Econ 204 (2015) - Final

08/19/2015

Instructions: This is a closed book exam. You have 3 hours. The weight of each question is indicated next to it. Write clearly, explain your answers, and be concise. You may use any result from class unless you are explicitly asked to prove it. Good luck!

1. (18pts) Suppose that (X,d) is a metric space and let $A \subset X$ be a subset. Let A_1 be the intersection of all closed sets that contain A. Let A_2 be the set of all $x \in X$ for which there exists a sequence $\{x_n\}$ in A such that $x_n \to x$. Show that $A_1 = A_2$, i.e., the two alternative definitions of the closure of A agree.

2. (16pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1,x_2) = x_1^4 x_2^5$. Give the second order Taylor expansion of f around $x^* = (1,1)$. What is the order of the error term?

3. (16pts) Define $F : \mathbb{R}^2 \to \mathbb{R}$ by:

$$F(x,a) = \sin^2(x) - a \qquad x, a \in \mathbb{R}.$$

For each of the following (x_0, a_0) values, state whether you can use the Implicit Function Theorem to conclude that there exist open sets $U, W \subset \mathbb{R}$ such that $x_0 \in U$, $a_0 \in W$, and a C^1 function $g : W \to U$ satisfying: (i) $g(a_0) = x_0$, and (ii) for every $a \in W$, $x = g(a)$ is the unique solution of $F(x,a) = 0$ for $x \in U$. If your answer is yes, find $g'(a_0)$.

   (a) $(x_0, a_0) = (\pi, 0)$.

   (b) $(x_0, a_0) = (\frac{\pi}{4}, \frac{1}{2})$.

   (c) $(x_0, a_0) = (\frac{5\pi}{4}, -\frac{1}{2})$.

4. (16pts) Find the solution $y : \mathbb{R} \to \mathbb{R}^2$ of the following initial value problem:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \text{ and } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

You can leave the solution in the form of a product of matrices and a matrix inverse.[1] In one sentence, explain how the solution behaves qualitatively.

[1] That is, you do not have to carry out the matrix multiplication and matrix inversion.

5. (16pts) Let X, Y, Z be vector spaces over the same field $\mathbb{F}$, and suppose that X and Y are finite dimensional. Show that for any $T \in L(X, Y)$ and $S \in L(Y, Z)$:

$$\dim(X) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T) \cap \mathrm{Ker}(S)) + \dim(\mathrm{Im}(S \circ T)).$$

6. (18pts) Let $(X, \|\cdot\|)$ be a normed vector space and assume that $B_1[0] = \{x \in X : \|x\| \leq 1\}$ is compact. Show that the Heine-Borel theorem applies to $(X, \|\cdot\|)$, that is, a subset $A \subset X$ is compact if and only if it is closed and bounded. Hint: Show first that for any $\beta \in \mathbb{R}$ and $x_0 \in X$, the function $f : X \to X$ defined by $f(x) = \beta x + x_0$ is continuous.

7. (Bonus, 20pts) Let $u : [0,1] \to \mathbb{R}$ be a continuous function, and $0 < \delta < 1$. Show that there exists a unique continuous function $V : [0,1] \to \mathbb{R}$, that solves the equation:

$$V(y) = \max_{c \in [0,y]} u(c) + \delta V(y - c) \quad \text{for all } y \in [0,1].$$