

## PROBLEM SOLVING

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Prove that

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

*Proof.*

$n = 1 \Rightarrow 1 < 2$  which is true.

Suppose for  $n = k$  we have

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} < 2\sqrt{k}.$$

Then for  $n = k + 1$  we have

$$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$$

Need to show:

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

If we multiply  $\sqrt{k+1}$  for the both sides, and simplify the formula, we can show that the left-hand-side is  $k^2 + k$ , and right-hand-side is  $k^2 + k + \frac{1}{4}$ . This completes the proof.  $\square$

Prove that

$$(0.1) \quad 2!4!\dots(2n)! \geq ((n+1)!)^n$$

*Proof.*  $n = 1 \Rightarrow 2! \geq (1+1)!$  which is true.

Suppose  $n = k, k \in \mathbb{Z}$  is true.

Need to show:

$$2!4!\dots(2k)!(2k+2)! \geq ((k+1)!)^k(2k+2)!$$

l.h.s. of (0.1) as  $n = k + 1$  is:

$$((k+1)!)^k 2(2k+2)!$$

r.h.s. of (0.1) as  $n = k + 1$  is:

$$((k+2)!)^{k+1} = (k+2)!((k+2)!)^k = (k+2)!(k+2)^k((k+1)!)^k$$

Thus, after canceling the common term, we only need to show  $(2k+2)! > (k+2)!(k+2)^k$ .

Since we can write the  $(2k+2)!$  into  $k$ -term of a product times a factorial  $(k+2)!$ , and replace the  $k$ -term product with a lower bound  $(k+2)^k$  as follows

$$(2k+2)! = (2k+2)(2k+1)\dots(k+3)(k+2)! > (k+2)^k(k+2)! = \text{r.h.s. of what we need to show.}$$

$\square$

Enumerate the number of ways that we can tile a  $2 \times n$  grid with  $2 \times 1$  dominoes.

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*Proof.* We can denote the number of ways to tile a  $2 \times n$  grid with  $2 \times 1$  dominoes as  $f_n$ . Then suppose we fill a  $2 \times 1$  domino on its left end, then to tile the remaining  $2 \times (n-1)$  grid, we have  $f_{n-1}$  ways. Secondly, if we tile a  $2 \times 2$  grid of the  $2 \times n$  grid on its left end, then there are  $f_{n-2}$  ways to tile the remaining grids. Thus,  $f_n = f_{n-1} + f_{n-2}$ . We also have the base cases:  $f_1 = 1$ , and  $f_2 = 2$ . It follows that it's a Fibonacci sequence, and it's general form is

$$f_n = \frac{\phi^n + \psi^n}{\sqrt{5}}$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

and

$$\psi = \frac{1 - \sqrt{5}}{2} = 1 - \phi.$$

□