Ec240a, Fall 2020

Professor Bryan Graham

Problem Set 2

Due: November 9th, 2020

Problem sets are due at 5PM. The GSI will provide instructions on how to turn in your problem set. You may work in groups, but each student should turn in their own write-up (including a "printout" of a narrated/commented and executed Jupyter Notebook if applicable). Please also e-mail a copy of any Jupyter Notebook to the GSI (if applicable).

1. Let $\mathcal{H}$ be an Hilbert space and y a fixed vector within it. Show that for each $\epsilon > 0$ that there exists a $\delta > 0$ such that

$$|\langle h_1, y\rangle - \langle h_2, y\rangle| \leq \epsilon$$

for all $h_1, h_2 \in \mathcal{H}$ where $\|h_1 - h_2\| \leq \delta$ (HINT: use the Cauchy-Schwarz Inequality).

2. The linear regression of Y into X is

$$\mathbb{E}^*\left[Y|\, X\right] = X'\gamma_0,\ \ \gamma_0 = \mathbb{E}\left[XX'\right]^{-1} \times \mathbb{E}\left[XY\right].$$

Let $X = (1, W')'$, with W a $K \times 1$ vector of linearly independent random variables. Show that

$$\mathbb{E}\left[XX'\right]^{-1} = \begin{bmatrix} 1 + \mathbb{E}\left[W\right]' \mathbb{V}\left(W\right)^{-1} \mathbb{E}\left[W\right] & -\mathbb{E}\left[W\right]' \mathbb{V}\left(W\right)^{-1} \\ -\mathbb{V}\left(W\right)^{-1} \mathbb{E}\left[W\right] & \mathbb{V}\left(W\right)^{-1} \end{bmatrix}$$

and hence also that

$$\gamma_0 = \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{bmatrix} \mathbb{E}\left[Y\right] - \mathbb{E}\left[W\right]' \beta_0 \\ \mathbb{V}\left(W\right)^{-1} \mathbb{C}\left(W, Y\right) \end{bmatrix}.$$

You may assume that all the expectations and variances in the expression above are well-defined.

3. Let X be a $K \times 1$ vector of covariates with a constant as first element. Let W be a $J \times 1$ vector of additional covariates (excluding a constant). Consider the **long regression** of Y onto X and W:

$$\mathbb{E}^*\left[Y|\, X, W\right] = X'\beta_0 + W'\gamma_0.$$

Further consider the **short regression** of Y onto X alone

$$\mathbb{E}^*\left[Y|\, X\right] = X'b_0.$$

Finally consider the **auxiliary linear** (multivariate) regression of W given X

$$\mathbb{E}^*\left[W|\, X\right] = \Pi_0 X.$$

Here Π_0 is the $J \times K$ coefficient matrix $\Pi_0 = \mathbb{E}\left[WX'\right] \times \mathbb{E}\left[XX'\right]^{-1}$. Let $U = Y - \mathbb{E}^*\left[Y|\, X, W\right]$.

(a) Use the Projection Theorem to show that $\mathbb{E}^*\left[U|\, X\right] = 0$.

(b) Use the Projection Theorem to show that $\mathbb{E}^*\left[X|\, X\right] = X$.

(c) Use you the results from (a) and (b) above as well as linearity of the projection operator to further show that

$$\mathbb{E}^*\left[Y|\, X\right] = X'\beta_0 + \mathbb{E}^*\left[W|\, X\right]'\gamma_0$$

and hence that

$$b_0 = \beta_0 + \Pi_0'\gamma_0.$$

(d) Interpret your result as an "omitted variable bias" (OVB) formula.

(e) Further argue that you have shown the **law of iterated linear predictors:**

$$\mathbb{E}^*\left[Y|\, X\right] = \mathbb{E}^*\left[\mathbb{E}^*\left[Y|\, X, W\right]|\, X\right].$$

4. Show that

$$\mathbb{V}\left(Y\right) = \mathbb{V}\left(Y - \mathbb{E}^*\left[Y|\, X\right]\right) + \mathbb{V}\left(\mathbb{E}^*\left[Y|\, X\right]\right).$$

5. Let $\mathbf{Y}$ be an $N \times 1$ vector of outcomes and $\mathbf{X}$ and $N \times K$ vector of covariates (which includes a constant in column 1). The projection of $\mathbf{Y}$ onto the column space of $\mathbf{X}$ coincides with the least squares fit

$$\hat{\mathbf{Y}} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y}.$$

Let $\hat{\mathbf{U}} = \mathbf{Y} - \hat{\mathbf{Y}}$ be the fitted residuals. Using vector space methods show that:

(a) $\mathbf{X}'\hat{\mathbf{U}} = 0$

(b) $\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right)'\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right) = \hat{\mathbf{U}}'\hat{\mathbf{U}} + \left(\hat{\mathbf{Y}} - \frac{\mathbf{Y}'\iota}{N}\right)'\left(\hat{\mathbf{Y}} - \frac{\mathbf{Y}'\iota}{N}\right)$

(c) $0 \leq R^2 \leq 1$ for $R^2 = 1 - \frac{\hat{\mathbf{U}}'\hat{\mathbf{U}}}{\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right)'\left(\mathbf{Y} - \frac{\mathbf{Y}'\iota}{N}\right)}$

6. Compute the following exercises from Hansen (2019): 2.4, 2.16, 3.2, 3.3, 3.6 (using a Projection Theorem argument), 3.8, 3.9