\in\mathbb{Z}_+} \beta^{-t}\rho(x_t, z_t) \le \sum_{t\in\mathbb{Z}_+} \beta^{-t}\rho(x_t, y_t) + \sum_{t\in\mathbb{Z}_+} \beta^{-t}\rho(y_t, z_t) \tag{24}$$

holds, and the triangle inequality falls out.

(b) Show that $(\mathbb{R}^\omega, d)$ is a bounded metric space.

It is a bounded metric because one can pick $x = (0, 0, \ldots)$. Then for every $y \in \mathbb{R}^\omega$

$$\rho(x,y) = \sum_{t\in\mathbb{Z}_+} \beta^{-t}\rho(0, y_t) \le \sum_{t\in\mathbb{Z}_+} \beta^{-t} = \frac{\beta}{\beta - 1} < \infty \tag{25}$$

(c) Is $[0,1]^\omega$ an open or closed subset of $\mathbb{R}^\omega$? (in either case present a proof)

It is a closed subset of $\mathbb{R}^\omega$. To see this, take the sequence $\left\{x^{(n)}\right\} \subset [0,1]^\omega$ and suppose $d\left(x^{(n)}, x\right) \to 0$ for some $x \in \mathbb{R}^\omega$. Since $\beta^{-t}\rho\left(x_t^{(n)}, x_t\right) \le d\left(x^{(n)}, x\right)$, then $\left|x_t^{(n)} - x_t\right| \to 0$ for all $t \in \mathbb{Z}_+$. Since $[0,1]$ is closed and $\left\{x_t^{(n)}\right\} \subset [0,1]$ then it has to be the case that $x_t \in [0,1]$ too. This in turn implies that $x \in [0,1]^\omega$, thereby $[0,1]^\omega$ is a closed subset.

(d) Is $(\mathbb{R}^\omega, d)$ a complete metric space? (prove if yes, otherwise provide a counterexample)

Let's take the Cauchy sequence $\left\{x^{(n)}\right\} \subset [0,1]^\omega$. By definition of Cauchy sequence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \ge N$: $d\left(x^{(n)}, x^{(m)}\right) < \varepsilon$. This in particular implies that $\beta^{-t}\rho\left(x_t^{(n)}, x_t^{(m)}\right) < \varepsilon$, and hence for every $t \in \mathbb{Z}_+$, $\left\{x_t^{(n)}\right\}$ is a Cauchy sequence in $\mathbb{R}$, thereby convergent to a point in $\mathbb{R}$ say x_t (remember that $\mathbb{R}$ is complete). This constructs a tuple $x \in \mathbb{R}^\omega$. It is only left to show $x^{(n)} \to x$ under the metric d. For given $\varepsilon > 0$, pick $T \in \mathbb{Z}_+$ large enough such that $\sum_{t>T} \beta^{-t} < \varepsilon/2$; this is possible because the

[3] For the definition of $\mathbb{R}^\omega$ please refer to Q2 of ps.1

infinite sum is bounded. Then

$$\begin{aligned} d\left(x^{(n)}, x\right) &= \sum_{t \le T} \beta^{-t} \rho\left(x_t^{(n)}, x_t\right) + \sum_{t > T} \beta^{-t} \rho\left(x_t^{(n)}, x_t\right) \\ &< \sum_{t \le T} \beta^{-t} \rho\left(x_t^{(n)}, x_t\right) + \varepsilon/2 \end{aligned} \tag{26}$$

The first term on the RHS of the last inequality is a sum over *finitely* many numbers which are all converging to 0. Therefore, for large enough n this sum can be made smaller than $\varepsilon/2$, and hence $d\left(x^{(n)}, x\right) < \varepsilon$; concluding the proof of $x^{(n)} \to x$. In part (c), we have seen that $[0,1]^\omega$ is a closed subset, therefore x as the limit point of $\left\{x^{(n)}\right\}$ is in $[0,1]^\omega$ as well. This verifies that $[0,1]^\omega$ is a complete subset.

We could have also proved that $\mathbb{R}^\omega$ is a complete metric space, and then conclude that $[0,1]^\omega$ must be complete because it is a closed subset of $\mathbb{R}^\omega$.

(e) Is $[0,1]^\omega$ a totally bounded subset under d? Is it a compact subset?

It is a totally bounded subset. Again for a given $\varepsilon > 0$ pick T large enough such that $\sum_{t>T} \beta^{-t} < \varepsilon/2$. Since $[0,1] \subset \mathbb{R}$ is totally bounded, for each $t \in \{0, 1, \ldots, T\}$ there exists a collection of points $\mathscr{A}_t^\varepsilon := \left\{a_1^t, a_2^t, \ldots, a_{p_t}^t\right\}$ such that:

$$[0,1] \subset \bigcup_{j=1}^{p_t} \left(a_j^t - \varepsilon\beta^t/2T, a_j^t + \varepsilon\beta^t/2T\right) \tag{27}$$

Now consider the following set of points:

$$\mathscr{A}^\varepsilon := \left\{(a_1, a_2, \ldots) \in [0,1]^\omega : a_t \in \mathscr{A}_t^\varepsilon \text{ for } t \le T \text{ and } a_t = 0 \text{ for } t > T\right\} \tag{28}$$

This is a finite set (why?) and can be considered as the centers of open ε-balls that further on will cover the $[0,1]^\omega$. Now for any point $x \in [0,1]^\omega$ one can find a point $a \in \mathscr{A}^\varepsilon$ such that:

$$\begin{aligned} d(x, a) &= \sum_{t \le T} \beta^{-t} \overbrace{\rho(x_t, a_t)}^{\le \varepsilon\beta^t/2T} + \overbrace{\sum_{t > T} \beta^{-t} \rho(x_t, a_t)}^{<\varepsilon/2} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned} \tag{29}$$

Therefore, $\left\{B_\varepsilon(a) : a \in \mathscr{A}^\varepsilon\right\}$ is a collection of finitely many open sets that covers $[0,1]^\omega$. Hence, $[0,1]^\omega$ is totally bounded. This lets us to invoke theorem 9 of lecture note 6, and deduce the compactness of set $[0,1]^\omega$ (since this subset is complete and totally bounded).