應數三作業二詳解

教師：陳恒榆
助教：蕭鎬澤

1. (a) $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$, $e^{z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k}$, $e^{-z} = \sum_{k=0}^{\infty} \frac{1}{k!}(-z)^{k}$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z}) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{1}{k!} z^{k} - \sum_{k=0}^{\infty} \frac{1}{k!}(-z)^{k}\right)$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty}\left(\frac{1}{(2k)!} z^{2k} + \frac{1}{(2k+1)!} z^{2k+1}\right) - \sum_{k=0}^{\infty}\left(\frac{1}{(2k)!}(-z)^{2k} + \frac{1}{(2k+1)!}(-z)^{2k+1}\right)\right\}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1} = z + \sum_{k=0}^{\infty} \frac{1}{(2k+3)!} z^{2k+3}$$

$$\frac{1}{z^{2}} \sinh z = \frac{1}{z^{2}}\left(z + \sum_{k=0}^{\infty} \frac{1}{(2k+3)!} z^{2k+3}\right) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(2k+3)!} z^{2k+1} \quad \#$$

1. (b) $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$, $e^{z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k}$, $e^{-z} = \sum_{k=0}^{\infty} \frac{1}{k!}(-z)^{k}$

$$\cosh \frac{1}{z} = \frac{1}{2}\left(e^{\frac{1}{z}} + e^{-\frac{1}{z}}\right) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z}\right)^{k} + \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{z}\right)^{k}\right)$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty}\left(\frac{1}{(2k)!}\left(\frac{1}{z}\right)^{2k} + \frac{1}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1}\right) + \sum_{k=0}^{\infty}\left(\frac{1}{(2k)!}\left(-\frac{1}{z}\right)^{2k} + \frac{1}{(2k+1)!}\left(-\frac{1}{z}\right)^{2k+1}\right)\right\}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{1}{z}\right)^{2k} = 1 + \frac{1}{2}\left(\frac{1}{z}\right)^{2} + \sum_{k=1}^{\infty} \frac{1}{(2k+2)!}\left(\frac{1}{z}\right)^{2k+2}$$

$$z^{3} \cosh \frac{1}{z} = z^{3}\left\{1 + \frac{1}{2}\left(\frac{1}{z}\right)^{2} + \sum_{k=1}^{\infty} \frac{1}{(2k+2)!}\left(\frac{1}{z}\right)^{2k+2}\right\} = z^{3} + \frac{1}{2} z + \sum_{k=1}^{\infty} \frac{1}{(2k+2)!}\left(\frac{1}{z}\right)^{2k-1} \quad \#$$

2. B_n 的定義， $\frac{x}{e^{x}-1} = \sum_{n=0}^{\infty} B_n \frac{1}{n!} x^{n}$

$f(z) = \frac{1}{e^{z}-1}$,

$$zf(z) = \frac{z}{e^{z}-1} = \sum_{k=0}^{\infty} B_k \frac{z^{k}}{k!} = B_0 + B_1 z + \sum_{k=2}^{\infty} B_k \frac{z^{k}}{k!}$$

$$= B_0 + B_1 z + \sum_{k=1}^{\infty}\left(B_{2k} \frac{z^{2k}}{(2k)!} + B_{2k+1} \frac{z^{2k+1}}{(2k+1)!}\right) \quad \because B_{2k+1} = 0 \text{ , } \forall k \geq 1$$

$$= B_0 + B_1 z + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}$$

$$f(z) = \frac{B_0}{z} + B_1 + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k-1}}{(2k)!} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k-1}}{(2k)!}$$

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$ #

3. $|a| < |z| < \infty \Rightarrow 0 < \frac{|a|}{|z|} < 1$

$$\frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-\frac{a}{z}} = \frac{a}{z} \cdot \left(1 + \frac{a}{z} + \left(\frac{a}{z}\right)^{2} + \left(\frac{a}{z}\right)^{3} \cdots\right) = \sum_{k=1}^{\infty}\left(\frac{a}{z}\right)^{k}$$

$z = e^{i\phi}$ 代入，

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$$\frac{a}{e^{i\phi}-a}=\sum_{k=1}^{\infty}(\tfrac{a}{e^{i\phi}})^k=\sum_{k=1}^{\infty}a^k\cdot(e^{-ik\phi})=\sum_{k=1}^{\infty}a^k\cdot(\cos k\phi-i\sin k\phi)$$

$$\text{左式}=\frac{a}{\cos\phi+i\sin\phi-a}=\frac{a\cdot(\cos\phi-a-i\sin\phi)}{1-2a\cos\phi+a^2}=\frac{a\cos\phi-a^2}{1-2a\cos\phi+a^2}+\frac{-ia\sin\phi}{1-2a\cos\phi+a^2}$$

$$\text{右式}=\sum_{k=1}^{\infty}a^k\cdot(\cos k\phi-i\sin k\phi)=\sum_{k=1}^{\infty}a^k\cos k\phi-i\sum_{k=1}^{\infty}a^k\sin k\phi$$

$$\begin{cases}\displaystyle\sum_{k=1}^{\infty}a^k\cos k\phi &=& \dfrac{a\cos\phi-a^2}{1-2a\cos\phi+a^2}\\ \displaystyle\sum_{k=1}^{\infty}a^k\sin k\phi &=& \dfrac{a\sin\phi}{1-2a\cos\phi+a^2}\end{cases}\quad\#$$

4. 令 $f_1(z)=6z^3$， $g_1(z)=z^7-2z^5-z+1$， $f(z)=f_1(z)+g_1(z)$，
因為在$|z|=1$時，$|f_1(z)|=6>4+1=|z^7|+|-2z^5|+|-z|+1\ge|g_1(z)|$，
而 $f_1(z)$ 在$|z|<1$內有三重根，由 Rouche's theorem 可知，$f(z)$ 在$|z|<1$內有三個根。

5. 令 $f_1(z)=-6z^2$， $g_1(z)=2z^5+z+1$， $f(z)=f_1(z)+g_1(z)$，
因為在$|z|=1$時，$|f_1(z)|=6>2+1+1=|2z^5|+|z|+1\ge|g_1(z)|$，
而 $f_1(z)$ 在$|z|<1$內有二重根，由 Rouche's theorem 可知，$f(z)$ 在$|z|<1$內有二個根。
令 $f_2(z)=2z^5$， $g_2(z)=-6z^2+z+1$， $f(z)=f_2(z)+g_2(z)$，
因為在$|z|=2$時，$|f_2(z)|=64>24+2+1=|-6z^2|+|z|+1\ge|g_2(z)|$，
而 $f_2(z)$ 在$|z|<2$ 內有五重根，由 Rouche's theorem 可知，$f(z)$ 在$|z|<2$ 內有五個根。
由於 $f(z)$ 在$|z|=1$與$|z|=2$ 時皆無根，所以 $f(z)$ 在$1\le|z|\le 2$ 範圍內有三個根。

6. 令 $f(z)=\frac{\pi\cot\pi z}{(b+z)^2+a^2}$

$$\lim_{z\to 0}(z-0)\cdot f(z)=\lim_{z\to 0}\frac{\pi}{(b+z)^2+a^2}\cdot\frac{z}{\tan\pi z}=\lim_{z\to 0}\frac{\pi}{b^2+a^2}\cdot\frac{1}{\pi\sec^2\pi z}=\frac{1}{b^2+a^2}$$

$$\lim_{z\to n}(z-n)\cdot f(z)=\lim_{z\to n}\frac{\pi}{(b+z)^2+a^2}\cdot\frac{z-n}{\tan\pi z}=\lim_{z\to n}\frac{\pi}{(b+z)^2+a^2}\cdot\frac{1}{\pi\sec^2\pi z}=\frac{1}{(b+n)^2+a^2}$$

$$\lim_{z\to -b+ai}(z+b-ai)\cdot f(z)=\lim_{z\to -b+ai}(z+b-ai)\cdot\frac{\pi\cot\pi z}{(b+z)^2+a^2}=\frac{\pi}{2ai}\cdot\cot(-b+ai)\pi\ -(1)$$

$$\lim_{z\to -b-ai}(z+b+ai)\cdot f(z)=\lim_{z\to -b-ai}(z+b+ai)\cdot\frac{\pi\cot\pi z}{(b+z)^2+a^2}=\frac{\pi}{-2ai}\cdot\cot(-b-ai)\pi\ -(2)$$

by $\cot x+\cot y=\frac{\cot x\cot y-1}{\cot(x+y)}$, $\tan(ix)=i\tanh x$

$\therefore$(1)+(2) $\frac{\pi}{2a}\tanh 2a\pi\,[\cot(-b+ai)\pi\cdot\cot(-b-ai)\pi-1]$

$$\int_{CN}dz\,f(z)=2\pi i\{\sum_{k\in z}\mathrm{Res}(f(z),k)+\sum_{j=1}^{m}\mathrm{Res}(f(z),z_j)\}=0$$

$$\sum_{k\in z}\mathrm{Res}(f(z),k)=-\sum_{j=1}^{m}\mathrm{Res}(f(z),z_j)$$

$$\Rightarrow\sum_{k=-\infty}^{k=\infty}\frac{1}{(z+k)^2+a^2}=\frac{\pi}{2a}\tanh 2a\pi\,[\cot(-b+ai)\pi\cdot\cot(-b-ai)\pi-1]\quad\#$$

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7. 令 $f(z)=\frac{\pi\cot\pi z}{z^{2k}}$

$$\cot x=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}\cdot B_{2n}}{(2n)!}x^{2n-1}\quad,\quad \cot\pi x=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}\cdot B_{2n}}{(2n)!}(\pi x)^{2n-1}$$

$$\lim_{z\to n}(z-n)\cdot f(z)=\lim_{z\to n}\frac{\pi}{z^{2k}}\cdot\frac{z-n}{\tan\pi z}=\lim_{z\to n}\frac{\pi}{z^{2k}}\cdot\frac{1}{\pi\sec^2\pi z}=\frac{1}{n^{2k}}\quad,\quad n\in Z\setminus\{0\}$$

$$\begin{aligned}\operatorname{Res}\{f(z),0\}&=\lim_{z\to 0}\frac{1}{(2k-1)!}\cdot\frac{d^{2k-1}}{dz^{2k-1}}(z^{2k}\cdot f(z))=\lim_{z\to 0}\frac{\pi}{(2k-1)!}\cdot\frac{d^{2k-1}}{dz^{2k-1}}\cot\pi z\\&=\frac{\pi}{(2k-1)!}\lim_{z\to 0}\frac{d^{2k-1}}{dz^{2k-1}}\{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}\cdot B_{2n}}{(2n)!}(\pi z)^{2n-1}\}\\&=\frac{\pi^{2n}}{(2k-1)!}\frac{(-1)^k\cdot 2^{2k}\cdot B_{2k}}{(2k)!}(2k-1)!\ ,\ (n=k)\\&=\frac{(-1)^k\cdot 2^{2k}\cdot B_{2k}}{(2k)!}\pi^{2n}\end{aligned}$$

$$\int_{CN}dz\ f(z)=2\pi i\{\sum_{n\in z}\operatorname{Res}(f(z),n)+\sum_{j=1}^{m}\operatorname{Res}(f(z),z_j)\}=0$$

$$\Rightarrow\sum_{n\in z}\operatorname{Res}(f(z),k)=0$$

$$\Rightarrow\sum_{n\in N}\frac{1}{n^{2k}}+\sum_{n\in N}\frac{1}{(-n)^{2k}}+\operatorname{Res}(f(z),0)=0$$

$$\Rightarrow\sum_{n=1}\frac{1}{n^{2k}}=-\frac{1}{2}\operatorname{Res}(f(z),0)=\frac{(-1)^{k+1}\cdot 2^{2k-1}\cdot\pi^{2n}}{(2k)!}B_{2k}$$

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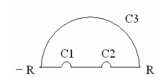
8. 令 $f(z)=\frac{1}{a^2-z^2}\cdot e^{iz}\quad,\quad g(x)=\frac{\cos x}{a^2-x^2}$



$$\oint dz\ f(z)=\int_{-R}^{-a-\varepsilon}dx\ g(x)+\int_{-a+\varepsilon}^{+a-\varepsilon}dx\ g(x)+\int_{+a-\varepsilon}^{+R}dx\ g(x)$$
$$+\int_{C_1,|z+a|<\varepsilon}dz\ f(z)+\int_{C_2,|z-a|<\varepsilon}dz\ f(z)+\int_{C_3,R\to\infty}dz\ f(z)=0$$

$$\begin{aligned}\int_{-\infty}^{\infty}dx\ \frac{\cos x}{a^2-x^2}&=-\int_{C_1,|z+a|<\varepsilon}dz\ \frac{1}{a^2-z^2}\cdot e^{iz}-\int_{C_2,|z-a|<\varepsilon}dz\ \frac{1}{a^2-z^2}\cdot e^{iz}\\&=\pi i[\lim_{z\to -a}\frac{z+a}{a^2-z^2}\cdot e^{iz}+\lim_{z\to a}\frac{z-a}{a^2-z^2}\cdot e^{iz}]\\&=\pi i[\frac{1}{2a}\cdot e^{-ia}-\frac{1}{2a}\cdot e^{ia}]\\&=\frac{1}{2a}\cdot\pi i[-i\sin a]\\&=\frac{\pi\sin a}{a}\end{aligned}$$

#

9. 由 $\sin x=0\Rightarrow x=0$ or $n\pi$, $n\in Z$

$\therefore \sin x=Ax(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\cdots=Ax\prod_{n=1}(x^2-n^2\pi^2)=x\prod_{n=1}(1-\frac{x^2}{n^2\pi^2})$

由 $\cos x=0\Rightarrow x=\frac{\pi}{2}+n\pi$, $n\in Z$

$\therefore \cos x=B(x-\frac{\pi}{2})(x+\frac{\pi}{2})(x-\frac{3\pi}{2})(x+\frac{3\pi}{2})\cdots=B\prod_{n=1}(x^2-\frac{(2n+1)^2\pi^2}{4})=\prod_{n=1}(1-\frac{4x^2}{(2n+1)^2\pi^2})$

$\cos x-\sin x=0\Rightarrow\cos x=\sin x\Rightarrow\tan x=0\Rightarrow x=\frac{\pi}{4}+n\pi$, $n\in Z$

$\cos x-\sin x=C(x-\frac{\pi}{4})(x+\frac{3\pi}{4})(x-\frac{5\pi}{4})(x+\frac{7\pi}{4})\cdots=C\prod_{n=1}(x+\frac{(-1)^n(2n-1)\pi}{4})=C^*\prod_{n=1}(1+\frac{(-1)^n\cdot 4x}{(2n-1)\pi})$

當 $x=0$ 時，$\cos 0-\sin 0=1=C^*\prod_{n=1}(1+\frac{(-1)^n\cdot 4\cdot(0)}{(2n-1)\pi})\quad,\quad C^*=1$ #

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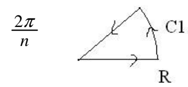
10. $\tan x+\cot x=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{2}{\sin 2x}$

$$\sin x=x\prod_{n=1}(1-\tfrac{x^2}{n^2\pi^2}) \quad , \quad \sin 2x=(2x)\prod_{n=1}(1-\tfrac{(2x)^2}{n^2\pi^2})$$

$$\tan x+\cot x=\frac{2}{\sin 2x}=\frac{2}{2x}\prod_{n=1}\frac{1}{(1-\frac{(2x)^2}{n^2\pi^2})}=\frac{1}{x}\prod_{n=1}\frac{n^2\pi^2}{n^2\pi^2-4x^2}=\frac{1}{x}\prod_{n=1}(1+\frac{4x^2}{n^2\pi^2-4x^2}) \quad \#$$

11. 令 $f(z)=\frac{1}{1+z^n}$, $z_0=e^{i\pi/n}$



$$\oint dz\, f(z)=2\pi i\mathrm{Res}(f(z),z_0)=2\pi i\lim_{z\to z_0}(z-z_0)f(z)$$

$$=2\pi i\lim_{z\to z_0}\frac{1}{n\cdot z^{n-1}}=\frac{2\pi i}{n}\lim_{z\to z_0}\frac{z}{z^n}=\frac{2\pi i}{n}\cdot\frac{e^{i\pi/n}}{e^{i\pi}}=\frac{-2\pi i}{n}e^{i\pi/n}$$

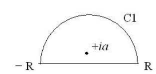
$$\oint dz\, f(z)=\int_0^R dx\, f(x)+\int_{C_1}dz\, f(z)+\int_R^0 d(x\cdot e^{2\pi i/n})\, f(x\cdot e^{2\pi i/n})$$

$$=\int_0^R dx\,\frac{1}{1+x^n}+0-e^{2\pi i/n}\int_0^R dx\,\frac{1}{1+x^n}$$

$$=(1-e^{2\pi i/n})\int_0^R dx\,\frac{1}{1+x^n}=\frac{-2\pi i}{n}e^{i\pi/n}$$

$$\Rightarrow\int_0^R dx\,\frac{1}{1+x^n}=\frac{\frac{-2\pi i}{n}e^{i\pi/n}}{(1-e^{2\pi i/n})}=\frac{-2\pi i}{n}\frac{1}{(e^{-i\pi/n}-e^{i\pi/n})}=\frac{-2\pi i}{n}\frac{1}{-2\sin(\pi/n)}=\frac{\pi}{n\sin(\pi/n)} \quad \#$$

12.(a) 令 $f(z)=\frac{e^{iz}}{(z^2+a^2)^2}$



$$\oint dz\, f(z)=2\pi i\mathrm{Res}(f(z),ia)=2\pi i\lim_{z\to ai}\frac{d}{dz}[\frac{e^{iz}}{(z+ia)^2}]$$

$$=2\pi i\lim_{z\to ai}[ie^{iz}(z+ia)^{-2}-2e^{iz}(z+ia)^{-3}]$$

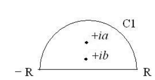
$$=2\pi i[ie^{-a}(2ai)^{-2}-2e^{-a}(2ai)^{-3}]$$

$$=2\pi i\frac{(1+a)e^{-a}}{4a^3 i}=\frac{(1+a)\pi}{2a^3e^a}$$

$$\oint dz\, f(z)=\int_{-\infty}^{\infty}dx\,\frac{\cos x}{(x^2+a^2)^2}+\underbrace{\int_{C_1,R\to\infty}dz\, f(z)}_{0}=\frac{(1+a)\pi}{2a^3e^a} \quad \#$$

$$\Rightarrow\int_{-\infty}^{\infty}dx\,\frac{\cos x}{(x^2+a^2)^2}=\frac{(1+a)\pi}{2a^3e^a}$$

12.(b) 令 $f(z)=\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$



$$\oint dz\, f(z)=2\pi i\{\mathrm{Res}(f(z),ia)+\mathrm{Res}(f(z),ib)\}$$

$$=2\pi i\{\lim_{z\to ai}(z-ai)f(z)+\lim_{z\to bi}(z-bi)f(z)\}$$

$$=2\pi i\{\lim_{z\to ai}\frac{e^{iz}}{(z+ai)(z^2+b^2)}+\lim_{z\to bi}\frac{e^{iz}}{(z^2+a^2)(z+bi)}\} \quad \#$$

$$=2\pi i\{\frac{e^{-a}}{(2ai)(-a^2+b^2)}+\frac{e^{-b}}{(-b^2+a^2)(2bi)}\}$$

$$=\frac{\pi}{(a^2-b^2)}\{\frac{1}{be^b}-\frac{1}{ae^a}\}$$

13. 令 $f(z)=\frac{(\log z)^2}{1+z^2}$, $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{i\pi/2}$, $-1=\cos\pi+i\sin\pi=e^{i\pi}$

$$x>0 \ , \ f(x)=\frac{(\log x)^2}{1+x^2} \ , \ f(-x)=\frac{(\log(-x))^2}{1+x^2}=\frac{[\log(x)+\log(-1)]^2}{1+x^2}=\frac{[\log(x)+i\pi]^2}{1+x^2}=\frac{[\log(x)]^2+i2\pi\log(x)-\pi^2}{1+x^2}$$

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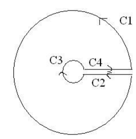
$$\oint dz\, f(z) = 2\pi i\{\mathrm{Res}(f(z),i)\} = 2\pi i \lim_{z\to i}(z-i)f(z)$$
$$= 2\pi i \lim_{z\to i}\tfrac{(\log z)^2}{z+i} = 2\pi i \tfrac{(\log i)^2}{2i}$$
$$= \pi(\tfrac{i\pi}{2})^2 = -\tfrac{\pi^3}{4}$$

$$\oint dz\, f(z) = \int_{-\infty}^{-\varepsilon} dx\, f(x) + \int_{C_2,|z-0|<\varepsilon} dz\, f(z) + \int_{\varepsilon}^{\infty} dx\, f(x) + \int_{C_1,R\to\infty} dz\, f(z)$$
$$= \int_0^\infty dx\, \tfrac{(\log x)^2}{1+x^2} + \int_0^\infty dx\, \tfrac{2i\pi\log x}{1+x^2} + \int_0^\infty dx\, \tfrac{-\pi^2}{1+x^2} + 0 + \int_0^\infty dx\, \tfrac{(\log x)^2}{1+x^2} + 0 = -\tfrac{\pi^3}{4}$$

$$\int_0^\infty dx\, \tfrac{(\log x)^2}{1+x^2} = -\tfrac{1}{2}\int_0^\infty dx\, \tfrac{2i\pi\log x}{1+x^2} + \tfrac{1}{2}\int_0^\infty dx\, \tfrac{\pi^2}{1+x^2} - \tfrac{\pi^3}{8}$$
$$= 0 + \tfrac{1}{2}\cdot\tfrac{\pi^3}{2} - \tfrac{\pi^3}{8} = \tfrac{\pi^3}{8}$$ #

14. 令 $f(z) = \frac{1}{1+z} z^{a-1}$



$$\oint dz\, f(z) = 2\pi i\{\mathrm{Res}(f(z),-1)\} = 2\pi i \lim_{z\to -1}(z+1)f(z)$$
$$= 2\pi i \lim_{z\to -1} z^{a-1} = 2\pi i \lim_{z\to -1}\tfrac{z^a}{z}$$
$$= (-2\pi i)(-1)^a = (-2\pi i)\{\cos a\pi + i\sin a\pi\}$$
$$= (-2\pi i)e^{ia\pi}$$

$$\oint dz\, f(z) = \int_{C_1} dz\, f(z) + \int_{C_2} dz\, f(z) + \int_{C_3} dz\, f(z) + \int_{C_4} dz\, f(z)$$
$$= \int_{C_1,R\to\infty} dz\, f(z) + \int_R^\varepsilon dx\, \tfrac{x^{a-1}}{1+x}\cdot e^{2\pi i(a-1)} + \int_{C_3,|z-0|<\varepsilon} dz\, f(z) + \int_\varepsilon^R dx\, \tfrac{x^{a-1}}{1+x}$$
$$= (1-e^{2\pi i(a-1)})\int_\varepsilon^R dx\, \tfrac{x^{a-1}}{1+x} = (-2\pi i)e^{ia\pi}$$

$$\int_\varepsilon^R dx\, \tfrac{x^{a-1}}{1+x} = \frac{(-2\pi i)e^{ia\pi}}{(1-e^{2\pi i(a-1)})} = \frac{(-2\pi i)e^{ia\pi}}{(1-e^{2\pi i a})} = \frac{-2\pi i}{e^{-ia\pi}-e^{i\pi a}} = \frac{\pi}{\sin a\pi}$$ #

15. 令 $n=2k$， $k\in Z$， $z=e^{i\phi}$， $dz = ie^{i\phi}d\phi = izd\phi$， $d\phi = \frac{1}{iz}dz$， $\cos\phi = \frac{1}{2}(z+\frac{1}{z}) = \frac{1}{2}(\frac{z^2+1}{z})$

$(2k)! = (2k-1)!!(2k)!! = (2k-1)!!(2^k)(k!)$， $(2k)!! = (2^k)(k!)$

$$\int_0^{2\pi} d\phi\,(\cos\phi)^n = \oint_{|z|=1}\tfrac{1}{iz}dz\,(\tfrac{1}{2}(\tfrac{z^2+1}{z}))^{2k}$$
$$= \oint_{|z|=1} dz\, \tfrac{(z^2+1)^{2k}}{iz^{2k+1}}(\tfrac{1}{2})^{2k}$$
$$= 2\pi i\{\mathrm{Res}(f(z), z=0)\}$$
$$= 2\pi i\{\lim_{z\to 0}\tfrac{1}{i(2k)!}\tfrac{d^{2k}}{dz^{2k}}(z^2+1)^{2k}(\tfrac{1}{2})^{2k}$$ #
$$= 2\pi\{\lim_{z\to 0}\tfrac{1}{(2k)!}(\tfrac{1}{2})^{2k}\tfrac{d^{2k}}{dz^{2k}}\sum_j C_j^{2k} z^{2j}\}$$
$$= 2\pi\cdot\tfrac{1}{(2k)!}\tfrac{(2k)!}{k!\cdot k!}(\tfrac{1}{2})^{2k}(2k)!\,,\quad j=k$$
$$= 2\pi\cdot\tfrac{1}{k!\cdot k!}(\tfrac{1}{2})^{2k}(2k-1)!!(2k)!!$$
$$= 2\pi\cdot\tfrac{(2k-1)!!}{(2k)!!}$$