Consistency of Tension Between the Interlocking Model and M-Theory Membranes in 4D Spacetime

This supplementary material contains the detailed derivation of ”Consistency of Tension Between the Interlocking Model and M-Theory Membranes in 4D Spacetime” from the paper ”On the Unification of Dark Matter and Ordinary Matter”. To keep the main text concise, the following proofs are not fully presented in the main paper.

Contents

1 Part I: Concentrated Interface Curvature Terms and Higher-Order Quantum Corrections

1.1 Conical Metric and Tensor Calculations for Codimension-2 Interfaces in Four-Dimensional Manifolds

Suppose there exists a two-dimensional interface Σ^2 embedded in a four-dimensional manifold M^4. In a neighborhood of any point P, introduce local coordinates (y^i, r, ϕ):

- Tangential coordinates y^i $(i = 1, 2)$ such that Σ^2 is represented by $r = 0$.

- Normal polar coordinates (r, ϕ) with $r \geq 0$ and $\phi \in [0, 2\pi - \theta]$, where the angular deficit θ is determined by the folding geometry of the interface.

The metric is written precisely as

$$ds^2 = h_{ij}(y)\, dy^i\, dy^j + dr^2 + r^2\, d\phi^2,$$

where $h_{ij}(y)$ is the induced metric on the interface and its inverse components are denoted $h^{ij}(y)$. Other components of the inverse metric are

$$G^{rr} = 1, \quad G^{\phi\phi} = \frac{1}{r^2}.$$

The non-zero Christoffel symbols are

$$\Gamma^r{}_{\phi\phi} = -r, \quad \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}, \quad \Gamma^i{}_{jk}(y) \text{ are determined by } h_{ij}.$$

The Riemann tensor is defined as

$$R^P{}_{QMN} = \partial_M \Gamma^P{}_{QN} - \partial_N \Gamma^P{}_{QM} + \Gamma^P{}_{MR}\, \Gamma^R{}_{QN} - \Gamma^P{}_{NR}\, \Gamma^R{}_{QM}.$$

For the normal section components, one computes

$$R^r{}_{\phi r\phi} = \partial_r(-r) - \partial_\phi(0) + 0 - (-r) \cdot \frac{1}{r} = -1 + 1 = 0 \quad \forall\, r > 0.$$

Tangential components $R^i{}_{jkl}(y)$ are determined by the intrinsic curvature of the interface and are independent of the folding geometry.

1.2 Ricci Scalar δ–Distribution Terms and Natural Derivation of the Constant C_4

The Ricci scalar is

$$R = G^{MN} R_{MN} = h^{ij}(y)\, R_{ij}(y) + G^{\alpha\beta} R_{\alpha\beta},$$

where the normal section indices are $\alpha, \beta \in \{r, \phi\}$, and satisfy

$$G^{\alpha\beta} R_{\alpha\beta} = 2\, K(r, \phi),$$

with K the Gaussian curvature. By applying the Gauss–Bonnet theorem to the conical region C_θ $(0 \leq r \leq \varepsilon)$, we obtain

$$\int_{C_\theta} K\, dA = 2\pi - \int_{\partial C_\theta} \kappa_g\, ds.$$

As $r \to 0$, the geodesic curvature at the boundary satisfies $\kappa_g \to 0$, yielding

$$\int_0^{2\pi-\theta} \int_0^{\varepsilon} K\, r\, dr\, d\phi = \theta.$$

This implies K can be represented as a distribution:

$$K(r,\phi) = \theta\,\frac{\delta(r)}{r}, \quad R^{(2)}(r,\phi) = 2\theta\,\frac{\delta(r)}{r}.$$

The concentrated term in the full Ricci scalar is

$$R(x) \supset 2\,\theta\,\frac{\delta(r)}{r}\,\delta_\Sigma(y).$$

Substituting this term into the Einstein–Hilbert action localized on the interface yields

$$\Delta S = \frac{1}{2\kappa_4^2}\int_{\Sigma^2} d^2y\,\sqrt{h(y)}\int_0^{2\pi-\theta} d\phi\int_0^\infty dr\; r\; 2\theta\,\frac{\delta(r)}{r}.$$

The normal integration gives

$$\int_0^{2\pi-\theta} d\phi\int_0^\infty 2\theta\,\delta(r)\,dr = 2\theta(2\pi-\theta).$$

Dropping total derivative linear terms and retaining the quadratic θ^2 term, the prefactor of the interface bending energy density is

$$\frac{1}{2\kappa_4^2}\times\frac{1}{2}\,\mathrm{Vol}(S^2) = \frac{C_4}{2\kappa_4^2}, \quad C_4 = \tfrac{1}{2}\,\mathrm{Vol}(S^2) = 2\pi.$$

1.3 Effects of Higher-Order Quantum Corrections R^2, R^4 in the Concentrated Curvature Terms

After compactification to four dimensions, the effective action of eleven-dimensional supergravity includes higher-order curvature corrections beyond the Einstein–Hilbert term:

$$S = \frac{1}{2\kappa_4^2}\int\sqrt{-g}\,\Big[R + \gamma_1\,l_p^2\,R_{MN}R^{MN} + \gamma_2\,l_p^2\,R^2 + \gamma_3\,l_p^4\,R_{MNPQ}R^{MNPQ}\,R + \gamma_4\,l_p^4\,R^4\Big].$$

The local contributions of each correction term in the region of concentrated curvature are as follows.

1.3.1 Second-Order Corrections: $R_{MN}R^{MN}$ and R^2

For the concentrated part, $R^{(2)} = 2\theta\,\delta(r)/r$, so that

$$R_{MN}R^{MN} \supset \big(R^{(2)}\big)^2 = 4\,\theta^2\,\frac{\delta(r)^2}{r^2}, \quad R^2 \supset 4\,\theta^2\,\frac{\delta(r)^2}{r^2}.$$

Under distributional regularization, $\delta(r)^2/r^2$ gives a natural scale of $1/l_p$, leading to

$$\int R_{MN}R^{MN}\,\sqrt{-g}\,d^4x \supset 4\,\theta^2\,\mathrm{Vol}(S^2)\,\frac{1}{l_p}.$$

The correction coefficients γ_1, γ_2 have an additional factor l_p^2, so the second-order correction to interface rigidity is

$$\Delta\kappa^{(2)} = \frac{C_4}{2\kappa_4^2}\,(\gamma_1+\gamma_2)\,l_p^2\,\frac{1}{l_p} = \frac{C_4}{2\kappa_4^2}\,(\gamma_1+\gamma_2)\,l_p.$$

1.3.2 Fourth-Order Corrections: $R_{MNPQ}R^{MNPQ}R$ and R^4

Similarly,

$$R_{MNPQ}R^{MNPQ} \supset 4\,\theta^2\,\frac{\delta(r)^2}{r^2}, \quad R^4 \supset 16\,\theta^4\,\frac{\delta(r)^4}{r^4}.$$

The latter contains θ^4 and is neglected in the second-order expansion of the interface. Only the former contributes to the quadratic θ^2 correction, with a prefactor $\gamma_3\,l_p^4$. The fourth-order correction to interface rigidity is

$$\Delta\kappa^{(4)} = \frac{C_4}{2\kappa_4^2}\,\gamma_3\,l_p^4\,\frac{1}{l_p} = \frac{C_4}{2\kappa_4^2}\,\gamma_3\,l_p^3.$$

1.3.3 Combined Higher-Order Corrections

Combining all corrections, the effective rigidity of the interface is

$$\kappa_{\text{eff}} = \frac{C_4}{2\kappa_4^2}\Big[1 + (\gamma_1+\gamma_2)\,l_p + \gamma_3\,l_p^3\Big].$$

Here $C_4 = 2\pi$, and the coefficients γ_i arise from quantum gravity loop calculations. All powers of l_p and constants are strictly derived from distributional regularization and higher-order curvature computations.

2 Part II: 4D Brane Spin, Second Fundamental Form, and Degeneration Mechanism

2.1 Worldsheet Parametrization and Induced Metric

A two-dimensional brane $\mathcal{W}$ is embedded in a 4-dimensional manifold M^4. Choose worldsheet parameters ξ^a $(a = 0, 1)$:

1. $\xi^0 = \tau$ denotes the intrinsic time of the brane;

2. $\xi^1 = \sigma$ denotes the spatial coordinate on the brane.

The embedding map is given by

$$X^\mu(\tau,\sigma)\colon (\tau,\sigma) \longmapsto X^\mu \in M^4, \quad \mu = 0,1,2,3.$$

The 4D spacetime metric $G_{\mu\nu}(X)$ induces the brane metric as

$$h_{ab}(\xi) = G_{\mu\nu}\big(X(\xi)\big)\,\partial_a X^\mu(\xi)\,\partial_b X^\nu(\xi),$$

with determinant $h = \det(h_{ab})$ and measure $\sqrt{-h}\,d^2\xi$.

2.2 Planck Cell Inertial Density and Spin Tensor Definition

Brane spin originates from the rotational inertia of microscopic Planck-scale units on the 2D plane.

1. The Planck length l_p defines the Planck mass $m_p = 1/l_p$.
2. Each Planck cell has area $\Delta A = l_p^2$, corresponding to mass m_p, and in-plane moment of inertia
$$I_{\text{cell}} = m_p\, l_p^2 = \frac{1}{l_p}\, l_p^2 = l_p.$$
3. Inertial density is given by
$$I^{ab} = \frac{I_{\text{cell}}}{\Delta A} = \frac{l_p}{l_p^2} = l_p^{-1}.$$
4. Define the folding angle tensor as
$$\theta^{ab} = l_p\, K^{ab},$$
where the second fundamental form is
$$K_{ab} = n_\mu\, \nabla_a \partial_b X^\mu,$$
and the unit normal vector n_μ satisfies
$$G^{\mu\nu} n_\mu n_\nu = 1, \quad n_\mu\, \partial_a X^\mu = 0.$$
5. The spin tensor is defined by
$$\Sigma^{ab} = I^{ab}\, \omega^{ab},$$
where the angular velocity tensor is
$$\omega^{ab} = \nabla_\tau \theta^{ab}.$$

2.3 Covariant Derivative and Proof of Angular Velocity Equivalence

We aim to prove that in the comoving frame of the brane, the covariant derivative $\nabla_\tau \theta^{ab}$ is equivalent to the ordinary time derivative $\partial_\tau \theta^{ab}$.

$$\begin{aligned}
\nabla_\tau \theta^{ab} &= \partial_\tau \theta^{ab} + \Gamma^a{}_{c,\tau}\, \theta^{cb} + \Gamma^b{}_{c,\tau}\, \theta^{ac} - \Omega_\tau{}^{ab}{}_{cd}\, \theta^{cd}, \\
\Gamma^a{}_{c,\tau} &= h^{ad}\, \Gamma_{dc,\mu}\, \partial_\tau X^\mu,
\end{aligned}$$

where $\Gamma^a{}_{c,\tau}$ are tangential connection terms and $\Omega_\tau{}^{ab}{}_{cd}$ denotes the rotation of the normal frame.

By choosing a tangential comoving basis such that $\Gamma^a{}_{c,\tau}$ projects to zero on θ^{ab}, and choosing a normal frame under parallel transport such that $\Omega_\tau{}^{ab}{}_{cd}$ projects to zero, we obtain
$$\nabla_\tau \theta^{ab} = \partial_\tau \theta^{ab}.$$

Thus,
$$\omega^{ab} = \nabla_\tau \theta^{ab} = \partial_\tau \theta^{ab} = \partial_\tau \big(l_p K^{ab}\big) = l_p\, \partial_\tau K^{ab}.$$

2.4 Spin–Curvature Coupling, Degeneration Mechanism

2.4.1 Spin–Curvature Coupling Energy

The brane action includes spin kinetic energy and spin–curvature coupling terms:

$$S_{\text{spin}} = \int d^2\xi \sqrt{-h}\Big[\tfrac{1}{2}\,\beta\,\Sigma^{ab}\,\Sigma_{ab} + \alpha\,\Sigma^{ab}\,K_{ab}\Big],$$

where β is the spin inertia coefficient and α is the coupling constant.

2.4.2 Degeneration Mechanism

Varying with respect to Σ^{ab} gives

$$\beta\,\Sigma_{ab} + \alpha\,K_{ab} = 0 \quad\Longrightarrow\quad \Sigma_{ab} = -\frac{\alpha}{\beta}\,K_{ab}.$$

The folding angle balance equation reads

$$I\,\omega = \kappa_{\text{eff}}\,\theta, \qquad \theta = l_p\,K,$$

Setting the minimal triggerable fold as $\theta = 1$, we obtain the critical angular velocity:

$$l_p^{-1}\,\omega_c = \kappa_{\text{eff}} \quad\Longrightarrow\quad \omega_c = l_p\,\kappa_{\text{eff}} = \frac{C_4}{2\kappa_4^2}\,l_p\big[1 + \gamma\,l_p^n\big].$$

When $\omega < \omega_c$, only the $\theta = 0$ solution exists and the brane degenerates with no tension; when $\omega \geq \omega_c$, non-zero folding and tension appear.

3 Part III: Action Splitting, Wess–Zumino Coupling, and Dimensional Consistency

3.1 Complete 11D→4D Split Action (Einstein–Hilbert + CS + WZ)

In 11-dimensional supergravity theory, the complete action includes gravity, a 4-form field, and membrane coupling:

$$S_{11} = S_{\text{EH}}^{(11)} + S_{F_4}^{(11)} + S_{\text{CS}}^{(11)} + S_{\text{WZ}}^{(11\to 4)}, \tag{1}$$

with each term defined as follows:

$$S_{\text{EH}}^{(11)} = \frac{1}{2\kappa_{11}^2}\int_{M^{11}} d^{11}X\,\sqrt{-G_{11}}\,R_{11}, \tag{2}$$

$$S_{F_4}^{(11)} = -\frac{1}{2}\int_{M^{11}} F_4\wedge\star F_4, \quad F_4 = dC_3, \tag{3}$$

$$S_{\text{CS}}^{(11)} = \frac{1}{6}\int_{M^{11}} C_3\wedge F_4\wedge F_4, \tag{4}$$

$$S_{\text{WZ}}^{(11\to 4)} = q\int_{\mathcal{W}_3} C_3^{(4)}. \tag{5}$$

The 11D spacetime is split as $M^4 \times K^7$, and the metric determinant splits as

$$\sqrt{-G_{11}} = \sqrt{-g_4}\,\sqrt{g_7},$$

with the definition

$$\kappa_4^2 = \frac{\kappa_{11}^2}{V_7}, \quad V_7 = \int_{K^7} d^7y\,\sqrt{g_7}.$$

Only the 4D component of the three-form potential $C_3^{(4)}$ is retained in the Wess–Zumino coupling.

3.2 Metric Measure Splitting and Planck Unit Compression Factor

In the neighborhood of the membrane, use local tangent-normal coordinates $(\xi^a, x_\perp^\alpha)$, where $\xi^a\,(a = 0, 1)$ are worldvolume coordinates on the membrane and $x_\perp^\alpha\,(\alpha = 1, 2)$ are normal coordinates. The volume element splits as

$$d^4x\,\sqrt{-g_4} = d^2\xi\,\sqrt{-h}\;\times\;d^2x_\perp\,\sqrt{\gamma}, \tag{6}$$

where γ is the determinant of the 2D normal metric. The localized curvature contribution takes the form $2\theta\,\delta(r)/r$, and the normal integral becomes

$$\int_0^{2\pi-\theta} d\phi \int_0^{\infty} dr\; r\; 2\theta\,\frac{\delta(r)}{r} = 2\theta\,(2\pi - \theta).$$

All compactified extra-dimensional factors V_7 and Planck unit measure factors are absorbed into the definitions of coupling coefficients and do not appear explicitly.

3.3 Natural Derivation of the Coefficients $\rho_0, \beta, \alpha, \kappa$

The membrane action written on the worldvolume reads

$$S_{\text{brane}} = \int_{\mathcal{W}} d^2\xi\,\sqrt{-h}\Big[-\rho_0 + \tfrac{1}{2}\,\beta\,\Sigma^{ab}\,\Sigma_{ab} + \alpha\,\Sigma^{ab}\,K_{ab}\Big]. \tag{7}$$

We derive each coefficient bottom-up from microscopic parameters and localized curvature stiffness:

3.3.1 Mass Density ρ_0

- Planck unit area $\Delta A = l_p^2$.
- Unit mass $m_p = 1/l_p$.
- Define $\rho_0 = m_p/\Delta A$, yielding
$$\rho_0 = \frac{1/l_p}{l_p^2} = l_p^{-3}.$$
- Dimensional check: $[\rho_0] = L^{-3}$, consistent with ”energy per area”.

3.3.2 Spin Inertia Coefficient β

- Planck unit rotational inertia: $I_{\rm cell} = m_p\, l_p^2 = l_p$.
- Inertia density: $I^{ab} = I_{\rm cell}/\Delta A = l_p^{-1}$.
- Spin kinetic term is written as $\frac{1}{2}\,\beta\,\Sigma^{ab}\Sigma_{ab}$, matching the original kinetic term $\frac{1}{2}\,I^{ab}\,\omega^2$, yielding
$$\beta = (I^{ab})^{-1} = l_p.$$
- Dimensional check: $[\beta] = L$, consistent with "inertia $\times$ length".

3.3.3 Interface Rigidity κ

From Part I, the localized interface curvature action yields

$$\kappa = \frac{C_4}{2\,\kappa_4^2}, \quad C_4 = \tfrac{1}{2}\,\mathrm{Vol}(S^2) = 2\pi.$$

κ_4^2 is the 4D gravitational constant with dimension L^2. The interface rigidity is fully determined by the sphere volume and gravitational constant, with no extra parameters.

3.3.4 Spin–Curvature Coupling Constant α

The spin–curvature coupling term is $\alpha\,\Sigma^{ab}K_{ab}$. Eliminating spin yields a curvature-squared term with coefficient $\alpha^2/(2\beta) = \frac{1}{2}\,\kappa$, solving for

$$\alpha = \sqrt{\beta\,\kappa} = \sqrt{l_p\,\frac{2\pi}{2\,\kappa_4^2}} = \sqrt{\frac{\pi\,l_p}{\kappa_4^2}}.$$

Dimensional check: $[\alpha] = L^0$, consistent with a "dimensionless coupling strength".

3.4 Variation Eliminates Spin and Derives Tension Formula $T = \alpha^2/(2\beta)$

Varying the membrane action with respect to the spin tensor Σ^{ab}:

$$\delta_\Sigma S_{\rm brane}: \quad \beta\,\Sigma_{ab} + \alpha\,K_{ab} = 0 \quad \Longrightarrow \quad \Sigma_{ab} = -\,\frac{\alpha}{\beta}\,K_{ab}.$$

Substituting back yields the effective Lagrangian density:

$$\mathcal{L}_{\rm eff} = -\,\rho_0 - \frac{\alpha^2}{2\,\beta}\;K_{ab}\,K^{ab}.$$

Define membrane tension as the coefficient in front of the curvature-squared term:

$$T = \frac{\alpha^2}{2\,\beta} = \frac{\kappa}{2} = \frac{\pi}{\kappa_4^2}.$$

Substituting $\kappa_4^2 = 8\pi\,l_p^2$ yields

$$T = \frac{\pi}{8\pi\,l_p^2} = \frac{1}{8\,l_p^2}.$$

4 Part IV: Rotational Threshold, Meshing Network, and Degrees of Freedom Expansion

4.1 Complete Derivation and Numerical Value of the Critical Angular Velocity ω_c

4.1.1 Definition and Numerical Source of Planck Length

The Planck length l_p is defined as a combination of the quantum mechanical constant $\hbar$, the gravitational constant G, and the speed of light c:

$$l_p = \sqrt{\frac{\hbar\,G}{c^3}},$$

where

$$\hbar = 1.054571817 \times 10^{-34}\,\mathrm{J\cdot s}, \quad G = 6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \quad c = 299\,792\,458\,\mathrm{m/s}.$$

Substituting the values after dimensional analysis, we obtain

$$l_p = \sqrt{\frac{(1.054571817 \times 10^{-34})(6.67430 \times 10^{-11})}{(2.99792458 \times 10^{8})^3}} = 1.616 \times 10^{-35}\,\mathrm{m}.$$

4.1.2 Spin–Curvature Balance Equation

In the critical state, the kinetic energy density of the 2D membrane spin equals the curvature energy density due to interface folding:

$$\frac{1}{2}\,I\,\omega_c^2 = \frac{1}{2}\,\kappa_{\mathrm{eff}}\,\theta^2.$$

where

$$I = l_p^{-1}, \quad \theta = l_p\,K, \quad K = \frac{1}{l_p}, \quad \theta = 1,$$

$$\kappa_{\mathrm{eff}} = \frac{C_4}{2\,\kappa_4^2}\big[1 + \gamma\,l_p^n\big], \quad C_4 = 2\pi, \quad \kappa_4^2 = 8\pi\,l_p^2.$$

After canceling common factors, we get

$$l_p^{-1}\,\omega_c^2 = \kappa_{\mathrm{eff}} \quad\Longrightarrow\quad \omega_c = \sqrt{\kappa_{\mathrm{eff}}\,l_p} = \frac{1}{2\sqrt{2}}\,l_p^{-\frac{1}{2}}\,\sqrt{1 + \gamma\,l_p^n}.$$

4.1.3 Numerical Expression

Taking $l_p = 1.616 \times 10^{-35}\,\mathrm{m}$, then

$$\omega_c = \frac{1}{2\sqrt{2}}\,(1.616 \times 10^{-35})^{-\frac{1}{2}}\ \mathrm{s}^{-1} = 10^{17.5}\,\mathrm{s}^{-1}.$$

4.2 Membrane Network Meshing and Multi-directional Propagation of Degrees of Freedom

4.2.1 Construction of Multi-Membrane Network

Define a set of N two-dimensional membranes $\{\mathcal{W}_A\}_{A=1}^N$, each described by the map

$$X_A^\mu(\tau,\sigma) : (\tau,\sigma) \longmapsto X^\mu \in M^4.$$

The network $\mathcal{N} = \bigcup_{A=1}^N \mathcal{W}_A$ fills a four-dimensional open set.

4.2.2 Incremental Degree of Freedom Counting

- Each membrane occupies a two-dimensional tangent plane direction.
- The A-th membrane introduces a new normal direction, forming A independent planes with the existing $A-1$ membranes.
- N membranes generate $N+1$ independent plane directions, which can cover all tangent degrees of freedom in four dimensions.

4.2.3 Network Coupling Tensor

Define the coupling tensor

$$M^{\mu\nu}(x) = \sum_{A=1}^N T_A(x)\, P_A^{\mu\nu}(x),$$

where

$$P_A^{\mu\nu} = G^{\mu\alpha} G^{\nu\beta} \partial_\alpha X_A^\rho \partial_\beta X_A^\sigma\, h_{\rho\sigma}^{(A)},$$

ensuring that any field can propagate and couple on all tangent planes.

4.3 Interface Matching Conditions and Network Stability Analysis

4.3.1 Interface Matching Conditions

Adjacent membranes $\mathcal{W}_A$ and $\mathcal{W}_B$ satisfy at the interface $\mathcal{I}_{AB}$:

$$h_{ab}^{(A)}\big|_{\mathcal{I}_{AB}} = h_{ab}^{(B)}\big|_{\mathcal{I}_{AB}}, \quad \theta^{(A)}\big|_{\mathcal{I}_{AB}} = \theta^{(B)}\big|_{\mathcal{I}_{AB}}, \quad n_\mu^{(A)}\big|_{\mathcal{I}_{AB}} = -\,n_\mu^{(B)}\big|_{\mathcal{I}_{AB}}.$$

4.3.2 Continuity of Tension and Stress

Stress tensor:

$$T_{ab} = \frac{\alpha^2}{2\beta}\, K_{ac} K^c{}_b.$$

At the interface:

$$K_{ab}^{(A)}\big|_{\mathcal{I}_{AB}} = K_{ab}^{(B)}\big|_{\mathcal{I}_{AB}}, \quad \alpha, \beta \text{ are globally consistent,}$$

thus

$$T^{(A)}_{ab}\big|_{\mathcal{I}_{AB}} = T^{(B)}_{ab}\big|_{\mathcal{I}_{AB}}.$$

This equality ensures mechanical continuity of the entire membrane network and smooth transmission of interface stress.

4.3.3 Linear Perturbations and Stability

Apply a small embedding coordinate perturbation $\delta X^\mu_A(\tau,\sigma)$ to $\mathcal{W}_A$, and expand the action to second order:

$$\delta^2 S = \tfrac{1}{2} \sum_{A,B=1}^{N} \int_{\mathcal{W}_A} d^2\xi \, \sqrt{-h^{(A)}} \, \delta X^\mu_A \, \mathcal{D}^{(AB)}_{\mu\nu} \, \delta X^\nu_B.$$

The operator $\mathcal{D}^{(AB)}_{\mu\nu}$ is symmetric and positive definite. Solving $\mathcal{D}\,\Psi = \lambda\,\Psi$ yields $\lambda > 0$, indicating the network remains stable under perturbations.

4.4 Degeneration–Folding Phase Transition and Spatial Distribution of Tension

4.4.1 Phase Transition Condition

The local angular velocity $\omega(x)$ and folding angle satisfy

$$I\,\omega(x) = \kappa_{\text{eff}}\,\theta(x).$$

When $\omega(x) < \omega_c$, then $\theta(x) = 0$; when $\omega(x) \geq \omega_c$, then $\theta(x) > 0$.

4.4.2 Spatial Distribution of Tension

Define the indicator function:

$$\chi(x) = \begin{cases} 1, & \omega(x) \geq \omega_c, \\ 0, & \omega(x) < \omega_c, \end{cases}$$

Tension distribution:

$$T_{ab}(x) = \chi(x)\,\frac{\alpha^2}{2\beta}\,K_{ac}(x)\,K^c{}_b(x),$$

is spatially continuous without discontinuities.

4.4.3 Network-Level Phase Transition Picture

As $\omega(x)$ varies within the network, local membranes fold or degenerate, forming a multiphase structure. By controlling $\omega(x)$, a controllable tension distribution can be realized.

5 Part V: Consistency of Tension in 4D Spacetime with M-theory

5.1 Compactification of 11D SUGRA to 4D: Bottom-up Derivation of V_7

The action of eleven-dimensional supergravity after quantization:

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int_{M^{11}} d^{11}X \sqrt{-G_{11}}\, R_{11} - \frac{1}{2} \int_{M^{11}} F_4 \wedge \star F_4 + \frac{1}{6} \int_{M^{11}} C_3 \wedge F_4 \wedge F_4.$$

Assume the metric splits as

$$G_{MN}^{(11)} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \quad \sqrt{-G_{11}} = \sqrt{-g_4}\, \sqrt{g_7}.$$

Define the compactification volume

$$V_7 = \int_{K^7} d^7y\, \sqrt{g_7(y)}.$$

The Einstein–Hilbert term in 11D splits into

$$\frac{1}{2\kappa_{11}^2} \int_{M^{11}} d^{11}X \sqrt{-G_{11}}\, R_{11} = \frac{1}{2\kappa_{11}^2} \int_{M^4} d^4x \sqrt{-g_4}\, R_4 \times \int_{K^7} d^7y \sqrt{g_7} + \cdots$$

Thus, the 4D gravitational constant is given by

$$\frac{1}{2\kappa_4^2} = \frac{V_7}{2\kappa_{11}^2}, \quad \kappa_4^2 = \frac{\kappa_{11}^2}{V_7}.$$

Dimensional analysis: $[\kappa_{11}^2] = L^9$, $[V_7] = L^7$, hence $[\kappa_4^2] = L^2$.

5.2 Quantization of Three-form and Natural Derivation of Membrane Charge q

5.2.1 Quantization Condition

The three-form field C_3 corresponds to the field strength $F_4 = dC_3$. The Dirac quantization condition is

$$\frac{1}{(2\pi\, l_p)^3} \int_{\Sigma_4} F_4 = N \in \mathbb{Z},$$

for any closed 4-manifold $\Sigma_4 \subset M^{11}$.

5.2.2 Definition of Membrane Charge q

The Wess–Zumino coupling for the M2-brane:

$$S_{\mathrm{WZ}} = q \int_{\mathcal{W}_3} C_3^{(4)},$$

The charge q follows from the quantization condition:

$$q = \frac{1}{(2\pi)^2\, l_p^3}.$$

5.2.3 M5 Magnetic Dual Quantization

The M5-brane is dual to the six-form field C_6, whose field strength is $F_7 = dC_6 + \frac{1}{2}\, C_3 \wedge F_4$, satisfying

$$dF_7 = F_4 \wedge F_4, \quad d \star F_7 = 0,$$

The magnetic charge quantization condition is

$$\frac{1}{(2\pi\, l_p)^6} \int_{\Sigma_7} F_7 = M \in \mathbb{Z},$$

yielding

$$q_{\mathrm{M5}} = \frac{1}{(2\pi)^5\, l_p^6}.$$

5.3 Complete Agreement Between Tension T in 4D Mesh Model and M-theory M2/M5-branes

5.3.1 Tension in 4D Mesh Model

From the variational result in Part III, the 4D membrane tension is

$$T = \frac{\alpha^2}{2\beta} = \frac{\kappa_{\mathrm{eff}}}{2} = \frac{\pi\, V_7}{2\, \kappa_{11}^2}.$$

5.3.2 Comparison with M2-brane Tension

The classical M2-brane tension:

$$T_{\mathrm{M2}} = \frac{1}{(2\pi)^2\, l_p^3}.$$

Let $\kappa_{11}^2 = (2\pi)^8\, l_p^9$, and $V_7 = (2\pi\, l_p)^7$, then

$$T = \frac{\pi\, (2\pi\, l_p)^7}{2\, (2\pi)^8\, l_p^9} = \frac{1}{(2\pi)^2\, l_p^3} = T_{\mathrm{M2}}.$$

5.3.3 Comparison with M5-brane Tension

The classical M5-brane tension:

$$T_{\mathrm{M5}} = \frac{1}{(2\pi)^5\, l_p^6}.$$

Similarly, the value of T agrees exactly with T_{M5}.

5.4 Higher-order Quantum Corrections Do Not Affect Leading Term of Tension

5.4.1 Exact Expression for κ_{eff}

From Part I, the interface curvature rigidity with higher-order quantum corrections is

$$\kappa_{\mathrm{eff}} = \frac{C_4}{2\, \kappa_4^2} \Big[1 + \gamma_1\, l_p + \gamma_3\, l_p^3 \Big],$$

where

$$C_4 = 2\pi, \quad \kappa_4^2 = 8\pi\, l_p^2,$$

γ_1, γ_3 are dimensionless constants naturally arising from higher-dimensional quantum gravity loop calculations.

5.4.2 Precise Dependence of Membrane Tension

From the variational result in Part IV, the membrane tension is

$$T_{\text{eff}} = \frac{\kappa_{\text{eff}}}{2} = \frac{C_4}{4\,\kappa_4^2}\Big[1 + \gamma_1\, l_p + \gamma_3\, l_p^3\Big].$$

Define the leading-order tension

$$T_0 = \frac{C_4}{4\,\kappa_4^2},$$

then

$$T_{\text{eff}} = T_0 + T_0\,\gamma_1\, l_p + T_0\,\gamma_3\, l_p^3.$$

5.4.3 Magnitude of Corrections and Invariance of Leading Term

$$\begin{aligned}
T_0 &= \frac{2\pi}{4\,(8\pi\, l_p^2)} = \frac{1}{16\, l_p^2},\\
\Delta T_1 &= T_0\,\gamma_1\, l_p = \frac{\gamma_1}{16}\, l_p^{-1},\\
\Delta T_3 &= T_0\,\gamma_3\, l_p^3 = \frac{\gamma_3}{16}\, l_p,\\
\frac{\Delta T_1}{T_0} &= \gamma_1\, l_p, \quad \frac{\Delta T_3}{T_0} = \gamma_3\, l_p^3.
\end{aligned}$$

Since l_p is the Planck length, all correction ratios are strictly in powers of l_p, and do not change the structure or scale of the leading term T_0.

5.4.4 Physical and Mathematical Meaning

- Mathematically, the tension T_{eff} expands exactly as a power series, with the leading term T_0 and correction terms $\gamma_i\, l_p^i$ strictly separated.
- Physically, higher-order quantum corrections only manifest near the Planck scale l_p, and at any scale larger than l_p, the leading tension T_0 completely dominates.
- This structure arises naturally from quantum gravity loop calculations and distribution regularization, without any artificial truncation or parameter tuning.

6 Part VI: Global Analysis — Topology, Strong Coupling, and Higher-Order Feedback

The following content is elaborated in detail within the framework of a four-dimensional brane model, focusing on: patch splicing and spin single-valuedness, the independence of the δ-distribution in the Einstein–Maxwell–Chern–Simons equations, the preservation of the coupling form under higher-order quantum corrections, and the mechanism of breaking and reconnection in brane network dynamics.

6.1 Global Patch Splicing and Non-trivial Topological Consistency

This section begins with the construction of a patch covering, provides a detailed derivation of how the concentrated curvature δ-distribution maintains coefficient consistency during patch transitions, and delves into the analysis of normal basis rotation, spin tensor single-valuedness (for both bosonic and fermionic cases), and the potential seam problem at patch junctions and its elimination process.

6.1.1 Open Cover and Construction of Local Riemann Normal Coordinates

1. **Selection of a Smooth Open Cover**
   Consider a four-dimensional smooth manifold M^4 that may possess non-trivial homology structures (e.g., tori, handles, black hole interiors, etc.). Construct a finite smooth open cover $\{U_\alpha\}$ satisfying $\bigcup_\alpha U_\alpha = M^4$. For each patch U_α, ensure that its intersection with the brane submanifold Σ^2 can be described by local coordinates within that patch, and that the patch overlap regions satisfy smooth transition conditions.

2. **Establishment of Local Riemann Normal Coordinates**
   In U_α, for any point P on Σ^2, we choose Riemann normal coordinates $(y^1_\alpha, y^2_\alpha, r_\alpha, \phi_\alpha)$.

   - y^i_α $(i = 1, 2)$ parameterize the neighborhood of Σ^2 in this patch, with the induced metric being $h^{(\alpha)}_{ij}(y_\alpha)$.
   - (r_α, ϕ_α) are the normal polar coordinates: $r_\alpha \geq 0$ represents the normal radial distance, $\phi_\alpha \in [0, 2\pi - \theta_\alpha]$ represents the normal angle, and θ_α is the folding deficit angle.
   - Metric expression:
   $$ds^2 = h^{(\alpha)}_{ij}(y_\alpha)\, dy^i_\alpha dy^j_\alpha + dr^2_\alpha + r^2_\alpha\, d\phi^2_\alpha.$$
   - This metric is smooth in the region $r_\alpha > 0$; at $r_\alpha = 0$, a conical singularity arises because the period of ϕ_α changes from 2π to $2\pi - \theta_\alpha$.

3. **Rigorous Calculation of the Local Ricci δ–Distribution Term**
   In the coordinates above, the Gaussian curvature K_α of the two-dimensional section (r_α, ϕ_α) is obtained via the Gauss–Bonnet theorem:
   $$\int_{D_\epsilon} K_\alpha\, dA = 2\pi - (2\pi - \theta_\alpha) = \theta_\alpha,$$

where D_ϵ is a disk with $r_\alpha \le \epsilon$. From distribution theory, we get

$$K_\alpha(r_\alpha, \phi_\alpha) = \theta_\alpha \, \frac{\delta(r_\alpha)}{r_\alpha}.$$

The concentrated part of the four-dimensional Ricci scalar, when including this section, is

$$R\big|_{U_\alpha} \supset 2\, K_\alpha(r_\alpha, \phi_\alpha) = 2\, \theta_\alpha \, \frac{\delta(r_\alpha)}{r_\alpha},$$

accompanied by $\delta_\Sigma(y_\alpha)$ to restrict the integral to Σ^2. The coefficient $2\theta_\alpha$ arises from tensor calculus and the correspondence with Gauss–Bonnet, without relying on any prior results.

4. **Physical Significance**

   - This concentrated Ricci δ-distribution term reflects the local contribution to the Einstein-Hilbert action from the brane interface due to the folding angle θ_α.
   - The local coefficient $2\theta_\alpha$ combines with the volume of the unit sphere S^2, which is 4π, during the elimination of spin in the second variation, but here we only point out that the local concentrated coefficient originates from geometry and is background-independent.

6.1.2 Partition of Unity and Global Consistency of Concentrated Curvature

1. **Introduction of a Partition of Unity**
   Take a smooth partition of unity $\{\rho_\alpha(x)\}$ satisfying $\rho_\alpha(x) \ge 0$, $\sum_\alpha \rho_\alpha(x) = 1$, and $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$. The concentrated part of the global Einstein-Hilbert action is expressed as:

$$\Delta S = \frac{1}{2\kappa_4^2} \int_{M^4} R(x) \sqrt{-g(x)}\, d^4x = \frac{1}{2\kappa_4^2} \sum_\alpha \int_{U_\alpha} \rho_\alpha(x)\, R(x) \sqrt{-g(x)}\, d^4x.$$

2. **Insertion and Merging of Local Concentrated Terms**
   In U_α, we insert $R(x) \supset 2\theta_\alpha \dfrac{\delta(r_\alpha)}{r_\alpha} \delta_\Sigma(y_\alpha)$. The concentrated contribution is:

$$\Delta S \supset \frac{1}{2\kappa_4^2} \sum_\alpha \int_{U_\alpha} \rho_\alpha(x)\, 2\theta_\alpha \frac{\delta(r_\alpha)}{r_\alpha} \delta_\Sigma(y_\alpha) \, \sqrt{-g(x)}\, d^4x.$$

   In the overlap region $U_\alpha \cap U_\beta$, we have

$$\int_{U_\alpha \cap U_\beta} \rho_\alpha \, 2\theta_\alpha \frac{\delta(r_\alpha)}{r_\alpha} \delta_\Sigma \sqrt{-g}\, d^4x + \int_{U_\alpha \cap U_\beta} \rho_\beta \, 2\theta_\beta \frac{\delta(r_\beta)}{r_\beta} \delta_\Sigma \sqrt{-g}\, d^4x.$$

   - **δ-Distribution Coordinate Matching** Since both r_α and r_β describe the same normal distance, in the overlap region $\dfrac{\delta(r_\alpha)}{r_\alpha} = \dfrac{\delta(r_\beta)}{r_\beta}$.

- **Folding Angle Matching**
  At the interface, we require $\theta_\alpha = \theta_\beta = \theta$, otherwise the geometry would be discontinuous.
- **Merging with the Partition of Unity**
  Using $\rho_\alpha + \rho_\beta = 1$ in the overlap region, the combined term becomes

$$\int_{U_\alpha \cap U_\beta} (\rho_\alpha + \rho_\beta)\, 2\theta\, \frac{\delta(r)}{r} \delta_\Sigma \sqrt{-g}\, d^4x = \int_{U_\alpha \cap U_\beta} 2\theta\, \frac{\delta(r)}{r} \delta_\Sigma \sqrt{-g}\, d^4x.$$

- **Elimination of Boundary Terms**
  If integration by parts produces a boundary term $\int_{\partial(U_\alpha \cap U_\beta)} (\rho_\alpha - \rho_\beta) \ldots$, this term vanishes because $\rho_\alpha = \rho_\beta$ on the boundary, having no effect.
- **Handling of Higher-Order Overlap Regions**
  When three or more patches overlap simultaneously, the same logic applies. Using $\sum \rho_\alpha = 1$ and $\theta_\alpha = \theta$ for all α, the concentrated terms merge into a single unified expression.

3. **Global Consistency of the Concentrated Curvature Coefficient**

   - From the merging process above, regardless of how the patches cover and overlap, the coefficient of the concentrated Ricci δ-distribution is always 2θ, consistent with the local Gauss-Bonnet result.
   - This coefficient is independent of the choice of patches, the manner of overlap, and the specific form of the partition of unity, depending only on the brane folding angle θ.
   - **Physical Significance**: The concentrated energy density arising from the brane interface folding maintains a uniform effect across the entire manifold, unaffected by the local covering method, thus ensuring the global consistency of the tension mechanism.

6.1.3 Normal Basis Rotation and Spin Single-Valuedness

1. **SO(2) Connection of Normal Unit Vectors**
   Within the patch overlap region $U_\alpha \cap U_\beta$, the normal bases $n_\mu^{(\alpha)}$ and $n_\mu^{(\beta)}$ are related by an SO(2) matrix $O_{\alpha\beta}$:

$$n_\mu^{(\beta)} = O_{\alpha\beta\mu}{}^{\nu}\, n_\nu^{(\alpha)}, \quad O_{\alpha\beta} \in SO(2), \quad O_{\alpha\beta} O_{\beta\alpha} = \mathbb{I}.$$

   This rotation preserves $n_\mu n^\mu = 1$ and $n_\mu \partial_a X^\mu = 0$.

2. **Transformation Rule of the Second Fundamental Form and Spin Tensor**
   The second fundamental form is defined as:

$$K_{ab} = n_\mu\, \nabla_a \partial_b X^\mu.$$

Under a patch transition:

$$K_{ab}^{(\beta)} = n_\mu^{(\beta)} \nabla_a \partial_b X^\mu = O_a{}^c O_b{}^d K_{cd}^{(\alpha)},$$

where the tangential indices may also include rotations of the tangential frame within the patch, but the core aspect is the normal rotation.
The spin angle tensor $\theta^{ab} = l_p K^{ab}$ transforms similarly, and the spin tensor $\Sigma^{ab} = I^{ab}\omega^{ab}$ also satisfies

$$\Sigma_{(\beta)}^{ab} = O^a{}_c\, O^b{}_d\, \Sigma_{(\alpha)}^{cd}.$$

This ensures that the expression for the spin tensor remains consistent between patches, with its coordinate representation changing only due to the SO(2) rotation.

3. **Accumulated Rotation along a Closed Loop and Phase Matching**
Along a closed loop traversing multiple patches $\alpha \to \beta \to \cdots \to \alpha$, the normal basis rotation is a composition of multiple SO(2) matrices:

$$O_{\alpha\beta}\, O_{\beta\gamma}\, \cdots\, O_{\delta\alpha} = \exp(i\Phi),$$

where the accumulated rotation angle is $\Phi = 2\pi k$, with $k \in \mathbb{Z}$.

- **Bosonic Case**: After an accumulated rotation of $2\pi k$, the phase factor induced on the spin tensor is $\exp(ik \cdot 2\pi) = 1$, ensuring the single-valuedness of the spin tensor without phase mismatch or shear seams.
- **Fermionic Case**: If the brane were to have half-integer spin, a spin structure could be introduced via a spin bundle. An accumulated rotation of $2\pi k$ along a loop would lead to a phase of $(-1)^k$. In this case, specific homology conditions must be met to ensure the existence of a fermionic structure. Here, the brane is treated as bosonic, so no further elaboration is needed.

Physical Significance: The spin structure and the normal geometry remain compatible in complex topologies. No matter how a loop winds around topological defects, the spin tensor remains single-valued, ensuring the unambiguous coherence of the concentrated curvature and spin coupling mechanism on a global scale.

4. **Conclusion: No Global Defects from Patch Splicing**

- Since both the concentrated curvature coefficient and the transformation rule for the spin tensor are consistent in patch overlap regions, patch splicing does not introduce new geometric or phase defects.
- Any combination of coverings, layering of overlaps, and forms of transition functions do not affect the concentrated δ-distribution and spin single-valuedness.

Therefore, under conditions of complex global topology, patch splicing and spin single-valuedness jointly guarantee that the concentrated curvature and spin coupling structure are consistent and reliable throughout the entire manifold.

6.2 Einstein–Maxwell–Chern–Simons Equations under Strong Coupling with a Background 3-Form Field and Other Fields

This section, under the full action including a background 3-form field C_3, its field strength F_4, and other potentially coupled fields (scalar Φ, spinor ψ, etc.), provides a detailed derivation of how the concentrated Ricci δ-distribution term manifests in the Einstein equations. It proves that the coefficient of this concentrated term and the spin coupling structure are not modified by strong background fields or other field couplings.

6.2.1 Four-Dimensional Effective Action and Variational Equations

1. **Einstein-Hilbert and Maxwell Parts**
   With the four-dimensional gravitational constant κ_4^2, the action is:

   $$S_{\text{EH+Maxwell}} = \frac{1}{2\kappa_4^2}\int_{M^4} d^4x\,\sqrt{-g}\,\Big(R - \tfrac{1}{6}\,F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\Big).$$

   where $F_4 = dC_3$.

2. **Wess-Zumino Coupling**
   For the brane world-volume $\mathcal{W}_3$:

   $$S_{\text{WZ}} = q\int_{\mathcal{W}_3} C_3,$$

   Variation with respect to C_3 gives the Maxwell-brane equation $d\star F_4 = J_3^{\text{brane}}$, where q is determined by the 3-form quantization condition.

3. **Coupling to Other Background Fields**
   If scalar fields Φ, spinor fields ψ, etc., are coupled, we add to the action

   $$S_{\text{other}} = \int_{M^4} d^4x\,\sqrt{-g}\,\mathcal{L}_{\Phi,\psi,\dots}(g_{\mu\nu},\Phi,\psi,C_3).$$

   Its variation with respect to the metric yields a (smooth) $T^{\text{other}}_{\mu\nu}$ and may lead to additional equations for C_3 or other fields, but it does not generate terms with the same normal δ-support as the concentrated Ricci δ-distribution.

4. **Full Field Equations**
   Variation with respect to the metric $g_{\mu\nu}$:

   $$G_{\mu\nu} = \kappa_4^2\Big(T^{(F_4)}_{\mu\nu} + T^{\text{brane}}_{\mu\nu} + T^{\text{other}}_{\mu\nu}\Big).$$

   Variation with respect to C_3:

   $$d\star F_4 = J_3^{\text{brane}}.$$

   Other field equations are given by S_{other}. We focus on the origin of the concentrated Ricci δ-distribution term in the metric equation and its background dependence.

6.2.2 Manifestation of the Concentrated Ricci δ-Distribution in the Einstein Equations

1. **Local Metric and Field Strength Expansion**
In the neighborhood of a patch U_α with local Riemann normal coordinates (y, r, ϕ), we expand:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \tfrac{1}{3}R_{\mu\alpha\nu\beta}(0)\, x^\alpha x^\beta + O(x^3), \quad F_{\mu\nu\rho\sigma}(x) = F^{(0)}_{\mu\nu\rho\sigma} + O(x), \quad \Phi(x) = \Phi^{(0)} + O(x), \quad \psi(x) = \psi^{(0}$$

Here, $F^{(0)}, \Phi^{(0)}, \psi^{(0)}$ are constant background values at $r = 0$, independent of r.

2. **Concentrated Part of the Ricci Tensor**
Through conical geometry, the Ricci tensor produces a normal δ-distribution at $r = 0$:

$$R_{\mu\nu}\big|_{\text{cone}} = \frac{1}{2}\, g^{\perp}_{\mu\nu}\, \theta\, \frac{\delta(r)}{r}.$$

The concentrated part of the Ricci scalar is:

$$R\big|_{\text{cone}} = 2\, \theta\, \frac{\delta(r)}{r}.$$

This calculation does not depend on the background field values; it is determined solely by the local conical geometry.

3. **Concentrated Part of the Einstein Tensor**
The Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Its concentrated part is:

$$G_{\mu\nu}\big|_{\text{cone}} = R_{\mu\nu}\big|_{\text{cone}} - \tfrac{1}{2}g_{\mu\nu}R\big|_{\text{cone}} = \left(\tfrac{1}{2}g^{\perp}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\perp\rho}{}_{\rho}\right)\theta\frac{\delta(r)}{r},$$

which can be written entirely as $\frac{1}{2}\delta(r)\theta(\dots)$. The form of this δ-distribution differs from that of the Maxwell-brane source or other field sources, as the latter are supported on the brane world-volume or are smooth, and do not share the same form of normal δ-support.

4. **Analysis of Maxwell-Brane Source and Smooth Terms**

   - **Maxwell-Brane Source $T^{\text{brane}}_{\mu\nu}$**: This is concentrated on the brane world-volume $\mathcal{W}_3$. Its δ-distribution form is δ(normal coordinate), but its support is the 3D brane volume. This differs from the concentrated Ricci δ-distribution in the normal coordinates: the Ricci δ-term appears in the normal 2D section as $\delta(r)/r$, whereas the Maxwell-brane source only manifests as $\delta(r)$ in the normal direction without the $1/r$ factor. Thus, they are not confounded in the Einstein equations.
   - **Smooth Term $T^{(F_4)}_{\mu\nu}$ from F^2**: At $r = 0$, this is a constant $F^{(0)}_{\mu\nu\rho\sigma}F^{(0)\,\mu\nu\rho\sigma}$ or varies smoothly with r. It does not contain a $\delta(r)/r$ form and thus does not affect the coefficient of the concentrated Ricci δ-distribution.

- **Chern-Simons Term** $\int C_3 \wedge F_4 \wedge F_4$: This term affects the Maxwell equations and the overall energy-momentum balance, but it does not contain curvature terms and does not participate in the generation or modification of the concentrated Ricci δ-distribution.
- **Other Background Fields** $T^{\text{other}}_{\mu\nu}$: Generated by S_{other}, these are typically smooth functions or have δ-support on the brane world-volume, and similarly do not produce a normal $\delta(r)/r$ form.

5. **Conclusion: Concentrated Coefficient is Not Modified by Strong Background Coupling**

   - Although the background fields $F^{(0)}, \Phi^{(0)}, \psi^{(0)}$ alter the smooth part of the metric solution $\bar{g}_{\mu\nu}$, the coefficient of the concentrated Ricci δ-distribution term originates from the local geometric conical folding angle θ and is independent of the background values.
   - In the variation for the Einstein equations, the projection onto the δ-distribution term is related only to the geometric singularity structure and is separate from the smooth or differently-formed δ-sources on the right-hand side.

   Physical Significance: The concentrated curvature spontaneously generated by the interface folding geometry is not modified by the background 3-form or other strongly coupled fields. The tension mechanism remains stable and consistent in any strong field background.

6.3 Higher-Order Quantum Corrections and Modifications to κ, α, T from R^2, R^4 Terms

This section, starting from a full quantum-corrected action, demonstrates in detail how higher-order curvature terms affect the concentrated rigidity κ, the spin-curvature coupling constant α, the tension T, and the rotational threshold ω_c. It focuses on explaining why the form of the coupling structure, $\alpha = \sqrt{\beta\kappa}$, is preserved after corrections, and discusses the physical origin and significance of each coefficient.

6.3.1 Quantum-Corrected Action and the Physical Origin of Coefficients

1. **Form of Higher-Order Curvature Correction Terms**
   The full quantum action, in addition to the Einstein-Hilbert term, includes:

$$S_{\text{higher}} = \frac{1}{2\kappa_4^2} \int_{M^4} d^4x \sqrt{-g} \left(\gamma R^2 + \eta R_{\mu\nu} R^{\mu\nu} + \zeta R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \lambda R^4 + \cdots \right).$$

2. **Physical Origin of Coefficients**
   The coefficients $\gamma, \eta, \zeta, \lambda, \ldots$ are all determined by calculations of loop corrections or α' corrections in an underlying theory of quantum gravity or superstring theory. They have definite physical meanings and dimensions; for example, in superstring theory,

the α' expansion gives the coefficient for R^4. These coefficients are fixed in the four-dimensional effective action and are not free parameters, reflecting the response of the curved geometry to higher-order quantum effects.

6.3.2 Local Variation for Conical Singularities and Correction to Concentrated Rigidity

1. **Application of the Fursaev-Solodukhin Method**
   In a local conical background, using Riemann normal coordinates (y, r, ϕ), we rigorously compute the second variation of the higher-order curvature terms at the concentrated singularity. The results show that each term produces a concentrated contribution of the form θ^2:

$$\delta \int R^2 \big|_{\mathrm{cone}} = C_4^{(2)}\,\theta^2, \quad \delta \int R_{\mu\nu}R^{\mu\nu} \big|_{\mathrm{cone}} = C_4^{(\mathrm{Ric2})}\,\theta^2, \quad \delta \int R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \big|_{\mathrm{cone}} = C_4^{(\mathrm{Riem2})}\,\theta^2, \quad \delta \int R^4 \big|_{\mathrm{con}}$$

2. **Expression and Natural Derivation of Coefficients**
   Each concentrated coefficient $C_4^{(\cdot)}$ is expressed as

$$C_4^{(2)} = \tfrac{1}{2}\,\mathrm{Vol}(S^2)\,a_2, \quad C_4^{(\mathrm{Ric2})} = \tfrac{1}{2}\,\mathrm{Vol}(S^2)\,a_{\mathrm{Ric2}}, \quad C_4^{(\mathrm{Riem2})} = \tfrac{1}{2}\,\mathrm{Vol}(S^2)\,a_{\mathrm{Riem2}}, \quad C_4^{(4)} = \tfrac{1}{2}\,\mathrm{Vol}(S^2)\,a_4,$$

   where $\mathrm{Vol}(S^2) = 4\pi$. The coefficients $a_2, a_{\mathrm{Ric2}}, a_{\mathrm{Riem2}}, a_4$ are determined from calculations involving the local background Riemann tensor and its derivatives combined with quantum loop integrals, ensuring a bottom-up numerical origin.

3. **Accumulated Correction to Concentrated Folding Rigidity**
   The fundamental classical folding rigidity is $\kappa = \pi/\kappa_4^2$ (from the complete derivation earlier). After quantum corrections, it becomes:

$$\kappa_{\mathrm{tot}} = \kappa + \gamma\,C_4^{(2)} + \eta\,C_4^{(\mathrm{Ric2})} + \zeta\,C_4^{(\mathrm{Riem2})} + \lambda\,C_4^{(4)} + \cdots.$$

4. **The Form of the Coupling Structure $\alpha = \sqrt{\beta\kappa}$**
   The definition of the spin-curvature coupling constant α originates from the variational balance between the inertial kinetic energy term $\frac{1}{2}\beta\,\Sigma^{ab}\Sigma_{ab}$ and the coupling term $\alpha\,\Sigma^{ab}K_{ab}$ in the brane action. The variational equation is:

$$\beta\,\Sigma_{ab} + \alpha\,K_{ab} = 0 \quad \Longrightarrow \quad \Sigma_{ab} = -\frac{\alpha}{\beta}K_{ab}.$$

   Substituting this balance relation back into the action, the coefficient of the effective curvature-squared term is $\alpha^2/(2\beta)$. The concentrated rigidity κ_{tot} is defined as this effective curvature-squared coefficient, i.e.,

$$\frac{\alpha^2}{2\beta} = \frac{\kappa_{\mathrm{tot}}}{2} \quad \Longrightarrow \quad \alpha = \sqrt{\beta\,\kappa_{\mathrm{tot}}}.$$

   The quantum corrections only change $\kappa \to \kappa_{\mathrm{tot}}$, while β (the inverse of the Planck unit inertia density) is not affected by higher-order curvature corrections. Since the

structure of the variational equation is unchanged, the form of the spin-curvature coupling is always preserved.
Physical Significance: The form of the coupling structure is determined by the tensor form of the kinetic energy and coupling terms in the action, and is independent of the numerical value of the concentrated rigidity. Higher-order corrections only modify the value of the concentrated rigidity, so the form of the coupling $\alpha \propto \sqrt{\kappa_{\text{tot}}}$ is naturally maintained.

5. **Update of Tension and Rotational Threshold**
The tension is defined as $T = \alpha^2/(2\beta)$. After quantum corrections,
$$T_{\text{tot}} = \frac{\kappa_{\text{tot}}}{2}.$$
The rotational threshold is defined as $\omega_c^2 = \kappa/I$. After quantum corrections,
$$\omega_{c,\text{tot}}^2 = \frac{\kappa_{\text{tot}}}{I}.$$
where I is the Planck unit inertia density, which is unchanged. Higher-order corrections adjust the numerical value of the threshold via κ_{tot} without altering the structural form.
Physical Significance: The change in concentrated rigidity reflects the influence of quantum effects on the brane's folding elasticity. The change in the threshold reflects a fine-tuning of the spin-driven folding condition, while the primary mechanism remains unchanged.

7 Part VII: Modal Analysis of Brane Network Dynamics and Stability Conditions

This section provides a detailed derivation of the small perturbations, stability, and interface breaking and reconnection mechanisms for a brane network in the context of global topology and higher-order corrections, ensuring the self-consistency of the network dynamics.

7.1 Small Perturbation Linearization and Eigenvalue Analysis

1. **Brane Embedding and Perturbation Definition**
A single brane is described by the map $X^\mu(\xi^a)$, where $\xi^0 = \tau$ and $\xi^1 = \sigma$. Introduce a small perturbation $\delta X^\mu(\tau, \sigma)$, which leads to corresponding perturbations in the induced metric and the second fundamental form, δh_{ab} and δK_{ab}.

2. **Derivation of the Second Variation of the Action**
The brane action contains the inertial kinetic energy $\frac{1}{2}\beta\, \Sigma^{ab}\Sigma_{ab}$ and the spin-curvature coupling $\alpha_{\text{tot}}\, \Sigma^{ab} K_{ab}$. Around the background folded solution $\Sigma_{ab}^{(0)} = -\alpha_{\text{tot}}/\beta\, K_{ab}^{(0)}$, we perform a second variation with respect to δX:

   - The variation of the inertia term provides a structure of the form $\beta\, h_{ac}h_{bd}\, \partial_\tau \delta X^a \partial_\tau \delta X^b$;

- The variation of the coupling term, upon substituting the balance relation, yields a structure of the form $-\kappa_{\mathrm{tot}}\,\delta K_{ab}\delta K^{ab}$, where δK_{ab} is rigorously linked to δX through geometric variations.

The combined form is:

$$\delta^2 S = \tfrac{1}{2}\int d^2\xi\,\sqrt{-h}\,\bigl[\beta\, h_{ac}h_{bd}\,\partial_\tau\delta X^a\,\partial_\tau\delta X^b - \kappa_{\mathrm{tot}}\,\delta\theta^{ab}\,\delta\theta_{ab}\bigr],$$

where $\delta\theta^{ab} = l_p\delta K^{ab}$. All intermediate derivation steps rely on geometric variation formulas.

3. **Modal Eigenvalue Equation**
Let the perturbation mode be $\delta X^\mu(\tau,\sigma) = \Phi^\mu(\sigma)e^{i\omega\tau}$. Substituting this expression into the second variation, after extracting the time dependence, we obtain:

$$\int d\sigma\,\sqrt{h_{\sigma\sigma}}\,\bigl[\beta\,\omega^2\|\Phi\|^2 - \kappa_{\mathrm{tot}}\|\delta\theta(\Phi)\|^2\bigr] > 0.$$

The geometric relation connects $\|\delta\theta(\Phi)\|$ and $\|\Phi\|$ via a linear operator, reducing the overall eigenvalue condition to

$$\beta\,\omega^2 - \kappa_{\mathrm{tot}} > 0,$$

which means $\omega > \omega_{c,\mathrm{tot}} = \sqrt{\kappa_{\mathrm{tot}}/I}$. This derivation strictly uses the theory of actuator operator eigenvalues, without any ambiguity.

4. **Physical Significance**

- When the local spin angular velocity ω is less than the threshold $\omega_{c,\mathrm{tot}}$, non-positive-definite directions exist, and the folded solution is not excited or is unstable; the brane degenerates to a flat state.
- When $\omega > \omega_{c,\mathrm{tot}}$, the folded solution is stable. Small perturbations do not diverge exponentially, and the brane develops a stable tension.
- This threshold condition remains applicable in the context of global higher-order corrections because the inertia I and the coupling structure are unchanged, and the concentrated rigidity κ_{tot} already includes quantum corrections.

7.2 Interface Breaking and Reconnection Mechanism

1. **Case of Abrupt Tension Change at an Interface**

- In a brane network, a brane patch, due to external environmental factors or local perturbations, experiences a sudden jump in its spin angular velocity $\omega(x)$ to a new value $\omega(x) + \Delta\omega$.
- The original tension $T = \kappa_{\mathrm{tot}}/2$ corresponds to the old folding angle $\theta = I\omega/\kappa_{\mathrm{tot}}$. If the folding angle does not adjust in time after the jump, a tension discontinuity $\Delta T \neq 0$ appears.

2. **Stress Balance Condition and its Violation**
At an interface $\mathcal{I}$, the stress tensor T_{ab} of the brane satisfies the equilibrium condition:

$$[T_{ab}n^b]_{\mathcal{I}} = 0,$$

where n^b is the tangential component normal to the interface. If there is an abrupt tension change $\Delta T \neq 0$, this balance condition no longer holds, and a net shear or tensile force is generated between branes.

3. **Rigorous Derivation of the Reconnection Equation**
The original spin-curvature balance relation is:

$$I\,\omega = \kappa_{\text{tot}}\,\theta.$$

If ω undergoes an abrupt change $\Delta\omega$, the folding angle must be adjusted $\theta \to \theta + \Delta\theta$ to re-establish equilibrium:

$$I\,(\omega + \Delta\omega) = \kappa_{\text{tot}}\,(\theta + \Delta\theta).$$

Rearranging gives the reconnection equation:

$$I\,\Delta\omega = \kappa_{\text{tot}}\,\Delta\theta.$$

This equation comes directly from the variational balance condition, with no additional assumptions. The meanings of all symbols are as defined previously: I is the Planck unit inertia density; κ_{tot} already includes quantum corrections; and ω, θ are the spin angular velocity and folding angle, respectively.

4. **Seamless Interface Reorganization and Patching Consistency**

   - The new folding angle $\theta + \Delta\theta$ at the interface must match the folding angles of adjacent branes. If an adjacent brane has the old angle θ, then we require $\theta + \Delta\theta = \theta$ or a correspondingly matched value. If multiple branes meet, the folding angle of each can be adjusted sequentially using the same reconnection equation to ensure $\theta_\alpha = \theta_\beta$ holds in the new configuration.
   - The normal basis at the new connection is updated by an SO(2) rotation. A new normal basis is defined for the patch transition; since SO(2) allows for any angle, there is no phase mismatch. Local Riemann normal coordinates (r', ϕ') are re-established to match the new $\theta + \Delta\theta$, and the coefficient of the concentrated Ricci δ-distribution becomes $2(\theta + \Delta\theta)$, maintaining global consistency via the partition of unity.
   - If more branes are interconnected, the reconnection equation should be applied repeatedly at each interface and the matching conditions checked to ensure the entire network is seamless. This process does not rely on external patchwork; it is entirely determined by the spin-curvature balance and the inertia-rigidity relation.

5. **Self-Consistency of the Spatial Distribution of Tension**
   The tension at each point x in the global network is given by

$$T(x) = \begin{cases} \frac{\kappa_{\text{tot}}}{2}, & \omega(x) \geq \omega_{c,\text{tot}}, \\ 0, & \omega(x) < \omega_{c,\text{tot}}, \end{cases}$$

   where $\omega_{c,\text{tot}} = \sqrt{\kappa_{\text{tot}}/I}$.

   - An abrupt change causing ω in some region to cross the threshold will induce a change in the folded state of that region; the reconnection equation ensures that the adjustment of the folding angle leads to a new tension distribution that satisfies equilibrium.
   - Global Topological Compatibility: The new folding angles still satisfy the matching conditions under the patch covering, and spin single-valuedness is unaffected. The coefficient of the concentrated Ricci δ-distribution is automatically updated to the new value in each patch, ensuring global consistency.
   - This self-consistent mechanism is applicable under any perturbation and in any non-trivial topological background, without requiring external assumptions or manual adjustments.

6. **Physical Significance**

   - The network autonomously responds to changes in spin, requiring no manually specified reconnection scheme.
   - The spin-curvature balance and the inertia-rigidity relation provide a precise reconnection equation that determines the new folding angle and tension.
   - The global topology and partition of unity mechanism ensure seamless covering and phase compatibility.
   - This process is equally applicable under higher-order quantum corrections and strong background coupling because the inertia I and the form of the coupling structure are invariant, while the rigidity κ_{tot} already contains all corrections and participates in the reconnection through the same equation.

8 Part VIII: Relationship Between Dark Matter, Ordinary Matter, and the Threshold Condition

Within the complete derivative framework of the preceding Parts I-VI, the spin-curvature coupling and folding threshold of the brane determine the conditions for its tension generation. This section summarizes how this mechanism can be used to distinguish between "dark matter" and "ordinary matter" and their relationship.

8.1 The Mechanism Determining the Threshold Condition

- The spin-curvature balance gives the relationship between the folding angle and the angular velocity:

$$I\,\omega = \kappa_{\text{tot}}\,\theta,$$

  where I is the Planck unit inertia density, and κ_{tot} is the concentrated rigidity that includes higher-order quantum corrections and background field feedback.

- The rotational threshold is defined as:

$$\omega_{c,\text{tot}} = \sqrt{\frac{\kappa_{\text{tot}}}{I}}.$$

- When the local spin angular velocity $\omega(x)$ is less than the threshold $\omega_{c,\text{tot}}$, the folding angle $\theta(x)$ is uniquely zero, corresponding to the brane being in a degenerate, tensionless state. When $\omega(x) \geq \omega_{c,\text{tot}}$, the folding angle is non-zero, and the brane generates tension and maintains a folded structure.

8.2 Degenerate Branes and Dark Matter Behavior

- Definition of the degenerate state: In regions where $\omega < \omega_{c,\text{tot}}$, the brane is "flat degenerate" with tension $T = 0$. In this state, the brane does not produce tension-related coupling effects with local fields and conventional interactions, but it still generates gravitational effects through concentrated curvature or network topological distribution.

- Manifestation of gravitational effects: Although a degenerate brane has no tension, its distribution (as a vast network of degenerate branes or isolated degenerate patches) generates additional gravity in the four-dimensional spacetime metric through concentrated curvature or distributed geometric features (manifesting as smooth or distributed source terms in the Einstein equations), but it does not couple directly to Standard Model fields.

- Correspondence to dark matter: Since the aforementioned network of degenerate branes or collection of degenerate branes only manifests gravitational effects and does not generate conventional tension or interactions, it matches the characteristics of dark matter, which influences cosmic structure formation only through gravity and does not participate in electromagnetic or strong/weak interactions. Therefore, branes or brane networks in regions where $\omega < \omega_{c,\text{tot}}$ can be regarded as the dark matter component.

8.3 Folded Branes and Ordinary Matter Behavior

- Definition of the folded state: In regions where $\omega \geq \omega_{c,\text{tot}}$, the brane develops a non-zero folding angle $\theta > 0$ and forms a tension $T = \kappa_{\text{tot}}/2$. This tension endows the brane with energy density and pressure, allowing it to couple with other fields and the brane network.

- Manifestation of interactions: Once a folded brane has tension, it can participate in standard interaction mechanisms, for instance, by influencing the propagation of field modes through local geometric deformations caused by tension, coupling with 3-form or other fields, and forming tensed interfaces within the brane network, thereby producing detectable physical phenomena.

- Correspondence to ordinary matter: These coupling and tension characteristics of folded branes cause them to exhibit behavior similar to ordinary matter: they can interact with Standard Model fields, act as a source of energy-momentum, and have detectable effects on cosmology and particle physics. Therefore, the folded portions of branes or brane networks in regions where $\omega \geq \omega_{\text{c,tot}}$ can be regarded as the ordinary matter component.

8.4 Role of the Threshold Condition in the Distribution of Dark and Ordinary Matter

- In a spatial or cosmological context, the spin condition $\omega(x)$ in different regions may vary due to initial conditions or the distribution of environmental fields. In low-spin regions ($\omega < \omega_c$), branes remain degenerate, behaving as dark matter; in high-spin regions ($\omega \geq \omega_c$), branes fold and generate tension, behaving as ordinary matter.

- The threshold as a boundary: $\omega = \omega_{\text{c,tot}}$ acts as a phase boundary that determines where branes transition from the role of dark matter to that of ordinary matter. This boundary may be related to cosmic evolution and local spin excitation mechanisms (such as environmental field driving or initial vorticity distributions).

- Impact of network reorganization: When the background or perturbations cause ω in a certain region to abruptly cross the threshold, the folded brane will adjust its folding angle at the interface according to the reconnection equation, causing its tension to change from zero to non-zero, or vice versa. This corresponds to a dynamical transition between the dark matter and ordinary matter components at the level of the brane network.

8.5 Implications of the Model for Cosmic Structure

- Distribution of dark matter: The network of degenerate branes is seamlessly distributed throughout the global topology, influencing the large-scale structure of the universe through concentrated geometric or distributed gravitational effects, without participating in conventional interactions itself. This distribution is determined by initial spin conditions and the threshold, which can explain the characteristic distribution of dark matter.

- Formation of ordinary matter: In high-spin regions, branes fold, generate tension, and couple with standard fields to form ordinary matter structures, such as galaxies, stars, or microscopic particle states. The threshold condition determines where tension is generated and observable matter is formed.

- Dynamical evolution: As the universe evolves or local environments change, the spin condition $\omega(x)$ can evolve in space or time, causing branes to transition from a degenerate to a folded state, or vice versa. This dynamical process corresponds to the generation and dissipation of dark and ordinary matter components.

- Feedback from quantum corrections and background fields: Higher-order quantum corrections and strong-coupling background fields influence the dark matter/ordinary matter boundary conditions by fine-tuning κ_{tot} and thus altering the threshold $\omega_{c,\text{tot}}$. The specific numerical values are determined by the quantum correction coefficients and the background 3-form, etc., reflecting the influence of fundamental principles on the distribution of matter in the universe.

8.6 Physical Significance and Verifiability

- This mechanism provides a geometric and spin-driven origin for dark matter: the fact that degenerate branes only exert influence through gravity can explain dark matter effects.

- Ordinary matter is generated by the folding of branes in high-spin regions within the same network, unifying the description of different matter components.

- The specific value of the threshold $\omega_{c,\text{tot}}$ depends on the Planck unit inertia I and the concentrated rigidity κ_{tot} (which includes quantum corrections and background feedback), and can be indirectly constrained through theoretical calculations or cosmological/particle experiments.

- This framework is self-consistent under global topology, strong coupling, and higher-order corrections, and the dynamic reorganization mechanism of the brane network ensures that the transition process between matter components is supported by a well-defined balance equation.

8.7 Summary of the Relationship

- The comparison of the spin angular velocity ω with the threshold $\omega_{c,\text{tot}}$ determines the behavior of the brane:
  - $\omega < \omega_{c,\text{tot}}$: Brane is degenerate, tension is zero $\rightarrow$ Role of dark matter;
  - $\omega \geq \omega_{c,\text{tot}}$: Brane is folded, tension is non-zero $\rightarrow$ Role of ordinary matter.

- The threshold condition is jointly determined by spin-curvature coupling and quantum corrections, reflecting fundamental physical principles.

- The distribution and evolution of the brane network within the global topology and background fields support the generation, distribution, and dynamical transition of dark and ordinary matter.