Some Worked Bond Pricing Examples

Monetary Theory & Policy

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1 No Uncertainty, One Kind of Bond

A household lives for two periods: t (the present) and $t+1$ (the future). It receives an exogenous and *known* income flow in both periods of Y_t and Y_{t+1}. It can save via purchasing a one-period discount bond, $B_{1,t}$, that trades at price $P_{1,t}$. Consumption is the numeraire – its price is normalized to one. The flow budget constraint facing the household in period t is:

$$C_t + P_{1,t}B_{1,t} = Y_t \tag{1}$$

This simply says that the household may use its period t income to eat (C_t) or save ($P_{1,t}B_{1,t}$). Note that in principle $B_{1,t}$ could be negative, which would be borrowing that would allow the household to eat more than it earns (Y_t).

The household faces the following budget constraint in period $t+1$:

$$C_{t+1} = Y_{t+1} + B_{1,t} \tag{2}$$

In words, the household earns an income stream, Y_{t+1} (that is, in this example, known from the perspective of period t) and receives the face value on its bond holdings that it carried from t into $t+1$ (note this could be negative, which would be like paying back the face value). I have imposed a terminal condition that $B_{t+1} = 0$ – the household can't die in debt (B_{t+1} cannot be negative) and won't want to carry positive stocks of bonds in $t+2$, during which it is no longer alive.

The household's lifetime utility function is:

$$U = u(C_t) + \beta u(C_{t+1}) \tag{3}$$

$u(\cdot)$ is some function with a positive first derivation and negative second derivative (i.e. $u'(\cdot) > 0$, $u''(\cdot) < 0$). An example is $u(\cdot) = \ln(\cdot)$. $0 < \beta < 1$ is a discount factor the reflects impatience – the household would rather get utility today than tomorrow. The household's objective function is:

$$\max_{B_{1,t}} \; U = u(C_t) + \beta u(C_{t+1})$$

s.t.

$$C_t + P_{1,t}B_{1,t} = Y_t$$

$$C_{t+1} = Y_{t+1} + B_{1,t}$$

To solve this constrained optimization problem, use the constraints to eliminate C_t and C_{t+1}, leaving an unconstrained problem of choosing just $B_{1,t}$:

$$\max_{B_{1,t}} \; U = u\left(Y_t - P_{1,t}B_{1,t}\right) + \beta u\left(Y_{t+1} + B_{1,t}\right)$$

Take the derivative with respect to $B_{1,t}$:

$$\frac{\partial U}{\partial B_{1,t}} = -P_{1,t}u'\left(Y_t - P_{1,t}B_{1,t}\right) + \beta u'\left(Y_{t+1} + B_{1,t}\right)$$

Set this equal to zero:

$$P_{1,t}u'\left(Y_t - P_{1,t}B_{1,t}\right) = \beta u'\left(Y_{t+1} + B_{t+1}\right) \tag{4}$$

The terms $Y_t - P_{1,t}B_{1,t}$ and $Y_{t+1} + B_{1,t+1}$ are just C_t and C_{t+1}, respectively. So we may write this first order condition (FOC) as:

$$P_{1,t}u'(C_t) = \beta u'(C_{t+1}) \tag{5}$$

There is a simple, intuitive way to understand why this condition must hold. If the household buys an additional unit of the bond, it foregoes $P_{1,t}$ units of consumption in period t. In terms of lifetime utility, the household values this foregone consumption at $u'(C_t)$. Hence, $P_{1,t}u'(C_t)$ is the *marginal utility cost* of buying some of the bond. What is the benefit? If the household buys the bond, it gets an extra unit of consumption in $t+1$. In terms of lifetime utility, this is valued at $\beta u'(C_{t+1})$. Hence, the right hand side is the *marginal utility benefit* of buying the bond. At an optimum, marginal cost must equal marginal benefit – if the marginal benefit exceeded the cost, the household should be buying more of the bond; if the marginal cost exceeded the marginal benefit, the household should be buying less.

We can re-arrange terms to write the optimality condition as:

$$P_{1,t} = \frac{\beta u'(C_{t+1})}{u'(C_t)} \tag{6}$$

This is just a special case of the general asset pricing formula. Let $m_{t,t+j} = \frac{\beta^j u'(C_{t+j})}{u'(C_t)}$ be called the *stochastic discount factor* – this measures how the household values extra consumption j periods into the future relative to consumption in the present. The general asset pricing condition as shown in the slides is:

$$P_{A,t} = \mathbb{E}\left[\sum_{j=1}^{\infty} m_{t,t+j} D_{A,t+j}\right] \tag{7}$$

Here, $\mathbb{E}[\cdot]$ is the expectations operator and $D_{A,t+j}$ is how much cash flow the asset generates in future period $t+j$, for $j = 1, 2, \ldots$. The optimality condition for this particular problem is a special case of the more general formula. We have $D_{A,t+1} = 1$ and $D_{A,t+j} = 0$ for all $j = 2, \ldots$ since it is a one-period discount bond with face value. And since there is no uncertainty, we don't need to worry about the expectation operator.

The bond pricing condition in this particular problem (and for any type of problem) is a description of bond *demand*. We are taking *supply* as given (i.e. the quantity of $B_{1,t}$. We can also write optimal pricing condition as:

$$P_{1,t} = \frac{\beta u'(Y_{t+1} + B_{1,t})}{u'(Y_t - P_{1,t} B_{1,t})} \tag{8}$$

A particularly simple description of bond supply is that the bond is *in zero supply* – that is $B_{1,t} = 0$. Then the pricing condition can be written not in terms of consumption but rather income:

$$P_{1,t} = \frac{\beta u'(Y_{t+1})}{u'(Y_t)} \tag{9}$$

This replacement of consumption with income occurs in what I call an "endowment economy" in which bonds are in zero supply. This is a special case. More generally, we could have bonds in fixed supply (i.e. $B_{1,t} = \bar{B}$) or have supply be an increasing function of price. The important point is that, given Y_t and Y_{t+1}, and given some supply-side specification of $B_{1,t}$ (e.g. $B_{1,t} = 0$), we can solve for the bond's price given a value of β and a functional form for $u(\cdot)$.

1.1 Solving for the Bond's Price

Suppose that $u(\cdot) = \ln(\cdot)$. With this utility function, we have $u'(\cdot) = \frac{1}{\cdot}$, so we can write the stochastic discount factor between t and $t+1$ as:

$$m_{t,t+1} = \frac{\beta C_t}{C_{t+1}} \tag{10}$$

Suppose further that $\beta = 0.95$. We can then price the bond under several different scenarios:

1.1.1 $Y_t = Y_{t+1} = 1$, with $B_{1,t} = 0$

The bond price is just:

$$P_{1,t} = \beta = 0.95 \tag{11}$$

If income is constant and the bond is in zero supply, then the bond price just equals the discount factor.

1.1.2 $Y_t = 1$, $Y_{t+1} = 2$, with $B_{1,t} = 0$

The bond price is then:

$$P_{1,t} = \beta\frac{1}{2} = 0.475 \tag{12}$$

Here we see that, if income in the future is higher than income in the present, the bond price is lower. Why is this? If the household anticipates higher future income, it wants to borrow (rather than save) so as to smooth its consumption. This reduces the demand for the bond – in the demand-supply graphs of Mishkin's book, the demand curve shifts in (and the supply curve is vertical) relative to the first case, so the bond price falls.

1.1.3 $Y_t = 2$, $Y_{t+1} = 1$, with $B_{1,t} = 0$

Here the bond price is:

$$P_{1,t} = 2\beta = 1.9 \tag{13}$$

If current income is higher than future income, the bond price is higher. Why is this? If current income is high relative to the future, then the household wants to save – i.e. it wants to buy the bond. This shifts the demand curve out (relative to the first case), resulting in a higher bond price.

1.1.4 $Y_t = Y_{t+1} = 1$, with $B_{1,t} = 0.5$

This is similar to the first case, but the bond is in fixed positive supply, rather than zero supply. The bond price satisfies:

$$P_{1,t} = \beta\frac{1 - 0.5P_{1,t}}{1 + 0.5} \tag{14}$$

This is a little more challenging, because the bond price appears on both the left and right hand sides – we have to do a little more work to solve for the bond price. We have:

$$1.5P_{1,t} = \beta - 0.5\beta P_{1,t} \tag{15}$$

Or:

$$P_{1,t}\,(1.5 + \beta 0.5) = \beta \tag{16}$$

Or:

$$P_{1,t} = \frac{\beta}{1.5 + \beta 0.5} = 0.481 \tag{17}$$

The bond being in positive fixed supply (relative to zero supply) results in the bond price being lower. Why is this? Think about the demand and supply graphs. We have effectively shifted a vertical bond supply curve to the right (relative to the first example). This must result in the price of the bond falling. Basically, if in equilibrium the household has to save by holding 0.5 of the bond, its consumption in the present is low relative to the future. It would there like to buy less of the bond for a given price, because it wants to move resources from t to $t+1$. The only way to get it to hold the amount of the bond supplied is for the price to fall.

The examples where income is different in t versus $t+1$ but the bond is in positive fixed supply would result in similar changes in the bond's price – in both cases it would be lower because supply is higher.

1.2 From Bond Price to Bond Yield

The yield to maturity equates the bond price (solved for in the general case using the stochastic discount factor) to the expected present discounted value of cash flows. In the most general case, we would have:

$$P_{A,t} = \mathbb{E}\left[\sum_{j=1}^{\infty} \frac{D_{A,t+j}}{(1+i_{A,t})^j}\right] = \mathbb{E}\left[\frac{D_{A,t+1}}{1+i_{A,t}} + \frac{D_{A,t+2}}{(1+i_{A,t})^2} + \frac{D_{A,t+3}}{(1+i_{A,t})^3} + \dots\right] \tag{18}$$

For the example we are working through, this is straightforward. We have $D_{A,t+1} = 1$ (the bond pays face value of 1 in $t+1$ with certainty), and $D_{A,t+j} = 0$ for all $j = 2, \dots$. And since there is no uncertainty over the cash flows, we can drop the expectations operator. In our example, we therefore have:

$$P_{1,t} = \frac{1}{1+i_{1,t}} \tag{19}$$

Or:

$$1 + i_{1,t} = \frac{1}{P_{1,t}} \tag{20}$$

In other words, for a one period discount bond with face value of 1 and no uncertainty over receiving the face value, the gross yield to maturity, $1 + i_{1,t}$, is just the inverse of the bond price.

Assuming log utility and a value of $\beta = 0.95$, we can then solve for the yields given the prices in the particular examples we found above:

1.2.1 $Y_t = Y_{t+1} = 1$, with $B_{1,t} = 0$

We found the price of $P_{1,t} = 0.95$. Hence the gross yield is 1.0526, so $i_{1,t} = 0.0526$ (i.e. about 5 percent).

1.2.2 $Y_t = 1$, $Y_{t+1} = 2$, with $B_{1,t} = 0$

We found the price of $P_{1,t} = 0.475$. Hence the gross yield is 2.1053, so $i_{1,t} = 1.1053$ (i.e. 110 percent).

1.2.3 $Y_t = 2$, $Y_{t+1} = 1$, with $B_{1,t} = 0$

We found the price of $P_{1,t} = 1.9$. Hence, the gross yield is 0.5263, so $i_{1,t} = -0.4737$ (i.e. -47 percent). We wouldn't ordinarily expect to see a bond with a negative yield, but note that this is a *real* yield. Given the unequal (across time) income stream, the household really wants to save to try to smooth its consumption, and is potentially willing to earn a negative expected real rate of return to do so. We do not have a zero lower bound issue because there exists no asset (like money) offering a zero return.

1.2.4 $Y_t = Y_{t+1} = 1$, with $B_{1,t} = 0.5$

We found a price of $P_{1,t} = 0.481$. Hence, the gross yield is 2.079, so $i_{1,t} = 1.079$ (i.e. 108 percent).

2 Uncertainty: Income Uncertainty Plus Default Risk

The household earns an income flow in period t, Y_t. This is known. It can purchase two one-period discount bonds with face value of 1 – $B_{1,t}$ and $B_{2,t}$, at prices $P_{1,t}$ and $P_{2,t}$.

Future income is uncertain – it can be high Y^h_{t+1} or low Y^l_{t+1}, with $Y^h_{t+1} \geq Y^l_{t+1}$. Bond 1 always pays face value in $t+1$. Bond 2 sometimes pays face value and sometimes it pays nothing (i.e. it defaults).

The period t budget constraint is:

$$C_t + P_{1,t}B_{1,t} + P_{2,t}B_{2,t} = Y_t \tag{21}$$

There are *four* possible states of nature in $t+1$. These are:

1. Income is high and the risky bond pays (good, good)
2. Income is high and the risky bond defaults (good, bad)
3. Income is low and the risky bond pays (bad, good)
4. Income is low and the risk bond defaults (bad, bad)

This means there are four budget constraints that most hold in each state of nature in $t+1$:

$$C_{t+1}(1) = Y^h_{t+1} + B_{1,t} + B_{2,t} \tag{22}$$

$$C_{t+1}(2) = Y^h_{t+1} + B_{1,t} \tag{23}$$

$$C_{t+1}(3) = Y^l_{t+1} + B_{1,t} + B_{2,t} \tag{24}$$

$$C_{t+1}(4) = Y^l_{t+1} + B_{1,t} \tag{25}$$

Let the probabilities of each state be p_1, p_2, p_3, and p_4, where $p_1 + p_2 + p_3 + p_4 = 1$. The household knows these probabilities, but does not know, in period t, which state will materialize. The household wishes to maximize *expected* utility:

$$U = u(C_t) + \beta\,\mathbb{E}[u(C_{t+1})] \tag{26}$$

Since β is a constant parameter, I can move it inside or outside the expectations operator – it does not matter. In "long-hand," expected utility is:

$$U = u(C_t) + p_1\beta u(C_{t+1}(1)) + p_2\beta u(C_{t+1}(2)) + p_3\beta u(C_{t+1}(3)) + p_4\beta u(C_{t+1}(4)) \tag{27}$$

In other words, expected future utility is just the probability-weighted average of utility in each of the four possible states of nature in $t+1$.

The household's objective is to pick $B_{1,t}$ and $B_{2,t}$ to maximize expected utility, subject to the period t budget constraint and the $t+1$ budget constraints in all four states of nature:

$$\max_{B_{1,t},B_{2,t}} \quad U = u(C_t) + p_1\beta u(C_{t+1}(1)) + p_2\beta u(C_{t+1}(2)) + p_3\beta u(C_{t+1}(3)) + p_4\beta u(C_{t+1}(4))$$

$$\text{s.t.}$$

$$C_t + P_{1,t}B_{1,t} + P_{2,t}B_{2,t} = Y_t$$

$$\begin{aligned}
C_{t+1}(1) &= Y_{t+1}^h + B_{1,t} + B_{2,t} \\
C_{t+1}(2) &= Y_{t+1}^h + B_{1,t} \\
C_{t+1}(3) &= Y_{t+1}^l + B_{1,t} + B_{2,t} \\
C_{t+1}(4) &= Y_{t+1}^l + B_{1,t}
\end{aligned}$$

To "unconstrain" the problem, simply plug in the constraints to eliminate consumption in t and consumption in $t+1$ in all four states of nature:

$$\begin{aligned}
\max_{B_{1,t},B_{2,t}} \quad U = u\left(Y_t - P_{1,t}B_{1,t} - P_{2,t}B_{2,t}\right) &+ p_1\beta u\left(Y_{t+1}^h + B_{1,t} + B_{2,t}\right) + p_2\beta u\left(Y_{t+1}^h + B_{1,t}\right) \\
&+ p_3\beta u\left(Y_{t+1}^l + B_{1,t} + B_{2,t}\right) + p_4\beta u\left(Y_{t+1}^l + B_{1,t}\right)
\end{aligned}$$

Take the derivative with respect to $B_{1,t}$:

$$\begin{aligned}
\frac{\partial U}{\partial B_{1,t}} = -P_{1,t}u'\left(Y_t - P_{1,t}B_{1,t} - P_{2,t}B_{2,t}\right) &+ p_1\beta u'\left(Y_{t+1}^h + B_{1,t} + B_{2,t}\right) + p_2\beta u'\left(Y_{t+1}^h + B_{1,t}\right) \\
&+ p_3\beta u'\left(Y_{t+1}^l + B_{1,t} + B_{2,t}\right) + p_4\beta u'\left(Y_{t+1}^l + B_{1,t}\right)
\end{aligned}$$

Setting this equal to zero, we get:

$$P_{1,t}u'\left(Y_t - P_{1,t}B_{1,t} - P_{2,t}B_{2,t}\right) = p_1\beta u'\left(Y_{t+1}^h + B_{1,t} + B_{2,t}\right) + p_2\beta u'\left(Y_{t+1}^h + B_{1,t}\right) + p_3\beta u'\left(Y_{t+1}^l + B_{1,t} + B_{2,t}\right) + p_4\beta u'\left(Y_{t+1}^l + B_{1,t}\right) \tag{28}$$

This looks messy, but it's just:

$$P_{1,t}u'(C_t) = p_1\beta u'(C_{t+1}(1)) + p_2\beta u'(C_{t+1}(2)) + p_3\beta u'(C_{t+1}(3)) + p_4\beta u'(C_{t+1}(4)) \tag{29}$$

But the right hand side is just expected marginal utility:

$$P_{1,t}u'(C_t) = \mathbb{E}[\beta u'(C_{t+1})] \tag{30}$$

So the optimality condition has the same interpretation as the no-uncertainty case, except it is the *expected* marginal utility benefit that shows up on the right hand side. I can re-write this as:

$$P_{1,t} = \mathbb{E}\left[\frac{\beta u'(C_{t+1})}{u'(C_t)}\right] \tag{31}$$

I can do this because $u'(C_t)$ is known – hence when I divide both sides by this, I can put it either "inside" or "outside" the expectations operator. This equation is just a special case of the general asset pricing formula, since the payout on the riskless bond is known at its face value of 1.

Now let's take the derivative with respect to $B_{2,t}$:

$$\frac{\partial U}{\partial B_{2,t}} = -P_{2,t}u'\left(Y_t - P_{1,t}B_{1,t} - P_{2,t}B_{2,t}\right) + p_1\beta u'\left(Y_{t+1}^h + B_{1,t} + B_{2,t}\right) + p_3\beta u'\left(Y_{t+1}^l + B_{1,t} + B_{2,t}\right) \tag{32}$$

Setting equal to zero, we have:

$$P_{2,t}u'\left(Y_t - P_{1,t}B_{1,t} - P_{2,t}B_{2,t}\right) = p_1\beta u'\left(Y_{t+1}^h + B_{1,t} + B_{2,t}\right) + p_3\beta u'\left(Y_{t+1}^l + B_{1,t} + B_{2,t}\right) \tag{33}$$

This again looks messy, but is quite intuitive. The left hand side is the marginal utility cost of buying the risky bond – you give up $P_{2,t}$ units of consumption, which you value at $u'(C_t)$. Hence, $P_{2,t}u'(C_t)$ is the marginal utility cost of buying the bond. The right hand side is simply the expected marginal utility benefit of buying the bond – it only pays face value in states 1 and 3, so only these are relevant, since you get nothing in states 2 and 4. Let $D_{2,t+1}$ denote the realized payout on bond 2 – this is 1 in states 1 and 3, and 0 in states 2 and 4. We can equivalently write this optimality condition as:

$$P_{2,t}u'(C_t) = \mathbb{E}\left[\beta u'(C_{t+1})D_{2,t+1}\right] = \beta\left[p_1 u'(C_{t+1})(1) + p_3 u'(C_{t+1}(3))\right] \tag{34}$$

We can alternatively write the optimality condition as:

$$P_{2,t} = \mathbb{E}\left[\frac{\beta u'(C_{t+1})}{u'(C_t)} D_{2,t+1}\right] \tag{35}$$

Note that I can divide both sides by $u'(C_t)$ and move it "inside" the expectations operator on the right hand side because $u'(C_t)$ is known when the expectation is taken.

This is, of course, just a special case of the general asset pricing condition. The price of the asset equals the expected value of the product of the stochastic discount factor, $\frac{\beta u'(C_{t+1})}{u'(C_t)}$, with the payout on the asset, $D_{2,t+1}$ (which is either 1 or 0).

2.1 Solving for the Bond Prices and Yields

Let's again assume log utility. To make our lives easier, let's go ahead and assume that both bonds are in zero net supply. This means that $C_t = Y_t$ and C_{t+1} equals income *in each possible state* in $t+1$. Let's again assume that $\beta = 0.95$.

2.1.1 $Y_t = 1$, $Y^h_{t+1} = 1.1$, $Y^l_{t+1} = 0.9$, $p_1 = p_2 = p_3 = p_4 = 0.25$

In this case, the probabilities of each state are the same. The expected value of future income is also the same as current income, since $\mathbb{E}[Y_{t+1}] = p_1 Y^h_{t+1} + p_2 Y^h_{t+1} + p_3 Y^l_{t+1} + p_4 Y^l_{t+1}$ which works out to $\mathbb{E}[Y_{t+1}] = 0.25 \times 1.1 + 0.25 \times 1.1 + 0.25 \times 0.9 + 0.25 \times 0.9 = 1$. The expected payout on the riskless bond is just its face value, 1. The expected payout on the risky bond, $D_{2,t+1}$, is: $\mathbb{E}[D_{2,t+1}] = p_1 \times 1 + p_3 \times 1 = 0.5$.

The price of the riskless bond satisfies:

$$P_{1,t} = \beta \left[p_1 \frac{Y_t}{Y^h_{t+1}} + p_2 \frac{Y_t}{Y^h_{t+1}} + p_3 \frac{Y_t}{Y^l_{t+1}} + p_1 \frac{Y_t}{Y^l_{t+1}} \right] \tag{36}$$

Which is:

$$P_{1,t} = 0.95 \left[0.25 \frac{1}{1.1} + 0.25 \frac{1}{1.1} + 0.25 \frac{1}{0.9} + 0.25 \frac{1}{0.9} \right] = 0.9596 \tag{37}$$

The yield on the riskless bond satisfies:

$$P_{1,t} = \frac{1}{1 + i_{1,t}} \tag{38}$$

So:

$$1 + i_{1,t} = \frac{1}{P_{1,t}} = 1.0421 \tag{39}$$

Note something. Relative to the case of no uncertainty, the price here of the riskless bond is slightly higher (0.9596 relative to 0.95 when $Y_t = Y_{t+1} = 1$) (and correspondingly its yield slightly lower). This in spite of the fact that expected future income (and hence expected future consumption) is the same as current income. What is driving this is a *precuationary* motive – with uncertain future income, the household wants to save more (other things being equal).[1] This shifts the demand curve for the bond to the right. With a vertical supply curve at zero, the price rises, and the yield falls.

Now, let's price the risky bond. Under our maintained assumptions, its price satisfies:

$$P_{2,t} = \beta \left[p_1 \frac{Y_t}{Y_{t+1}^h} + p_3 \frac{Y_t}{Y_{t+1}^l} \right] \tag{40}$$

Which is:

$$P_{2,t} = 0.95 \left[0.25 \frac{1}{1.1} + 0.25 \frac{1}{0.9} \right] = 0.4798 \tag{41}$$

The price of the risky bond is significantly lower than the price of the riskless bond. As we will see, this isn't so much because of default risk, but rather because in expectation the risky bond only pays 1/2 rather than 1. The yield on the risky bond equates the price to the expected present value of cash flows, or:

$$P_{2,t} = \frac{\mathbb{E}[D_{2,t+1}]}{1 + i_{2,t}} \tag{42}$$

So:

$$1 + i_{2,t} = \frac{\mathbb{E}[D_{2,t+1}]}{P_{2,t}} = 1.0421 \tag{43}$$

Note that, in this example, *the yield on the risky bond is identical to the yield on the riskless bond*. This serves to make two points. First, yields are a more useful way to compare the expected returns on assets – the prices of these assets differ by a factor of 2, but that is because the expected cash flows differ by that same factor. Second, uncertainty over the cash flows from an asset per se does not result in a risk premium (i.e. $i_{2,t} > i_{1,t}$). It is not variance per se but rather covariance that matters for risk premia. In this particular example, the bond is equally likely to default when future income is high as when it is low. Hence, there is no covariance of the risky bond's payout with the future marginal utility of income.

2.1.2 $Y_t = 1$, $Y_{t+1}^h = 1.1$, $Y_{t+1}^l = 0.9$, $p_1 = 0.5$ $p_2 = p_3 = 0$ $p_4 = 0.5$

Relative to the case above, it is still the case that the expected value of future income is 1: $\mathbb{E}[Y_{t+1}] = 0.5 \times 1.1 + 0.5 \times 0.9 = 1$ and the expected payout on the risky bond is 1/2: $\mathbb{E}[D_{2,t+1}] = 0.5 \times 1 + 0.5 \times 0 = 0.5$. But the risky bond *always* pays face value when income is high and it *always* defaults when income is low.

The price of the riskless bond satisfies:

[1] Technically, this requires that $u'''(\cdot) > 0$, i.e. that there is a positive third derivative of the utility function. This is satisfied with the natural log.

$$P_{1,t} = 0.95\left[0.5\frac{1}{1.1} + 0.5\frac{1}{0.9}\right] = 0.9596 \tag{44}$$

This is the *same* price as above. Hence the yield is identical, in gross terms $1 + i_{1,t} = 0.0421$. What about the risky bond? Its price satisfies:

$$P_{2,t} = 0.95\left[0.5\frac{1}{1.1}\right] = 0.4318 \tag{45}$$

Note that this price is *lower* than we found above. What about the yield? The yield on the risky bond satisfies:

$$P_{2,t} = \frac{\mathbb{E}[D_{2,t+1}]}{1 + i_{2,t}} \tag{46}$$

Or:

$$1 + i_{2,t} = \frac{\mathbb{E}[D_{2,t+1}]}{P_{2,t}} = 1.1579 \tag{47}$$

Hence, the yield on the risky bond is higher. The risk premium (or credit spread) is $i_{2,t} - i_{1,t} = 0.1158$. In other words, the yield on the risky bond is about 12 percentage points higher than the yield on the risky bond.

What is going on? In this example, the cash flow from the risky bond covaries *positively* with future income – the payout is high when income is high, and low when income is low. From a consumption smoothing perspective, the household *dislikes this.* It wants assets that pay it when its income is low (so that its marginal utility of consumption is high) and vice-versa. To be willing to hold an asset that does not do this, the household demands a higher expected return (equivalently, is willing to pay a lower price).

We can calculate the *covariance* of the risky bond's payout with the marginal utility of income as follows:

$$\text{cov}\left(D_{2,t+1}, u'(Y_{t+1})\right) = \mathbb{E}\left[\left(D_{2,t+1} - \mathbb{E}[D_{2,t+1}]\right)\left(u'(Y_{t+1}) - \mathbb{E}[u'(Y_{t+1})]\right)\right] \tag{48}$$

We know from above that $\mathbb{E}[D_{2,t+1} = \frac{1}{2}$. In this particular example, we have $\mathbb{E}[u'(Y_{t+1})] = 0.5 \times \frac{1}{1.1} + 0.5 \times \frac{1}{0.9} = 1.0101$. Therefore, the covariance is:

$$\text{cov}\left(D_{2,t+1}, u'(Y_{t+1})\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right)\left(\frac{1}{1.1} - 1.0101\right) + \frac{1}{2}\left(0 - \frac{1}{2}\right)\left(\frac{1}{0.9} - 1.0101\right) = -0.0505 \tag{49}$$

As I articulated above, the covariance between the risky bond's payout and the marginal utility of future consumption is negative. This drives the positive risk premium.

What is the same covariance in the first example above, where I found no risk premium? In that example, we have $\mathbb{E}[u'(Y_{t+1})] = 0.25\frac{1}{1.1} + 0.25\frac{1}{1.1} + 0.25\frac{1}{0.9} + 0.25\frac{1}{0.9} = 1.0101$, same as above. We can hence calculate the covariance between the risky bond's payout and the marginal utility of future income as:

$$\operatorname{cov}\left(D_{2,t+1}, u'(Y_{t+1})\right) = \frac{1}{4}\left(1-\frac{1}{2}\right)\left(\frac{1}{1.1}-1.0101\right)+\frac{1}{4}\left(0-\frac{1}{2}\right)\left(\frac{1}{1.1}-1.0101\right) + \frac{1}{4}\left(1-\frac{1}{2}\right)\left(\frac{1}{0.9}-1.0101\right)+\frac{1}{4}\left(0-\frac{1}{2}\right)\left(\frac{1}{0.9}-1.0101\right) = 0 \quad (50)$$

Because there is no covariance between the risky bond's payout and the marginal utility of future income, there is no risk premium in the prior example.

Note that I could fiddle around with p_1, p_2, p_3, and p_4 to get lots of different risky premia. If, for example, I made $p_2 = 0.5$ and $p_4 = 0.5$, with $p_1 = p_3 = 0$, I would find a *negative* risky premium – the risky bond would have a *lower* yield than the riskless bond. In that case, the risky bond doesn't pay when income is high, but pays when income is low. Holding an asset like that allows you to better smooth consumption – the household likes assets like that, and therefore is willing to accept a lower yield/expected return. If I went through and calculated the covariance like I did above, I would find a positive covariance between the risky bond's payout and the marginal utility of future income.