Problem 1.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Define $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(x, \omega) = f(x) + \omega$$

Show that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$ there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

Solution

If we can show that the Jacobian of F with respect to all of its arguments has full rank whenever $F(x, \omega) = 0$, then the Transversality theorem gives us that there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$, $D_x F(x_0, \omega_0)$ has full rank as well.

Taking the "full" Jacobian of F, and denoting the Jacobian of f by Df,we have that:

$$DF(x, \omega) = \begin{bmatrix} Df(x) & I_n \end{bmatrix}$$

since the Jacobian of F with respect to ω is the identity matrix, I_n.

I_n has rank n, so $DF(x, \omega)$ must also have rank n, so the condition of the Transversality theorem is satisfied. Thus, there is a set $\Omega_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that, if $\omega \notin \Omega_0$, then for each x_0 satisfying $F(x_0, \omega_0) = 0$, $|DF_x(x_0, \omega_0)| \neq 0$.

We now finish the problem by applying the implicit function theorem. This tells us directly that, whenever $F(x_0, \omega_0) = 0$ and $|DF_x(x_0, \omega_0)| \neq 0$, there is an open set U containing x_0, an open set V containing ω_0, and a C^1 function $h : V \to U$ such that for all $\omega \in V$, $x = h(\omega)$ is the unique element of U satisfying $F(x, \omega) = 0$.

Problem 2.

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$. Then, show that exactly one of the following two conditions hold:

- $\exists x \in \mathbb{R}^n$ such that $Ax = b$, with $x \geq 0$;
- $\exists y \in \mathbb{R}^{1\times m}$ such that $A'y \geq 0$, and $y'b < 0$.

Hint: you may want to use the following definition and its properties. If $v_1, v_2, ..., v_n$ are the columns of A, define

$$Q = \text{cone}(A) \equiv \left\{ s \in \mathbb{R}^m : s = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \forall i \right\},$$

i.e., Q is the set of all conic combinations of the columns of A. Note that Q is non-empty ($0 \in Q$), and assume it is closed and convex (you should be able to prove this!).

Solution

This is known as the Farkas' Lemma. First we show that both conditions cannot hold simultaneously. Note that $y'Ax = y'(Ax) = y'b < 0$ (since the first condition implies that $Ax = b$ and the second that $y'b < 0$). Likewise, note that $y'Ax = (y'A)x = (A'y)'x \geq 0$ (since the first condition implies that $x \geq 0$ and the second that $A'y \geq 0$). Then, both conditions cannot hold simultaneously.

Then, we show that if the first condition fails, then the second has to hold. Consider $Ax = \sum_{i=1}^{n} x_i v_i = b$. From the hint we know that $x \geq 0$ if and only if $b \in Q$. So if the first condition fails, $x < 0$ and, then, $b \notin Q$.

Since Q is non-empty, closed, and convex, we can use the strong version of the separating hyperplane theorem: there exists $\alpha \in \mathbb{R}^m$, with $\alpha \neq 0$, and some $\beta \in \mathbb{R}$ such that $\alpha' b > \beta$ and $\alpha' s < \beta$, $\forall s \in Q$. Since $0 \in Q$, we know that $\beta > 0$. Note also that $\lambda v_i \in Q$, $\forall \lambda > 0$. Then, since $\alpha' s < \beta \; \forall s \in Q$, we have that $\alpha'(\lambda v_i) < \beta \; \forall \lambda > 0$, so $\alpha' v_i < \beta/\lambda \; \forall \lambda > 0$.

Letting $\lambda \to \infty$, this implies $\alpha' v_i \leq 0$. Then, setting $y = -\alpha$, we obtain that $y'b < 0$ and $y'v_i \geq 0$, $\forall i$. Since the v_i are the columns of A, we get that $A'y \geq 0$. Then, condition 2 holds.

Problem 3.

Call a vector $\pi \in \mathbb{R}^n$ a *probability vector* if

$$\sum_{i}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0 \ \forall i$$

We say there are n states of the world, and π_i is the probability that state i occurs. Suppose there are two traders (trader 1 and trader 2) who each have a set of prior probability distributions (Π_1 and Π_2) which are nonempty, convex, and compact. Call a *trade* a vector $f \in \mathbb{R}^n$, which denotes the net transfer trader 1 receives in each state of the world (and thus $-f$ is the net transfer trader 2 receives in each state of the world). A trade is *agreeable* if

$$\inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i > 0 \quad \text{and} \quad \inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i(-f_i) > 0$$

Prove that there exists an agreeable trade if and only if there is no common prior (that is, $\Pi_1 \cap \Pi_2 = \varnothing$).

Solution

($\Longrightarrow$) Let $f \in \mathbb{R}^n$ be an agreeable trade. Towards contradiction, suppose $\exists \pi^* \in \Pi_1 \cap \Pi_2$. Then

$$\sum_i \pi_i^* f_i \geq \inf_{\pi \in \Pi_1} \sum_i \pi_i f_i > 0$$

But then

$$\sum_i \pi_i^*(-f_i) < 0 \implies \inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i(-f_i) < 0$$

a contradiction.

($\Longleftarrow$) Suppose that $\Pi_1 \cap \Pi_2 = \varnothing$. Because Π_1 and Π_2 are convex, compact, and disjoint sets in $\mathbb{R}^n$, the Strict Separating Hyperplane Theorem guarantees the existence of a nonzero trade $f \in \mathbb{R}^n$ and some $k \in \mathbb{R}$ such that $f \cdot x > k$ for all $x \in \Pi_1$ and $f \cdot x < k$ for all $x \in \Pi_2$. This implies that the trade $g \in \mathbb{R}^n$ given by $g_i = f_i - k$ for all $i = 1, 2, \ldots, n$ satisfies

$$\inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i g_i = \inf_{\pi \in \Pi_1} \sum_{i=1}^{n} (\pi_i f_i - \pi_i k) = \inf_{\pi \in \Pi_1} \sum_{i=1}^{n} \pi_i f_i - k > 0$$

and

$$\inf_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i(-g_i) = -\sup_{\pi \in \Pi_2} \sum_{i=1}^{n} (\pi_i f_i - \pi_i k) = -\sup_{\pi \in \Pi_2} \sum_{i=1}^{n} \pi_i f_i - k > 0.$$

Problem 4.

(a) Let $A \subset \mathbb{R}^n$ be a convex set, and $\lambda_1, \lambda_2, ..., \lambda_p \geq 0$, with $\sum_{i=1}^{p} \lambda_i = 1$. Prove that, if $x_1, x_2, ..., x_p \in A$, then $\sum_{i=1}^{p} \lambda_i x_i \in A$.

(b) The sum $\sum_{i=1}^{p} \lambda_i x_i$ defined in (a) is called a convex combination. The convex hull of a set S, denoted by $co(S)$, is the intersection of all convex sets which contain S. Prove that the set of all convex combinations of the elements of S is exactly $co(S)$.

Solution

(a) We prove this by induction. Since A is convex, the condition holds trivially for $p = 1$ and $p = 2$. We assume it holds for $p = r$ (induction hypothesis) and use that to prove it for $p = r + 1$.

Define $\Lambda := \sum_{i=1}^{r} \lambda_i$, so $1 - \Lambda = \lambda_{r+1}$. Then

$$\sum_{i=1}^{r+1} \lambda_i x_i = \left(\sum_{i=1}^{r} \lambda_i x_i \right) + \lambda_{r+1} x_{r+1} = \Lambda \left(\sum_{i=1}^{r} \frac{\lambda_i}{\Lambda} x_i \right) + (1 - \Lambda) x_{r+1}.$$

Note that $\sum_{i=1}^{r} \frac{\lambda_i}{\Lambda} = 1$, so $\left(\sum_{i=1}^{r} \frac{\lambda_i}{\Lambda} x_i \right) \in A$ (induction hypothesis). Then, the RHS expression is a convex combination of two elements of A, so it is in A.

(b) Denote by $C(S)$ the set of all convex combinations of the elements of S. Let $x \in C(S)$. By definition, x is a convex combination of points of S. Note that $S \subset co(S)$, and $co(S)$ is convex. Then, using the result of item (a), we conclude that $x \in co(S)$ and, therefore, $C(S) \subset co(S)$.

To prove that $co(S) \subset C(S)$ holds, it is sufficient to show that $C(S)$ is a convex set since $co(S)$ is, by definition, the smallest convex set containing the points of S. Let $x, y \in C(S)$. By the definition of $C(S)$, there are $x_1, x_2, ..., x_p \in S$, $\{\mu_i\}_{i=1}^{p}$, $y_1, y_2, ..., y_r \in S$, $\{\nu_i\}_{i=1}^{r}$, with $\sum_{i=1}^{p} \mu_i = \sum_{i=1}^{r} \nu_i = 1$, and $\mu_i \geq 0 \ \forall i = 1, ..., p$, $\nu_i \geq 0 \ \forall i = 1, ..., r$, such that $x = \sum_{i=1}^{p} \mu_i x_i$ and $y = \sum_{i=1}^{r} \nu_i y_i$.

Let $0 \leq \lambda \leq 1$. Then

$$\begin{aligned} \lambda x + (1 - \lambda) y &= \lambda \sum_{i=1}^{p} \mu_i x_i + (1 - \lambda) \sum_{i=1}^{r} \nu_i y_i, \\ &= \sum_{i=1}^{p} \lambda \mu_i x_i + \sum_{i=1}^{r} (1 - \lambda) \nu_i y_i. \end{aligned}$$

Then, since $\sum_{i=1}^{p} \lambda \mu_i + \sum_{i=1}^{r} (1 - \lambda) \nu_i = \lambda \sum_{i=1}^{p} \mu_i + (1 - \lambda) \sum_{i=1}^{r} \nu_i = \lambda + (1 - \lambda) = 1$, the RHS is a convex combination of elements of S, and therefore is in $C(S)$. Then, $C(S)$ is convex.

Problem 5.

a) *Berge's Maximum Theorem:* Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \twoheadrightarrow X$. Define $v(y) = \max_{x \in \Gamma(y)} f(x, y)$ and $\Omega(y) = \arg\max_{x \in \Gamma(y)} f(x, y)$. Suppose f and Γ are continuous, and that Γ has non-empty compact values. Show that v is continuous and Ω is uhc with non-empty compact values. *Hint*: you may find useful to use the sequential definitions of uhc and lhc.

b) Assume that Γ also has convex values. Show that if f is quasi-concave in x, Ω has convex values.[1]

c) Let $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ denote a *social game*, where I is the (finite) set of players, and $u^i : \prod_{j \in I} S^j \to \mathbb{R}$ is the objective function of player $i \in I$ defined over $s = (s^j; j \in I) \in \prod_{j \in I} S^j$, with $S^j \subset \mathbb{R}^{n_j}$, $n_j > 0$. Each player i chooses $s^i \in \arg\max_{s \in \Gamma^i(s_{-i})} u^i(s, s_{-i})$, with $s_{-i} := (s_j; j \in I \setminus \{i\})$, and $\Gamma^i(s_{-i}) \subset S^i$. Define an equilibrium for the social game $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ as a vector $\bar{s} = (\bar{s}^i; i \in I)$ such that, $\forall i \in I$, $u^i(\bar{s}) \geq u^i(s, \bar{s}_{-i})$, $\forall s \in \Gamma^i(\bar{s}_{-i})$, where $\bar{s}_{-i} := (\bar{s}^j; j \neq i)$.

   Assume S^i is convex, compact, and non-empty for each $i \in I$, and that u^i is continuous and quasi-concave in s^i for each $i \in I$. Use the previous parts of this question to show that, if $\{\Gamma^i\}_{i \in I}$ are continuous and have compact, convex, and non-empty values, then an equilibrium for $\mathcal{S}(I, (u^i, S^i, \Gamma^i)_{i \in I})$ exists.

Solution

a) Since Γ has non-empty compact values, continuity of f implies that, $\forall y \in Y$, $\Omega(y) \neq \emptyset$. Given $y \in Y$, let $\{x_n\}_{n \in \mathbb{N}} \in \Omega(y)$ be a sequence that converges to $x \in X$. Fix $x' \in \Gamma(y)$. Since $x_n \in \Omega(y)$ for each n, $f(x_n, y) \geq f(x', y)$ for each n. Since $x_n \to x$ and f is continuous, this implies $f(x, y) \geq f(x', y)$. Since $x' \in \Gamma(y)$ was arbitrary, this implies that $x \in \Omega(y)$. Thus $\Omega(y)$ is closed. Since $\Omega(y) \subset \Gamma(y)$ and $\Gamma(y)$ is compact, $\Omega(y)$ is compact.

   To show that Ω is uhc, let's use the sequential characterization (that can be used because Ω has compact values). Fix $y \in Y$ and take a sequence $\{y_n\}_{n \in \mathbb{N}}$ that converges to $y \in X$. Given a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $x_n \in \Omega(y_n)$, $\forall n \in \mathbb{N}$, we know that $x_n \in \Gamma(y_n)$, $\forall n \in \mathbb{N}$. Then, given that Γ is uhc, we know that exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ that converges to some $x \in \Gamma(y)$. So it suffices to show that $x \in \Omega(y)$. To that end, let $x' \in \Gamma(y)$. Since Γ is lhc, there exists a sequence $x'_n \to x$ such that $x'_n \in \Gamma(y_n)$ for each n. Then for each k, $f(x_{n_k}, y_{n_k}) \geq f(x'_{n_k}, y_{n_k})$. Letting $k \to \infty$, this implies $f(x, y) \geq f(x', y)$. Since $x' \in \Gamma(y)$ was arbitrary, this implies $x \in \Omega(y)$. Thus Ω is uhc.

   Finally, to show that v is continuous, consider a sequence $\{y_n\} \subset Y$ that converges to $y \in Y$. We know that, $\forall n \in \mathbb{N}$, exists $x_n \in \Gamma(y_n)$ such that $v(y_n) = f(x_n, y_n)$. That is, exists $x_n \in \Omega(y_n)$. Since Ω is uhc, there is a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ that converges to some point $x \in \Omega(y)$. Likewise, since Ω is lhc, there exists a sequence $\{\tilde{x}_n\}$ such that $\tilde{x}_n \to x$

[1] A function $f : X \to \mathbb{R}$ is quasi-concave if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $f(\lambda x_1 + (1 - \lambda) x_2) \leq \max\{f(x_1), f(x_2)\}$.

and $\tilde{x}_n \in \Omega(y_n)$ for each n. Since f is continuous, $v(y_n) = f(\tilde{x}_n, y_n) \to f(x, y) = v(y)$. Thus v is continuous.

b) Given $y \in Y$, take two points x_1 and x_2 in $\Omega(y)$. Since $\Gamma(y)$ has convex values, $\forall \lambda \in (0,1)$, $z_\lambda := \lambda x_1 + (1-\lambda)x_2 \in \Gamma(y)$. Also, since f is quasi-concave in x, $f(z_\lambda, y) \geq \min\{f(x_1, y), f(x_2, y)\}$. Then, $z_\lambda \in \Omega(y)$, $\forall \lambda \in (0,1)$. Then, Ω has convex values.

c) Identify the set I with $\{1, ..., \kappa\}$ for some $k \in \mathbb{N}$ (which we can do without loss of generality since I is finite). Define, for all $i \in I$, the correspondence $\Phi^i : \prod_{j \neq i} S^j \twoheadrightarrow S^i$ as $\Phi^i(s_{-i}) := \arg\max_{s \in \Gamma^i(s_{-i})} u^i(s, s_{-i})$. Given Berge's Maximum Theorem, we know that, $\forall i \in I$, Φ^i is uhc and has compact, convex, and non-empty values. Then, the correspondence $\Phi : \prod_{i=1}^{\kappa} S^i \twoheadrightarrow \prod_{i=1}^{\kappa} S^i$ defined by $\Phi(s) = \Phi^1(s_{-1}) \times ... \times \Phi^\kappa(s_{-\kappa})$ satisfies the hypotheses of Kakutani's fixed point theorem. Then, there is $\bar{s} = (\bar{s}^i; i \in I)$ such that $\bar{s} \in \Phi(\bar{s})$. This fixed point $\bar{s}$ is an equilibrium for the social game.

Problem 6.

Solve the following differential equation: $y'' - 5y' + 4y = e^{4x}$. Concretely, provide (i) the general solution of the homogeneous differential equation, and (ii) the particular and general solutions of the inhomogeneous differential equation. Solve explicitly for the constants using the following initial conditions: $y(0) = 3$, $y(0)' = \frac{19}{3}$.

Solution

The characteristic equation is $k^2 - 5k + 4 = 0$. The solutions are given by $k_1 = 4$ and $k_2 = 1$. Then, the general solution of the homogeneous equations is $y_0(x) = C_1 e^{4x} + C_2 e^x$.

For the particular inhomogeneous solution, we conjecture that $y_1(x) = xAe^{4x}$. Then

$$\begin{aligned} y_1'(x) &= (A + 4Ax)e^{4x}, \\ y_2''(x) &= (8A + 16Ax)e^{4x}. \end{aligned}$$

This implies that $(8A + 16Ax)e^{4x} - 5(A + 4Ax)e^{4x} + 4xAe^{4x} = e^{4x}$. Solving for A yields $A = \frac{1}{3}$ and, therefore, $y_1(x) = \frac{x}{3}e^{4x}$.

The general inhomogeneous solution is $y_0(x) + y_1(x) = C_1 e^{4x} + C_2 e^x + \frac{x}{3}e^{4x}$.

Finally, using the initial conditions, we have that $C_1 + C_2 = 3$ and $4C_1 + C_2 + \frac{1}{3} = \frac{19}{3}$. Then, $C_1 = 1$ and $C_2 = 2$.