

Problem sets are due at 5PM in the GSIs mailbox. You may work in groups, but each student should turn in their own write-up (including a printout of a narrated/commented and executed Jupyter Notebook if applicable). Please also e-mail a copy of any Jupyter Notebook to the GSI (if applicable).

1 Multivariate normal distribution

Let $\mathbf{Y} = (Y_1, \dots, Y_K)'$ be a $K \times 1$ random vector with density function

$$f(y_1, \dots, y_K) = (2\pi)^{-K/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2} (\mathbf{y} - \mu)' \Sigma^{-1} (\mathbf{y} - \mu)\right),$$

for Σ a symmetric positive definite $K \times K$ matrix and μ a $K \times 1$ vector. We say that \mathbf{Y} is a multivariate normal random variable with mean μ and covariance Σ or

$$\mathbf{Y} \sim \mathcal{N}(\mu, \Sigma).$$

The multivariate normal distribution arises frequently in econometrics and a mastery of its basic properties is essential for both applied and theoretical work in econometrics. This problem provides an opportunity for you to review and/or learn some of these properties.

Prove the following properties of the multivariate normal distribution.

1. For

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

with $\Sigma_{12} = \Sigma'_{21}$ show that

$$\mathbf{Y}_1 \sim \mathcal{N}(\mu_1, \Sigma_{11}).$$

2. Likewise show that

$$\mathbf{Y}_2 | \mathbf{Y}_1 = \mathbf{y}_1 \sim \mathcal{N}(\mu_2 + \Sigma_{21} \Sigma_{11}^{-1} (\mathbf{y}_1 - \mu_1), \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}).$$

3. If $\Sigma_{12} = 0$, show that \mathbf{Y}_1 and \mathbf{Y}_2 are independent.

4. Let $\mathbf{Z} = \mathbf{A} + \mathbf{B}\mathbf{Y}$ for \mathbf{A} and \mathbf{B} non-random and \mathbf{B} with full row rank, show that

$$\mathbf{Z} \sim N(\mathbf{A} + \mathbf{B}\mu, \mathbf{B}\Sigma\mathbf{B}').$$

5. Set $\mathbf{A} = 0$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Show that for $\mathbf{Z} = \mathbf{A} + \mathbf{B}\mathbf{Y} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ that

- (a) Z_1 and Z_2 are univariate normal random variables;

- (b) their joint distribution is *not* bivariate normal;
- (c) Explain.
6. Let $\{\mathbf{Y}_i\}_{i=1}^N$ be a random sample of size N drawn from the multivariate normal population described above. Show that $\sqrt{N}(\bar{\mathbf{Y}} - \mu)$ is a $\mathcal{N}(0, \Sigma)$ random variable for $\bar{\mathbf{Y}} = \frac{1}{N} \sum_{i=1}^N \mathbf{Y}_i$, the sample mean (HINT: Use independence of the $i = 1, \dots, N$ draws and your result in Problem 4 above).
7. Let $W = (\mathbf{Y} - \mu)' \Sigma^{-1} (\mathbf{Y} - \mu)$. Show that $W \sim \chi_K^2$.
8. Let $\mathbf{W} = N \cdot (\bar{\mathbf{Y}} - \mu)' \Sigma^{-1} (\bar{\mathbf{Y}} - \mu)$. Show that $\mathbf{W} \sim \chi_K^2$ (i.e., \mathbf{W} is a chi-square random variable with K degrees of freedom).
9. Let $\chi_K^{2, 1-\alpha}$ be the $(1 - \alpha)^{th}$ quantile of the χ_K^2 distribution (i.e., the number satisfying the equality $\Pr(\mathbf{W} \leq \chi_K^{2, 1-\alpha}) = 1 - \alpha$ with \mathbf{W} a chi-square random variable with K degrees of freedom). Let D be a $P \times K$ ($P \leq K$) matrix of rank P and d a $P \times 1$ vector of constants. Consider the hypothesis

$$\begin{aligned} H_0 : D\mu &= d \\ H_1 : D\mu &\neq d. \end{aligned}$$

Maintaining H_0 derive the sampling distribution of $D\bar{\mathbf{Y}}$ as well as that of

$$\mathbf{W} = N \cdot (D\bar{\mathbf{Y}} - d)' (D\Sigma D)^{-1} (D\bar{\mathbf{Y}} - d).$$

You observe that, for the sample in hand, $\mathbf{W} > \chi_P^{2, 1-\alpha}$ for $\alpha = 0.05$. Assuming H_0 is true, what is the ex ante (i.e., pre-sample) probability of this event? What are you inclined to conclude after observing \mathbf{W} in the sample in hand?

2 Exercises

1. [Adapted from exercise 18.1 from Goldberger (1991)]. Suppose that $\mathbf{Y} \sim \mathcal{N}(\mu, \Sigma)$ with

$$\mu = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

- (a) Calculate $\mathbb{E}[Y_3 | Y_1 = y_1, Y_2 = y_2]$ and $\mathbb{V}(Y_3 | Y_1 = y_1, Y_2 = y_2)$
- (b) Calculate $\mathbb{E}[Y_3 | Y_1 = y_1]$ and $\mathbb{V}(Y_3 | Y_1 = y_1)$
- (c) Calculate $\Pr(-1 \leq Y_3 \leq 2)$
2. Let Y_1 and Y_0 respectively denote child and parent height. Assume that

$$Y_t \sim \mathcal{N}(\mu, \sigma^2)$$

for $t = 0, 1$ (so that the distribution of height is the *same* across the two generations). Let $\rho = \mathbb{C}(Y_1, Y_0) / \sqrt{\mathbb{V}(Y_1)} \sqrt{\mathbb{V}(Y_0)}$ equal the correlation between Y_0 and Y_1 . Show the following:

- (a) $\mathbb{E}[Y_1 | Y_0 = y_0] = (1 - \rho)\mu + \rho y_0$

- (b) Under what conditions would you expect a child's height to exceed that of their parents? The opposite. Why is this called regression to mean?
 - (c) Prove that, in general, $0 \leq \rho^2 \leq 1$.
3. Let θ be some parameter of interest. For example the average number of hours per week a graduate student in economics spends studying. Upon arriving in graduate school you summarize your beliefs/uncertainty about θ by assuming that $\theta \sim \mathcal{N}\left(\bar{\theta}, \frac{1}{\rho_\theta}\right)$.
- (a) If $\bar{\theta} = 10$ and $\rho_\theta = 1/10$, then what is the probability that you assign to the possibility that θ exceeds 40 hours a week? That is is less than 10 hours per week?
 - (b) Let $S_t = \theta + \epsilon_t$ with $\epsilon_t \sim \mathcal{N}\left(0, \frac{1}{\rho_\epsilon}\right)$. Compute the conditional distribution of θ given S_1 .
 - (c) You observe the additional (independent) signals S_2, \dots, S_T . Compute the conditional distribution of θ given all T signals.