應數三作業四詳解 Q1~3

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1. $f(x)=e^{ax}, \qquad -L<x<L$

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos\tfrac{n\pi x}{L}+b_n\sin\tfrac{n\pi x}{L}$$

$$a_0=\tfrac{1}{2L}\int_{-L}^{L}e^{ax}dx=\tfrac{1}{2aL}e^{ax}\Big|_{-L}^{L}=\tfrac{1}{2aL}(e^{aL}-e^{-aL})$$

$$a_n=\tfrac{1}{L}\int_{-L}^{L}e^{ax}\cos\tfrac{n\pi x}{L}dx=\tfrac{1}{aL}\int_{-L}^{L}\cos\tfrac{n\pi x}{L}de^{ax}$$

$$=\tfrac{1}{aL}e^{ax}\cos\tfrac{n\pi x}{L}\Big|_{-L}^{L}-\tfrac{1}{aL}\int_{-L}^{L}e^{ax}d\cos\tfrac{n\pi x}{L}$$

$$=\tfrac{1}{aL}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]+\tfrac{n\pi}{aL^2}\int_{-L}^{L}e^{ax}\sin\tfrac{n\pi x}{L}dx$$

$$=\tfrac{1}{aL}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]+\tfrac{n\pi}{a^2L^2}\int_{-L}^{L}\sin\tfrac{n\pi x}{L}de^{ax}$$

$$=\tfrac{1}{aL}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]+\tfrac{n\pi}{a^2L^2}e^{ax}\sin\tfrac{n\pi x}{L}\Big|_{-L}^{L}-\tfrac{n\pi}{a^2L^2}\int_{-L}^{L}e^{ax}d\sin\tfrac{n\pi x}{L}$$

$$=\tfrac{1}{aL}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]+0-\tfrac{n^2\pi^2}{a^2L^2}\tfrac{1}{L}\int_{-L}^{L}e^{ax}\cos\tfrac{n\pi x}{L}dx$$

$$\Rightarrow (1+\tfrac{n^2\pi^2}{a^2L^2})\tfrac{1}{L}\int_{-L}^{L}e^{ax}\cos\tfrac{n\pi x}{L}dx=\tfrac{1}{aL}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]$$

$$\Rightarrow a_n=\tfrac{1}{L}\int_{-L}^{L}e^{ax}\cos\tfrac{n\pi x}{L}dx=\tfrac{aL}{a^2L^2+n^2\pi^2}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]$$

$$b_n=\tfrac{1}{L}\int_{-L}^{L}e^{ax}\sin\tfrac{n\pi x}{L}dx=\tfrac{1}{aL}\int_{-L}^{L}\sin\tfrac{n\pi x}{L}de^{ax}$$

$$=\tfrac{1}{aL}e^{ax}\sin\tfrac{n\pi x}{L}\Big|_{-L}^{L}-\tfrac{1}{aL}\int_{-L}^{L}e^{ax}d\sin\tfrac{n\pi x}{L}$$

$$=-\tfrac{n\pi}{a^2L^2}\int_{-L}^{L}e^{ax}\cos\tfrac{n\pi x}{L}dx$$

$$=-\tfrac{n\pi}{a^2L^2}\int_{-L}^{L}\cos\tfrac{n\pi x}{L}de^{ax}$$

$$=-\tfrac{n\pi}{a^2L^2}e^{ax}\cos\tfrac{n\pi x}{L}\Big|_{-L}^{L}+\tfrac{n\pi}{a^2L^2}\int_{-L}^{L}e^{ax}d\cos\tfrac{n\pi x}{L}$$

$$=\tfrac{n\pi}{a^2L^2}\left[(-1)^{n+1}e^{aL}-(-1)^{n+1}e^{-aL}\right]-\tfrac{n^2\pi^2}{a^2L^2}\tfrac{1}{L}\int_{-L}^{L}e^{ax}\sin\tfrac{n\pi x}{L}dx$$

$$\Rightarrow (1+\tfrac{n^2\pi^2}{a^2L^2})\tfrac{1}{L}\int_{-L}^{L}e^{ax}\sin\tfrac{n\pi x}{L}dx=\tfrac{n\pi}{a^2L^2}\left[(-1)^{n+1}e^{aL}-(-1)^{n+1}e^{-aL}\right]$$

$$\Rightarrow b_n=\tfrac{1}{L}\int_{-L}^{L}e^{ax}\sin\tfrac{n\pi x}{L}dx=\tfrac{n\pi}{a^2L^2+n^2\pi^2}\left[(-1)^{n+1}e^{aL}-(-1)^{n+1}e^{-aL}\right]$$

$$f(x)=\tfrac{1}{2aL}(e^{aL}-e^{-aL})+\sum_{n=1}^{\infty}\tfrac{aL}{a^2L^2+n^2\pi^2}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]\cos\tfrac{n\pi x}{L}$$
$$+\sum_{n=1}^{\infty}\tfrac{n\pi}{a^2L^2+n^2\pi^2}\left[(-1)^{n+1}e^{aL}-(-1)^{n+1}e^{-aL}\right]\sin\tfrac{n\pi x}{L}$$

$$g(x)=e^{-ax}=a_0^*+\sum_{n=1}^{\infty}a_n^*\cos\tfrac{n\pi x}{L}+b_n^*\sin\tfrac{n\pi x}{L}$$
$$=\tfrac{1}{2aL}(e^{aL}-e^{-aL})+\sum_{n=1}^{\infty}\tfrac{aL}{a^2L^2+n^2\pi^2}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]\cos\tfrac{n\pi x}{L}$$
$$+\sum_{n=1}^{\infty}\tfrac{n\pi}{a^2L^2+n^2\pi^2}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]\sin\tfrac{n\pi x}{L}$$

$$\cosh(ax)=\tfrac{1}{2}(e^{ax}+e^{-ax})=\tfrac{1}{2}(f(x)+g(x))$$
$$=\tfrac{1}{2aL}(e^{aL}-e^{-aL})+\sum_{n=1}^{\infty}\tfrac{aL}{a^2L^2+n^2\pi^2}\left[(-1)^n e^{aL}-(-1)^n e^{-aL}\right]\cos\tfrac{n\pi x}{L}$$

$$\sinh(ax)=\tfrac{1}{2}(e^{ax}-e^{-ax})=\tfrac{1}{2}(f(x)-g(x))$$
$$=\sum_{n=1}^{\infty}\tfrac{n\pi}{a^2L^2+n^2\pi^2}\left[(-1)^{n+1}e^{aL}-(-1)^{n+1}e^{-aL}\right]\sin\tfrac{n\pi x}{L}$$

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2. $f(x)=x(\pi-x)\,,\qquad 0<x<\pi$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n=\tfrac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin nx\,dx=\tfrac{2}{\pi}\int_0^{\pi} \pi x\sin nx-x^2\sin nx\,dx$$

$$=-\tfrac{2}{n}\int_0^{\pi} x\,d\cos nx+\tfrac{2}{\pi}\int_0^{\pi} x^2\,d\cos nx$$

$$=-\tfrac{2}{n}x\cos nx\Big|_0^{\pi}+\tfrac{2}{n}\int_0^{\pi}\cos nx\,dx+\tfrac{2}{\pi n}x^2\cos nx\Big|_0^{\pi}-\tfrac{2}{\pi n}\int_0^{\pi}\cos nx\,dx^2$$

$$=-\tfrac{2\pi}{n}(-1)^n+\tfrac{2}{n^2}\sin nx\Big|_0^{\pi}+\tfrac{2\pi}{n}(-1)^n-\tfrac{4}{\pi n}\int_0^{\pi}x\cos nx\,dx$$

$$=0+0+0+0-\tfrac{4}{\pi n^2}\int_0^{\pi}x\,d\sin nx$$

$$=-\tfrac{4}{\pi n^2}x\sin nx\Big|_0^{\pi}+\tfrac{4}{\pi n^2}\int_0^{\pi}\sin nx\,dx$$

$$=-\tfrac{4}{\pi n^3}\cos nx\Big|_0^{\pi}=-\tfrac{4}{\pi n^3}\left[(-1)^n-1\right]=\tfrac{4}{\pi n^3}\left[1+(-1)^{n+1}\right]$$

$$=\begin{cases} 0 & ,\quad n=even \\ \tfrac{8}{\pi n^3} & ,\quad n=odd \end{cases}$$

let $n=2m-1,\qquad m\in Z$

then, $b_m^*=\frac{8}{\pi(2m-1)^3}$

$$f(x)=x(\pi-x)=\sum_{m=1}^{\infty} b_m^*\sin(2m-1)x=\sum_{m=1}^{\infty}\tfrac{8}{\pi(2m-1)^3}\sin(2m-1)x$$

$$f(x=\tfrac{\pi}{2})=\tfrac{\pi}{2}(\pi-\tfrac{\pi}{2})=\sum_{m=1}^{\infty}\tfrac{8}{\pi(2m-1)^3}\sin\tfrac{(2m-1)\pi}{2}$$

$$\Rightarrow \tfrac{\pi^2}{4}=\tfrac{8}{\pi}\left[1-\tfrac{1}{3^3}+\tfrac{1}{5^3}-\tfrac{1}{7^3}+\cdots+\tfrac{(-1)^n}{(2n+1)^3}+\cdots\right]$$

$$\Rightarrow 1-\tfrac{1}{3^3}+\tfrac{1}{5^3}-\tfrac{1}{7^3}+\cdots+\tfrac{(-1)^n}{(2n+1)^3}+\cdots=\tfrac{\pi^3}{32}$$

3. We know

$$\int_{-\pi}^{\pi}\sin nx\,dx=\int_{-\pi}^{\pi}\cos nx\,dx=\int_{-\pi}^{\pi}\sin nx\cos mx\,dx=0$$

$$\tfrac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\cos mx\,dx=\tfrac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\sin mx\,dx=\delta_{nm}$$

$$f(t)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt+b_n\sin nt\,,$$

$$g(t)=\tfrac{1}{2}c_0+\sum_{m=1}^{\infty}c_m\cos mt+d_m\sin mt$$

$$f(t)g(t)=\tfrac{1}{4}a_0c_0+\tfrac{1}{2}a_0\times(\sum_{m=1}^{\infty}c_m\cos mt+d_m\sin mt)+\tfrac{1}{2}c_0\times(\sum_{n=1}^{\infty}a_n\cos nt+b_n\sin nt)$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\begin{pmatrix} a_nc_m\cos nt\cos mt+a_nd_m\cos nt\sin mt \\ +b_nd_m\sin nt\sin mt+b_nc_m\sin nt\cos mt\end{pmatrix}$$

$$\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(t)g(t)\,dt=\tfrac{1}{\pi}\int_{-\pi}^{\pi}\tfrac{1}{4}a_0c_0\,dt+\tfrac{1}{\pi}\int_{-\pi}^{\pi}\tfrac{1}{2}a_0\times(\sum_{m=1}^{\infty}c_m\cos mt+d_m\sin mt)\,dt$$

$$+\tfrac{1}{\pi}\int_{-\pi}^{\pi}\tfrac{1}{2}c_0\times(\sum_{n=1}^{\infty}a_n\cos nt+b_n\sin nt)\,dt$$

$$+\tfrac{1}{\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\begin{pmatrix} a_nc_m\cos nt\cos mt+a_nd_m\cos nt\sin mt \\ +b_nd_m\sin nt\sin mt+b_nc_m\sin nt\cos mt\end{pmatrix}dt$$

$$=\tfrac{1}{2}a_0c_0+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_nc_m+b_nd_m)\delta_{nm}=\tfrac{1}{2}a_0c_0+\sum_{n=1}^{\infty}(a_nc_n+b_nd_n)$$

Solutions of Apply mathematics(III) HW#4

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Problem 4

Replacing $z = e^{i\phi}$, we have

$$\log(1 + e^{i\phi}) = \log(e^{\frac{i\phi}{2}}(2\cos(\frac{\phi}{2})))$$

So, we get

$$\mathrm{Re}\{\log(1 + e^{i\phi})\} = \log|e^{\frac{i\phi}{2}}(2\cos(\frac{\phi}{2}))| = \log|2\cos(\frac{\phi}{2})|$$

R.H.S of equation is

$$\log|2\cos(\frac{\phi}{2})| = \sum_{n=1}^{\infty} \mathrm{Re}\{\frac{(-1)^{n+1}e^{in\phi}}{n}\} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(n\phi)}{n}$$

Changing variable by $\phi/2 = \pi/2 - \phi'/2$, we have

$$\log|2\sin(\frac{\phi'}{2})| = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(n(\pi - \phi'))}{n} = -\sum_{n=1}^{\infty} \frac{\cos(n\phi')}{n}$$

Finally, we can deduce

$$\begin{aligned}
\log|\tan(\frac{\phi}{2})| &= \log|\sin(\frac{\phi}{2})| - \log|\cos(\frac{\phi}{2})| \\
&= -\sum_{n=1}^{\infty} \frac{\cos(n\phi)}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(n\phi)}{n} \\
&= -\sum_{n=1}^{\infty} (1 - (-1)^{n+1}) \frac{\cos(n\phi)}{n} \\
&= -2\sum_{k=1}^{\infty} \frac{\cos((2k-1)\phi)}{(2k-1)}
\end{aligned}$$

Because it is combined to condition, the domain of ϕ is the intersection of $-\pi < \phi < \pi$ and $0 < \phi < \pi$. ■

Problem 5

Because $\delta(x-a) - \delta(x+a)$ is an odd function, it has no $\cos((n\pi x)/L)$ contribution. i.e.

$$\delta(x-a) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

By orthogonal relation, we have

$$c_n = \frac{2}{L}\int_0^L \delta(x-a)\sin\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\sin\left(\frac{n\pi a}{L}\right)$$

So, we have

$$\delta(x-a) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

By definition, we have

$$\frac{d}{dx}H(x-a) = \delta(x-a)$$

So, $f(x)$ can be written as

$$\begin{aligned}
f(x) = H(x-a) &= \int_0^x \delta(x'-a)dx' \\
&= \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\int_0^x \sin\left(\frac{n\pi x'}{L}\right)dx' \\
&= \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\frac{L}{\pi}(1-\cos\left(\frac{n\pi x}{L}\right)) \\
&= \frac{2}{\pi}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)(1-\cos\left(\frac{n\pi x}{L}\right))
\end{aligned}$$

The average value I is the following integral

$$I = \frac{1}{L}\int_0^L f(x)dx$$

Since we know

$$\frac{1}{L}\int_0^L dx(1-\cos\left(\frac{n\pi x}{L}\right))=1$$

So we get

$$I=\frac{2}{\pi}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)$$

■

6.
from problem (3),

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dt f(t)g(t) = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty}(a_n c_n + b_n d_n)$$

When $f(t) = g(t)$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dt[f(t)]^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$f(x) = |\sin x| \leftarrow$even function. $\therefore b_n = 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} [\sin(1+n)x + \sin(1-n)x] dx \end{aligned}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin x dx \\ &= -\frac{2}{\pi}\cos x\Big|_0^{\pi} \\ &= \frac{4}{\pi} \end{aligned}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x dx = -\frac{1}{2\pi}\cos 2x\Big|_0^{\pi} = 0$$

$n \geq 2$,

$$\begin{aligned} a_n &= -\frac{1}{\pi}(\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1})\Big|_0^{\pi} \\ &= -\frac{1}{\pi}(\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}) + \frac{1}{\pi}(\frac{1}{n+1} - \frac{1}{n-1}) \\ &= \frac{(-1)^n}{\pi}(\frac{-2}{n^2-1}) + \frac{1}{\pi}(\frac{-2}{n^2-1}) \\ &= \frac{-2}{\pi}(\frac{1}{n^2-1})((-1)^n + 1) \\ &= \begin{cases} 0 & n \,\mathrm{odd} \\ -\frac{4}{\pi}\frac{1}{n^2-1} & n \,\mathrm{even} \end{cases} \end{aligned}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx[f(x)]^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \sin^2 x = 1$$

$$1 = \frac{1}{2}\frac{16}{\pi^2} + \sum_{m=2}^{\infty}\frac{16}{\pi^2}\frac{1}{((2m)^2-1)^2} = \frac{8}{\pi^2} + \frac{16}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{((2m)^2-1)^2}$$

$$\therefore \sum_{m=1}^{\infty}\frac{1}{((2m)^2-1)^2} = \frac{\pi^2}{16} - \frac{1}{2}$$

7.

$$e^{iat} = f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad -\pi \le t \le \pi$$

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dt f(t) e^{-int} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} dt e^{i(a-n)t} \\
&= \frac{1}{2\pi}\frac{1}{i(a-n)}(e^{i(a-n)\pi} - e^{-i(a-n)\pi}) \\
&= \frac{\sin a\pi \cos n\pi}{\pi(a-n)}
\end{aligned}$$

$$\int_{-\pi}^{\pi} dt(f(t))^2 = \int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n c_m^* e^{i(n-m)t} dt = \sum_{n=-\infty}^{\infty}|c_n|^2 \cdot 2\pi$$

$$\Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} dt(f(t))^2 = \sum_{n=-\infty}^{\infty}|c_n^2|$$

$$\int_{-\pi}^{\pi} dt(f(t))^2 = \int_{-\pi}^{\pi} dt = 2\pi$$

$$\Longrightarrow 1 = \sum_{n=-\infty}^{\infty}\frac{\sin^2 \pi a}{\pi^2(a-n)^2}$$

$$\Longrightarrow \frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty}\frac{1}{(a-n)^2}$$