Financial Econometrics Econ 40357
Volatility, ARCH, GARCH

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Brooks, Chapter 9.

Volatility

Financial returns are not normally distributed. They exhibit

1. Leptokurtotic (fat tails)
2. Volatility clusters
3. The **unconditional** distribution of short-horizon returns aren't normal. But their **conditional** distributions could be normal.

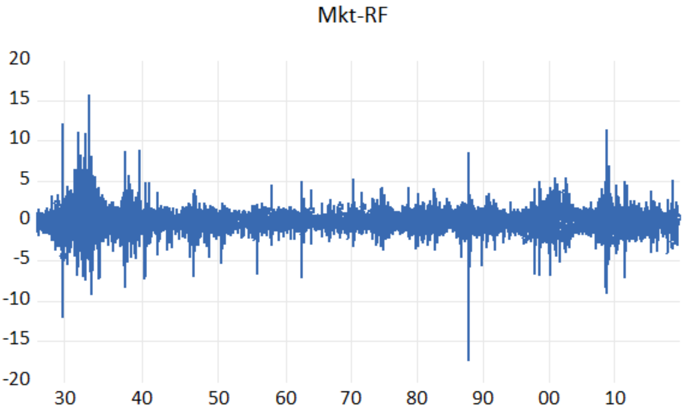
We want to model return volatility. Why?

1. Estimate the value of market risk. (Sharpe ratios).

$$Sharpe = \frac{r_p - r_f}{\sigma_p}$$

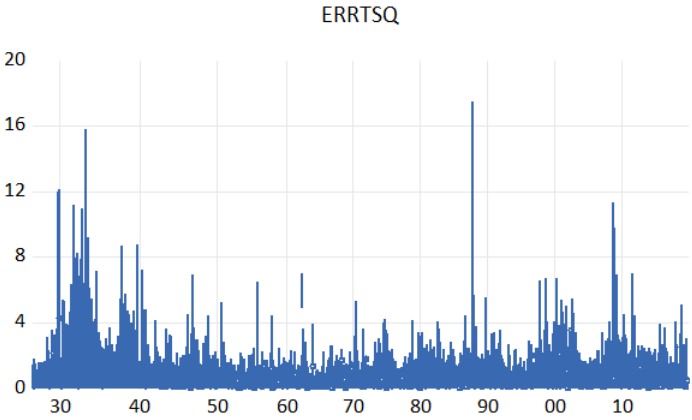
   where $r_p - r_f$ is portfolio excess return and σ_p is portfolio volatility. Sharpe ratio is the average portfolio return per unit of volatility (a risk concept).
2. Volatility is a key parameter for pricing financial derivatives. All modern option pricing techniques rely on a volatility parameter for price evaluation.
3. Volatility is used for risk management assessment and in general portfolio management. Financial institutions want to know the current value of the volatility of the managed assets.
4. They also want to predict their future values. Volatility forecasting is important for institutions involved in options trading and portfolio management.
5. Volatility changes over time, which makes these pricing examples conditional on the current environment (high, low volatility). We want to model how volatility changes and what it depends on.

Market excess return



Let r^e_{mt} be the market excess return. Suppose we have only one observation. How would you form the sample variance? The sample standard deviation?

Square root of squared daily market excess returns



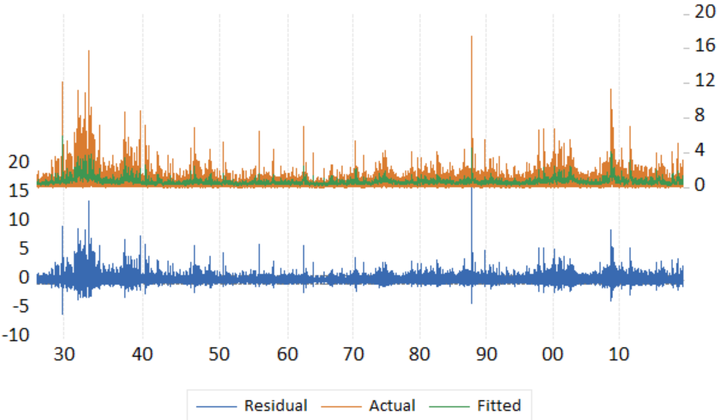
Does staring at this picture make you want to regress it on lags of itself?

Dependent Variable: ERRTSQ
Method: Least Squares
Sample (adjusted): 7/02/1926 9/30/2019
Included observations: 24578 after adjustments

Variable	Coeff	Std. Error	t-Statistic	Prob.
C	0.483965	0.006482	74.66786	0.0000
ERRTSQ(-1)	0.292132	0.006101	47.88619	0.0000
R-squared	0.085343	Mean dependent var		0.683687

Dependent Variable: ERRTSQ
Sample (adjusted): 7/13/1926 9/30/2019
Included observations: 24571 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.165840	0.007648	21.68515	0.0000
ERRTSQ(-1)	0.091617	0.006369	14.38482	0.0000
ERRTSQ(-2)	0.134604	0.006383	21.08746	0.0000
ERRTSQ(-3)	0.110312	0.006409	17.21309	0.0000
ERRTSQ(-4)	0.092554	0.006413	14.43328	0.0000
ERRTSQ(-5)	0.104587	0.006413	16.30962	0.0000
ERRTSQ(-6)	0.100466	0.006409	15.67642	0.0000
ERRTSQ(-7)	0.062992	0.006383	9.868447	0.0000
ERRTSQ(-8)	0.060376	0.006369	9.479599	0.0000
R-squared	0.227199	Mean dependent var		0.683789



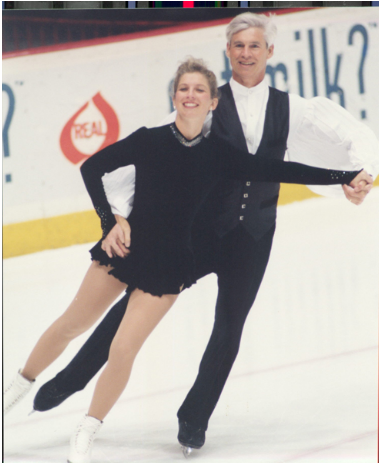
The ARCH/GARCH class of models

- Popular way to model is with ARCH (autoregressive conditional heteroskedasticity) and GARCH (generalized ARCH).
- ARCH was invented by Robert Engle. The Nobel committee gave him the economics prize in part for this.
- GARCH was invented by Tim Bollerslev, who was Engle's student at UCSD.
- There's also,
    - EGARCH (exponential GARCH)
    - IGARCH (integrated GARCH)
    - STARCH (smooth-transition ARCH)
    - TARCH (threshold ARCH)
    - FIGARCH (fractionally integrated GARCH)
    - SWARCH (switching ARCH).

Robert Engle Nobel Laureat

Nobel Prize citation: “for methods of analyzing economic time series with time- varying volatility (ARCH)”

Robert Engle Does Ice Dancing!



The ARCH/GARCH class of models

- Return on some asset

$$\begin{aligned} r_t &= a + \beta x_t + u_t \\ u_t &\sim N\left(0, \sigma_t^2\right) \end{aligned}$$

  Notice t subscript on variance. σ_t^2 is the **conditional** variance of u_t. Conditional on past observations of u_t

$$\sigma_t^2 = E\left[\left(u_t - E_t\left(u_t\right)\right)^2 | u_{t-1}, u_{t-2}, \ldots\right] = \text{Var}\left(u_t | u_{t-1}, u_{t-2}, \ldots\right)$$

  This says the conditional variance changes over time. It is time-varying. It **moves around** over time. ARCH is a **parametric model** of the conditional variance.

- Intuition: remember how we want to think of conditional expectation as regression?
- Estimation done by **maximum likelihood**

ARCH

ARCH(1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2$$

ARCH(2)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2$$

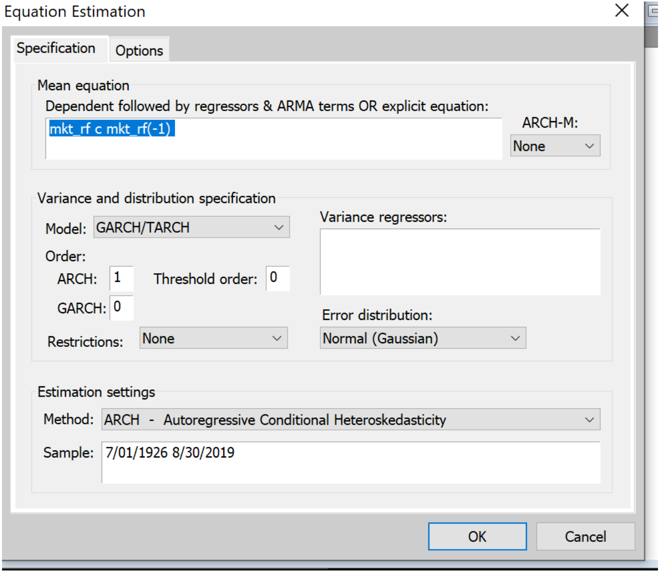
ARCH(q)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots \alpha_q u_{t-q}^2$$

Test for ARCH effects

- Run the main regression
$$r_t = \hat{a} + \hat{\beta} x_t + \hat{u}_t$$
save the residuals $\hat{u}_t$
- Regress the squared residuals $\hat{u}_t^2$ on q lags of itself (to test for ARCH(q)).
$$\hat{u}_t^2 = b_0 + b_1 \hat{u}_{t-1}^2 + \cdots b_q \hat{u}_{t-q}^2 + v_t$$
where v_t is the error term. You can do an $F-$test on the coefficients.
- You can also do a Lagrange multiplier (LM) test. Get the R^2 from this regression.
$$TR^2 \sim \chi_q^2$$
- What does the F-test and LM test test?
$$H_0 : (b_1 = 0) \cap (b_2 = 0) \cap \cdots (b_q = 0)$$
The alternative is $H_A :$ NOT H_0.

Test for and Estimate ARCH model in EViews



Test for and Estimate ARCH model in EViews

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 15:12
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 11 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.028996	0.004540	6.386850	0.0000
MKT_RF(-1)	0.246425	0.002153	114.4349	0.0000
		Variance Equation		
C	0.656324	0.003491	188.0134	0.0000
RESID(-1)^2	0.473991	0.008016	59.13110	0.0000

R-squared	-0.028109	Mean dependent var	0.029196
Adjusted R-squared	-0.028151	S.D. dependent var	1.062371
S.E. of regression	1.077221	Akaike info criterion	2.768679
Sum squared resid	28494.91	Schwarz criterion	2.770000
Log likelihood	-33992.61	Hannan-Quinn criter.	2.769107
Durbin-Watson stat	2.323311		

Test for and Estimate ARCH model in EViews

Equation: EQ02 Workfile: VOLATILITY::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 15:13
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 23 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*RESID(-2)^2 + C(6)*RESID(-3)^2 + C(7)*RESID(-4)^2 + C(8)*RESID(-5)^2 + C(9)*RESID(-6)^2 + C(10)*RESID(-7)^2 + C(11)*RESID(-8)^2 + C(12)*RESID(-9)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.053366	0.004332	12.31774	0.0000
MKT_RF(-1)	0.127875	0.006636	19.27105	0.0000
		Variance Equation		
C	0.160815	0.002808	57.26586	0.0000
RESID(-1)^2	0.122085	0.004743	25.74071	0.0000
RESID(-2)^2	0.113064	0.004983	22.69077	0.0000
RESID(-3)^2	0.110880	0.006016	18.43183	0.0000
RESID(-4)^2	0.120884	0.005092	23.74082	0.0000
RESID(-5)^2	0.094493	0.005543	17.04883	0.0000
RESID(-6)^2	0.095819	0.005632	17.01452	0.0000
RESID(-7)^2	0.057432	0.004902	11.71529	0.0000
RESID(-8)^2	0.091738	0.004974	18.44226	0.0000
RESID(-9)^2	0.078895	0.004899	16.10544	0.0000

R-squared	-0.000094	Mean dependent var	0.029196
Adjusted R-squared	0.000135	S.D. dependent var	1.062371

Oy! Too many parameters!

GARCH

GARCH(1,1)

$$
\begin{aligned}
r_t &= a + \beta x_t + u_t \\
u_t &\sim N\left(0, \sigma_t^2\right)
\end{aligned}
$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2$$

The (1,1) refers to number of lags of u^2 and σ^2, and where

$$0 \leq \beta \leq 1$$

(class: why do we need this?)

GARCH

GARCH(1,1) is constrained infinite ordered ARCH. Observe,

$$\begin{aligned}
\sigma_{t-1}^2 &= \alpha_0 + \alpha_1 u_{t-2}^2 + \beta\sigma_{t-2}^2 \\
\sigma_{t-2}^2 &= \alpha_0 + \alpha_1 u_{t-3}^2 + \beta\sigma_{t-3}^2
\end{aligned}$$

substitute this into previous

$$\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\underbrace{\left(\alpha_0 + \alpha_1 u_{t-2}^2 + \beta\sigma_{t-2}^2\right)}_{\sigma_{t-1}^2} \\
&= \alpha_0\left(1+\beta\right) + \alpha_1 u_{t-1}^2 + \alpha_1\beta u_{t-2}^2 + \beta^2\sigma_{t-2}^2 \\
&= \alpha_0\left(1+\beta\right) + \alpha_1\left(u_{t-1}^2 + \beta u_{t-2}^2\right) + \beta^2\left(\alpha_0 + \alpha_1 u_{t-3}^2 + \beta\sigma_{t-3}^2\right) \\
&= \alpha_0\left(1+\beta+\beta^2\right) + \alpha_1\left(u_{t-1}^2 + \beta u_{t-2}^2 + \beta^2 u_{t-3}^2\right) + \beta^3\sigma_{t-3}^2
\end{aligned}$$

Keep going. $\beta^k = 0$ as $k \to \infty$.

$$\sigma_t^2 = \frac{\alpha_0}{1-\beta} + \frac{\alpha_1}{\beta}\sum_{j=1}^{\infty}\beta^j u_{t-j}^2$$

GARCH

GARCH(2,1)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \beta \sigma_{t-1}^2$$

GARCH(1,2)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-1}^2$$

Usually, GARCH(1,1) does the job.

GARCH

Equation: EQ02 Workfile: VOLATILITY::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 15:25
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 26 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.047571	0.004436	10.72388	0.0000
MKT_RF(-1)	0.126446	0.006698	18.87856	0.0000
		Variance Equation		
C	0.012641	0.000497	25.43512	0.0000
RESID(-1)^2	0.103414	0.002076	49.82175	0.0000
GARCH(-1)	0.886041	0.002373	373.4385	0.0000

R-squared	0.000338	Mean dependent var	0.029196
Adjusted R-squared	0.000298	S.D. dependent var	1.062371
S.E. of regression	1.062213	Akaike info criterion	2.428462
Sum squared resid	27706.47	Schwarz criterion	2.430113
Log likelihood	-29814.08	Hannan-Quinn criter.	2.428997
Durbin-Watson stat	2.107261		

Forecasting Volatility

Notice the sample

Equation: EQ02 Workfile: VOLATILITY::Untitled\

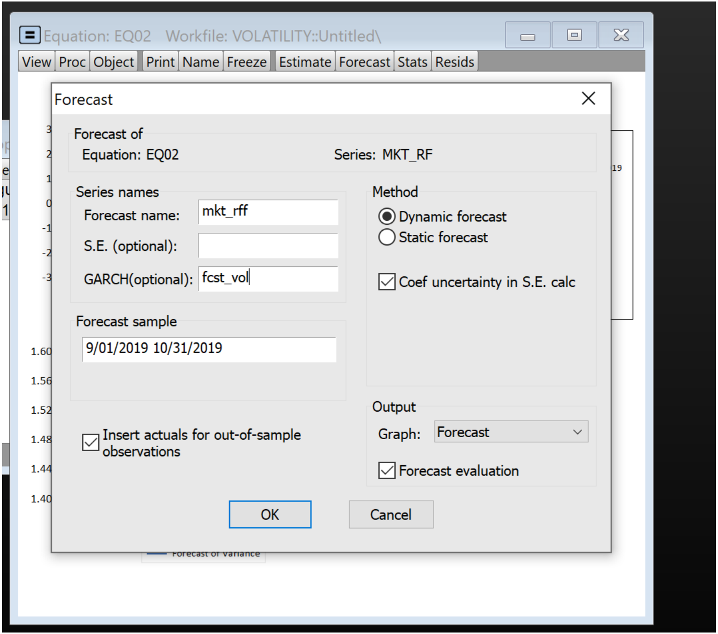
View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/18/19 Time: 19:56
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 26 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.047571	0.004436	10.72388	0.0000
MKT_RF(-1)	0.126446	0.006698	18.87856	0.0000
		Variance Equation		
C	0.012641	0.000497	25.43512	0.0000
RESID(-1)^2	0.103414	0.002076	49.82175	0.0000
GARCH(-1)	0.886041	0.002373	373.4385	0.0000

R-squared	0.000338	Mean dependent var	0.029196
Adjusted R-squared	0.000298	S.D. dependent var	1.062371
S.E. of regression	1.062213	Akaike info criterion	2.428462
Sum squared resid	27706.47	Schwarz criterion	2.430113
Log likelihood	-29814.08	Hannan-Quinn criter.	2.428997
Durbin-Watson stat	2.107261		

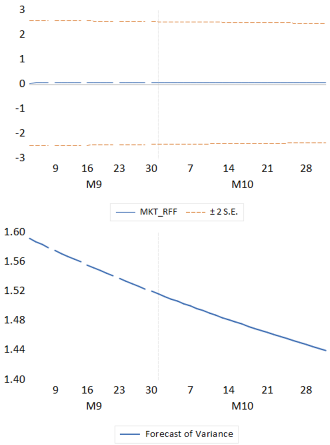
Forecasting Volatility



Forecasting Volatility

Equation: EQ02 Workfile: VOLATILITY::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids



Forecast: MKT_RFF
Actual: MKT_RF
Forecast sample: 9/03/2019 10/31/2019
Included observations: 47

Root Mean Squared Error	0.603903
Mean Absolute Error	0.453813
Mean Abs. Percent Error	159.2341
Theil Inequality Coef.	0.911622
Bias Proportion	0.000910
Variance Proportion	0.997522
Covariance Proportion	0.001568
Theil U2 Coefficient	1.111137
Symmetric MAPE	171.9425

How to forecast the conditional variance in Eviews

1. Equation Window $\rightarrow$ view $\rightarrow$Garch Graph
2. Equation Window $\rightarrow$ Proc $\rightarrow$ Make GARCH Variance Series
3. Equation Window $\rightarrow$ Proc $\rightarrow$ Forecast (give an name for GARCH(optional) forecast, such as garchf. This will be the forecasted GARCH process

Both static and dynamic forecasting use the original estimated coefficients at every step.
Static computes a sequence of one-step ahead forecasts using the actual (not forecasted) values of lagged deppendent variables. Dynamic forecasting uses only information available at the beginning of the forecast period. i.e., no updating the rhs variables. It forecasts the rhs variables.

ARCH-M, GARCH-M (in the mean)

Is higher volatility associated with higher or lower returns? Here is GARCH-M example

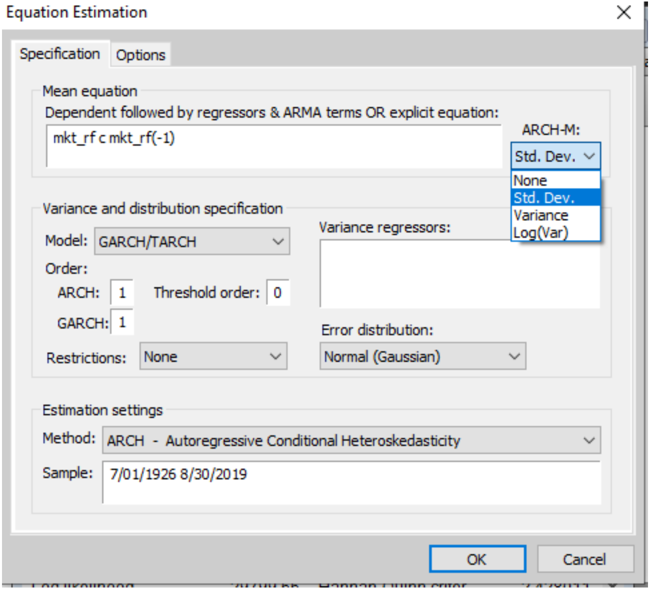
$$r_t^e = a + b\sigma_t + u_t$$

$$u_t \sim N\left(0, \sigma_t^2\right)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2$$

- Use volatility as ‘regressor’ to preserve units.
- $b > 0$, high volatility, r^e expected to be large. $b < 0$, high volatility, r^e expected to be small.
- Estimation is by maximum likelihood.
- To implement, choose the option in EViews

ARCH-M/GARCH-M in Eviews



ARCH-M/GARCH-M in Eviews

Equation: EQ02 Workfile: VOLATILITY::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: MKT_RF
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/19/19 Time: 11:09
Sample (adjusted): 7/02/1926 8/30/2019
Included observations: 24558 after adjustments
Convergence achieved after 25 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
@SQRT(GARCH)	0.096047	0.016573	5.795300	0.0000
C	-0.016379	0.011622	-1.409401	0.1587
MKT_RF(-1)	0.126616	0.006788	18.65283	0.0000
		Variance Equation		
C	0.012760	0.000527	24.19541	0.0000
RESID(-1)^2	0.104404	0.002114	49.38275	0.0000
GARCH(-1)	0.884931	0.002414	366.6144	0.0000

R-squared	-0.002482	Mean dependent var	0.029196
Adjusted R-squared	-0.002564	S.D. dependent var	1.062371
S.E. of regression	1.063733	Akaike info criterion	2.427369
Sum squared resid	27784.64	Schwarz criterion	2.429350
Log likelihood	-29799.66	Hannan-Quinn criter.	2.428011