應數三作業五詳解 Q1~2

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1. Fourier Transform

$$\mathscr{F}[f(x)] = \hat{f}(k) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \qquad \mathscr{F}^{-1}\left[\hat{f}(k)\right] = f(x) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk$$

Fourier Cosine Transform

$$\mathscr{F}_c[f_c(x)] = \hat{f}_c(k) = \sqrt{\tfrac{2}{\pi}}\int_0^{\infty} f_c(x)\cos kx\,dx, \qquad \mathscr{F}_c^{-1}\left[\hat{f}_c(k)\right] = \hat{f}_c(x) = \sqrt{\tfrac{2}{\pi}}\int_0^{\infty} f_c(k)\cos kx\,dk$$

Fourier Sine Transform

$$\mathscr{F}_s[f_s(x)] = \hat{f}_s(k) = \sqrt{\tfrac{2}{\pi}}\int_0^{\infty} f_s(x)\sin kx\,dx, \qquad \mathscr{F}_s^{-1}\left[\hat{f}_s(k)\right] = \hat{f}_s(x) = \sqrt{\tfrac{2}{\pi}}\int_0^{\infty} f_s(k)\sin kx\,dk$$

$$\begin{aligned}
\mathscr{F}_c\left[e^{-ax}\right] &= \sqrt{\tfrac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos kx\,dx = \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\int_0^{\infty}\cos kx\,de^{-ax}\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\left[e^{-ax}\cos kx\Big|_0^{\infty} - \int_0^{\infty} e^{-ax}d\cos kx\right]\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\left[-1 + k\int_0^{\infty} e^{-ax}\sin kx\,dx\right]\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\left[-1 + (\tfrac{-k}{a})\int_0^{\infty}\sin kx\,de^{-ax}\right]\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\left[-1 + (\tfrac{-k}{a})e^{-ax}\sin kx\Big|_0^{\infty} + (\tfrac{k}{a})\int_0^{\infty} e^{-ax}d\sin kx\right]\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{1}{a}) + 0 - \sqrt{\tfrac{2}{\pi}}(\tfrac{k^2}{a^2})\int_0^{\infty} e^{-ax}\cos kx\,dx
\end{aligned}$$

$$\Rightarrow (1+\tfrac{k^2}{a^2})\sqrt{\tfrac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos kx\,dx = \sqrt{\tfrac{2}{\pi}}(\tfrac{1}{a})$$

$$\Rightarrow \mathscr{F}_c\left[e^{-ax}\right] = (1+\tfrac{k^2}{a^2})\sqrt{\tfrac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos kx\,dx = \sqrt{\tfrac{2}{\pi}}(\tfrac{a}{k^2+a^2})$$

$$\begin{aligned}
\mathscr{F}_s\left[e^{-ax}\right] &= \sqrt{\tfrac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin kx\,dx = \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\int_0^{\infty}\sin kx\,de^{-ax}\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-1}{a})\left[e^{-ax}\sin kx\Big|_0^{\infty} - \int_0^{\infty} e^{-ax}d\sin kx\right]\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{k}{a})\int_0^{\infty} e^{-ax}\cos kx\,dx\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-k}{a^2})\int_0^{\infty}\cos kx\,de^{-ax}\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{-k}{a^2})\left[e^{-ax}\cos kx\Big|_0^{\infty} - \int_0^{\infty} e^{-ax}d\cos kx\right]\\
&= \sqrt{\tfrac{2}{\pi}}(\tfrac{+k}{a^2}) + \sqrt{\tfrac{2}{\pi}}(\tfrac{-k^2}{a^2})\int_0^{\infty} e^{-ax}\sin kx\,dx
\end{aligned}$$

$$\Rightarrow (1+\tfrac{k^2}{a^2})\sqrt{\tfrac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin kx\,dx = \sqrt{\tfrac{2}{\pi}}(\tfrac{k}{a^2})$$

$$\Rightarrow \mathscr{F}_s\left[e^{-ax}\right] = (1+\tfrac{k^2}{a^2})\sqrt{\tfrac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin kx\,dx = \sqrt{\tfrac{2}{\pi}}(\tfrac{k}{k^2+a^2})$$

$$\begin{aligned}
\int_0^{\infty}\tfrac{1}{k^2+a^2}\cos kx\,dk &= \tfrac{1}{2}\int_{-\infty}^{\infty}\tfrac{1}{k^2+a^2}e^{ikx}dk = \tfrac{2\pi i}{2}\lim_{k\to ia}(k-ia)\tfrac{1}{k^2+a^2}e^{ikx}\\
&= \pi i\tfrac{1}{2ia}e^{-ax} = \tfrac{\pi}{2a}e^{-ax}
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty}\tfrac{k}{k^2+a^2}\sin kx\,dk &= \tfrac{1}{2i}\int_{-\infty}^{\infty}\tfrac{k}{k^2+a^2}e^{ikx}\,dk = \tfrac{2\pi i}{2i}\lim_{k\to ia}(k-ia)\tfrac{k}{k^2+a^2}e^{ikx}\\
&= \pi\tfrac{ia}{2ia}e^{-ax} = \tfrac{\pi}{2}e^{-ax}
\end{aligned}$$

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2. $\mathscr{F}\left[\phi(x)\right]=\hat{\phi}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{ikx}dx$

$$\begin{aligned}
\mathscr{F}\left[\phi'(x)\right]&=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi'(x)e^{-ikx}dx=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}d\phi(x)\\
&=\tfrac{1}{\sqrt{2\pi}}\phi(x)e^{-ikx}\Big|_{-\infty}^{\infty}-\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)de^{-ikx}\\
&=0+ik\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ikx}dx\\
&=ik\mathscr{F}\left[\phi(x)\right]
\end{aligned}$$

$$\begin{aligned}
\mathscr{F}\left[\phi''(x)\right]&=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi''(x)e^{-ikx}dx=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}d\phi'(x)\\
&=\tfrac{1}{\sqrt{2\pi}}\phi'(x)e^{-ikx}\Big|_{-\infty}^{\infty}-\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi'(x)de^{-ikx}\\
&=0+ik\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi'(x)e^{-ikx}dx\\
&=ik\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}d\phi(x)\\
&=\tfrac{ik}{\sqrt{2\pi}}\phi(x)e^{-ikx}\Big|_{-\infty}^{\infty}-ik\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)de^{-ikx}\\
&=0-k^2\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-ikx}dx\\
&=-k^2\mathscr{F}\left[\phi(x)\right]
\end{aligned}$$

$$\begin{aligned}
\mathscr{F}\left[f(x)\right]&=\mathscr{F}\left[\phi''(x)-s^2\phi(x)\right]\\
&=\mathscr{F}\left[\phi''(x)\right]-s^2\mathscr{F}\left[\phi(x)\right]\\
\hat{f}(k)&=-k^2\mathscr{F}\left[\phi(x)\right]-s^2\mathscr{F}\left[\phi(x)\right]\\
&=-(k^2+s^2)\mathscr{F}\left[\phi(x)\right]
\end{aligned}$$

$$\Rightarrow\mathscr{F}\left[\phi(x)\right]=\tfrac{-1}{k^2+s^2}\hat{f}(k)$$

$$\begin{aligned}
\Rightarrow\phi(x)&=\mathscr{F}^{-1}\left[\mathscr{F}\left[\phi(x)\right]\right]=\mathscr{F}^{-1}\left[\tfrac{-1}{k^2+s^2}\hat{f}(k)\right]\\
&=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tfrac{-1}{k^2+s^2}\hat{f}(k)e^{ikx}dx\\
&=\tfrac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tfrac{1}{k^2+s^2}\hat{f}(k)e^{ikx}dx
\end{aligned}$$

3.

$$\hat{f}(k) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx} dx = \sqrt{\frac{2}{\pi}} \frac{\sin k}{k}$$

$$f(x) \star f(x) = \int_{-\infty}^{\infty} f(x-y) f(y) dy = \int_{-1}^{1} f(x-y) dy = \begin{cases} 2 - |x| & \text{if } |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}[f(x) \star f(x)] = \sqrt{2\pi} \hat{f}^2(k) = \sqrt{2\pi} \frac{2}{\pi} \frac{\sin^2 k}{k^2}$$

Using Plancherel Identity,

$$\int_{-\infty}^{\infty} dx |f(x)|^2 = \int_{-\infty}^{\infty} dk |\hat{f}(k)|^2$$

$$2 = \int_{-\infty}^{\infty} \frac{2}{\pi} \frac{\sin^2 k}{k^2} dk$$

$$\int_{-\infty}^{\infty} dx |f \star f|^2 = \int_{-\infty}^{\infty} dk \cdot 2\pi \frac{4}{\pi^2} \frac{\sin^4 k}{k^4}$$

$$\text{LHS} = \int_{-2}^{2} (2 - |x|)^2 dx = 2 \int_{0}^{2} (2-x)^2 dx = \frac{16}{3}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin^4 k}{k^4} dk = \frac{2}{3} \pi$$

4.

Expand$\sum_{n=-\infty}^{\infty} \delta(x + nL)$ on $[-\frac{L}{2}, \frac{L}{2}]$

$$\sum_{n=-\infty}^{\infty} \delta(x + nL) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{i2\pi nx}{L}}$$

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{i\frac{2\pi nx}{L}} \sum_n \delta(x + nL) = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{i\frac{2\pi nx}{L}} \delta(x) = \frac{1}{L}$$

$$\therefore \sum_n \delta(x + nL) = \frac{1}{L} \sum_n e^{-\frac{i2\pi nx}{L}}$$

Expand $\sum_n f(x + nL)$ on $[-\frac{L}{2}, \frac{L}{2}]$

$$\sum_n f(x + nL) = \sum_n c_n e^{\frac{i2\pi nx}{L}}$$

$$
\begin{aligned}
c_n &= \frac{1}{L}\int_{-L/2}^{L/2} dx e^{-\frac{i2\pi nx}{L}} \sum_{m=-\infty}^{\infty} f(x+mL) \\
&= \frac{1}{L}\sum_{l}\int_{-L/2}^{L/2} dx e^{-\frac{i2\pi nx}{L}} f(x+mL) \\
&= \frac{1}{L}\sum_{l} e^{i2\pi m}\int_{L(m-\frac{1}{2})}^{L(m+\frac{1}{2})} du e^{-\frac{2\pi nu}{L}} f(u) \\
&= \frac{1}{L}\int_{-\infty}^{\infty} du e^{-\frac{i2\pi n}{L}u} f(u) \\
&= \frac{\sqrt{2\pi}}{L}\hat{f}(\frac{2\pi n}{L}),
\end{aligned}
$$

which completes the proof.

5.

$$
J_0 = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{ix\cos\theta}
$$

$$
\begin{aligned}
g(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx e^{-ikx} J_0(x) \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\theta \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ix(k-\cos\theta)} \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\theta \frac{1}{2\pi} 2\pi\delta(\cos\theta - k) \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\theta \delta(\cos\theta - k)
\end{aligned}
$$

$$
\delta(f(\theta)) = \sum_a \frac{1}{|f'(\theta_a)|}\delta(\theta-\theta_a),\ \ f(\theta_a) = 0
$$

If $|k| < 1$, $\cos\theta = k$, $\sin\theta = \pm\sqrt{1-k^2}$

$$
\begin{aligned}
g(k) &= \frac{1}{\sqrt{2\pi}}\{\int_0^{2\pi} d\theta\delta(\theta-\theta_1)\frac{1}{\sqrt{1-k^2}} + \int_0^{2\pi} d\theta\delta(\theta-\theta_2)\frac{1}{\sqrt{1-k^2}}\} \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1-k^2}}
\end{aligned}
$$

Otherwise, $g(k) = 0$.

Solutions of Apply mathematics(III) HW#5

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2012.6.10

Conventions of Fourier transform

$$
\begin{aligned}
\mathcal{F}[u(t)] &\equiv \int_{-\infty}^{\infty} dt e^{-i\omega t} u(t) \equiv U(\omega) \\
\mathcal{F}^{-1}[U(\omega)] &\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} U(\omega)
\end{aligned}
$$

Problem 6

It is easy to find

$$
\mathcal{F}[f(x)] = \frac{2a}{a^2 + k^2}, \ \mathcal{F}[g(x)] = \frac{2b}{b^2 + k^2}
$$

The convolution $f * g$ satisfies

$$
f(x) * g(x) = \mathcal{F}^{-1}[\mathcal{F}[f(x)]\mathcal{F}[g(x)]]
$$

To verify it easier, we set $x > 0$. We have

$$
\begin{aligned}
f * g &= \int_{-\infty}^{0} dy e^{-a(x-y)} e^{by} + \int_{0}^{x} dy e^{-a(x-y)} e^{-by} + \int_{x}^{\infty} e^{-a(y-x)} e^{-by} \\
&= \frac{e^{-ax}}{a+b} + \frac{e^{-bx} - e^{-ax}}{a-b} + \frac{e^{-bx}}{a+b} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \frac{4ab}{(a^2+k^2)(b^2+k^2)}
\end{aligned}
$$

Put $x = 0^+$, we get

$$
\frac{2}{a+b} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{4ab}{(a^2+k^2)(b^2+k^2)}
$$

■

Problem 7

The integral equation can be represented as

$$f * f = e^{-ax^2}$$

i.e.

$$(F(k))^2 = \mathcal{F}[e^{-ax^2}] = \sqrt{\frac{\pi}{a}} e^{-k^2/4a}$$

So we have

$$F(k) = \pm \left(\frac{\pi}{a}\right)^{1/4} e^{-k^2/8a}$$

Apply inverse Fourier transform

$$f(x) = \pm \left(\frac{4a}{\pi}\right)^{1/4} e^{-2ax^2}$$

$\blacksquare$

Problem 8

Replacing $x(t) = \int_{-\infty}^{\infty} dt' G(t-t') f(t')$ to the differential equation, we have

$$\int_{-\infty}^{\infty} dt' f(t') \frac{d^2 G(t-t')}{dt^2} + 2\gamma \frac{dG(t-t')}{dt} + q^2 G(t-t') = \int_{-\infty}^{\infty} dt' f(t') \delta(t-t')$$

Comparing two sides and setting $t' = 0$, we have

$$\frac{d^2 G(t)}{dt^2} + 2\gamma \frac{dG(t)}{dt} + q^2 G(t) = \delta(t)$$

Applying Fourier transfrom

$$(-\omega^2 + 2i\gamma\omega + q^2)\mathcal{F}[G(t)] = \mathcal{F}[\delta(t)] = 1$$

So we have

$$\mathcal{F}[G(t)] = \frac{1}{-\omega^2 + 2i\gamma\omega + q^2}$$

We get $G(t)$ by applying inverse transform

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega e^{i\omega t}}{-\omega^2 + 2i\gamma\omega + q^2}$$

$\blacksquare$