Implications of Lee-Yang Theorem In Quantum Gravity

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(Dated: January 15, 2019)

The contributions of this note are twofold: First, it gives a generic recipe to apply Lee-Yang Theorem to solutions of Einstein field equations. Secondly, this existence of the applicability of Lee-Yang Theorem on a partition function of spacetime manifolds might also shed some light on the connection between the number theory, gravity, and gauge field theory. The connection to the Riemann Zeta function is quite interesting when one is also studying the distribution of non-trivial zeroes of the Riemann Zeta function[1], or its generic form (Dirichlet L-function).

PACS numbers: 98.80.Cq, 04.70.Dy, 11.10.Kk

I. INTRODUCTION

The first previous version of this note was written in Fall 2012, and presented as my final project in Cosmological Physics (Dark Energy and Dark Matter) class at National Taiwan University. My primary goal in writing this note was to create a new generic framework which enables exact and fast evaluation of candidates of quantum gravity as long as its partition function is well defined. I was inspired by the idea of Regge calculus that to study General Relativity by descretizing Einstein-Hilbert action and using methods in statistical mechanics, and statistical field theory. Thus, the starting point in the previous version is simple: by making no microscopic assumptions about the spacetime geometry; we impose Lee-Yang Theorem, if we can derive a non-analytic result on the real axis (i.e., the partition function can vanish on the real axis), then there is a phase transition. We can know there should be some spacetime "molecules" or "atoms" by showing there exists a phase transition over all configurations of spacetime. However, in order to apply Lee-Yang Theorem, a system should be described by a lattice-gas model, and that is not the case of the model used in the previous versions (unless we discretize spacetime manifolds and redefine the measure, $\mu[\lambda]$, by labeling spin to element). Hence, the partition function in the previous versions was ill-defined. An alternative way to do this is instead of taking thermodynamic limit on the gravity side (and gravitational field itself), we should take the limit on the boundary side by applying the holographic principle as follows: According to what we have learned from the example of the holographic principle - the Ads/CFT correspondence[2], then we assume the following identity is valid in the previous editions:

$$Z_{QFT}[\phi_0] = Z_{gravity}[\phi \to \phi_0] \tag{1}$$

where the left-hand side in (1) is a generating functional on the boundary

$$Z_{QFT}[\phi_0] = \left\langle \exp\left[\int \phi_0 \mathcal{O}\right]\right\rangle_{QFT}, \tag{2}$$

and the right-hand side in (1), for the sake of simplicity, let's only consider a scalar field ϕ in AdS

$$Z_{gravity}[\phi \to \phi_0] = \sum_{\phi \to \phi_0} e^{S_{gravity}}. \tag{3}$$

The right-hand side in equation (1) is not only a path integral, but also a partition function on the gravity side and it sums over all possible configurations which have the value ϕ_0 on the boundary. By doing so, although we don't have a well-defined gravity theory that can be applied Lee-Yang Theorem, but we can have a well defined Ising model on the gauge theory side, and we can study phase transition phenomena based on holographic principle. Another resolution is to prove a generalized Lee-Yang Theorem that can be applied to all possible configurations of the solution of Einstein field equations. Furthermore, a recent development in Ads/MERA correspondence and ds/MERA[3, 4] should be even better examples. Recently, most of research has put a lot of effort into this direction, for instance, studying quantum gravitational effects of a phase transition in the spacetime topology, which is a transition from a black hole solution to a no-black hole solution, i.e., the Hawking-Page phase transition, in the boundary CFT with supersymmetry.

In the next section, we shall give a brief recipe of a general framework for classifying a given path integral (i.e., partition function, and with a well-defined measure) of spacetime geometry into whether it has phase transition directly.

II. A GENERAL RECIPE FOR APPLYING LEE-YANG THEOREM ON A PARTITION OF SPACETIME GEOMETRY

We should make at each stage the most optimistic possible assumption. Before that, we should address a

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common problem for writing down a well-defined partition function of the Einstein-Hilbert Action (including actions with higher derivatives). Since the gravitational field action, $S_{gravity}$, after taking the summation in discrete cases (or integration in continuum cases) is a number, and this number can be arbitrarily small or even negative. Thus, when deciding the measure of the path integral or the weight for each path of the partition function in each configuration, if we apply the variational method with Lagrange multipliers to determine the specific partitioning, which provides a maximal weight, then we can derive the partition function in the denominator of this probability. Hence, a partition function is a sum over all possible configurations[5] of a solution of Einstein field equations:

$$\sum e^{-S_{gravity}}. \tag{4}$$

However, those configurations with $S_{gravity} < 0$ lead to a divergent path integral. One way to solve this problem is by adding new terms into $S_{gravity}$, the second way suggested by Hawking[6] is imposing a conformal gauge on the path integral where $S_{gravity} < 0$, and then integrating over all conformal factors,

$$e^{iS_{gravity}} \times \text{[gauge fixing terms]}. \tag{5}$$

We can see that due to the way we derive the partition function, there is always an exponential factor, in other words, to write a generic recipe of partition function, we shall consider a generic form e^{z_k} where z_k represents all possible alternatives, including $-S_{gravity}$, $S_{gravity}$, $iS_{gravity}$, $\exp\{\exp(-S_{gravity})\}$, etc.

We knew that if the partition function is a finite polynomial, then does not exist discontinuous behavior, and hence no phase transitions. Therefore, in our purpose, we only need to focus on the remaining unclassified cases, and those are infinite polynomials. Thus, we rewrite this *generic function* into an infinite product with an arithmetic function, the Möbius function, in its exponent:

$$e^{z_k} = \prod_{n=1}^{\infty} (1 - z_k^n)^{\frac{-\mu(n)}{n}}. \tag{6}$$

Furthermore, we can rewrite it with some nice arithmetic functions in the exponent of each term[7]:

$$e^{z_k} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{\tau^n}\right)^{\left(\frac{\mu(n)-\phi(n)}{n}\right)\cdot z_k} \tag{7}$$

where

$$\tau = \frac{1+\sqrt{5}}{2}.$$

After doing so, the next step is to find the zeros of the partition function:

$$\mathcal{Z} = \sum_k e^{z_k}. \tag{8}$$

Then, once the given model is well-defined, meaning that it satisfies all assumptions in Lee-Yang Theorem, then we can use the above identities as tools to check whether the given model has a phase transition by solving the following equation:

$$\lim_{V\to\infty} \left(\frac{1}{V} \log \mathcal{Z}(z_k, V, T)\right) = 0, \tag{9}$$

where T can be derived by using wick rotation, and V is the total volume of the system. In other words, after taking the thermodynamic limit (or macroscopic limit), and solving the above equation. If there is a zero located on the real axis, meaning it is shifted from the imaginary world (the complex plane) to the real world, then we know the given model has a phase transition.

III. CONCLUSION

This note shows that as long as a given model is well-defined, then we can apply Lee-Yang Theorem to the exponential class of the partition function of the universe.

However, there are at least two important open questions should be focused on: First, to make a further investigation of this work to study whether a phase transition really happens in the spacetime manifolds. Furthermore, a possible connection to the Riemann Zeta function is quite interesting: it may connect gravity and gauge field theory to number theory, since by studying this distribution of zeros, we can learn the information of the phase transition of solutions of Einstein field equations.

It's due to the generic framework we have demonstrated in this note, this work may shed some light on using General Relativity (Einstein-Hilbert Action, and Einstein field equations and solutions) to study number theory. This holographic principle approach may help us to build a new intuition on the zeroes of partition functions (path integrals) that comes from our understandings in gravity. Finally, an outlook on future work: first, we shall investigate the distribution of zeros of the partition functions (path integrals) of gravity, or on its boundary side by imposing the holographic principle. Second, we shall investigate ways of generalizing Lee-Yang theorem for solutions of Einstein field equations. Third, alternative, we should also investigae the possibility of applying Lee-Yang theorem to MERA models. Fourth, a further study on connection of this holographic approach and the distribution of non-trivial zeroes of the Riemann Zeta function[1], or in general, Dirichlet L-functions from a partition function perspective. Finally, a rigorous proof of identity (1) is necessary.

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