Atiyah-Singer Index Theorem

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1/10 Why we need Atiyah-Singer index theorem?

Reason 1

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Reason 3

Now days, there are several aspects still active:

classical point sources, closed timelike curves, the classical and quantum behavior of point particles, matter couplings, supergravity, asymptotic behavior and global charges, the radial gauge, topologically massive gravity, gauge theory, the construction of observables from topological field theories, ...and so on and so forth.

2/10 What is A-S index theorem?

Scientists describe the world by measuring quantities and forces that vary over time and space. The rules of nature are often expressed by formulas, called differential equations, involving their rates of change. Such formulas may have an ”index,” the number of solutions of the formulas minus the number of restrictions that they impose on the values of the quantities being computed. i.e. **The analytical index is equal to its topological index.** The Atiyah-Singer index theorem calculated this number in terms of the geometry of the surrounding space.

2/10 What is A-S index theorem?

Formal Description [Nakahara]

Let (E, D) be an elliptic complex over an m-dimensional compact manifold M without a boundary. The index of this complex is given by

$$ind(E, D) = (-1)^{m(m+1)/2} \int_M ch\left(\bigoplus_r (-1)^r E_r\right)\left(\frac{Td(TM^{\mathbb{C}})}{e(TM)}\right)_{vol}. \tag{1}$$

In the integrand of the RHS, only m-forms are picked up, so that the integration makes sense.

Remarks: The division by e(TM) can really be carried out at the formal level. If m is an odd integer, the index vanishes identically, see below. Original references are Atiyah and Singer (1968a, b), Atiyah and Segal (1968).

2/10 How to derive it? (Example on the board!)

Basic Requiremets

smooth manifolds,and algebraic topology(especially cohomology). Some familiarity with basic notions of functional analysis: Hilbert spaces, bounded linear operators, L^2-spaces

2/10 How to derive it? (Example on the board!)

Method 1 Pseudodifferential operators

Method 2 Cobordism

Method 3 K theory

Method 4 Heat equation

Method 5 Using Supersymmetry for Dirac Operator form

Method 6 For a Spin Complex form

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Property: The field equations are generally covariant; that is, they're invariant under the action of the group of diffeomorphisms of the spacetime M, which can be view as a gauge group.

7/10 Example 2: the Hirzebruch signature theorem

Properties:
Since the vacuum state now is locally flat,
i. there are no gravitational waves in the classical theory (Weyl tensor is also vanishes.)
ii. upon quantization, there are no quantum graviton (c.f. topologically massive gravity by adding CS-term)
iii. Source produce curvature, but only locally at the location in space-time

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