5.

First identity:

Recognizing w being out of integration contour and Eq.2= 0.(6 pts.)

The rest.(10 pts.)

Second identity.(4 pts.)

$$f(z) = \frac{1}{2\pi i}\oint_{c_r}\frac{f(\zeta)d\zeta}{\zeta - z} \tag{1}$$

$$\frac{1}{2\pi i}\oint_{c_r}\frac{f(\zeta)d\zeta}{\zeta - w} = 0 \tag{2}$$

with $w = \frac{r^2}{\bar{z}}$, $|z| < r$.

Eq.(2)= 0 because $|w| > r$

Eq.(1)−Eq.(2) $\Longrightarrow$

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{c_r}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right)f(\zeta)d\zeta \\
(\text{Let}\,\zeta = re^{i\phi}) &= \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{\zeta\bar{\zeta}}{\bar{z}}}\right)f(\zeta)i\zeta d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{z} - \bar{\zeta}}\right)f(\zeta)d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left[\frac{\zeta + z}{\zeta - z} + \frac{\bar{\zeta} + \bar{z}}{\bar{\zeta} - \bar{z}}\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi f(re^{i\phi})\Re\left[\frac{re^{i\phi} + z}{re^{i\phi} - z}\right]
\end{aligned}$$

$$\begin{aligned}
\Re\left[\frac{re^{i\phi} + \rho}{re^{i\phi} - \rho}\right] &= \frac{1}{2}\left(\frac{re^{i\phi} + \rho}{re^{i\phi} - \rho} + \frac{re^{-i\phi} + \rho}{re^{-i\phi} - \rho}\right) \\
&= \frac{r^2 - \rho^2}{r^2 - 2\rho\cos\phi + \rho^2}
\end{aligned}$$

8.

Obtaining the stationary pts. (2 pts.)
Contribution from each saddle pt. (7 pts. for each)
The rest. (4 pts.)

$$F(z) = \int_{-\infty}^{\infty} dt \exp[is(\frac{1}{5}t^5 + t)]$$

$$\phi(t) \equiv \frac{1}{5}t^5 + t$$

$$\phi'(t) = t^4 + 1 = 0 \implies t = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$$

By inspecting the behavior of $\Re[i\phi(t)]$ or by the direction of steepest descents at each saddle pt. We conclude that we only have to include the contributions from $e^{i\frac{\pi}{4}}$ and $e^{i\frac{3\pi}{4}}$.

(i)$t = e^{i\frac{\pi}{4}}$,

$$\begin{aligned}\phi(t) &\approx \phi(e^{i\frac{\pi}{4}}) + \frac{\phi"(e^{i\frac{\pi}{4}})}{2}(t - e^{i\frac{\pi}{4}})^2 \\ &= \frac{1}{5}e^{i\frac{5}{4}\pi} + e^{i\frac{\pi}{4}} + 2e^{i\frac{3}{4}\pi}(t - e^{i\frac{\pi}{4}})^2\end{aligned}$$

$$\begin{aligned}F_1 &= \exp[is(\frac{1}{5}e^{i\frac{5}{4}\pi} + e^{i\frac{\pi}{4}})]\int_{-\infty}^{\infty} dt \exp[s \cdot i \cdot 2e^{i\frac{3}{4}\pi}(t - e^{i\frac{\pi}{4}})^2] \\ &= \exp[is(\frac{1}{5}e^{i\frac{5}{4}\pi} + e^{i\frac{\pi}{4}})]\int_{-\infty}^{\infty} dre^{i\phi}\exp[2sr^2e^{i(\frac{\pi}{2}+\frac{3}{4}\pi+2\phi)}]\end{aligned}$$

Determining ϕ such that $\frac{\pi}{2} + \frac{3}{4}\pi + 2\phi = \pi$.

$$\begin{aligned}F_1 &= \exp[ise^{i\frac{\pi}{4}}(\frac{1}{5}e^{i\pi} + 1)]e^{-\frac{1}{8}\pi i}\sqrt{\frac{\pi}{2s}} \\ &= \exp(-\frac{4}{5\sqrt{2}}s)\exp(i\frac{4}{5\sqrt{2}}s)e^{-\frac{\pi}{8}i}\sqrt{\frac{\pi}{2s}}\end{aligned}$$

(ii)Similarly, when $t = e^{i\frac{3}{4}\pi}$,

$$F_2 = \exp(-\frac{4}{5\sqrt{2}}s)\exp[-i\frac{4}{5\sqrt{2}}s]e^{\frac{1}{8}\pi i}\sqrt{\frac{\pi}{2s}}$$

Comining the contributions from 2 saddle pts, we have

$$F_1 + F_2 = e^{-\beta s}\cos(\beta s - \frac{\pi}{8})\sqrt{\frac{2\pi}{s}},\ \beta = \frac{4}{5\sqrt{2}}$$

10.

Locating the poles and realizing that the rectangular contour integral becomes 0. (4 pts.)
I_1 & I_3 (5 pts.)
I_2 and taking the limit. (4 pts.)
I_4 (5 pts.)
The rest. (2 pts.)

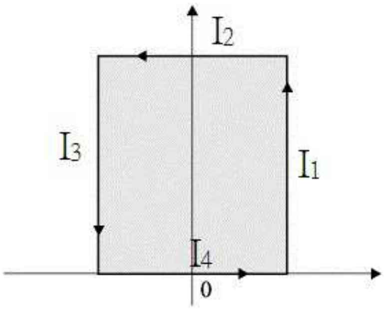
$$f(z) = \frac{z}{a - e^{-iz}}$$

First check that if there is any pole enclosed by the contour.

$$a = e^{-iz} = e^{-i(x+iy)} = e^{y}e^{-ix}$$

$$\therefore \quad -\infty < y = \ln a < 0$$
$$x = 2n\pi$$

So there is no pole in the upper complex plane.



$$I_1 + I_2 + I_3 + I_4 = 0$$

$$I_1 = \int_0^R \frac{\pi + iy}{a - e^{-i(\pi+iy)}} i dy = \int_0^R \frac{-y + i\pi}{a + e^y} dy$$

$$I_3 = \int_R^0 \frac{-\pi + iy}{a - e^{-i(-\pi+iy)}} i dy = \int_R^0 \frac{-y - i\pi}{a + e^y} dy = \int_0^R \frac{y + i\pi}{a + e^y} dy$$

$$\begin{aligned}
I_1 + I_3 &= 2\pi i \int_0^R \frac{dy}{a + e^y} \\
&= 2\pi i \int_0^R \frac{e^{-y} dy}{ae^{-y} + 1} \\
&= -\frac{2\pi i}{a} \int_0^R \frac{d(ae^{-y} + 1)}{ae^{-y} + 1} \\
&= -\frac{2\pi i}{a} [\ln(ae^{-R} + 1) - \ln(a + 1)] \\
&\to \frac{2\pi i}{a} \ln(1 + a) \text{ as } R \to \infty
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_\pi^{-\pi} \frac{x + iR}{a - e^{-i(x+iR)}} dx \\
&= \int_\pi^{-\pi} \frac{-x - iR}{a - e^{R}e^{-ix}} dx \to 0 \text{ as } R \to \infty
\end{aligned}$$

$$I_4 = \int_{-\pi}^{\pi} \frac{xdx}{a - e^{-ix}}$$

$$\begin{aligned}
\Im[I_4] &= \frac{1}{2i}\int_{-\pi}^{\pi}[\frac{x}{a-e^{-ix}} - \frac{x}{a-e^{ix}}]dx \\
&= \frac{1}{i}\int_{0}^{\pi}[\frac{x}{a-e^{-ix}} - \frac{x}{a-e^{ix}}]dx \\
&= \int_{0}^{\pi}\frac{-2x\sin x}{a^2-2a\cos x+1}dx \\
&= -\Im[I_1+I_3] \\
&= -\frac{2\pi}{a}\ln(1+a)
\end{aligned}$$

$$\therefore \int_0^{\pi} dx\frac{x\sin x}{1-2a\cos x+a^2} = \frac{\pi}{a}\ln(1+a),\ \ 0<a<1$$