Examples of Prime Geodesic Theorem

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1 Abstract

Te goal of this paper is to answer the following questions:

- what are we counting in prime geodesic theorem?
- how to prove prime geodesic theorem?
- why does prime geodesic theorem matter?

by showing explicitly details and using some examples run through definitions and theorems in the whole paper.

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2 List of Symbols

- $\ell(T)$ is called the displacement length of T, T is an element of a linear fractional group.
- Γ denotes Fuchsian groups.
- $\mathbf{H}/\Gamma$ is upper half plane modulo a Fuchsian group. Points in this set are in the fundamental domain of Γ.

3 Introduction

In Riemann's 1859 paper "On the Number of Primes Less Than a Given magnitude[1]," he derived an explicit formula which involves a sum over all non-trivial zeros of Riemann Zeta function, and it implies that the real part of non-trivial zeroes of Riemann Zeta function controlls the oscillation of primes around their expectation locations. In a letter to Andrew Odlyzko, Pólya made a conjecture that the imaginary part of non-trivial zeros of Riemann zeta function is correspounding to eigenvalues of an unbounded self-adjoint operator. Analogue to the parition function of primes, since each hyperbolic plane naturally can obtain a laplacian (Laplace-Beltrami operator). In order to classify compact Riemann surfaces, Selberg wrote down a partition function, based one the inspiration from Riemann zeta function, for the lenghth of prime closed geodesics (also called prime orbits) on Riemann surfaces (hyperbolic planes quotient some topological discrete groups)[2], and he also derived a trace formula based on the idea of Riemann explicit formula and poisson summation formula.

With the help of Selberg trace formula, Huber proved a theorem that the lenghth spectrum of prime closed geodesic is bijective to the eigenvalue specturm of the laplace equation of the hyperbolic plane. Then, in 1970 Grigory Margulis proved a parallel result of prime number theorem. Furthermore, in 1980, Peter Sarnak proved an analogue Chebotarev's density theorem in his PhD thesis.

Conjectured by Gauss, proved by Hadamard, de la Vallée–Poussin (1896), the prime number theorem says that

$$\#\{\text{ primes } \leq x\} \sim \frac{x}{\ln x} \text{ as } x \to \infty.$$

The proof of this theorem to Riemann Zeta function $\zeta(s)$, whose only singularity is a simple pole at the point $s = 1$. It can also be proved that $\zeta(s)$ has no zeros or poles on the line $\Re(s) = 1$.

Based on this one can prove prime number theorem by using Wiener-Ikehara Tauberian theorem.

As an analogy, we can have a similar developement on a even larger theory of prime number theorem.

In order to state what we are going to prove, let's introduce some notations and terminolgies, details will be provided in the later sections:

Definition 1 (Primitive Element) *Let Γ be a Fuchsian group, then $T \in \Gamma$ is primitive, if there does not exist S in the group of T, i.e. Γ, such that $T = S^k$, $|k| > 1$.*

Let $x \in \mathbf{R}$ be a given exponent lenght of primitive oriented closed geodesics.

Definition 2 *Let Γ be a Fuchsian group, i.e. a discrete subgroup of $PSL_2(\mathbf{R})$. Let P_0 be a primitive element in Γ,*

$$\pi_X(x) = \#\left\{ \textit{distinct}\{P_0\} : e^{\ell(P_0)} \leq x \in \mathbf{R} \right\}.$$

Definition 3 *Let X be a hyperbolic surface. Let γ be a primitive oriented closed geodesic on X, and let $\ell(\gamma)$ denote the length of γ. The primitive length spectrum on compact hyperbolic surface X with genus $g \geq 2$ is definied as a set with multiplicities$\mathcal{L}_X = \{\ell(\gamma)\}$.*

Hence, as an analogy e^{ℓ} plays the role of a prime number.

Then by applying Selberg Zeta function to the lemma, we can have Prime Geodesic Theorem:

Theorem 1 (Prime Geodesic Theorem) *For X a geometrically non-elementary hyperbolic surface with critical exponent δ, the asymptotic approximation of $\pi_X(t)$ is*

$$\pi_X(t) \sim \frac{\delta t}{\delta t} \textit{ as } t \to \infty.$$

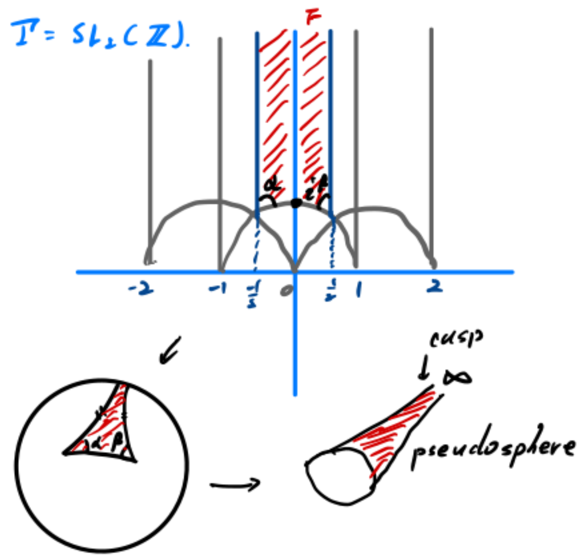
In this paper, we are going to start our investigations from small examples to build up our understandings and intuitions on these progressions. There are four pilars of this paper: algebra (number theory and group theory), analysis (real and complex), topology, and geometry (Euclidean and differential). Assuming what is Euclidean geometry is known for readers of this paper, let's set it as our starting point of this journey.

3.1 Disk model and Upper half-plane model

Since Greek many mathematicians have attempted to prove the parallel postulate (the fifth postulate) from other four axioms, but they all failed. Around 1792, Gauss is the first to discover a possibility of geometry outside of Euclidian geometry. Gauss concealed his discovery, because he thought it is unthinkable that a triangle with infinetly length (whose three angles at the three vertices A, B, C are all zero) can have a finite area which he proved is proportional to $\pi - (\angle A + \angle B + \angle C)$. Hence later on Lobachevsky (1826) and Bolyai (1829) independetly discovered this again. Most of Lobachevsky's results are based on the same idea of a triangle with infinite length in at least two edges. He did this in spherical coordinate and built hyperbolic trignometry from spherical trignometry. For instance an arc of a circle is made by swipping a radius with finite length with a finite angle, he defined horoccycle as the arc made by swipping a radius with infinite length with a finite angle.

Considering curved surfaces in $\mathbf{R}^3$, where the geometries are called non-Euclidean, these geometries are self-consistent without the fifth postulate which failed in these non-Euclidean geometries.

In 1968, Eugenio Beltrami discovered the first model which convinced mathematicians that Lobachevsky's non-Euclidean geometry is as consistent as Euclidean geometry, because Beltrami used a cured discovered by Claude Perrault in 1670, and studied by Newton (1676) and Huygens (1692) called tractrix.



Beltrami rotated tractirx with 2π in 3d Euclidean space to form a pseudosphere which he proved that pseudosphere has a negative constant Gaussian curvature, and it's a portion of the entire non-Euclidean space. Klein in 1872 coined *hyperbolic geometry* as its formulae can be derived from Lobachevsky's spherical geometry by replacing trigonometric functions by the hyperbolic functions. Klein also proposed a disk model that contains the entire hyperbolic plane in it. However, this model is not conformal.

The real great thing of Beltrami's pseudosphere model is although it's a portion of the entire hyperbolic plane, but it's isometric in 3d Euclidean space, hence it made hyperbolic geometry bacome convincing for mathematicians, because since then we can use Euclidean metric to study hyperbolic geometry.

In 1881, Poincaré extended the idea of Beltrami, by considering pseudosphere as a portion of hyperbolic plane in a disk with infinity in the boundary of the disk, he found that the isometries of Beltrami's pseudosphere model are just the Möbius transformations from the disk $\mathbf{D}$ to itself.

He then defined the infinitesimal length element and metric in for hyperbolic geometry in this disk model

Poincaré also extended this 2d hyperbolic plane $\mathbf{H}^2$ in to 3 dimensions and this extension is the fundation for some important works in 3-manifold done by Dehn (1912), Artin(1924), Thurston (1975), and Perleman (2002).

Back to Beltrami's pseudosphere model, in addition to these developement, in 1901, Hilbert closed a question that whether it's possible to embed entire hyperbolic plane into Euclidean space. He proved that it is not possible to isometrically embed the entire hyperbolic plane in to Euclidean space as a surface, that is there does exist a smooth surface in $\mathbf{R}^3$ with Euclidean topology whose geodesic geometry can represent entire hyperbolic plane.

Since we can use Möbius transformations such as

$$z \mapsto \frac{z-i}{z+i} = w$$

conformally map a unit disk of Poincaré's disk model of a hyperbolic plane into upper half-plane, and have it's inverse

$$w \mapsto \frac{w+1}{w-1}(-i) = \frac{w+1}{w-1} e^{\frac{3\pi i}{2}}$$

to map w in the upper half-plane back to the Poincaré's disk model. Thus, the disk model and upper

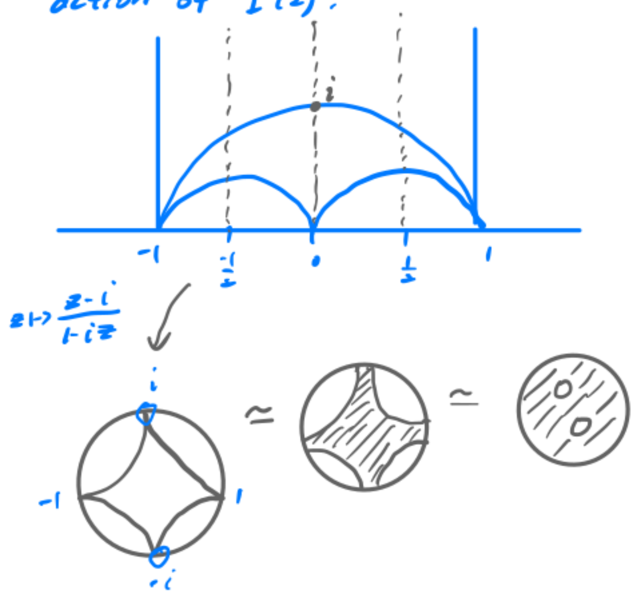
$\Gamma(2) \subset PSL_2(\mathbb{Z}) \subset PSL_2(\mathbb{R})$.

If $T \in \Gamma(2)$, $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$ad - bc = 1$; $a, b, c, d \in \mathbb{Z}$.

a, d are odd, b, c are even.

A fundamental domain for the action of $\Gamma(2)$:



halpf-plane model are equivalent.

Denote the upper half-plane $\mathbf{H}_{>0} := \{z \in \mathbf{C} : \Im(z) > 0\}$.

Then the infinitesimal length element of hyperbolic geometry in $H_{>0}$ is

$$ds = \frac{|dz|}{\Im(z)} = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

Example 1 *Want to show that with the above metric, the Gaussian curvature of this hyperbolic plane is a negative constant, -1.*

From differential geometry, recall the notation of metric $ds^2 = g_{11}dx^2 + g_{22}dy^2 = \dfrac{dx^2 + dy^2}{y^2}$, *and wee have Gaussian curvature*

$$K = \frac{-1}{\sqrt{g_{11}g_{22}}} \left(\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{g_{11}}} \frac{\partial \sqrt{g_{22}}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{g_{22}}} \frac{\partial \sqrt{g_{11}}}{\partial y} \right) \right)$$

then substitue $g_{11} = g_{22} = \dfrac{1}{y^2}$, *it follows that* $K = -1$.

3.2 Topological Groups

We have to notice that the group γ is a topological discrete group. This means we cannot randomly pick some real numbers to be a 2 by 2 matrix to be either a generator or an element of γ. Hence, we have to build some intuitions on topological discrete groups.

To begin with, let G be a group, and f be the group operation, g be a map that maps $x \in G$ to the inverse of x, $g(x) = x^{-1}$.

Definition 4 (Continuity.) *f and g are continuous, if for any open set $U \subseteq G$, $f^{-1}(U)$ and $g^{-1}(U)$ is open in the domains of f and g.*

Definition 5 (Topological group.) *G is a topological group, if G is a topological space for which f and g are continuous.*

Definition 6 (General Leaner Group.) *Let $\mathbf{F}$ be a field, then the general linear group $GL_n(\mathbf{F})$ is the group of invertible n by n matrices with entrices in $\mathbf{F}$ under matrix multiplication.*

Definition 7 (Special Leaner Group.) *Let $\mathbf{F}$ be a field, then the general linear group $SL_n(\mathbf{F})$ is the group of invertible n by n matrices with entrices in $\mathbf{F}$ under matrix multiplication, and with determinant 1.*

$SL_n(\mathbf{F})$ is a subgroup of $GL_n(\mathbf{F})$.

Definition 8 (Projective Special Leaner Group.) *Let $\mathbf{F}$ be a field, then the projective special linear group $PSL_n(\mathbf{F})$ is the quotient group of $SL_n(\mathbf{F})$ by its center $Z(SL_n(\mathbf{F}))$.*

For instance, if $\mathbf{F} = \mathbf{R}$, then $Z(SL_n(\mathbf{R})) = \{\pm \mathrm{id}\}$.

Example 2 *We can see that $PSL_2(\mathbf{R})$ and $SL_2(\mathbf{R})$ are topological groups.*

Example 3 *We can see that $\left\langle \begin{pmatrix} 1 & \sqrt{5} \\ 0 & 1 \end{pmatrix} \right\rangle < PSL_2(\mathbf{R})$ is a subgroup of $PSL_2(\mathbf{R})$, hence it is also topological group.*

3.3 Topological Discrete Groups

Definition 9 *A subgroup H, denoted as $H < SL_2(\mathbf{R})$, is discrete if H has no accumulation points in $SL_2(\mathbf{R})$.*

Example 4 *We can see that $\left\langle \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \right\rangle$ is a discrete topological group.*

Non-Example 1 *We can see that $\left\langle \begin{pmatrix} 1 & \sqrt{5} \\ 0 & 1 \end{pmatrix} \right\rangle$ is not a discrete topological group, because $\sqrt{5}$ is irrational, and hence this group has accumulation points.*

Lemma 1 *The following three statements are equivalent for a subgroup G of $SL_2(\mathbf{R})$:*

- *(i) There does not exist accumulation points in G;*
- *(ii) G has no accumulation points in $SL_2(\mathbf{R})$;*
- *(iii) The identity is an isolated point of G.*

If we define the following definition, then we can have some alternative ways to check whether a subgroup of $PSL_2(\mathbf{R})$ is discrete.

Definition 10 *G acts properly discontinuously on $\mathbf{H}_{>0}$ if for all compact subsets $E \subset H_{>0}$, then $gE \cap E = \emptyset$ for all but finitely many $g \in G$. That is, if any compact set E of $\mathbf{H}_{>0}$ only intersects with finitely many orbit points of the action, then the action of G is properly discontinuous.*

Lemma 2 *The following three statements are equivalent for a subgroup G of $SL_2(\mathbf{R})$:*

- *(i) G does not act properly discoutinuously on $H_{>0}$;*
- *(ii) Some G orbit in $H_{>0}$ has accumulation points in $H_{>0}$;*
- *(iii) All orbits of G in $H_{>0}$ have accumulation points in $H_{>0}$, except the case that one orbit that consists of a single point fixed by all elements in G.*

Theorem 2 *If G acts properly discontinuously on $\mathbf{H}_{>0}$, then G is a discrete subgroup of $SL_2(\mathbf{R})$.*

Lemma 3 *Let G be an abelian subgroup of $Aut\mathbf{H}_{>0}$ which acts properly discontinuously and that contains an eliptic transformation or a parabolic transformation is cyclic.*

Theorem 3 *Let G be an abelian discrete subgroup of $Isom^{+}\mathbf{H}_{>0}$, then G is cyclic.*

Lemma 4 *Let G be a discrete subgroup of $Isom^{+}\mathbf{H}_{>0}$. Let T be an element in G, T is hyperbolic. Let $S \in G$, and S is parabolic, then S and T does not have a common fixed point.*

Theorem 4 *G is a discrete subgroup of $PSL_2(\mathbf{R})$ if and only if the action of G on $\mathbf{H}_{>0}$ is properly discontinuously.*

3.4 Fuchsian Groups

Definition 11 (Fuchsian Group) *A Fuchsian group is a discrete subgroup of $SL_2(\mathbf{R})$.*

Example 5 *The first example is $SL_2(\mathbf{Z})$, for instance, we can see that $\left\langle \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \right\rangle$ is a discrete topological group, and it's a subgroup of $SL_2(\mathbf{Z})$. Hence it is Fuchsian. Furthermore, this group generator is parabolic. Then all of congruence subgroups of $SL_2(\mathbf{Z})$ are also Fuchsian.*

Definition 12 *A fundamental domain $\mathcal{F} \subset \mathbf{H}_{>0}$ for a Fuchsian group Γ is a closed set such that*

$$\Gamma\mathcal{F} := \bigcup_{T \in \Gamma} T\mathcal{F} = \mathbf{H}_{>0}$$

and for each $T \neq id$, the intersection of the interior of $\mathbf{F}$ and the interior of $T\mathbf{F}$ is empty.

Definition 13

$$\Psi_0(s) := \sum_{k=1}^{\infty} \frac{\ln(\xi_k)}{\xi_k^s}, \Re(s) > 1.$$

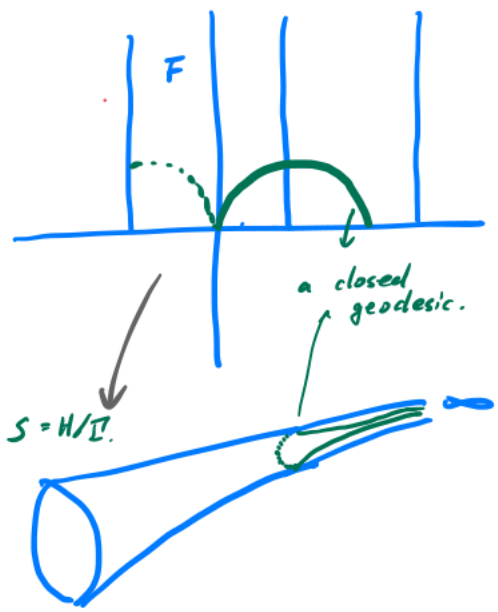
Definition 14

$$\theta_0(x) := \sum_{\xi_k \leq x} \ln(\xi_k) \sim x.$$

4 Selberg trace formulae

One of the original motivations for Selberg to write derive this formula is to classify compact Riemann surface.

Theorem 5 *If X is a Riemann surface with genus greater than 1, Γ is a strictly hyperbolic Fuchsian group such that the fundamentl group $\pi_1(X) = \Gamma$, then $X \cong \mathbf{H}_{>0}/\Gamma$.*



Definition 15 (Lapalce-Beltrami operator) *Let ∇^2 be an operator on $X \cong \mathbf{H}_{>0}/\Gamma$ where X has a C^∞ metric,*

$$ds^2 = \sum_{ij} g_{ij} dx_i dx_j.$$

∇^2 is a self-adjoint operator with compact resolvent and is called Lapalce-Beltrami operator and is defined as follows:

$$\nabla^2 \phi := \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ik} \frac{\partial \phi}{\partial x_k} \right)$$

where ϕ is an nonzero eigenfuncion.

Then the Laplacian eigenvalue problem on compact Riemann surfaces is to find λ that satisfies

$$\nabla^2 \phi + \lambda \phi = 0.$$

Since X is compact, there is no boundary condition, then it can be proved that ∇^2 is self-adjoint and has compact resolvent. The inverse of ∇^2 is denoted as $(\nabla^2)^{-1}$ which is self-adjoint and compact. It follows from spectrum theorem that the eigenvalues of $(\nabla^2)^{-1}$ form a real sequence converging to zero. Then we can order these eigenvalues in a increasing sequence:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...,$$

notice that $\lim_{n \to \infty} \lambda_n = \infty$.

What are we counting?

$$e^{l_n} \xleftrightarrow[\text{by Huber Thm.}]{\text{1-1 correspondence}} \lambda_n$$

e^{l_n}: primes i.e. exponential length of primitive geodesic

λ_n: eigenvalues of Laplace equation associated to Riemann surface $X = \mathbb{H}/\Gamma$

$$\Downarrow \qquad\qquad \Downarrow$$

$$Z_X(s) = \prod_{l \in L_X} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)l}\right) \longleftrightarrow \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

Theorem 6

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} = \frac{\mu(\mathcal{F})}{4\pi t} + \sum_{k=0}^{N} b_n t^n + \mathcal{O}(t^{N+1})$$

where $\mu(\mathcal{F})$ *denotes the area of* $\mathcal{F}$ *and* b_n *are constants.*

Theorem 7 (Tauberian) *Let* μ *be a positive measure on* $[0, \infty)$, $\alpha \in (0, \infty)$. *Assume we are given*

$$\int_0^{\infty} e^{-\lambda t} d\mu(\lambda) \sim \alpha t^{-\alpha}, t \to 0,$$

$$\int_0^{x} d\mu(\lambda) \sim \frac{\alpha x^{\alpha}}{\Gamma(\alpha + 1)}, x \to \infty.$$

Let $\mathcal{H}$ be a Hilbert space, and let A be an operato in $\mathcal{L}(\mathcal{H})$.

Definition 16 *A is called positive if* $\langle Ax, x \rangle \geq 0, \forall x \in \mathcal{H}$, *and it is denoted as* $A \geq 0$.

Definition 17 A^* *is the adjoint operator of* A.

Let $\{e_n\}_{k=0}^{\infty}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$.
From functional analysis,

Theorem 8 *If we let* A *be a positive operator.* $\sum_{k=0}^{\infty} \langle e_n, Ae_n \rangle$ *is independent of the choice of orthonormal basis.*

Definition 18 *Then,* $\sum_{k=0}^{\infty} \langle e_n, Ae_n \rangle$ *is defined as the trace of the operator* A, *and is denoted by* $Tr(A) := \sum_{k=0}^{\infty} \langle e_n, Ae_n \rangle$.

Definition 19 *An operator* $A \in \mathcal{L}(\mathcal{H})$ *is called a trace class operator if and only if* $Tr(A) < \infty$.

For our own purpose, let's consider the space

$$L^2(X) = L^2(\mathbf{H}_{>0}/\Gamma)$$
$$= \left\{ f \in L^2(\mathcal{F}) : f(Tz) = f(z), \forall T \in \Gamma, z \in \mathbf{H}_{>0} \right\}.$$

One can show that $L^2(S)$ is a Hilbert space equipped with the inner product

$$\langle f_1, f_2 \rangle = \int_{\mathcal{F}} f_1(z)\overline{f_2(z)} d\mu(z).$$

Theorem 9 (Selberg trace formula) *Let $h(z)$ be analytic on $\left\{ z \in \mathbf{H}_{>0} : \Im(z) \leq \frac{1}{2} + \epsilon \right\}$ such that*

$$h(z) = h(-z), |h(z)| \leq C(1 + |r|)^{-2-\epsilon},$$

with $C, \epsilon > 0$, and define $g(x), x \in \mathbf{R}$ by the Fourier transformation of h:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(z) e^{-izx} dz.$$

Then we have

$$\sum_{k=0}^{\infty} h(r_k) = \frac{\mu(\mathcal{F})}{4\pi} \int_{-\infty}^{\infty} z h(z) \tan h(\pi z) dz$$
$$+ \sum_{\{T\}} \frac{\ell(T_0) g[\ell(T)]}{e^{\ell(T)/2} - e^{-\ell(T)/2}}$$

where $\{T\}$ is taken over all distinct conjugacy class in Γ except the class of identity. The sums and integrals are absolutely convergent.

Theorem 10 *If $S, T \in \Gamma$, and $\exists P \in PSL_2(\mathbb{R})$ such that $T = S^{-1}TS$, then $tr(T) = tr(S)$.*

Example.

$T \in \Gamma \subset PSL_2(\mathbb{R})$, T is hyperbolic

$T := \begin{bmatrix} e^{\ell(T)/2} & 0 \\ 0 & e^{-\ell(T)/2} \end{bmatrix}$ $\quad T: z \mapsto e^{\ell} z$

where $\ell := \inf_{z \in H} (z, Tz)$

= the shortest lengh from z to Tz.

$$tr(T) = e^{\ell/2} + e^{-\ell/2}$$
$$= 2\cosh(\ell/2)$$
$$\Rightarrow \frac{\ell}{2} = \operatorname{arccosh}\left(\frac{1}{2} tr(T)\right)$$
$$\Rightarrow \ell(T) = 2\cosh^{-1}\left(\frac{1}{2} tr(T)\right).$$

Theorem 11 *There is a injection between the length of primitive closed oriented geodesics on* $S \cong \mathbf{H}_{>0}/\Gamma$. *and the cojugacy classes* $\{T\}$ *in* Γ

5 Proof of The Prime Orbit Theorem

Theorem 12 (Newman's Tauberian Theorem) *Assume that* $f(t)$ *is bounded, and locally integrable on* $[0, \infty)$, *and*

$$g(z) := \int_0^\infty f(t)e^{-zt}dt$$

exists for $\Re(z) > 0$. *If* g *extends to a holographic function on a neighborhood of* $\{z : \Re(z) \geq 0\}$, *then* f *is integrable on* $[0, \infty)$, *and*

$$\int_0^\infty f(t)dt = g(0) = \lim_{T\to\infty} g_T(x) = \lim_{T\to\infty} \int_0^T f(t)e^{0\cdot t}dt$$

Proof 1 *By assumption, we already have*

$$g(z) := \int_0^\infty f(t)e^{-zt}dt.$$

Want to show $f(t)$ *is integrable on* $[0, \infty)$, *and*

$$\int_0^\infty f(t)dt = \int_0^\infty f(t)e^{-0\cdot t}dt = g(0).$$

To show this, let's define

$$g_T(z) := \int_0^T f(t)e^{-zt}dt.$$

Then it is equivalent to show the following limit exists:

$$\lim_{T\to\infty} g_T(0) = g_\infty(0) = \int_0^\infty f(t)e^{-zt}dt|_{z=0} = \int_0^\infty f(t)dt = g(0).$$

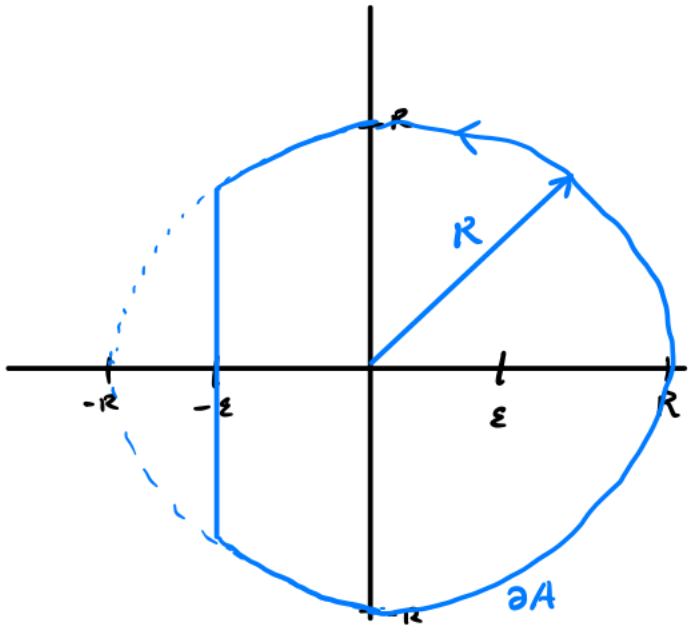
In brief, we need to show

$$\lim_{T\to\infty} g_T(0) = g(0).$$

Since $g(z)$ *is given and by the assumption* $g(z)$ *has an analytic extension. Hence* $\forall R > 0$, *take* $\epsilon > 0$ *such that* g *is analytic on*

$$A = \{|z| \leq R, \Re(z) \geq -\epsilon\}$$

To estimate $g(0) - g_T(0)$, *first, take* $\epsilon > 0$, *and consider a contour as follows:*



By Cauchy integral formula and Residue theorem, since $\dfrac{e^{zT}}{z}$ *has only one pole at* $z=0$ *on the oriented closed loop* ∂A*, hence*

$$\frac{1}{2\pi i}\int_{\partial A}\frac{e^{zT}}{z}=1.$$

Since e^{zT} *is entire and* $\dfrac{ze^{zT}}{R^2}$

$$\Rightarrow \frac{1}{2\pi i}\int_{\partial A}\left(\frac{R^2e^{zT}}{zR^2}+\frac{ze^{zT}}{R^2}\right)dz=1+0=1.$$

That is

$$\frac{1}{2\pi i}\int_{\partial A}\left(1+\frac{z^2}{R^2}\right)\frac{e^{zT}}{z}dz=1.$$

By Cauchy integral formula

$$g^{(n)}(a)=\frac{n!}{2\pi i}\int_{\partial A}\frac{g(z)dz}{(z-a)^{n+1}}$$

hence

$$g^{(0)}(0)=g(0)=\frac{1}{2\pi i}\int_{\partial A}\frac{g(z)dz}{z}.$$

Similarly,

$$g_T^{(0)}(0)=g_T(0)=\frac{1}{2\pi i}\int_{\partial A}\frac{g_T(z)dz}{z}$$

and this implies

$$g(0)-g_T(0)=\frac{1}{2\pi i}\int_{\partial A}[g(z)-g_T(z)]\frac{dz}{z}.$$

Since we have proved the integral $\left(1+\dfrac{z^2}{R^2}\right)e^{zT}$ *has contribution equal to 1, so by putting everything together,*

$$g(0)-g_T(0)=\frac{1}{2\pi i}\int_{\partial A}[g(z)-g_T(z)]\left(1+\frac{z^2}{R^2}\right)e^{zT\frac{dz}{z}}.$$

Let $M := \sup_{t\in[0,t]} |f|$.
(i) On A_+*, since* f *is bounded*

$$|g(z) - g_T(z)| = |\int_T^\infty f(t)e^{-zt}dt| \leq \frac{M}{2\pi}\left(\frac{e^{-\Re(z)T}}{\Re(z)}\right)$$
$$\Rightarrow |I_(T)| \leq \frac{M}{2\pi}\int_{A_+}\frac{1}{\Re(z)}\left|1+\frac{z^2}{R^2}\right|\frac{|dz|}{R}$$

For $z \in A_+$*, we have* $|1+\frac{z^2}{R^2}| = \frac{2\Re(z)}{R}$*, and the length of* A_+ *is* πR*. Hence,* $|I_+(T)| \leq \frac{M}{R}$.
(ii) On A_-*, decompose integral over* g *and* g_T*. Since* g_T *is analytic, we can deform the contour to the arc of a circle tht is the continuation on* A_+

$$\int_{A_-} g_T(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z} = \int_{A_R} g_T(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)$$

where $A_R := Re^{i\theta}$ *for* $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.
Since $|g_T(z)| \leq M\frac{e^{|\Re(z)|T}-1}{|\Re(z)|}$*, hence* $\left|\int_{A_-} g_T(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| \leq \frac{M}{R}$.

On the other hand, we have $\lim_{T\to\infty}\left|\int_{A_-} g(z)e^{zT}\left(1+\frac{z^2}{R^2}\right)\frac{dz}{z}\right| = 0$*. Since for* $\Re(z) < 0, e^{zT} \to 0$ *as* $T\to\infty$ *uniformly on compact set* $\{z : \Re(z) < 0\}$*. Therefore* $\limsup_{T\to\infty} |A-(T)| \leq \frac{M}{R} + 0 = \frac{M}{R}$.
(iii) Combining (i) and (ii) of $I_\pm(T)$

$$\limsup_{T\to\infty} |g(0) - g_T(0)| \leq \frac{2M}{R}.$$

Since R *is arbitrarily chosen, so* R *can be arbitrarily small that is*

$$\lim_{T\to\infty} g_T(0) = g(0),$$

as promised.

We can use this theorem to prove prime number theorem by defining

$$f(t) := \sum_{p \text{ is prime}} \ln p$$

and with $g(z)$ given by logarithmic derivative of the Riemann zeta function by $\frac{\zeta'(z)}{\zeta(z)}$.
To prove Prime Geodesic Theorem, instead of usin Riemann zeta funcion, we use Selberg Zeta function and follow the same procedure.

Proof 2 (Prime Geodesic Theorem) *Define* $\eta(s) := \sum_{\ell\in\mathcal{L}_x} \ell e^{-s\ell}$*. By Selberg Zeta function, for* $\Re(s) > \delta$*, we have*

$$\frac{Z_X^{'}(s)}{Z_X(s)} = \sum_{\ell\in\Gamma}\sum_{k=0}^{\infty}\frac{\ell}{e^{(s+k)\ell}-1}.$$

Then

$$\frac{Z_X^{'}(s)}{Z_X(s)} - \eta(s) = \sum_{\ell\in\Gamma}\frac{\ell}{e^{s\ell}\left(e^{s\ell}-1\right)} + \sum_{\ell\in\Gamma}\sum_{k=1}^{\infty}\frac{\ell}{e^{(s+k)\ell}-1}.$$

The first sum is holomorphic for $s \in \{s : \Re(s) > \delta/2\}$. *The second sum is holomorphic for* $s \in \{s : \Re(s) > \delta - 1\}$.

Hence, eta(s) has a meromorphic extension to $\{s : \Re(s) > \delta/2\}$ *and the region has the same poles and residues as* $\dfrac{Z_X'(s)}{Z_X(s)}$.

By lemma, $\eta(s) - \dfrac{1}{s-\delta}$ *is holomorphic for* $s \in \{s : \Re(s) \geq \delta\}$. *Define* $\mu(t) := \sum_{\ell \in \mathcal{L}_X, \ell \leq t} \ell$. *Then*

$$\eta(s) = \sum_{\ell \in \mathcal{L}_X} \ell e^{-s\ell} = \int_0^\infty e^{-st} d\mu(t).$$

Since $\pi_X(t) = \mathcal{O}(e^t)$, *for* $\Re(s) > 1$, *by integrating by parts*

$$\eta(s) = s \int_0^\infty e^{-st} \theta(t) dt.$$

Set $g(z) := \dfrac{\eta(z+\delta)}{z+\delta} - \dfrac{1}{\delta z}$.

$$\Rightarrow g(z) = \int_0^\infty e^{-(z+\delta)t} \theta(t) dt - \frac{1}{\delta z} = \int_0^\infty e^{-zt} \left[e^{-\delta t} \theta(t) - \frac{1}{\delta} \right] dt.$$

Because $\eta(s) - \dfrac{1}{s-\delta}$ *is holomorphic for* $\Re(s) \geq \delta$, *so* $g(z)$ *extends holomorphically to* $\Re(z) \geq 0$. *By Newman's Tauberian Theorem with* $f(t) = e^{-\delta t}\theta(t) - \dfrac{1}{\delta}$

$$\int_0^\infty \left(e^{-\delta t}\theta(t) - \frac{1}{\delta} \right) dt = g(0) < \infty.$$

Want to show the above impies $e^{-st}\theta(t) \sim \dfrac{1}{\delta}$ *as* $t \to \infty$, *i.e.,* $\theta(t) \sim \dfrac{e^{st}}{\delta}$ *as* $t \to \infty$.

Let $\epsilon > 0$ *be given. Assume there exists a sequence* $\{t_j\} \subset [0, \infty)$ *such that* $\theta(t_j) \leq \dfrac{e^{\delta(t_j+\epsilon)}}{\delta}$

$$\Rightarrow I(t_j) := \int_{t_j}^{t_j+\epsilon} \left[e^{-\delta t}\theta(t) - \frac{1}{\delta} \right] dt \geq \int_{t_j}^{t_j+\epsilon} \left[e^{-st}\theta(t_j) - \frac{1}{\delta} \right] dt$$

$$= \theta(t_j) \int_{t_j}^{t_j+\epsilon} \left[e^{-\delta t - \frac{1}{\theta(t_j)\delta}} \right] dt.$$

Since θ *is increasing, and* $t_j \to \infty$, *hence*

$$\frac{1}{\theta(t_j)\delta} \to 0$$

and

$$\int_{t_j}^{t_j+\epsilon} e^{-\delta t} dt = \frac{-1}{\delta} e^{-\delta t}\Big|_{t_j}^{t_j+\epsilon} = \frac{-1}{\delta} \left(e^{-\delta(t_j+\epsilon) - e^{-\delta t_j}} \right)$$

$$= \frac{e^{-\delta t_j} - e^{-\delta(t_j+\epsilon)}}{\delta}.$$

$$\Rightarrow I(t_j) = \theta(t_j) \left(\frac{1}{e^{\delta t_j}} - \frac{1}{e^{\delta(t_j+\epsilon)}} \right) \delta \geq \frac{e^{\delta(t_j+\epsilon)}}{\delta} \left(\frac{1}{e^{\delta t_j}} - \frac{1}{e^{\delta(t_j+\epsilon)}} \right) \delta = \left(\frac{e^{\delta(t_j+\epsilon)}}{e^{\delta t_j}} - 1 \right) = e^{\delta\epsilon} - 1.$$

Since $\epsilon > 0$ *is arbitrarily chosen, hence* $e^{\delta\epsilon}$ *can be arbitrarily large and this contradicts to* $\int_0^\infty \left(e^{-\delta t}\theta(t) - \dfrac{1}{\delta} \right) dt < \infty$.

Thus $\forall t_j \in [0,\infty)$, $\nexists t_j \to \infty$ *such that* $\theta(t_j) \geq e^{\delta(t_j+\epsilon)}$. *That is*

$$\limsup_{t\to\infty} \theta(t) \leq \limsup_{t\to\infty} \frac{e^{\delta t}}{\delta}$$
$$\Rightarrow \limsup_{t\to\infty} \delta e^{-\delta t}\theta(t) \leq 1.$$

Similarly, we can have $\liminf\limits_{t\to\infty} \delta e^{-\delta t}\theta(t) \geq 1$.

Since $\theta(t) \leq t\pi_X(t)$ *by its definition, so* $\liminf\limits_{t\to\infty} \dfrac{t\pi_X(t)}{\theta(t)} \geq 1$.

"(≥)"

For any $\lambda > 1$, $\theta(t) \geq \sum\limits_{\lambda t \leq \ell \leq t} \ell \geq \lambda t\left[\pi_X(t) - \pi_X(\lambda t)\right]$. *By the definition of* δ *as the exponent of component of* $\sum\limits_{\ell} e^{-s\ell}$, *we have* $\sum\limits_{\ell} e^{-s\ell} < \infty$ *for any* $s > \delta$. *Since*

$$\sum_{\ell \leq t} e^{-st} \geq \pi_X(t)e^{-st}$$
$$\Rightarrow \pi_X(t) = \mathcal{O}(e^{(\delta+\epsilon)t}), \forall \epsilon > 0.$$
$$\Rightarrow \theta(t) \geq \lambda t(\pi_X(t) - \pi_X(\lambda t))$$
$$\Rightarrow \lambda^{-1} \geq \frac{t\pi_X(t)}{\theta(t)} - \frac{t\pi_X(\lambda t)}{\theta(t)}$$
$$\Rightarrow \frac{t\pi_X(t)}{\theta(t)} \leq \lambda^{-1} + t\left(\frac{\pi_X(\lambda t)}{\theta(t)}\right) \leq \lambda^{-1} + t\frac{c' e^{(\delta+\epsilon)\lambda t}}{\theta(t)} = \lambda^{-1} + t\left(\frac{c'\delta}{k}\right)e^{(\delta+\epsilon)\lambda t - \delta t}$$
$$= \lambda^{-1} + tce^{(\delta+\epsilon)\lambda t - \delta t}$$

where $\theta(t) := k\dfrac{e^{\delta t}}{\delta}$ *and let* $c = \dfrac{c'\delta}{k}$, c' *is a constant.*
Take

$$\epsilon < \frac{\delta(1-\lambda)}{\lambda}$$
$$\Rightarrow \delta + \epsilon < \frac{\delta\lambda + \delta(1-\lambda)}{\lambda}$$
$$\Rightarrow \lambda(\delta+\epsilon) - \delta < \delta\lambda + \delta - \delta - \delta\lambda - \delta = 0$$
$$\Rightarrow e^{\alpha t} < 1, \forall t \in [0,\infty), \text{ where } \alpha = \lambda(\epsilon+\delta) - \delta < 0$$

and $e^{\alpha t} \to 0$ *as* $t \to \infty$.

$$\Rightarrow \limsup_{t\to\infty} \left(\frac{t\pi_X(t)}{\theta(t)}\right) \leq \lambda^{-1}, \forall \lambda > 1$$
$$\Rightarrow \theta(t) \sim t\pi_X(t) \text{ as } t \to \infty.$$

Since $\theta(t) \sim \dfrac{e^{\delta t}}{\delta}$ *as* $t \to \infty$, *hence* $t\pi_X(t) \sim \dfrac{e^{\delta t}}{\delta}$ *as* $t \to \infty$

$$\pi_X(t) \sim \frac{e^{\delta t}}{\delta t} \text{ as } t \to \infty.$$

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