Second Midterm Review Sheet, Part I

Ec240a – Second Half, Fall 2018

In preparing for the exam review your lecture notes, assigned course readings, problem sets and prepare answers to the questions on the review sheet(s). You may bring to the exam a single 8.5×11 inch sheet of paper with notes on it. No calculation aides are allowed in the exam (e.g., calculators, phones, laptops etc.). You may work in pencil or pen, however I advise the use of pencil. Please be sure to bring sufficient blue books with you to the exam if possible. Scratch paper will be provided.

[1] Available is a random sample of $i = 1, \ldots, N$ farmers. We have measures of output, Y_t, and (labor) input, X_t, for each farmer (on a per hectare basis) for each of $t = 1, \ldots, T$ years. The sample is

$$\left\{(Y_{i1}, \ldots, Y_{iT}, X_{i1}, \ldots, X_{iT})'\right\}_{i=1}^{N}.$$

We assume that, for $Y_t = \ln O_t$ and $X_t = \ln L_t$, output per hectare, O_t, equals

$$O_t = L_t^{\beta} Q^{\gamma} \exp(U_t)$$

with L_t labor, Q_t soil quality – which is unobserved by the econometrician – and U_t a stochastic, and also unobserved, input outside of the farmer's control (e.g., rainfall).

Our behavioral assumption is that the farmer chooses L_t to maximize expected profits. She knows period t output and input prices, respectively P_t and W_t, as well as her soil quality, Q.

$$L_t = \arg\max_{l} \mathbb{E}\left[\left. P_t l^{\beta} Q^{\gamma} \exp(U_t) - W_t l \right| P_t, W_t, Q\right]$$

She does not know rainfall, assumed non-forecastable by anything in her information set, with marginal distribution

$$U_t \sim N\left(0, \sigma_U^2\right).$$

You may also assume that (P_t, W_t) varies independently of all other variables in the model and is also independently and identically distributed across farms and over time.

[a] Show that the farmer's $t$labor input is

$$X_t = \mu + \frac{1}{1-\beta} A + V_t$$

with $\mu = \frac{1}{1-\beta}\left(\ln \beta + \frac{\sigma_U^2}{2}\right)$, $A = \gamma \ln Q$ and $V_t = \frac{1}{1-\beta} \ln\left(\frac{P_t}{W_t}\right)$. How does rainfall risk affect the farmer's chosen labor input level? Soil quality?

[b] Show that

$$\mathbb{E}\left[\left. Y_t \right| X_t, A\right] = \beta X_t + A.$$

Why is conditioning on land quality alone sufficient to identify β? What maintained assumption is important for this result?

[c] Show that, for $\sigma_A^2 = \mathbb{V}(A)$, and $\sigma_V^2 = \mathbb{V}(V_t)$ for $t = 1, \ldots, T$, that

$$\begin{aligned}
\mathbb{C}(A, X_t) &= \frac{1}{1-\beta}\sigma_A^2 \\
\mathbb{V}(X_t) &= \left(\frac{1}{1-\beta}\right)^2 \sigma_A^2 + \sigma_V^2 \\
\mathbb{E}[X_t] &= \mu + \frac{1}{1-\beta}\mathbb{E}[A] + \mathbb{E}[V_t].
\end{aligned}$$

[d] Next use your results in part [c] to show that

$$\mathbb{E}^*[A|X_t] = \eta_0 + \eta_1 X_t$$

where

$$\begin{aligned}
\eta_1 &= (1-\beta)\left[1 + (1-\beta)^2 \frac{\sigma_V^2}{\sigma_A^2}\right]^{-1} \\
\eta_0 &= E[A] - \eta_1 \left[\mu + \frac{1}{1-\beta}\mathbb{E}[A] + \mathbb{E}[V_t]\right].
\end{aligned}$$

[e] Using your results from parts [b] and [d] solve for the coefficient, say b, on X_t in $\mathbb{E}^*[Y_t|X_t]$. Does $b = \beta$? Discuss economic conditions under which $b \approx \beta$ as well as conditions where $b >> \beta$.

[f] Let $X = (X_1, \ldots, X_T)'$. Solve for $\mathbb{V}(X)$, $\mathbb{C}(A, X)$ and hence the coefficients $\delta = (\delta_1, \ldots, \delta_T)'$ on $X_1, \ldots, X_T$ in the linear regression

$$\mathbb{E}^*[A|X] = \lambda + X'\delta.$$

[g] Let $Y = (Y_1, \ldots, Y_T)'$ and $W = (\mathbf{1}, X')'$ Solve for the multivariate linear predictor

$$\mathbb{E}^*[Y|X] = \Pi W.$$

Specifically, express the elements of the $T \times (1+T)$ matrix Π in terms of β, λ and δ.

[h] Let $\theta = (\beta, \lambda, \delta')'$. Let $\pi = \text{vec}(\Pi')$ be the $T(T+1) \times 1$ vector constructed by vertically stacking the columns of Π. Let G be a $T(T+1) \times (1+1+T)$ matrix such that

$$\pi = G\theta.$$

Derive the form of G.

[i] Consider the estimator

$$\hat{\pi} = \left[\frac{1}{N}\sum_{i=1}^{N}(I_T \otimes W_i)(I_T \otimes W_i)'\right]^{-1} \times \left[\frac{1}{N}\sum_{i=1}^{N}(I_T \otimes W_i) Y_i\right].$$

Further define $U = Y - (I_T \otimes W_i)'\pi$ and

$$\Gamma = I_T \otimes \mathbb{E}[WW']$$

and

$$\Omega = \mathbb{E}\left[(I_T \otimes W_i)\, UU'\, (I_T \otimes W_i)'\right].$$

Argue that $\hat{\pi} \xrightarrow{p} \pi$ and furthermore also argue that $\sqrt{N}\,(\hat{\pi} - \pi) \xrightarrow{D} \mathcal{N}(0, \Lambda)$ for $\Lambda = \Gamma^{-1}\Omega\Gamma^{-1}$.

[j] Construct a minimum distance estimate of θ and derive its asymptotic sampling distribution.

[2] Let $\mathbf{Y} = (Y_1, \ldots, Y_N)'$ be N independent measurements of the same outcome, each distributed

$$Y_i \sim \mathcal{N}\left(\mu, \sigma_i^2\right).$$

Let $\mathbf{c}$ be an $N \times 1$ vector of constants. Consider estimates of μ in the family

$$\hat{\mu} = \mathbf{c}'\mathbf{Y}. \tag{1}$$

[a] Show that the sample mean $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ is a member of (1).

[b] Show that the mean squared error minimizing choice of $\mathbf{c}$ is

$$\mathbf{c} = \mu^2 \left(\operatorname{diag}\left\{\sigma_1^2, \ldots, \sigma_N^2\right\} + \mu^2 \iota_N \iota_N'\right)^{-1} \iota_N$$

with ι_N an $N \times 1$ vector of ones and $\operatorname{diag}\left\{\sigma_1^2, \ldots, \sigma_N^2\right\}$ denoting a diagonal matrix.

[c] Further show that the i^{th} element of $\mathbf{c}$ is

$$c_i = \frac{\mu^2}{\sigma_i^2} \frac{\left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right]^{-1}}{\left[\sum_{i=1}^{N} \frac{1}{\sigma_i^2}\right]^{-1} + \mu^2}.$$

HINT: For A an invertible matrix, u and v column vectors and b a scalar:

$$(A + buv')^{-1} = A^{-1} - \frac{b}{1 + bv'A^{-1}u} A^{-1}uv'A^{-1}.$$

[d] Assume that $\sigma_i^2 = \sigma^2$ for all $i = 1, \ldots, N$. Show that in this case the mean squared error minimizing estimate of μ is

$$\hat{\mu} = \frac{\mu^2}{\frac{\sigma^2}{N} + \mu^2} \bar{Y}$$

Prove that this estimate converges in mean square to μ (and hence also converges in probability).

[e] The estimate in part [d] is infeasible. Assume that σ^2 is known and consider the feasible estimator

$$\hat{\mu} = \left(1 - \frac{\frac{\sigma^2}{N}}{\bar{Y}^2}\right) \bar{Y}.$$

Provide a justification for this estimate. *Argue* that $\hat{\mu} \xrightarrow{p} \mu$. Do you think its mean squared error will be lower than that of the sample mean's in finite samples? Why?

[f] Rebut the assertion that "the sample mean's day has come and gone".