

1. Factor analysis: Is the econometric way to model cross-sectional correlations in a group of time series.
2. Empirical finance bros are always talking about Factors.
3. (Common) Factors can be observed (finance people like this) or unobserved. Single Factor representation of the returns.  $f_t$  is the factor,  $\epsilon_{t,i}$  is the idiosyncratic component of returns (lets say it's iid).

$$r_{t,1} = \alpha_1 + \beta_1 f_t + \epsilon_{t,1}$$

$$r_{t,2} = \alpha_2 + \beta_2 f_t + \epsilon_{t,2}$$

4. In finance, risk is not just volatility, but it is also covariance (of return) with something you care about (factor). Riskier assets (we believe), must pay higher returns to compensate for bearing that risk.
5. In the cross-section, why do some assets pay higher average returns and others pay lower average returns. We believe it is because the ones that pay more, are riskier. Exposure to risk is measured by the 'betas' above. Betas are the slope coefficients in regressions of returns on factor(s).
6. In a framework where factors are unobserved, we can estimate  $f_t$ , but interpretation of the factor is not straightforward. Save for a future class.
7. Alternatively, we can propose what the factor is, and evaluate whether it 'works' or not as a factor. This is what we do today.
8. (multi-factor model)

$$r_{t,1} = \alpha_1 + \beta_1 f_{t,1} + \gamma_1 f_{t,2} + \epsilon_{t,1}$$

$$r_{t,2} = \alpha_2 + \beta_2 f_{t,1} + \gamma_2 f_{t,2} + \epsilon_{t,2}$$

$$r_{t,3} = \alpha_3 + \beta_3 f_{t,1} + \gamma_3 f_{t,2} + \epsilon_{t,3}$$

$\vdots$

Sometimes people refer to these slope coefficients as factor loadings.

9. The beta-risk model: We assume excess returns on assets  $i = 1, \dots, n$  are driven by the factor structure

$$r_{t,i}^e = \alpha_i + \beta_i f_t + \epsilon_{t,i}$$

$$E(r_{t,i}^e) = \bar{r}_i^e = \beta_i \lambda$$

where  $\beta_i$  is asset  $i$ 's exposure to the risk factor  $f_t$ , and  $\lambda = E(f_t)$  is known as the 'price' of risk. Notice if we take the mean of the first equation,  $E(r_{t,i}^e) = \alpha_i + \beta_i E(f_t) + E(\epsilon_{t,i}) = \alpha_i + \beta_i \lambda$ . The theory says  $\alpha_i = 0$ , a key implication of the model. The main focus of the framework is a cross-sectional relationship.

10. Interpretation of non-zero alphas. Say we reject the hypothesis that the alphas are zero. If the assets are individual security returns, then we must not have the correct model. Maybe we have the wrong candidate as the factor, or maybe there are additional factors. If we are looking at portfolio returns (say managed returns), well, it could be that we don't have the correct model, but it also could be that the portfolio manager has some superior ability, and that the manager deserves extra compensation.
11. Every finance model on asset pricing can be represented in this beta-risk framework. Start with the Euler equation on investments. Let  $p_{t,i}$  be the price of asset  $i$ ,  $c_t$  is the consumption of the representative consumer/investor, who has utility  $u(c_t)$ , with marginal utility  $\partial u(c_t)/\partial c_t = u'(c_t)$ . The investor contemplates buying another share of firm  $i$  stock. The rule of rational life is to engage in an activity until the marginal benefit equals the marginal cost—this is the Euler equation. Let  $x_{t+1,i} = p_{t+1,i} + d_{t+1,i}$  be the payoff,  $0 < \beta < 1$  is the subjective discount factor,

$$\underbrace{p_{t,i} u'(c_t)}_{\text{mc}} = \underbrace{\beta E_t(u'(c_{t+1}) x_{t+1,i})}_{\text{expected marginal benefit}}$$

Since the left hand side variables are known at  $t$ , divide both sides by them. (SDF=stochastic discount factor).

$$1 = E_t \left( \underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)}}_{\text{SDF}} \underbrace{\frac{x_{t+1,i}}{p_{t,i}}}_{\text{gross return}} \right)$$

This is just rewriting the Euler equation in returns form.

12. Finance bros like to write the SDF like this,

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

and write the gross return as

$$1 + r_{t+1,i} = \frac{x_{t+1,i}}{p_{t,i}}$$

So now let's regroup with this notation. The Euler equation becomes

$$1 = E_t(m_{t+1}(1 + r_{t+1,i}))$$

This has to hold for all traded assets and the risk-free asset (with rate of return  $r_t^f$ ) is traded, so

$$1 = E_t(m_{t+1}(1 + r_t^f))$$

Subtract the Euler equation for the risk-free asset from the one for asset  $i$ .

$$\begin{aligned} 0 &= E_t \left( m_{t+1} (1 + r_{t+1,i}) - m_{t+1} (1 + r_t^f) \right) \\ 0 &= E_t \left( m_{t+1} (r_{t+1,i}^e) \right) \end{aligned}$$

where  $r_{t+1,i}^e = r_{t+1,i} - r_t^f$  is the excess return on asset  $i$ .

13. What we can do now is assume a one-factor representation for the SDF

$$m_t = 1 - b(f_t - \mu_f)$$

where  $\mu_f = E(f_t)$ . make this substitution,

$$\begin{aligned} 0 &= E_t \left( r_{t+1,i}^e (1 - b(f_{t+1} - \mu_f)) \right) \\ &= E_t r_{t+1,i}^e - b E_t \left( r_{t+1,i}^e (f_{t+1} - \mu_f) \right) \\ &= E_t r_{t+1,i}^e - \underbrace{\frac{\text{Cov}_t(r_{t+1,i}^e, f_{t+1})}{\text{Var}(f_{t+1})}}_{\beta_i} \underbrace{b \text{Var}(f_{t+1})}_{\lambda} \end{aligned}$$

$$E_t r_{t+1,i}^e = \lambda \beta_i$$