

Solutions of Apply mathematics(III) HW#1

2012.3.14

Problem 1

$$T'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2}$$

Because $ad - bc \neq 0$, we have

$$\lim_{|z| \rightarrow \infty} T'(z) = 0$$

■

Problem 2

Since z^n is entire function, the linear combination also entire function. ■

Problem 3

We have $f(z) = u(x, y) + iv(x, y)$, and we may set

$$\overline{f(z)} = u(x, y) + iv(x, y) = U(x, y) - iV(x, y)$$

Where $U(x, y) = u(x, y)$ and $V(x, y) = -v(x, y)$. By Cauchy-Riemann eq, $\overline{f(z)}$ shows

$$\partial_x U = \partial_y V, \partial_y U = -\partial_x V$$

It means

$$\partial_x u = -\partial_y v, \partial_y u = \partial_x v$$

But consider $f(z)$, we have

$$\partial_x u = \partial_y v, \partial_y u = -\partial_x v$$

The last two eq. is consisted when $\partial_x u = \partial_x v = 0$. i.e. $f'(z) = 0$. It means $f(z) = \text{constant}$ in \mathcal{S} ■

Problem 4

(a)

$$u(x, y) = x^2 - y^2$$

$$\partial_x u = 2x = \partial_y v \Rightarrow v = 2xy + C(x)$$

$$\partial_x u = 2y = \partial_y v \Rightarrow v = 2xy + C(y)$$

It means $v = 2xy + c$

(b)

$$u(x, y) = \cosh y \sin x$$

$$\partial_x u = \cosh y \cos x = \partial_y v \Rightarrow v = \sinh y \cos x + C(x)$$

$$\partial_x u = \sinh y \sin x = -\partial_y v \Rightarrow v = \sinh y \cos x + C(y)$$

It means $v = \sinh y \cos x + c$

(c)

$$u(x, y) = 2x^2 + x + 1 - 2y^2$$

$$\partial_x u = 4x + 1 = \partial_y v \Rightarrow v = 4xy + y + C(x)$$

$$\partial_x u = -4y = -\partial_y v \Rightarrow v = 4xy + C(y)$$

It means $v = 4xy + y + c$

(d)

$$u(x, y) = \frac{x}{x^2 + y^2}$$

$$\partial_x u = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \partial_y v \Rightarrow v = \frac{-y}{x^2 + y^2} + C(x)$$

$$\partial_x u = \frac{2xy}{(x^2 + y^2)^2} = -\partial_y v \Rightarrow v = \frac{-y}{x^2 + y^2} + \frac{1}{2y} + C(y)$$

We may set $\frac{1}{2y} + C(y) = C'(y)$ and get $v = \frac{-y}{x^2 + y^2} + c$ ■

Problem 5

We have $x = r \cos(\phi)$, $y = r \sin(\phi)$. By Chain rule

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \phi}$$

change (x, y) to $(r \cos(\phi), r \sin(\phi))$. We get

$$\partial_r v = \partial_x v \cos(\phi) + \partial_y v \sin(\phi)$$

$$\partial_\phi v = -r \partial_x v \sin(\phi) + r \partial_y v \cos(\phi)$$

Replace $\partial_x v, \partial_y v$ by Cauchy-Riemann eq $\partial_x u = \partial_y v, \partial_y u = -\partial_x v$.

$$\partial_r v = -\partial_y u \cos(\phi) + \partial_x u \sin(\phi)$$

$$\partial_\phi v = -r \partial_y u \sin(\phi) + r \partial_x u \cos(\phi)$$

Compair equations of real part

$$\partial_r u = \partial_x u \cos(\phi) + \partial_y u \sin(\phi)$$

$$\partial_\phi u = -r \partial_x u \sin(\phi) + r \partial_y u \cos(\phi)$$

We get

$$r \partial_r u = \partial_\phi v, \partial_\phi u = -r \partial_r v$$

■

Problem 6

By $r \partial_r u = \partial_\phi v, \partial_\phi u = -r \partial_r v$, we have

$$\frac{\partial^2 u}{\partial \phi^2} = -r \frac{\partial^2 v}{\partial r \partial \phi}, \frac{\partial u}{\partial r} + r \frac{\partial^2 u}{\partial r^2} = r \frac{\partial^2 v}{\partial r \partial \phi}$$

Replace RHS, we get

$$r^2 \frac{\partial^2 u}{\partial \phi^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$$

By similiarly operation, v is also satisfied

$$r^2 \frac{\partial^2 v}{\partial \phi^2} + r \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2}$$

■

Problem 7

Set $\omega = f(z)$. We have $z = f^{-1}(\omega)$

(a)

$$\omega = \frac{az + b}{cz + d} \Rightarrow z = \frac{a'\omega + b'}{c'\omega + d'}$$

Where $a' = -d$, $b' = b$, $c' = c$, $d' = -a$. It's easy to check

$$a'd' - b'c' = (-d)(-a) - bc \neq 0$$

(b)

$$z = \frac{az + b}{cz + d} \Rightarrow cz^2 + (d - a)z - b = 0$$

The maximum power of eq. is two, which infers at most two solutions of z .

(c)

We have composition of translation and dilatation $Z = cz + d$. There is also inversion transform $W = \frac{1}{Z}$. Möbius transform can be written as

$$\omega = \frac{a}{c} + \frac{bc - ad}{c}W$$

(d)

Since translation and dilation is trivial, we may choose

$$z = \frac{r}{\omega} + \xi$$

Putting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, $\alpha = \frac{a+ib}{2}$, straight line $ax + by = c$ can be written as

$$\bar{\alpha}z + \alpha\bar{z} = c$$

Applying the transformation, it becomes

$$\bar{\alpha}\frac{r}{\omega} + \alpha\frac{r}{\bar{\omega}} = c - \xi(\alpha + \bar{\alpha}) \Rightarrow \bar{\alpha}\bar{\omega} + \alpha\omega = \frac{c - \xi(\alpha + \bar{\alpha})}{r}|\omega|^2$$

Replacing $\omega = u + iv$, we get

$$\frac{c - \xi(\alpha + \bar{\alpha})}{r}(u^2 + v^2) = au - bv$$

$c - \xi(\alpha + \bar{\alpha}) = 0$, it becomes straight line, or it will be a circle. So as a circle

$$k(x^2 + y^2) + ax + by + c = 0 \Rightarrow k|z|^2 + \bar{\alpha}z + \alpha\bar{z} = c - \xi(\alpha + \bar{\alpha})$$

By the transform, we get

$$kr^2 + \bar{\alpha}r\bar{\omega} + \alpha r\omega = (c - \xi(\alpha + \bar{\alpha}))|\omega|^2 \Rightarrow kr^2 + r(au - bv) = (c - \xi(\alpha + \bar{\alpha}))(u^2 + v^2)$$

As the forward discussion, it is circle or straight line.

(e)

We may set

$$x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2i}, u = \frac{\omega + \bar{\omega}}{2}, v = \frac{\omega - \bar{\omega}}{2i}$$

x,y-axis are represented as $z = \bar{z} = 0$. The two straight lines, $u \pm v = 0$ can be written as

$$(1 - i)\omega - (1 + i)\bar{\omega} = 0$$

$$(1 - i)\omega + (1 + i)\bar{\omega} = 0$$

It implies $c(1 - i)\omega = 0 = z$, where c is a constant. Because $|z| = |\omega| = 1$, the constant $c = 1/\sqrt{2}$. i.e.

$$\omega = \frac{(1 + i)}{\sqrt{2}}z = e^{i\frac{\pi}{4}}z$$

■

Problem 8

Length of curve

$$L = \int_b^a dt |\gamma'(t)|^2$$

(a)

$$\gamma(t) = 3t + i \Rightarrow \gamma'(t) = 3 \Rightarrow \int_{-1}^1 3dt = 6$$

(b)

$$\gamma(t) = i \sin t \Rightarrow \gamma'(t) = i \cos t \Rightarrow \int_{-\pi}^{\pi} dt |\cos t| = 4 \int_0^{\pi/2} dt \cos t = 4$$

■

Problem 9

(a)

$$\gamma(t) = (1 - i)t, 0 \leq t \leq 1$$

$$\int_{\gamma} dz e^{3z} = \int_0^1 d[(1 - i)t] e^{(3-3i)t} = \frac{e^{(3-3i)} - 1}{3}$$

(b)

Since e^{3z} is entire function, any integral of close path should be zero. i.e.

$$\oint_{|z|=3} dz e^{3z} = 0$$

(c)

By the argument from (b), the integral is equal a straight line from $x = 0$ to $x = 1$. Set $\bar{\gamma}(t) = (1 + i)t, 0 \leq t \leq 1$

$$\int_{\gamma} dz e^{3z} = \int_{\bar{\gamma}} dz e^{3z} = \int_0^1 dt e^{3(1+i)t} = \frac{e^{3(1+i)} - 1}{3}$$

■

Problem 10

We have $z = \gamma(t)$, and

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

Integrates both part

$$f(z)g(z) \Big|_b^a = \int_{\gamma} dz f(z)g'(z) + f'(z)g(z)$$

We get

$$f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) = \int_{\gamma} dz f(z)g'(z) + f'(z)g(z)$$

■

Problem 11

$P(z)$ is analytic function. Any close contour integral is zero.

Problem 12

$$I = \oint_{C_r} \frac{dz}{z^2 - 2z - 8} = \oint_{C_r} \frac{dz}{(z - 4)(z + 2)}$$

$r = 1$, no pole inside the circle. $I = 0$

$r = 3$, one pole at $z = -2$

$$I = \frac{2\pi i}{-2 - 4} = -\frac{\pi i}{3}$$

$r = 5$, two poles at $z = -2$ and $z = 4$

$$I = \frac{2\pi i}{-2 - 4} + \frac{2\pi i}{4 + 2} = 0$$

■

Problem 13

(1)

no pole on the z -plane. The integral is zero.

(2)

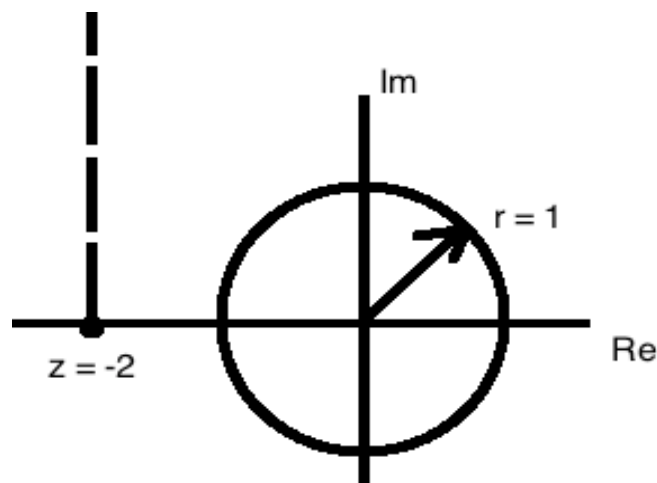
The nearest poles are $\pm \frac{\pi}{2}$. No pole inside the circle. The integral is zero.

(3)

There are two poles at $-1 \pm i$. Because $|-1 \pm i| = \sqrt{2}$, no pole inside the circle. The integral is zero.

(4)

The branch cut starts from $z = -2$. We can choose it as the following figure



The dash line is branch cuts. No pole in the circle. The integral is zero. ■

Problem 14

(1)

$$I = \oint_{\gamma} \frac{dz}{z^2 - 4} = \oint \frac{dz}{(z-2)(z+2)} = 2\pi i \left(\frac{1}{4} + \frac{-1}{4} \right) = 0$$

(2)

$$I = \oint_{\gamma} \frac{dz e^z}{(z-1)(z-2)} = 2\pi i \left(\frac{e}{1-2} + \frac{e^2}{2-1} \right) = 2\pi i e(e-1)$$

(3)

$$\begin{aligned} I &= \oint_{\gamma} \frac{dz}{(z+4)(z^2+1)} = 2\pi i \left(\frac{1}{(i+4)(i+i)} + \frac{1}{(-i+4)(-i-i)} \right) \\ &= \frac{\pi}{i+4} + \frac{\pi}{i-4} = \frac{2\pi i}{17} \end{aligned}$$

Note: here $z = -4$ is outside the circle.

(4) No pole inside the circle.

$$I = \oint_{\gamma} dz i^{z-3} = 0$$

■

Problem 15

Since we have

$$\oint_{|z|=1} \frac{e^{az}}{z} = 2\pi i$$

LHS also can be written as

$$\begin{aligned} \oint_{|z|=1} \frac{e^{az}}{z} &= \int_{-\pi}^{\pi} d(e^{i\phi}) \frac{e^{ae^{i\phi}}}{e^{i\phi}} = \int_{-\pi}^{\pi} d\phi i e^{a(\cos \phi + i \sin \phi)} \\ &= i \int_{-\pi}^{\pi} d\phi e^{a \cos \phi} \cos(a \sin \phi) - \int_{-\pi}^{\pi} d\phi e^{a \cos \phi} \sin(a \sin \phi) \end{aligned}$$

Here, $e^{a \cos \phi} \sin(a \sin \phi)$ is odd function, $e^{a \cos \phi} \cos(a \sin \phi)$ is even function.
We have

$$2\pi i = 2i \int_{-\pi}^{\pi} d\phi e^{a \cos \phi} \cos(a \sin \phi) \Rightarrow \int_{-\pi}^{\pi} d\phi e^{a \cos \phi} \cos(a \sin \phi) = \pi$$

■

Problem 16

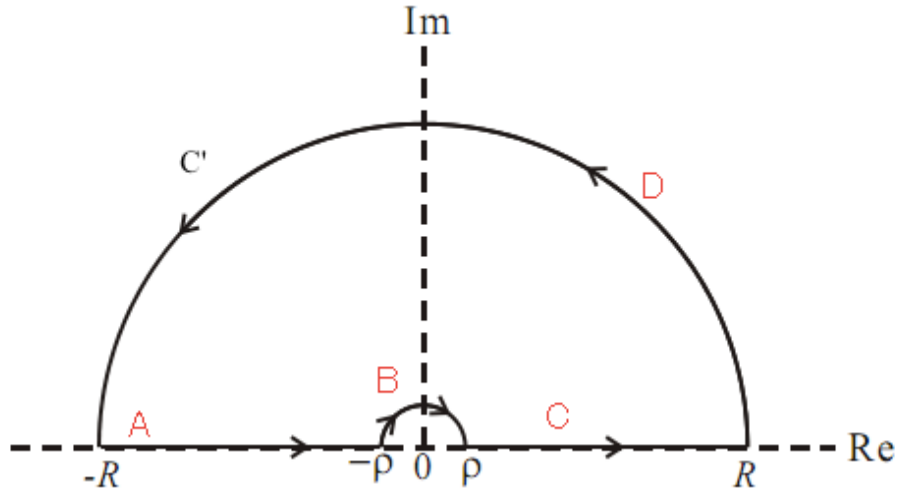
We want to evaluate the integral

$$I = \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x}$$

Consider the following case

$$\oint_C dz \frac{e^{iz}}{z}$$

$C' = A + B + C + D$, as show in the following figure



Where $R \rightarrow \infty$ and $\rho \rightarrow 0$. Since no pole in the contour C'

$$\oint_C dz \frac{e^{iz}}{z} = 0 = \int_A \frac{e^{iz}}{z} + \int_B \frac{e^{iz}}{z} + \int_C \frac{e^{iz}}{z} + \int_D \frac{e^{iz}}{z}$$

Two of contours integral A and C can be written as

$$\int_A \frac{e^{iz}}{z} + \int_C \frac{e^{iz}}{z} = \int_{\rho}^R dr \frac{e^{ir}}{r} + \int_{-R}^{-\rho} dr \frac{e^{ir}}{r} = I$$

By Jordan's lemma

$$\lim_{R \rightarrow \infty} \int_D \frac{e^{iz}}{z} = 0$$

So we get

$$I = - \int_B \frac{e^{iz}}{z} = \lim_{\rho \rightarrow 0} \int_0^\pi i d\phi e^{i\rho e^{i\phi}} = i \int_0^\pi d\phi = i\pi$$

We get

$$\begin{aligned} \Re\{I\} &= \int_{-\infty}^{\infty} dx \frac{\cos x}{x} = 0 \\ \Im\{I\} &= \int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \pi \end{aligned}$$

■

Bonus Question

We have showed any polynormial of degree $n > 0$ exist at most one zero, i.e.

$$P_n(z_0) = 0$$

Where z_0 is a zero of $P_n(z)$. As we know

$$\begin{aligned} P_n(z) &= P_n(z) - P_n(z_0) = \sum_{k=0}^n a_n(z^k - z_0^k) \\ &= \sum_{k=0}^n a_n(z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z z_0^{k-2} + z_0^{k-1}) \\ &= (z - z_0)P_{n-1}(z) \end{aligned}$$

By the argument, we have

$$\begin{aligned} P_n(z) &= (z - z_0)P_{n-1}(z) \\ &= (z - z_0)(z - z_1)P_{n-2}(z) = \cdots \\ &= (z - z_0)(z - z_1) \cdots (z - z_n)P_0 \end{aligned}$$

Where P_0 is a constant.

■

Percentage Weighted

Problem 1,2 (5%)

Problem 3 (5%)

Problem 4,8,9 (10%)

Problem 5 (10%)

Problem 6 (10%)

If you just show

$$\nabla^2 u(r, \phi) = 0$$

you'll get (5%).

Problem 7 (20%, each part 4%)

For part (e), Claiming it is rotation without calculation, you'll get 2%. Writing answer directly, you'll get no point.

Problem 10,11 (5%)

Problem 12,13,14 (10%)

For Problem 13(4), It is wrong that claiming the integral is zero without point out branch cut.

Problem 15 (5%)

Problem 16 (8%, 1.(3%), 2.(5%))

If you write this

$$\int_{-\infty}^{\infty} \frac{e^z}{z} = i\pi$$

then take real/imag part. You'll get no point. It's also no points writing down

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z} = i\pi$$

without checking contour.

Bonus Question (12%)

If you try to answer it, at most you can get 4%.

Point out at least one zero, and write down

$$P(z) = (z - z_0)Q(z)$$

You'll get 7%.

If you also show the maximum power of $Q(z)$ is z^{n-1} , then you'll get full score.