A PROOF OF THE IRRATIONALITY OF $\zeta(3)$

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Contents

1. The Main Theorem

Theorem 1. *$\zeta(3)$ is irrational.*

2. Idea of proving the main theorem

The idea of this approach is to show that the two sequences we generated by using a second-order recurrence relation to form a sequence of rational numbers, and the ratio of the two sequences converges to $\zeta(3)$ with the rate of the order $\frac{1}{b_n^2}$, and after we use a_n and b_n to define p_n and q_n, then it invokes the irrationality criterion (IC), and then we can estimate the δ which is defined in IC.

Our first step is to prove the following criterion:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{1+\delta}}, \tag{1}$$

The details for the meaning of the above inequality is in (13).

Once we prove the irrationality criterion (1), the next step is to prove the following identity (2) by using Zeilberger telescoping.

For all $A_1, A_2, ...$ where A_k are integers with form $-k^2, k \in \mathbb{Z}_{\geq 1}$, and $x = n^2, n \in \mathbb{Z}_{\geq 1}$.

$$\sum_{k=1}^{M} \frac{A_1 A_2 ... A_{k-1}}{(x + A_1)(x + A_2)...(x + A_k)}$$

$$= \frac{1}{x} - \frac{A_1 A_2 ... A_M}{x(x + A_1)(x + A_2)...(x + A_M)} \tag{2}$$

After proving (2), we can substitute $A_k = -k^2, k \in \mathbb{Z}_{\geq 1}$, and $x = n^2, n \in \mathbb{Z}_{\geq 1}$ so that we can derive the following equation (3). We know: $\sum_{n=1}^{\infty} \frac{1}{n^3} := \zeta(3)$. We will prove the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} \tag{3}$$

After preparing the first three steps, the next step is to prove that we can construct two sequences $\{a_n\}$ and $\{b_n\}$, where $a_0 = 0$, $a_1 = 6$, and $b_0 = 1$, $b_1 = 5$. The point is both sequences that starting with the different initial conditions satisfy the recurrence relation (8).

Furthermore, we can prove the two sequences have the following general forms:

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}. \tag{4}$$

We use the following notation to abbreviate $\{a_n\}$, i.e., the first sequence:

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}}. \tag{5}$$

The second sequence with a different initial condition can have the following form:

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{6}$$

Creative Telescoping. This terminology was introduced by v. d. Poorten [1]. One of his step is to write down b_n as the sum of $b_{n,k}$ (please see Appendix 6.3) which satisfies the other sequence a_n. To explicitly derive the second-order recurrence relation. This second-order recurrence relation is highly-non-trivial as mentioned by v. d. Poorten, "Neither Cohen nor I had been able to prove this in the intervening two months", until Zagier found a way, which is called Creative Telescoping, and we explicitly done in Appendix 6.3. This technique, combined with W. Gosper algorithm for indefinite hypergeometric summation is at the heart of D. Zeilbergers algorithm.

On the other hand, we have the following recurrence relation:

$$n^3 u_n + (n-1)^3 u_{n-2} = f(n) u_{n-1}, n \geq 2 \tag{7}$$

where u_n is an integer. $f(n) = (34n^3 - 51n^2 + 27n - 5)$ which is derived from the general from of the continued fraction of $\zeta(3)$.

Moreover, we can prove (7) is equivalent to the following recurrence relation:

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5) u_n + n^3 u_{n-1} = 0, n \geq 1 \tag{8}$$

where u_n is an integer.

This implies immediately that $\frac{a_n}{b_n}$ is the n-th convergent of the following continued fraction:

$$\zeta(3) = \frac{6}{5-}\frac{1}{117-}\cdots\frac{n^6}{34n^3+51n^2+27n+5-}\cdots$$
$$= \cfrac{6}{5-\cfrac{1}{117-\cfrac{64}{535-\cdots\cfrac{n^6}{34n^3+51n^2+27n+5-\cdots}}}}$$
$$= \cfrac{6}{p(0)-\cfrac{1^6}{p(1)-\cfrac{2^6}{p(2)-\cfrac{3^6}{\cdots}}}}$$

where $p(n) := 34n^3 + 51n^2 + 27n + 5$.

Let $\{a_n\}$, and $\{b_n\}$ be given, and they satisfy the recurrence relation

$$n^3 u_n - (n-1)^3 u_{n-2} = f(n) u_{n-1}, n \geq 2$$

where

$$f(n) = 34n^3 - 51n^2 + 27n - 5.$$

Then we have

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3} \tag{9}$$

where a_n, and b_n are defined as (4), and (6).

$$c_{n,k} \longrightarrow \zeta(3) \textit{ as } n \longrightarrow \infty \textit{ uniformly in } k. \tag{10}$$

Then we prove (11), and use (11) to prove (12).

$$\frac{a_n}{b_n} \longrightarrow \zeta(3) \textit{ as } n \longrightarrow \infty. \tag{11}$$

On the other hand, we prove the following:

If $a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}$, then

$$\zeta(3) - \frac{a_n}{b_n} = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}. \tag{12}$$

The final step, after we prove (11) and (12), we can combine them into the criterion (1) to prove the Theorem 1.

3. Step 1: Prove the irrationality criterion

Definition

A real number α is **approximable** to order n, if it suffices the inequality
$$\left|\alpha - \frac{h}{k}\right| < \frac{C}{k^n}$$
where C is a real constant, and $h, k \in \mathbb{Z}$.

It is clear to see that a rational number is approximable to order $n = 1$ but not higher. It follows that if we have a candidate, α, that we want to prove it is irrational, then all we need to show is to construct any infinite sequence of distinct rational numbers, $\frac{p_n}{q_n}$ such that we can show that for some $\delta > 0$
$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{C}{q_n^{1+\delta}}.$$
However, let's still prove it in the following theorem, to be self-contained.

Theorem

Any rational number is approximable to order 1, but not to any higher order.

Proof. Take $\frac{a}{b} \in \mathbb{Q}$, and $(a, b) = 1, b \geq 1$. There are infinitely many integral solution for the equation $ax - by = 1$. Suppose $x = x_0$, and $y = y_0$ is one solution, then the general solution is
$$x = x_0 + bt, y = y_0 + at, t \in \mathbb{Z}.$$
It's clear that $x_0 + bt > 0$ for all t. Hence there are infinitely many $(x, y), x > 0$ to the equation $ax - by = 1$. Then we have
$$\left|\frac{a}{b} - \frac{y}{x}\right| = \frac{1}{bx} < \frac{2}{x} \ \textit{where}\ (b \geq 1).$$
It means $\frac{a}{b}$ is approximable to order 1, with $n = 1, \frac{h}{k} = \frac{y}{x}, \alpha = \frac{a}{b}, C = 2$. That is to say, for any fraction $\frac{y}{x} \neq \frac{a}{b}$ we have
$$\left|\frac{a}{b} - \frac{y}{x}\right| = \left|\frac{ax - by}{bx}\right| \geq \frac{1}{bx}.$$
There doesn't exist a constant C such that $\frac{1}{bx} < \frac{C}{x^2}$ for infinitely many integers x, and so $\frac{a}{b}$ is not approximable for $n > 1$. ☐

Irrationality Criterion

Corollary. If there exists $\delta > 0$ and infinite pairs $p_n, q_n \in \mathbb{Z}$ and $(p_n, q_n) = 1$ such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}, \tag{13}$$

then $\alpha \in \mathbb{R}$ is irrational.

We prove an equivalent statement: Suppose $\alpha = \frac{a}{b}$ is a rational number. For any $\delta > 0$, there exist only finitely many rational numbers $\frac{p}{q}$ such that (13) is true. Notice that the way to use this criterion is to assume α is rational; however, if we can derive infinitely many rational numbers $\frac{p}{q}$ satisfy the inequality, hence it's a contradiction to the equivalent statement, and hence in this case α is not rational number.

For our purpose, in the previous theorem, C could be chosen as 1. This can be checked as follows.

Proof.

$$\Rightarrow \alpha - \frac{1}{q^{1+\delta}} < \frac{p}{q} < \alpha + \frac{1}{q^{1+\delta}}.$$

Suppose the solution for this double inequalities exist. Then we must have the following:

$$\alpha + \frac{1}{q^{1+\delta}} > \alpha + \frac{1}{bq} \Rightarrow q^{\delta} < b$$

or

$$\alpha - \frac{1}{q^{1+\delta}} < \alpha - \frac{1}{bq} \Rightarrow q^{\delta} < b.$$

Thus $q < b^{\frac{1}{\delta}}$, therefore, the possible values of q are $\left\{1, 2, ..., \lfloor b^{\frac{1}{\delta}} \rfloor\right\}$, meaning that there are only finitely many. Also, for each such q the value of p is constructed by the condition

$$\frac{|aq - bp|}{bq} < \frac{1}{q^{1+\delta}}$$

or

$$|aq - bp| < \frac{b}{q^{\delta}},$$

hence,

$$\frac{aq}{b} - \frac{1}{q^{\delta}} < p < \frac{aq}{b} + \frac{1}{q^{\delta}}.$$

It follows that, only a finite number of values of q will satisfy to each of them. There are only a finite number of values of the corresponding p, and this completes the proof. $\square$

Continued Fraction Expansion

Now, we divide both sides by $q_n \cdot q_{n-1}$, let $x = \frac{p_{n-1}}{q_{n-1}}$. Then we know we already found a smaller bound by using continued fraction, compared to Dirichlet (existential) Theorem. This method converges in a running time of order $\log n$ (the last section of our proof is an example for showing this time complexity):

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_{n-1} q_n}.$$

Form this result, we can estimate the convergence rate of the $\frac{a_n}{b_n}$.

Remark 1. The essence of continued fraction is Euclid Algorithm (which may remind us the $SL(2, \mathbb{Z})$ structure). The key idea in Euclid's algorithm is that the remainders form a strictly decreasing sequence of non-negative integers; the essence in continued fraction is: the denominators also form a strictly decreasing sequence of positive integers.

Remark 2. There is a significant difference between this result and Dirichlet Theorem:
The pigeonhole principle is not constructive. It does not tell us what $\frac{p}{q}$ exactly is. However, the continued fraction expansion is constructive. We can determine $\frac{p}{q}$ very efficiently. In fact, the convergents are the best rational approximations to the value of the the continued fraction of α. **That is, if $\frac{p_n}{q_n}$ is a convergent of α, then the best rational approximation to α with denominator at most q_n is $\frac{p_n}{q_n}$.**

Remark 3. Furthermore, the proof of Dirichlet Theorem is linear in time complexity, meaning that it takes n computations to determine an approximation. On the other hand, the continued fraction method has time complexity in $log(n)$.

Remark 4. Notice that the p_n and q_n are going to be our a_n and b_n times d_n^3, which is defined in step 9, that are systematically generated by the sequences that satisfy the second-order recurrence relation. And, from the notions of continued fraction, we know they are convergents (the truncated sequence), of a real number α. If this sequence is finite, then α is rational, and if not, then α is irrational.

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4. Step 9: Prove Theorem 1.

Proof. Starting from

$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}.$$

we have

$$\zeta(3) - \frac{a_n}{b_n} = O\left(\left(\frac{1}{b_n}\right)^2\right)$$

The recurrence relation satisfied by b_n:

$$b_n - (34 - 51n^{-1} + 27n^{-2} - 5n^{-3})b_{n-1} + (1 - 3n^{-1} + 3n^{-2} - n^{-3})b_{n-2} = 0$$

Consider the recurrence relation:

$$n^3 b_n + (n-1)^3 b_{n-2} = (34n^3 - 51n^2 + 27n - 5)b_{n-1}$$

which can be rewritten as

$$b_n + \left(\frac{n-1}{n}\right)^3 b_{n-2} = \left(34 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3}\right) b_{n-1}$$

and as n is large enough, we can obtain

$$b_n + b_{n-2} = 34b_{n-1}$$

which is a second-order recurrence relation with constant coefficients, meaning that we can solve it by setting $b_n = B^n$ for some constant B. Substituting this into the above equation we get

$$B^n + B^{n-2} = 34B^{n-1}$$

which is

$$B^2 + 1 = 34B.$$

Hence we can get the positive root as $B = 17 + 12\sqrt{2} = (1 + \sqrt{2})^4$. Therefore, for large n, we have $b_n \sim (1 + \sqrt{2})^{4n} := \alpha^n$.

Hence we have $b_n = O(\alpha^n)$. We also already knew all a_n are rational numbers with denominators which is divided by d_n^3 where $d_n = lcm(1, 2, 3, ..., n)$.

Since a_n is not integers, so we need the following steps to fit in the irrationality criterion. Next, define two new sequences p_n and q_n such that $\zeta(3) \longrightarrow \frac{p_n}{q_n}$ as $n \longrightarrow \infty$. Define

$$p_n = 2d_n^3 a_n, q_n = 2d_n^3 b_n$$

where $p_n, q_n \in \mathbb{Z}$.

If we don't define p_n, q_n, and we only use:

$$\zeta(3) - \frac{a_n}{b_n} = O\left(\frac{1}{b_n^2}\right)\textbf{, then by IC } \Rightarrow \frac{1}{b_n^2} < \frac{1}{b_n^{1+\delta}}$$
$$\Rightarrow 0 < \delta < 1\textbf{, and we are done?}$$

Since a_n is not integer, so we can't directly invoke irrationality criterion (IC). That's why we make it become an integer: $p_n := 2d_n^3 a_n$ where $d_n = lcm(1, 2, 3, ..., n)$. But, we don't want to change the estimation of $\left|\zeta(3) - \frac{a_n}{b_n}\right|$, so, we also did this to b_n, and obtain: $q_n := 2d_n^3 b_n$.

Now, we can use the irrationality criterion, we also notice the second reason why we can't directly use $\frac{1}{b_n^{1+\delta}}$ on the r.h.s of the IC, but $\frac{1}{q_n^{1+\delta}}$ instead: From the criterion, if the following is true then $\zeta(3)$ is irrational:

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| = \left|\zeta(3) - \frac{a_n}{b_n}\right| = O\left(\frac{1}{b_n^2}\right) < \frac{1}{q_n^{1+\delta}} = \frac{1}{(2d_n^3 b_n)^{1+\delta}}$$
$$\Rightarrow \frac{1}{b_n^2} < \frac{1}{(2d_n^3 b_n)^{1+\delta}}$$

It follows: (1) *if we take* $\delta = 1$ **the inequality isn't valid**, (2) $0 < \delta < 1$ which is what we need (and we can do a further estimation to find out the exact maximum value of $\delta = 0.080....$

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| = \left|\zeta(3) - \frac{a_n}{b_n}\right| = O\left(\frac{1}{b_n^2}\right) = O(\alpha^{-2n}).$$

Take

$$0 < \delta = \frac{\log\alpha - 3}{\log\alpha + 3} = 0.080529....$$

We obtain

$$\log\alpha = \frac{3(1+\delta)}{1-\delta} \Rightarrow \alpha^{-1+\delta} = e^{-3(1+\delta)}$$
$$\Rightarrow \alpha^{-2}\alpha^{1+\delta} = e^{-3(1+\delta)}$$

When n is large enough, we can have the following inequality:

$$\Rightarrow (\alpha^n 2d_n^3)^{-(1+\delta)} = \frac{1}{q_n^{1+\delta}} > \alpha^{-2n} = \frac{1}{(\alpha^n e^{3n})^{1+\delta}} = O\left(\frac{1}{b_n^2}\right)$$

Therefore, this δ is the maximum of all possible values to make the inequality be true. It follows that

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}$$

and this implies $\zeta(3)$ is irrational. □

5. References

[1] van der Poorten, Alfred (1979), "A proof that Euler missed ... Apréy's proof of the irrationality of (3)." The Mathematical Intelligencer, 1 (4): 195-203.

[2] A. Ya. Khinchin, (1964), "Continued Fractions." The University of Chicago Press.

6. Appendix

6.1. **Step 2: Prove (2).**

Proof. Let

$$\varsigma_0 = \frac{1}{x},$$

and

$$\varsigma_M = \frac{A_1 A_2 ... A_M}{x(x+A_1)(x+A_2)...(A_M)}.$$

Then

$$\frac{A_1 A_2 A_3 ... A_{k-1}}{(x+A_1)(x+A_2)...(x+A_M)}$$

$$= \frac{A_1 A_2 A_3 ... A_{k-1}}{x(x+A_1)(x+A_2)...(x+A_{k-1})} - \frac{A_1 A_2 ... A_k}{(x+A_1)(x+A_2)...(x+A_k)}$$

$$= \varsigma_{k-1} - \varsigma_k.$$

$$\Rightarrow \sum_{k=1}^{M} \frac{A_1 A_2 ... A_{k-1}}{(x+A_1)(x+A_2)...(x+A_k)}$$

$$= \sum_{k=1}^{M} (\varsigma_{k-1} - \varsigma_k) = \varsigma_0 - \varsigma_M.$$

□

6.2. **Step 3: Prove (3).**

Proof. Let $x = n^2$, and $A_k = -k^2$. where $1 \leq M \leq n-1$.

$$\Rightarrow \sum_{k=1}^{M} \frac{(-1)^2(-2)^2(-3)^2...(-(k-1))^2}{(n^2-1)(n^2-2^2)...(n^2-k^2)} := \Psi_n$$

$$= \sum_{k=1}^{M} \frac{(-1)^{k-1} 1^2 2^2 ... (k-1)^2}{(n^2-1)...(n^2-k^2)}$$

$$= \sum_{k=1}^{n-1} \frac{(-1)^{k-1}((k-1)!)^2}{(n^2-1)...(n^2-k^2)}$$

$$= \frac{1}{n^2} - \frac{2n^2(-1)^{n-1}((n-1)!)^2}{n^2((2n)!)^2}$$

$$= \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}$$

Hence

$$\sum_{n=1}^{L} \Psi_n = \sum_{n=1}^{L} \left(\frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}} \right)$$

Now, let

$$\varpi_{n,k} = \frac{1}{2} \frac{(k!)^2(n-k)!}{k^3(n+k)!}.$$

Since

$$(-1)^k(\varpi_{n,k} - \varpi_{n-1,k})n = \frac{(-1)^{k-1}((k-1)!)^2}{(n^2-1^2)...(n^2-k^2)}$$

Thus, we can obtain the following:

$$\sum_{n=1}^{L} \sum_{k=1}^{n-1} (-1)^k(\varpi_{n,k} - \varpi_{n-1,k})$$

$$= \sum_{k=1}^{L} (-1)^k(\varpi_{L,k} - \varpi_{0,k})$$

$$= \sum_{k=1}^{L} \frac{(-1)^k}{2k^3\binom{L+k}{k}\binom{L}{k}} + \frac{1}{2}\sum_{k=1}^{L} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.$$

$$= \sum_{n=1}^{L} \Psi_n$$

$$= \sum_{n=1}^{L} \frac{1}{n^3} - 2\sum_{n=1}^{L} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

It follows that as we take $L \longrightarrow \infty$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.$$

$\square$

6.3. **Step 4: Prove (4) and (6) satisfy (8). Need to show:** a_n **satisfies (8)**.

Proof. In the following we denote the labels n, k, μ, and ν as positive integers. It is clear that once we change n to $n+1$ we can derive (8) from (7). Next, by substituting a_μ into u_n within the left hand side of (8) we obtain:

$$(\mu+1)^3 \sum_{k=1}^{\mu} b_{\mu+1,k} c_{\mu+1,k} - f(\mu) \sum_{k=1}^{\mu} b_{\mu,k} c_{\mu,k} + \mu^3 \sum_{k=1}^{\mu} b_{\mu-1,k} c_{\mu-1,k}, \text{ and } (\mu \geq 1)$$

Because for each $r > \mu$, we have $\binom{\mu}{r} = 0$. Therefore we obtain the following:

Claim:

(14)
$$\sum_{\nu=0}^{\mu+1} \left[f(\mu) b_{\mu,\nu} c_{\mu,\nu} - \mu^3 b_{\mu-1,\nu} c_{\mu-1,\nu} - (\mu+1)^3 b_{\mu+1,\nu} c_{\mu+1,\nu} \right] = 0$$

where $n \geq 1$.

We need to show the equality holds.

Denote $b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$, and define

$$\Xi_{\mu,\nu} = \binom{\mu}{\nu}^2 \binom{\mu+\nu}{\nu}^2 (8\mu+4) \left(\nu(2\nu+1) - (2\nu+1)^2\right).$$

Then we can use $b_{n,k}$ to rewrite $\Xi_{\mu,\nu}$ as

$$\Xi_{\mu,\nu} = \binom{\mu}{\nu}^2 \binom{\mu+\nu}{\nu}^2 (8\mu+4) \left(\nu(2\nu+1) - (2\nu+1)^2\right) \tag{15}$$

$$= b_{\mu,\nu} (8\mu+4) \left(\nu(2\nu+1) - (2\nu+1)^2\right) \tag{16}$$

Also,

$$\Xi_{\mu,\nu} - \Xi_{\mu,\nu-1} = \binom{\mu+1}{\nu}^2 \binom{\mu+\nu+1}{\nu}^2 (n+1)^3$$
$$-f(n) \binom{\mu}{\nu}^2 \binom{\mu+\nu}{\nu}^2 + \binom{\mu-1}{\nu}^2 \binom{\mu+\nu-1}{\nu}^2 n^3.$$

Hence we derive the following two equations:

$$\Xi_{\mu,\nu} - \Xi_{\mu,\nu-1} = (\mu+1)^3 b_{\mu+1,\nu} - f(\mu) b_{\mu,\nu} + \mu^3 b_{\mu-1,\nu}$$

and

$$f(\mu)b_{\mu,\nu}c_{\mu,\nu} = (\mu+1)^3 b_{\mu+1,\nu}c_{\mu,\nu} + \mu^3 b_{\mu-1,\nu}c_{\mu,\nu} - (\Xi_{\mu,\nu} - \Xi_{\mu,\nu-1})c_{\mu,\nu}.$$

Substituting into the l.h.s. of (14), we have equation (17) as follows:

$$\sum_{\nu=0}^{\mu+1} \left[\mu^3 b_{\mu-1,\nu}(c_{\mu-1,\nu} - c_{\mu,\nu}) + (\mu+1)^3 b_{\mu+1,\nu}(c_{\mu+1,\nu} - c_{\mu,\nu}) + (\Xi_{\mu,\nu} - \Xi_{\mu,\nu-1})c_{\mu,\nu}\right]. \tag{17}$$

By definition of $c_{\mu,\nu}$ in (5), we can derive:

$$\begin{aligned} c_{\mu,\nu} - c_{\mu-1,\nu} &= \sum_{\lambda=1}^{\mu} \frac{1}{\lambda^3} + \sum_{\lambda=1}^{\nu} \frac{(-1)^{\lambda-1}}{2\lambda^3\binom{\mu}{\lambda}\binom{\mu+\lambda}{\lambda}} - \sum_{\lambda=1}^{\mu-1} \frac{1}{\lambda^3} + \sum_{\lambda=1}^{\nu} \frac{(-1)^{\lambda-1}}{2\lambda^3\binom{\mu-1}{\lambda}\binom{\mu+\lambda-1}{\lambda}} \\ &= \frac{1}{\mu^3} + \sum_{\lambda=1}^{\nu} \frac{(\mu-\lambda-1)!(\lambda-1)!(-1)^{\lambda}}{(\mu+\lambda)!} \\ &= \frac{1}{\mu^3} + \sum_{\lambda=1}^{\nu} \frac{(-1)^{\lambda}((\lambda-1)!)^2(\mu-\lambda-1)!(\lambda^2+\mu^2-\lambda^2)}{\mu^2(\mu+\nu)!} \\ &= \frac{1}{\mu^3} + \sum_{\lambda=1}^{\nu} \frac{(\mu-\lambda-1)!(\lambda!)^2(-1)^{\lambda}}{\mu^2(\mu+\lambda)!} - \sum_{\lambda=1}^{\nu} \frac{(\mu-\lambda)!((\lambda-1)!)^2(-1)^{\lambda+1}}{\mu^2(\mu+\lambda-1)!} \\ &= \frac{1}{\mu^3} - \left(\frac{(\mu-1)!}{\mu!\mu^2} + \left(\frac{(-1)^{\nu}(\nu!)^2(\mu-\nu-1)!}{(\mu+\nu)!\mu^2}\right)\right). \end{aligned} \tag{18}$$

Next, we define

$$\Upsilon_{\mu,\nu} := \Xi_{\mu,\nu}c_{\mu,\nu} + \frac{(10\mu+5)(-1)^{\nu-1}\nu}{\mu+1}\binom{\mu}{\nu}\binom{\mu+\nu}{\nu}. \tag{19}$$

Now, by using equation (16), (18), and (19) we can rewrite the l.h.s. of the recurrence relation as follows:

$$\Upsilon_{\mu,\nu} - \Upsilon_{\mu,\nu-1} = -\mu^3(c_{\mu,\nu} - c_{\mu-1,\nu})b_{\mu-1,\nu} + (\mu+1)^3 b_{\mu-1,\nu}(c_{\mu-1,\nu} - c_{\mu,\nu}) + c_{\mu,\nu}(\Xi_{\mu,nu} - \Xi_{\mu,\nu-1}). \tag{20}$$

Recall (17), and substitute (20) into it, then we can know that (17) is nothing else but the following summation:

$$\sum_{\nu=0}^{\mu+1}(\Upsilon_{\mu,\nu} - \Upsilon_{\mu,\nu-1}) = \Upsilon_{\mu,\mu+1} - \Upsilon_{\mu,-1} \tag{21}$$

We know that

$$\Upsilon_{\mu,\mu+1} = \Xi_{\mu.\mu+1}c_{\mu,\mu+1} + \frac{(\mu+1)(-1)^{\mu}(10\mu+1)\binom{\mu}{\mu+1}\binom{2\mu+1}{\mu}}{(\mu+1)\mu} = 0$$

It is due to if $r > \mu$, we have $\binom{\mu}{r} = 0$. Likewise, $\Upsilon_{\mu,-1} = 0$. Therefore, in (14), l.h.s.=r.h.s., hence a_μ satisfies the recurrence relation. □

Need to show: b_n satisfies (8).

Proof. Let's define the following notation so that we can rewrite $\Xi_{\mu,\nu}$ in a series.

$$\Theta_{\mu,\nu} := \Xi_{\mu,\nu} - \Xi_{\mu,\nu-1} = (\mu+1)^3 b_{\mu+1,\nu} - f(\mu)b_{\mu,\nu} - \mu^3 b_{\mu-1,\nu} \tag{22}$$

Thus,

$$\Xi_{\mu,\nu} = \Theta_{\mu,\nu} + \Xi_{\mu,\nu-1} = \Theta_{\mu,\nu} + \Theta_{\mu,\nu-1} + \Xi_{\mu,\nu-2}$$
$$= \Theta_{\mu,\nu} + \Theta_{\mu,\nu-1} + \Theta_{\mu,\nu-2} + ...(\text{ so on and so forth })...$$

Next, we apply the result that, if $\nu < 0$, then $\Xi_{\mu,\nu} = 0$. Hence,

$$\Xi_{\mu,\nu} = \sum_{\chi=0}^{\nu} \Theta_{\mu,\chi}.$$

We also know $b_{\mu,\mu+1} = 0$, $b_{\mu-1,\mu+1} = 0$, and $b_{\mu-1,\mu} = 0$. Therefore,

$$\Xi_{\mu,\mu+1} = \sum_{\chi=0}^{\mu+1} \Theta_{\mu,\chi} = (\mu+1)^3 \sum_{\chi=0}^{\mu+1} b_{\mu+1,\chi} - f(\mu) \sum_{\chi=0}^{\mu} b_{\mu,\chi} - \mu^3 \sum_{\chi=0}^{\mu-1} b_{\mu-1,\chi}$$

Thus, be recalling the definition of $b_{\mu,\nu}$, we have

$$\Xi_{\mu,\mu+1} = (\mu+1)^3 b_{\mu+1} - f(\mu)b_\mu - \mu^3 b_{\mu-1}.$$

But, $\Xi_{\mu,\chi} = 0$, if $\mu < \chi$. It follows that

$$(\mu+1)^3 b_{\mu+1} - f(\mu)b_\mu - \mu^3 b_{\mu-1} = 0.$$

In other words, b_μ satisfies the recurrence relation, hence it competes the proof. □

6.4. **Step 5: Prove (9).**

Proof. Since a_n and b_n satisfy the recurrence relation, we can substitute a_n and b_n into it as follows:

$$n^3 a_n + (n-1)^3 a_{n-2} = f(n)a_{n-1},$$

and multiply this equation on both sides by b_{n-1}.

$$n^3 b_n + (n-1)^3 b_{n-2} = f(n)b_{n-1}.$$

and multiply this equation on both sides by a_{n-1}, and subtract the previous one:

$$(a_n b_{n-1} - b_n a_{n-1})n^3 = (a_{n-2}b_{n-1} - b_{n-2}a_{n-1})(-1)(n-3)^3.$$

Then

$$a_n b_{n-1} - b_n a_{n-1} = \frac{(n-1)^3}{n^3}(a_{n-1}b_{n-2} - b_{n-1}a_{n-2})$$

$$= \frac{(n-1)^3}{n^3}\frac{(n-2)^3}{(n-1)^3}(a_{n-2}b_{n-3} - a_{n-3}b_{n-2})$$

$$= \textit{so on and so forth}$$

$$= \frac{1}{n^3}(a_{n-(n-1)}b_{n-n} - a_{n-n}b_{n-(n-1)})$$

$$= \frac{1}{n^3}(a_1 b_0 - a_0 b_1) = \frac{6}{n^3}.$$

□

6.5. **Step 6: Prove (10).**

Proof.

$$|c_{n,k} - \zeta(3)| = \left| -\sum_{m=n+1}^{\infty} \frac{1}{m^3} - \sum_{m=1}^{k} \frac{(-1)^m}{2m^3\binom{n}{m}\binom{n+m}{m}} \right|$$

$$\leq \left| \sum_{k=n+1}^{\infty} \frac{1}{k^3} \right| + \left| \sum_{m=1}^{k} \frac{(-1)^m}{2m^3\binom{n}{m}\binom{n+m}{m}} \right|$$

We know $\sum_{m=1}^{\infty} \frac{1}{m^3}$ is convergent, so let $\epsilon > 0$ be given, $\forall n > N_1$, we have

$$\left| \sum_{k=n+1}^{\infty} \frac{1}{k^3} \right| < \epsilon$$

so we only need to focus on the second term:

$$\left| \sum_{m=1}^{k} \frac{(-1)^m}{2m^3\binom{n}{m}\binom{n+m}{m}} \right| \leq \frac{k}{2n^2 + 2n}$$

$$\leq \frac{n}{2n^2 + 2} \leq \frac{1}{2n} < \epsilon, \forall n > \frac{1}{\epsilon}.$$

Hence we can let $N_2 = \frac{1}{2\epsilon}$, and $N := \{N_1, N_2\}$, then $\forall n > N$, the following is true:

$$|c_{n,k} - \zeta(3)| \leq 2\epsilon + \epsilon = 3\epsilon.$$

This convergence is independent of k so the convergence is uniform. □

6.6. **Step 7: Prove (11).**

Proof. Consider

$$a_n = \sum_{k=1}^{n} b_{n,k} c_{n,k},$$

$$b_n = \sum_{k=1}^{n} b_{n,k}$$

Consider $\varrho_{n,k}, \sigma_{n,k} \in \mathbb{R}$ and $\sigma_{n,k} \longrightarrow L$ as $n \longrightarrow \infty$ (uniformly in n) then

$$\frac{\sum_{k=1}^{n} \varrho_{n,k}\sigma_{n,k}}{\sum_{k=1}^{n} \varrho_{n,k}} \longrightarrow L \ as\ n \longrightarrow \infty$$

Therefore, we have

$$\frac{a_n}{b_n} \longrightarrow \zeta(3) \ as\ n \longrightarrow \infty.$$

□

6.7. **Step 8: Prove (12).**

Proof. We start with

$$\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} = \frac{6}{n^3 b_n b_{n+1}}.$$

Hence

$$\zeta(3) - \frac{a_n}{b_n} = \frac{6}{(n+1)^3 b_{n+1} b_n} + \varepsilon_{n+1}$$

$$= \frac{6}{(n+1)^3 b_{n+1} b_n} + \frac{6}{(n+2)^3 b_{n+2} b_{n+1}} + \ldots + \frac{6}{(n+m)^3 b_{n+m} b_{n+m-1}} + \varepsilon_{n+m}$$

$$= \sum_{k=n+1}^{m} \frac{6}{k^3 b_k b_{k-1}} + \varepsilon_{n+m}.$$

We know the first term has a power higher than the harmonic series, and we also know $\zeta(3) - \frac{a_n}{b_n} \longrightarrow 0$ so it converges to zero as $n \longrightarrow \infty$ and thus the second term goes away and we get

$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}.$$

□

6.8. **Dirichlet Theorem.**

To prove Dirichlet Theorem, we need the following definition.

Denote $\lfloor x \rfloor$ as the integral part of a real number x, and $(x) := x - \lfloor x \rfloor$ as the fractional part of x.

Dirichlet Theorem

Theorem 2. *Let $\xi' \in \mathbb{R}$ be given. If ξ' is real, then there exists rational numbers p, and q, and $(p,q) = 1$, such that $\left|\xi' - \frac{p}{q}\right| \leq \frac{1}{q^2}$.*

Proof. Let $\xi = (\xi')$. Let $m > 1$ be a fixed integer. Consider the following $m+1$ elements in a sequence: 0, (ξ), (2ξ), (3ξ), (4ξ), ..., $(m\xi)$. Secondly, consider m intervals: $\left[\frac{i}{m}, \frac{i+1}{m}\right)$ where $i \in \{0, 1, 2, ..., m-1\}$.

By the pigeon hole principle, since we have $m+1$ elements, but only m intervals, hence some two fractional parts fall into the same interval, $a\xi$ and $b\xi$ with $0 \leq a < b \leq m$, and that interval let's denote it as $\left[\frac{j}{m}, \frac{j+1}{m}\right)$.

Recall a fact that suppose $x \in \mathbb{R}$, and $y \in \mathbb{R}$, then $\exists n \in \mathbb{Z}$ such that $|x - y| = n + d$, where $d = |(x) - (y)|$. Here, we let $n = p$, $x = a\xi$, and $y = b\xi$.

Then by using this fact we can estimate the distance from the number $|x - y|$ to some integer p as follows:

$$||x - y| - p| = |\,|a - b| \cdot \xi - p| = |d| < \frac{1}{m}.$$

That is $|a - b| \cdot \xi$ is less that $\frac{1}{m}$ distant from some integer p.

Notice that $|a - b|$ is a positive integer not greater than m. Let's choose it to be q, then we just found that

$$|\,|a - b| \cdot \xi - p| = |q\xi - p| < \frac{1}{m}$$

$$\Rightarrow |q\xi - p| < \frac{1}{m} \leq \frac{1}{q}.$$

Then once we divide both sides by q, this completes the theorem.

$\square$

Remark. Additionally, if ξ is irrational, then there exist an infinite sequence $\frac{p_n}{q_n}$. To see this, consider that thus far, we have found one such q. However, why can we claim it's infinitely many? It follows from an observation that this proof actually can be seen as an algorithm that generates a $q \in [1, m]$ such that $|q\xi - p| < \frac{1}{m}$.

Then, once we choose a m which is large enough such that $\frac{1}{m} < |q\xi - p|$, the same reasoning, same procedure of the proof must construct a new different p and q in the same fashion.

Fundamental Recurrence Relation

Theorem 3. *Let p_n and q_n be the convergents. Then*

$$\det\left(\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix}\right) = p_n q_{n-1} - p_{n-1} q_n = (-1)^n, \text{ for all } n \geq 0.$$

Proof. We can proof this statement by mathematical induction. First of all, we check the base case, as $n = 0$, and $n = 1$:

$$p_0 q_{-1} - p_{-1} q_0 = 1(1) - 0(0) = 1 = (-1)^0$$

which is okay.
For $n = 1$, we obtain:

$$p_1 q_0 - p_0 q_1 = (a_1 p_0 + p_{-1})(0) - 1(a_1 q_0 + q_1)$$

$$= 0 - 1(0 + 1) = -1 = -(1)^1.$$

Both case are valid.
Thirdly, we assume it holds for all $n \leq k$, so we nee to show this is true as $n = k + 1$.

$$\begin{aligned} p_{k+1} q_k - p_k q_{k+1} &= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\ &= a_{k+1} p_k q_k + p_{k-1} q_k - a_{k+1} p_k q_k - p_k q_{k-1} \\ &= p_{k-1} q_k - p_k q_{k-1} \\ &= -(p_k q_{k-1} - p_k - 1 q_k) \\ &= -(-1)^k \\ &= (-1)^{k+1}. \end{aligned}$$

This completes the proof. ∎