final solution 1, 4, 7

2012.6.23

Problem 1

(7%)By orthogonal relations

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos(nx)\cos(mx) = \delta_{nm}$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos(nx) = 2\delta_{n0}$$

So, we have

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} dx (f(x))^2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} dx \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\right)\left(\frac{a_0}{2} + \sum_{m=1}^{\infty} a_n \cos(mx)\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} dx \left(\frac{a_0}{2}\right)^2 + a_0\left(\sum_{n=1}^{\infty} a_n \frac{1}{\pi}\int_{-\pi}^{\pi} dx \cos(nx)\right) \\
&+ \sum_{n,m=1}^{\infty} \frac{a_m a_n}{\pi}\int_{-\pi}^{\pi} dx \cos(nx)\cos(mx) \\
&= \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2
\end{aligned}$$

Consider $f(x) = x^2$, we have

$$\begin{aligned}
&(4\%)\frac{1}{\pi}\int_{-\pi}^{\pi} dx (f(x))^2 = \frac{2\pi^4}{5} \\
&(4\%)a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x^2 = \frac{2\pi^2}{3} \\
&(4\%)a_n = \frac{4(-1)^n}{n^2}
\end{aligned}$$

(1%)Replacing to original formula

$$\frac{2\pi^4}{5} - \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 = \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)^2$$

We have

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

$\blacksquare$

Problem 4

(6%)We have

$$\begin{aligned}
\mathcal{L}[f(x)] &= F(s) \\
\mathcal{L}[f'(x)] &= sF(s) - f(0) \\
\mathcal{L}[f''(x)] &= s^2F(s) - sf(0) - f'(0) \\
&\vdots \\
\mathcal{L}[f^{(n)}] &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)
\end{aligned}$$

(4%)What we need to solve

$$\frac{d^2y(t)}{dt^2} + 4y(t) = 16te^{-2t}$$

With initial conditions $y(0) = 1, y'(0) = 0$. We get

$$\mathcal{L}[y] = Y = \frac{16 + s(s+2)^2}{(s+2)^2(s^2+4)}$$

(10%)Applying inverse Fourier transform

$$\begin{aligned}
y(t) &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dze^{zt}Y(z) \\
&= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dze^{zt}\frac{16 + z(z+2)^2}{(z+2)^2(z^2+4)} \\
&= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dze^{zt}\left(\frac{16}{(z+2)^2(z^2+4)} - \frac{z}{(z^2+4)}\right) \\
&= (2t+1)e^{-2t} - \cos(2t) + \cos(2t) \\
&= (2t+1)e^{-2t}
\end{aligned}$$

$\blacksquare$

Problem 7

(3%)It is obvious

$$\mathcal{F}\left[\frac{d^2 f(x)}{dx^2}\right] = -k^2 \hat{f}(k)$$
$$-x^2 f(x) = \mathcal{F}\left[\frac{d^2 \hat{f}(k)}{dk^2}\right]$$

(3%)Put $f_n(x) = p_n(x)e^{-\frac{x^2}{2}}$, we get $p_n(x)$ satisfied

$$\frac{d^2 p_n(x)}{dx^2} - 2x\frac{dp_n(x)}{dx} - (\mu^2 + 1)p_n(x) = 0$$

(4%)Since $x \in \mathbf{R}$, we can set

$$p_n(x) = \sum_{n=0}^{\infty} h_n x^n$$

Replacing to the differential eq., we get recursion relation

$$(n+1)(n+2)h_{n+2} = (2n + \mu^2 + 1)h_n$$

(3%)This relations shows that $h_{n+2} = 0$ when $\mu = \mu_n$, which satisfies

$$2n + \mu_n^2 + 1 = 0$$

(5%)Because $\hat{f}(k)$ and $f(x)$ satisfies the same ODE, $\hat{f}(k)$ should be

$$\hat{f}(k) = q_n(k)e^{-\frac{k^2}{2}}$$

(2%)For any n, we know $c_n f_n(x)$ also satisfies the ODE. We can deduce that

$$\mathcal{F}[f(x)] = c_n f_n(x)$$

i.e. c_n is the eigenvalue of the operator $\mathcal{F}[\cdots]$.
(There is a typo in this problem, so you will get full credit by point $f(x)$ and $\hat{f}(k)$ satisfied the same ODE) $\blacksquare$

應數三期末考 Q2Q5Q8 詳解

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2. Bessel function

$J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\tau}d\tau$,

$$
\begin{aligned}
\mathscr{L}[J_0(x)] &= \int_0^\infty e^{-sx}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\tau}d\tau\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{-x(s-i\sin\tau)}dx\,d\tau \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{-1}{s-i\sin\tau}e^{-x(s-i\sin\tau)}\Big|_0^\infty d\tau \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{s-i\sin\tau}d\tau && \text{(2分)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{s+i\sin\tau}{s^2+\sin^2\tau}d\tau && \text{(2分)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{s}{s^2+\sin^2\tau}d\tau+\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{i\sin\tau}{s^2+\sin^2\tau}d\tau}_{=0,\text{ odd function}} && \text{(2分)}
\end{aligned}
$$

由積分公式 $\int_0^\infty dx\,\frac{1}{a+b\sin^2 x}=\frac{sign(a)}{\sqrt{a(a+b)}}\arctan(\sqrt{\frac{a+b}{a}}\tan x)$ ，限制條件 $(\frac{b}{a}>1)$

令 $a=s^2$, $b=1$ 帶入

$$
\begin{aligned}
\mathscr{L}[J_0(x)] &= \frac{s}{2\pi}\frac{sign(s^2)}{\sqrt{s^2(s^2+1)}}\arctan(\sqrt{\frac{s^2+1}{s^2}}\tan x)\Big|_{-\pi}^{\pi} \\
&= \frac{s}{2\pi}\frac{1}{s\sqrt{(s^2+1)}}\cdot 2\pi \\
&= \frac{1}{\sqrt{(s^2+1)}} && \text{(4分)}
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L}[e^{ax}(\cos kx+i\sin kx)] &= \mathscr{L}[e^{ax+ikx}] \\
&= \int_0^\infty e^{-sx}e^{ax+ikx}\,dx \\
&= \int_0^\infty e^{-x(s-a-ik)}\,dx \\
&= -\frac{1}{s-a-ik}e^{-x(s-a-ik)}\Big|_0^\infty \\
&= \frac{(s-a)+ik}{(s-a)^2+k^2} && \text{(2分)}
\end{aligned}
$$

$\mathscr{L}[e^{ax}\cos kx]=\frac{(s-a)}{(s-a)^2+k^2}$ (2 分)

$\mathscr{L}[e^{ax}\sin kx]=\frac{k}{(s-a)^2+k^2}$ (2 分)

$\mathscr{L}[f(x)]=\frac{1-2s}{s^2+4s+5}=\frac{-2(s+2)}{(s+2)^2+1^2}+\frac{5}{(s+2)^2+1^2}$ (2 分)

$f(x)=\mathscr{L}^{-1}[\mathscr{L}[f(x)]]=-2e^{-2x}\cos x+5e^{-2x}\sin x$ (2 分)

應數三期末考 Q2Q5Q8 詳解

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5. Fourier Transform

$$\mathscr{T}[f(x)]=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\hat{f}(k),\qquad \mathscr{T}[g(x)]=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-ikx}dx=\hat{g}(k)$$

$$f*g(x)=\int_{-\infty}^{\infty}dy\, f(x-y)g(y)$$

$$\mathscr{T}[f*g(x)]=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\int_{-\infty}^{\infty}f(x-y)g(y)\,dydx$$

$$=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ikx}f(x-y)g(y)\,dydx$$

令 $x-y=t$， $y=s$ 則， $x=t+s$， $dxdy=dtds$ 帶入

$$\mathscr{T}[f*g(x)]=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ik(t+s)}f(t)g(s)\,dtds$$

$$=\sqrt{2\pi}\{\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-ikt}\,dt\}\{\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(s)e^{-iks}ds\}$$

$$=\sqrt{2\pi}\hat{f}(k)\hat{g}(k) \qquad (4分)$$

$$\mathscr{T}[f_1(x)]=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_1(x)e^{-ikx}\,dx$$

$$=\tfrac{1}{\sqrt{2\pi}}\int_{-1/2}^{1/2}e^{-ikx}\,dx$$

$$=\tfrac{1}{\sqrt{2\pi}}\tfrac{1}{-ik}e^{-ikx}\Big|_{-1/2}^{1/2}$$

$$=\tfrac{1}{\sqrt{2\pi}}\tfrac{2}{k}\sin(\tfrac{k}{2}) \qquad (4分)$$

$$\mathscr{T}[f_2(x)]=\mathscr{T}[f_1*f_1(x)]$$

$$=\sqrt{2\pi}\hat{f}_1(k)\hat{f}_1(k)$$

$$=\sqrt{2\pi}\tfrac{1}{2\pi}(\tfrac{2}{k})^2\sin^2(\tfrac{k}{2})$$

$$=\tfrac{1}{\sqrt{2\pi}}\tfrac{4}{k^2}\sin^2(\tfrac{k}{2}) \qquad (4分)$$

$$f_1(x)=\begin{cases}1 & ,\quad |x|\le\tfrac{1}{2}\\ 0 & ,\quad |x|>\tfrac{1}{2}\end{cases}$$

$$f_2=f_1*f_1(x)=\int_{-\infty}^{\infty}dy\,f_1(x-y)f_1(y)=\int_{-1/2}^{1/2}dy\,f_1(x-y)$$

令 $t=(-y)+x$ ， $dt=-dy$

$$f_2=\int_{-1/2+x}^{1/2+x}dt\,f_1(t)=\begin{cases}1-|x| & ,\quad |x|\le 1\\ 0 & ,\quad |x|>1\end{cases} \qquad (4分)$$

Parseval's theorem： $\int_{-\infty}^{\infty}f(x)\overline{g(x)}\,dx=\int_{-\infty}^{\infty}\hat{f}(k)\overline{\hat{g}(k)}\,dk$

$$\int_{-\infty}^{\infty}[\hat{f}_2(k)]^2dk=\int_{-\infty}^{\infty}[f_2(x)]^2dx$$

$$\Rightarrow \tfrac{1}{2\pi}\int_{-\infty}^{\infty}(\tfrac{2}{k})^4\sin^4(\tfrac{k}{2})dk=\int_{-1}^{1}[1-|x|]^2dx$$

$$\Rightarrow \tfrac{1}{\pi}\int_{-\infty}^{\infty}(\tfrac{2}{k})^4\sin^4(\tfrac{k}{2})d\tfrac{k}{2}=2\int_0^1(1-x)^2dx=2\int_0^1 1-2x+x^2dx=\tfrac{2}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty}(\tfrac{2}{k})^4\sin^4(\tfrac{k}{2})d\tfrac{k}{2}=\tfrac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty}(\tfrac{1}{y})^4\sin^4(y)dy=\tfrac{2\pi}{3} \qquad (4分)$$

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8. Fourier Inverse Transform

$$g(x)=\mathscr{F}^{-1}\left[\hat{g}(k)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k)e^{ikx}\,dk$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{k(1+ix)}-e^{-k(1-ix)}\,dk$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{1+ix}e^{k(1+ix)}\Big|_{-1}^{1}+\frac{1}{1-ix}e^{-k(1-ix)}\Big|_{-1}^{1}\right] \quad (2分)$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(1+ix)}-e^{-(1+ix)}}{1+ix}+\frac{e^{-(1-ix)}-e^{(1-ix)}}{1-ix}\right] \quad (2分)$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{1+x^2}\begin{bmatrix}(1-ix)[\cos x+i\sin x]e-(1-ix)[\cos x-i\sin x]e^{-1}\\+(1+ix)[\cos x+i\sin x]e^{-1}-(1+ix)[\cos x-i\sin x]e\end{bmatrix} \quad (2分)$$

$$=\frac{2i}{\sqrt{2\pi}}\frac{1}{1+x^2}\left[e\sin x+e^{-1}\sin x-ex\cos x+e^{-1}x\cos x\right] \quad (2分)$$

$$=\frac{4i}{\sqrt{2\pi}}\frac{1}{1+x^2}\left[\sin x\cosh 1-x\cos x\sinh 1\right]$$

Laplace equation

$$\begin{cases}\nabla^2 u(x,y)=\frac{\partial^2}{\partial x^2}u(x,y)+\frac{\partial^2}{\partial y^2}u(x,y)=0\\ u(x,y)=g(x)\end{cases}$$

$$\Rightarrow\begin{cases}\mathscr{F}\left[\nabla^2 u(x,y)\right]=\mathscr{F}\left[\frac{\partial^2}{\partial x^2}u(x,y)\right]+\mathscr{F}\left[\frac{\partial^2}{\partial y^2}u(x,y)\right]=-k^2\hat{u}(k,y)+\frac{d^2}{dy^2}\hat{u}(k,y)=0\\ \mathscr{F}\left[u(x,y)\right]=\mathscr{F}\left[g(x)\right]=\hat{g}(k)\end{cases}$$

由上式可知：$\hat{u}(k,y)=A(k)e^{ky}+B(k)e^{-ky}$ (2 分)

由題目條件：$\hat{u}(k,0)=A(k)+B(k)=e^{k}-e^{-k}$

$\hat{u}(k,1)=A(k)e^{k}+B(k)e^{-k}=0$

可得：$A(k)=-e^{-k}$ ， $B(k)=e^{k}$ (2 分)

所以：$\hat{u}(k,y)=e^{k(1-y)}-e^{-k(1-y)}$

$$u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,y)e^{ikx}\,dk$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{k(1-y+ix)}-e^{-k(1-y-ix)}\,dk$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-y+ix}e^{k(1-y+ix)}\Big|_{-1}^{1}+\frac{1}{1-y-ix}e^{-k(1-y-ix)}\Big|_{-1}^{1}\right] \quad (2分)$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(1-y+ix)}-e^{-(1-y+ix)}}{1-y+ix}+\frac{e^{-(1-y-ix)}-e^{(1-y-ix)}}{1-y-ix}\right] \quad (2分)$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{(1-y)^2+x^2}\begin{bmatrix}(1-y-ix)e^{(1-y)}[\cos x+i\sin x]\\-(1-y-ix)e^{-(1-y)}[\cos x-i\sin x]\\+(1-y+ix)e^{-(1-y)}[\cos x+i\sin x]\\-(1-y+ix)e^{(1-y)}[\cos x-i\sin x]\end{bmatrix} \quad (2分)$$

$$=\frac{2i}{\sqrt{2\pi}}\frac{1}{(1-y)^2+x^2}\begin{bmatrix}e^{(1-y)}(1-y)\sin x-e^{(1-y)}x\cos x\\+e^{-(1-y)}(1-y)\sin x+e^{-(1-y)}x\cos x\end{bmatrix} \quad (2分)$$

$$=\frac{4i}{\sqrt{2\pi}}\frac{1}{(1-y)^2+x^2}\left[(1-y)\sin x\cosh(1-y)-x\cos x\sinh(1-y)\right]$$

Problem 3 Since $f(x)$ is even, $b_n = 0$.

$$f(x) = e^{\alpha x} + e^{-\alpha x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0/2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(e^{\alpha x} + e^{-\alpha x})dx = \frac{2}{\pi\alpha}(e^{\alpha\pi} - e^{\alpha\pi})$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(e^{\alpha x} + e^{-\alpha x})\cos nx dx \\ &= \frac{1}{\pi}\Re[\int_{-\pi}^{\pi}(e^{\alpha x} + e^{-\alpha x})e^{inx}dx] \\ &= \frac{(-1)^n}{\pi}\frac{2\alpha}{\alpha^2+n^2}(e^{\alpha\pi} - e^{-\alpha\pi}) \end{aligned}$$

$$e^{\alpha x} + e^{-\alpha x} = \frac{1}{\pi\alpha}(e^{\alpha\pi} - e^{-\alpha\pi}) + \sum_{n=1}^{\infty}\frac{e^{\alpha\pi} - e^{-\alpha\pi}}{\pi}(-1)^n\frac{2\alpha}{\alpha^2+n^2}\cos nx$$

$$\Longrightarrow \frac{\pi}{2}\frac{e^{\alpha x} + e^{-\alpha x}}{e^{\alpha\pi} - e^{-\alpha\pi}} = \frac{1}{2\alpha} + \sum_{n=1}^{\infty}(-1)^n\frac{\alpha}{\alpha^2+n^2}\cos nx$$

14%

Setting $x = \pi$, $t = 2\alpha\pi$

$$\frac{\pi}{2}(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}) = \frac{\pi}{t} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\frac{\frac{t}{2\pi}}{\frac{t^2}{4\pi^2}+n^2}\cos n\pi$$

And

$$\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{2}{1-e^{-t}} - 1$$

$$\therefore \frac{1}{t}[\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}] = 2\sum_{n=1}^{\infty}\frac{1}{4\pi^2n^2+t^2}$$

6%

Problem 6 Expand $F(x)$ on $[-\pi, \pi]$

$$\sum_{n=-\infty}^{\infty} f(x + n2\pi) = \sum_{n=\infty}^{\infty} c_n e^{inx}$$

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx e^{-inx}\sum_m f(x+m2\pi) \\
&= \frac{1}{2\pi}\sum_m \int_{-\pi}^{\pi} dx e^{-inx} f(x+m2\pi) \\
&= \frac{1}{2\pi}\sum_m e^{i2\pi m}\int_{2\pi(m-1/2)}^{2\pi(m+1/2)} du e^{-nu} f(u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du e^{-inu} f(u) \\
&= \frac{1}{\sqrt{2\pi}}\hat{f}(n)
\end{aligned}
$$

$$
\therefore \sum_{n=-\infty}^{\infty} f(x+n\cdot 2\pi) = \sum_{k=-\infty}^{\infty}\hat{f}(k)e^{ikx}
$$

8%

Let $f(x) = e^{-a|x|}$, $a > 0$ ($|f(x) \to 0|$ as $x \to \infty$)

$$
\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\cdot 2\int_0^{\infty} e^{-ax}e^{-ikx}dx = \sqrt{\frac{2}{\pi}}\frac{a-ik}{a^2+k^2}
$$

6%

Let $x = 0$

$$
\Longrightarrow \sum_{n=-\infty}^{\infty} e^{-2\pi|n|a} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{a-ik}{a^2+k^2} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{a^2+k^2}
$$

$$
\begin{aligned}
\coth \pi a &= \frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}} = 1 + \frac{2}{e^{2\pi a}}\frac{1}{1-e^{-2\pi a}} = 1 + \frac{2}{e^{2\pi a}}\sum_{n=1}^{\infty} e^{-2\pi a n} \\
&= \sum_{n=-\infty}^{\infty} e^{-2\pi|n|a} \quad \text{for a>0}
\end{aligned}
$$

6%

Problem 9

$$
\begin{aligned}
\mathcal{L}[\frac{1}{\sqrt{t}}e^{-\frac{\lambda^2}{4-t}}] &= \int_0^{\infty}\frac{dt}{\sqrt{t}}e^{-(st+\frac{\lambda^2}{4t})} = \int_0^{\infty}\frac{dt}{\sqrt{t}}e^{-(\sqrt{st}-\frac{\lambda}{2\sqrt{t}})^2-\lambda\sqrt{s}} = 2\int_0^{\infty} dy e^{-\beta(y-y^{-1})^2}\sqrt{\beta}\cdot\frac{e^{-\lambda\sqrt{s}}}{\sqrt{s}} \\
& (\beta = \frac{\lambda}{2}\sqrt{s},\ y = \sqrt{\frac{st}{\beta}})
\end{aligned}
$$

$$
\begin{aligned}
2\int_0^\infty dy e^{-\beta(y-y^{-1})^2} &= \int_{-\infty}^\infty dy e^{-\beta(y-y^{-1})^2} = (\int_0^\infty + \int_{-\infty}^0) dy e^{-\beta(y-y^{-1})^2} \\
&= \int_0^\infty dy e^{-\beta(y-y^{-1})^2} + \int_0^\infty \frac{dy}{y^2} e^{-\beta(y-y^{-1})^2} \\
&= \int_0^\infty d(y-\frac{1}{y}) e^{-\beta(y-y^{-1})^2} \\
&= \int_{-\infty}^\infty dz e^{-\beta z^2} = \sqrt{\frac{\pi}{\beta}}
\end{aligned}
$$

$$
\implies \mathcal{L}[\frac{1}{\sqrt{t}} e^{-\frac{\lambda^2}{4-t}}] = \sqrt{\frac{\pi}{s}} e^{-\lambda\sqrt{s}}
$$

Differentiate w.r.t. λ,

$$
\mathcal{L}[\frac{\lambda}{2\sqrt{t^3}} e^{-\frac{\lambda^2}{4-t}}] = \sqrt{\pi} e^{-\lambda\sqrt{s}}
$$

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$$
\partial_t h(x,t) = \kappa^2 \partial_x^2 h(x,t)
$$

Laplace transform the above differential equation,

$$
sH(x,s) - H(x,0)(=T_0) = \kappa^2 \partial_x^2 H(x,s)
$$

$$
\implies H(x,s) = Ae^{-\frac{\sqrt{s}}{\kappa}x} + Be^{\frac{\sqrt{s}}{\kappa}x} + \frac{T_0}{s}
$$

Matching the boundary conditions, we get

$$
H(x,s) = -\frac{T_0}{s} e^{-\frac{\sqrt{s}}{\kappa}x} + \frac{T_0}{s}
$$

Apply the inverse Laplace transform to get the answer in real space

$$
\mathcal{L}^{-1}[\frac{T_0}{s}] = T_0
$$

$$
\mathcal{L}^{-1}[-\frac{T_0}{s} e^{-\frac{\sqrt{s}}{\kappa}x}] =?
$$

Define $-\frac{T_0}{s} \equiv F(s)$

$$
\begin{aligned}
\mathcal{L}^{-1}[-\frac{T_0}{s} e^{-\frac{\sqrt{s}}{\kappa}x}] &= \int_0^t dt f(t-\tau) \frac{x}{2\kappa\sqrt{\pi t^3}} e^{-\frac{x^2}{4\kappa^2}\frac{1}{\tau}} \\
&= 1 - erf(\frac{x}{2\kappa\sqrt{t}})
\end{aligned}
$$

$$
\begin{aligned}
\implies h(x,t) &= -T_0(1 - erf(\frac{x}{2\kappa\sqrt{t}})) + T_0 \\
&= T_0 erf(\frac{x}{2\kappa\sqrt{t}})
\end{aligned}
$$

10%