Econ 204 – Problem Set 5

Due Friday, August 14

1. Let be X the set of at most second degree polynomials and $T : X \to X$ a linear transformation defined by $T(f(x)) = 2f(x) + xf'(x)$. Compute the matrix representation of T with respect to the basis $V = \{1, x, x^2\}$, compute $\ker T$, characterize $X/\ker T$, compute the eigenvalues and the corresponding eigenvectors of T. Is T diagonalizable?

   Solution For the matrix representation note that

   $$\begin{aligned} T(1) &= 3 \\ T(x) &= 3x + 1 \\ T(x^2) &= 4x^2 + 1 \end{aligned}$$

   hence

   $$crd_V(T(1)) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$
   $$crd_V(T(x)) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$
   $$crd_V(T(x^2)) = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

   hence

   $$Mtx_V(T) = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

   For the kernel of T compute $\ker Mtx_V(T) = \{x \in \mathbb{R}^3 : Mtx_V(T)x = 0\}$. Solving this linear system gives the only solution is the zero vector, hence $\ker Mtx_V(t) = \{0\}$ is just the zero vector, therefore $\ker T = \{0\}$ is just the constant zero function.

   Since the kernel of T is just the zero element of X, then $[x] = \{x\}$ itself only.

   For the eigenvalues of T compute the eigenvalues of $Mtx_V(t)$, hence the roots of the characteristic polynomial of

   $$Mtx_V(T) - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

   what is $\lambda_1 = 3$ with multiplicity of two and $\lambda_2 = 4$ with multiplicity of one.

The eigenvectors of $Mtx_V(T)$ corresponding to $\lambda_1 = 3$ solve the equation $(A-3I)x = 0$ hence has the form $E_3 = \left\{ \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$. Hence the corresponding eigenvectors of T are the polynomials $E_3 = \{f(x) = a+ax^2 : a \in \mathbb{R}\}$. And the eigenvectors of $Mtx_V(T)$ corresponding to $\lambda_2 = 4$ solve the equation $(A-4I)x = 0$ hence has the form $E_4 = \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ hence the corresponding eigenvectors of T are the polynomials $E_4 = \{f(x) = a : a \in \mathbb{R}\}$.

Finally since the eigenspace corresponding the eigenvalue with multiplicity two has a dimension of one, since $dim(E_3) = 1$ then T is not diagonalizable.

2. For the following functions, determine at what points the derivative exists, and if the derivative function is continuous (you may use that the derivative of $\sin x$ is $\cos x$):

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \quad g(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution For $x \neq 0$ we can find the derivatives of f and g using the simple properties of derivatives:

$$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}, \quad g'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

At $x = 0$ we use directly the definition of the derivative. Note that for $h \neq 0$ we have

$$\frac{f(h)-f(0)}{h} = \sin\frac{1}{h}, \quad \frac{g(h)-g(0)}{h} = h\sin\frac{1}{h}$$

Since $\lim_{h\to 0}\sin\frac{1}{h}$ is not defined, $f'(0)$ is not defined. However, note $\left|h\sin\frac{1}{h}\right| \leq |h|$ so $g'(0) = 0$ and the derivative of g exists everywhere. But $\lim_{x\to 0}\cos\frac{1}{x}$ is not defined, so $\lim_{x\to 0} g'(x) \neq g'(0)$, ie g' is not continuous at $x = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that $f'(\mathbb{R})$, the image of the derivative function, is an interval (possibly a singleton).

Solution To prove the claim it suffices to show that for any $a, b \in f'(\mathbb{R})$ with $a < b$, and any $c \in (a, b)$, we have $c \in f'(\mathbb{R})$. Note that if there are no two distinct values, since f is differentiable we have $f'(\mathbb{R}) \neq \varnothing$; so $f'(\mathbb{R}) = \{c\}$ and we're done (this occurs if f is a constant function).

Choose x_1, x_2 such that $f'(x_1) = a, f'(x_2) = b$, and assume without loss of generality that $x_1 < x_2$. Define the function $g : \mathbb{R} \to \mathbb{R}$ where $g(x) =$

$f(x) - cx$. g is also a differentiable function with $g'(x) = f'(x) - x$. This implies that g is continuous, hence by the Extreme Value Theorem g attains its minimum (and maximum) on the closed interval $[x_1, x_2]$.

Now note that $g'(x_1) = a - c < 0$, which says

$$\lim_{h \to 0} \frac{g(x_1 + h) - g(x_1)}{h} < 0$$

So for some $h' > 0$ we have that for every $0 < \varepsilon < h'$, $\frac{g(x_1+\varepsilon)-g(x_1)}{\varepsilon} < 0 \implies g(x_1 + \varepsilon) - g(x_1) < 0 \implies g(x_1 + \varepsilon) < g(x_1)$, so $g(x_1)$ is not a minimum of $g([x_1, x_2])$. A similar argument shows that since $g'(x_2) = b - c > 0$, $g(x_2)$ is not a minimum either. So g attains its minimum at some $x_0 \in (x_1, x_2)$, and the same argument implies that $g'(x_0) = 0$. Thus we have $f'(x_0) = c \iff c \in f'(\mathbb{R})$.

Remark: Note that we can't use the Intermediate Value Theorem since we can't assume f' is a continuous function.

4. If $a_0 + \frac{1}{2}a_1 + \cdots + \frac{1}{n}a_{n-1} + \frac{1}{n+1}a_n = 0$, where $a_0, \ldots, a_n$ are real constants, prove that the equation

$$a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = 0$$

has at least one real root between 0 and 1.

Solution Let $f(x) = a_0 x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$. This function is clearly differentiable everywhere, with $f'(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$. Applying the mean value theorem we have $f(1) - f(0) = f'(c)(1 - 0)$ for some $c \in (0, 1)$. Clearly $f(0) = 0$ and from how the coefficients were constructed we also have $f(1) = 0$. Thus we must have $f'(c) = 0$.

5. Compute the second-order Taylor expansion of $f(x) = \sin^2 x + \cos x \sin x$ around the point $x_0 = \frac{\pi}{2}$

Solution The derivatives are

$$\begin{aligned} f'(x) &= \cos^2 x - \sin^2 x + 2\cos x \sin x \\ f''(x) &= 2\cos^2 x - 2\sin^2 x - 4\cos x \sin x \end{aligned}$$

hence $f'(x_0) = -1$ and $f''(x_0) = -2$, therefore

$$f(x) = -1 - \left(x - \frac{\pi}{2}\right) - \left(x - \frac{\pi}{2}\right)^2$$