# Solutions of Apply mathematics(III) HW#1

2012.3.14

## Problem 1

$$T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

Because $ad - bc \neq 0$, we have

$$\lim_{|z|\to\infty} T'(z) = 0$$

$\blacksquare$

## Problem 2

Since $z^n$ is entire function, the linear combination also entrie function. $\blacksquare$

## Problem 3

We have $f(z) = u(x,y) + iv(x,y)$, and we may set

$$\overline{f(z)} = u(x,y) + iv(x,y) = U(x,y) - iV(x,y)$$

Where $U(x,y) = u(x,y)$ and $V(x,y) = -v(x,y)$. By Cauchy-Riemann eq, $\overline{f(z)}$ shows

$$\partial_x U = \partial_y V,\ \partial_y U = -\partial_x V$$

It means

$$\partial_x u = -\partial_y v,\ \partial_y u = \partial_x v$$

But consider $f(z)$, we have

$$\partial_x u = \partial_y v,\ \partial_y u = -\partial_x v$$

The last two eq. is consisted when $\partial_x u = \partial_x v = 0$. i.e. $f'(z) = 0$. It means $f(z) = \textit{constant}$ in $\mathcal{S}$ $\blacksquare$

# Problem 4

(a)
$u(x,y) = x^2 + y^2$

$$\partial_x u = 2x = \partial_y v \Rightarrow v = 2xy + C(x)$$

$$\partial_x u = 2y = -\partial_y v \Rightarrow v = -2xy + C(y)$$

It inferrs $v(x,y) = 0$ and $f'(z)$ doesn't exist at any nozero point.
(b)
$u(x,y) = \cosh y \sin x$

$$\partial_x u = \cosh y \cos x = \partial_y v \Rightarrow v = \sinh y \cos x + C(x)$$

$$\partial_x u = \sinh y \sin x = -\partial_y v \Rightarrow v = \sinh y \cos x + C(y)$$

It means $v = \sinh y \cos x + c$
(c)
$u(x,y) = 2x^2 + x + 1 - 2y^2$

$$\partial_x u = 4x + 1 = \partial_y v \Rightarrow v = 4xy + y + C(x)$$

$$\partial_x u = -4y = -\partial_y v \Rightarrow v = 4xy + C(y)$$

It means $v = 4xy + y + c$
(d)
$u(x,y) = \frac{x}{x^2+y^2}$

$$\partial_x u = \frac{y^2 - x^2}{(x^2+y^2)^2} = \partial_y v \Rightarrow v = \frac{-y}{x^2 + x^y} + C(x)$$

$$\partial_x u = \frac{2xy}{(x^2+y^2)^2} = -\partial_y v \Rightarrow v = \frac{-y}{x^2+y^2} + \frac{1}{2y} + C(y)$$

We may set $\frac{1}{2y} + C(y) = C'(y)$ and get $v = \frac{-y}{x^2+y^2} + c$ ■

# Problem 5

We have $x = r\cos(\phi)$, $y = r\sin(\phi)$. By Chain rule

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \phi}$$

change $(x, y)$ to $(r\cos(\phi), r\sin(\phi))$. We get

$$\partial_r v = \partial_x v \cos(\phi) + \partial_y v \sin(\phi)$$

$$\partial_\phi v = -r\partial_x v \sin(\phi) + r\partial_y v \cos(\phi)$$

Replace $\partial_x v$, $\partial_y v$ by Cauchy-Riemann eq $\partial_x u = \partial_y v$, $\partial_y u = -\partial_x v$.

$$\partial_r v = -\partial_y u \cos(\phi) + \partial_x u \sin(\phi)$$

$$\partial_\phi v = -r\partial_y u \sin(\phi) + r\partial_x u \cos(\phi)$$

Compair equations of real part

$$\partial_r u = \partial_x u \cos(\phi) + \partial_y u \sin(\phi)$$

$$\partial_\phi u = -r\partial_x u \sin(\phi) + r\partial_y u \cos(\phi)$$

We get

$$r\partial_r u = \partial_\phi v,\ \partial_\phi u = -r\partial_r v$$

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# Problem 6

By $r\partial_r u = \partial_\phi v$, $\partial_\phi u = -r\partial_r v$, we have

$$\frac{\partial^2 u}{\partial \phi^2} = -r\frac{\partial^2 v}{\partial r \partial \phi},\ \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = r\frac{\partial^2 v}{\partial r \partial \phi}$$

Replace RHS, we get

$$r^2\frac{\partial^2 u}{\partial \phi^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2}$$

By similiarly operation, $v$ is also satisfied

$$r^2\frac{\partial^2 v}{\partial \phi^2} + r\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial \phi^2}$$

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# Problem 7

Set $\omega = f(z)$. We have $z = f^{-1}(\omega)$
(a)

$$\omega = \frac{az+b}{cz+d} \Rightarrow z = \frac{a'\omega+b'}{c'\omega+d'}$$

Where $a' = -d$, $b' = b$, $c' = c$, $d' = -a$. It's easy to check

$$a'd' - b'c' = (-d)(-a) - bc \neq 0$$

(b)

$$z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0$$

The maxium power of eq. is two, which inferrs at most two solutions of $z$.
(c)
We have composition of translation and dilatation $Z = cz + d$. There is also inversion transform $W = \frac{1}{Z}$. Möbius transform can be written as

$$\omega = \frac{a}{c} + \frac{bc-ad}{c}W$$

(d)
Since translation and dilation is trivial, we may choose

$$z = \frac{r}{\omega} + \xi$$

Puting $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$, $\alpha = \frac{a+ib}{2}$, straight line $ax + by = c$ can be written as

$$\overline{\alpha}z + \alpha\overline{z} = c$$

Applying the transformation, it becomes

$$\overline{\alpha}\frac{r}{\omega} + \alpha\frac{r}{\overline{\omega}} = c - \xi(\alpha+\overline{\alpha}) \Rightarrow \overline{\alpha\omega} + \alpha\omega = \frac{c-\xi(\alpha+\overline{\alpha})}{r}|\omega|^2$$

Replacing $\omega = u + iv$, we get

$$\frac{c-\xi(\alpha+\overline{\alpha})}{r}(u^2+v^2) = au - bv$$

$c - \xi(\alpha+\overline{\alpha}) = 0$, it becomes straight line, or it will be a circle. So as a circle

$$k(x^2+y^2) + ax + by + c = 0 \Rightarrow k|z|^2 + \overline{\alpha}z + \alpha\overline{z} = c - \xi(\alpha+\overline{\alpha})$$

By the transform, we get

$$kr^2+\overline{\alpha}r\overline{\omega}+\alpha r\omega = (c-\xi(\alpha+\overline{\alpha}))|\omega|^2 \Rightarrow kr^2+r(au-bv) = (c-\xi(\alpha+\overline{\alpha}))(u^2+v^2)$$

As the foward discussion, it is circle or straight line.
(e)
We may set

$$x = \frac{z+\overline{z}}{2}\, , y = \frac{z-\overline{z}}{2i}\, , u = \frac{\omega+\overline{\omega}}{2}\, , v = \frac{\omega-\overline{\omega}}{2i}$$

x,y-axis are represented as $z = \overline{z} = 0$. The two straight lines, $u \pm v = 0$ can be written as

$$\begin{aligned}(1+i)\omega - (1+i)\overline{\omega} = 0\\ (1-i)\omega + (1+i)\overline{\omega} = 0\end{aligned}$$

It implies $c(1-i)\omega = 0 = z$, where $c$ is a constant. Because $|z| = |\omega| = 1$, the constant $c = 1/\sqrt{2}$. i.e.

$$\omega = \frac{(1+i)}{\sqrt{2}}z = e^{i\frac{\pi}{4}}z$$

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# Problem 8

Length of curve

$$L = \int_b^a dt|\gamma'(t)|^2$$

(a)

$$\gamma(t) = 3t+i \Rightarrow \gamma'(t) = 3 \Rightarrow \int_{-1}^{1} 3dt = 6$$

(b)

$$\gamma(t) = i\sin t \Rightarrow \gamma'(t) = i\cos t \Rightarrow \int_{-\pi}^{\pi} dt|\cos t| = 4\int_0^{\pi/2} dt\cos t = 4$$

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# Problem 9

(a)
$\gamma(t) = (1-i)t$, $0 \le t \le 1$

$$\int_\gamma dze^{3z} = \int_0^1 d[(1-i)t]e^{(3-3i)t} = \frac{e^{(3-3i)-1}}{3}$$

(b)
Since $e^{3z}$ is entire function, any integral of close path should be zero. i.e.

$$\oint_{|z|=3} dze^{3z} = 0$$

(c)
By the arguement from (b), the integral is equal a straight line form $x = 0$ to $x = 1$. Set $\bar{\gamma}(t) = t$, $0 \le t \le 1$

$$\int_\gamma dze^{3z} = \int_{\bar{\gamma}} dze^{3z} = \int_0^1 dte^{3t} = \frac{e^3 - 1}{3}$$

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# Problem 10

We have $z = \gamma(t)$, and

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

Integrates both part

$$f(z)g(z)\Big|_b^a = \int_\gamma dzf(z)g'(z) + f'(z)g(z)$$

We get

$$f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) = \int_\gamma dzf(z)g'(z) + f'(z)g(z)$$

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# Problem 11

$P(z)$ is analytic function. Any close contour integral is zero.

## Problem 12

$$I = \oint_{Cr} \frac{dz}{z^2 - 2z - 8} = \oint_{Cr} \frac{dz}{(z-4)(z+2)}$$

$r = 1$, no pole inside the circle. $I = 0$
$r = 3$, one pole at $z = -2$

$$I = \frac{2\pi i}{-2-4} = -\frac{\pi i}{3}$$

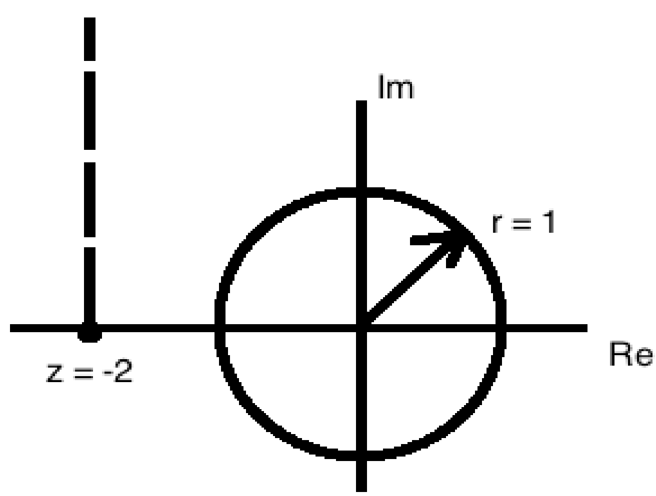
$r = 5$, two poles at $z = -2$ and $z = 4$

$$I = \frac{2\pi i}{-2-4} + \frac{2\pi i}{4+2} = 0$$

$\blacksquare$

## Problem 13

(1)
no pole on the $z$-plane. The integral is zero.
(2)
The nearest poles are $\pm\frac{\pi}{2}$. No pole in side the circle. The integral is zero.
(3)
There are two pole at $-1 \pm i$. Because $|-1 \pm i| = \sqrt{2}$, no pole in side the circle. The integral is zero.
(4)
The branch cut start from $z = -2$. We can choose it as the following figure



The dash line is branch cuts. No pole in the circle. The integral is zero. ■

# Problem 14

(1)

$$I = \oint_\gamma \frac{dz}{z^2-4} = \oint \frac{dz}{(z-2)(z+2)} = 2\pi i\left(\frac{1}{4}+\frac{-1}{4}\right) = 0$$

(2)

$$I = \oint_\gamma \frac{dze^z}{(z-1)(z-2)} = 2\pi i\left(\frac{e}{1-2}+\frac{e^2}{2-1}\right) = 2\pi i e(e-1)$$

(3)

$$\begin{aligned} I &= \oint_\gamma \frac{dz}{(z+4)(z^2+1)} = 2\pi i\left(\frac{1}{(i+4)(i+i)}+\frac{1}{(-i+4)(-i-i)}\right) \\ &= \frac{\pi}{i+4}+\frac{\pi}{i-4} = \frac{2\pi i}{17} \end{aligned}$$

Note: here $z=-4$ is outside the circle. ■

# Problem 15

Since we have

$$\oint_{|z|=1} \frac{e^{az}}{z} = 2\pi i$$

LHS also can be written as

$$\begin{aligned} \oint_{|z|=1} \frac{e^{az}}{z} &= \int_{-\pi}^{\pi} d(e^{i\phi})\frac{e^{ae^{i\phi}}}{e^{i\phi}} = \int_{-\pi}^{\pi} d\phi i e^{a(\cos\phi+i\sin\phi)} \\ &= i\int_{-\pi}^{\pi} d\phi e^{a\cos\phi}\cos(a\sin\phi) - \int_{-\pi}^{\pi} d\phi e^{a\cos\phi}\sin(a\sin\phi) \end{aligned}$$

Here, $e^{a\cos\phi}\sin(a\sin\phi)$ is odd function, $e^{a\cos\phi}\cos(a\sin\phi)$ is even function. We have

$$2\pi i = 2i\int_{-\pi}^{\pi} d\phi e^{a\cos\phi}\cos(a\sin\phi) \Rightarrow \int_{-\pi}^{\pi} d\phi e^{a\cos\phi}\cos(a\sin\phi) = \pi$$

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## Problem 16

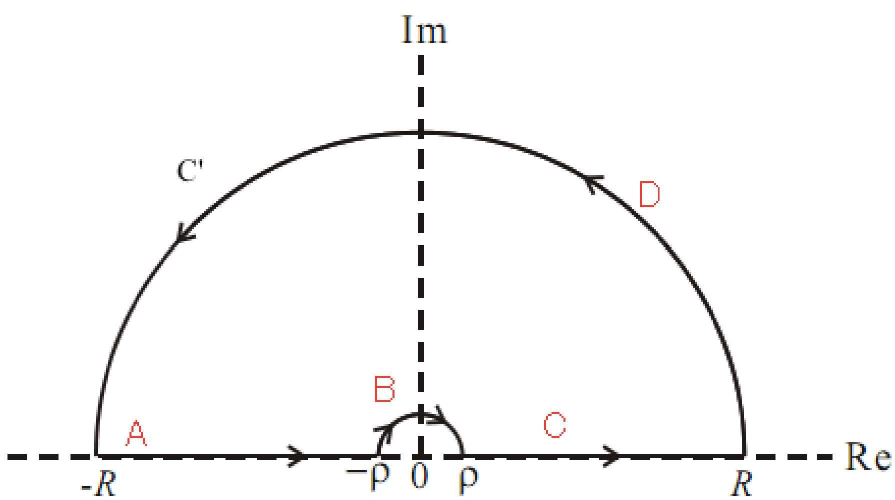
We want to evaluate the integral

$$I = \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x}$$

Consider the following case

$$\oint_C dz \frac{e^{iz}}{z}$$

$C' = A + B + C + D$, as show in the following figure




Where $R \to \infty$ and $\rho \to 0$. Since no pole in the contour $C'$

$$\oint_C dz \frac{e^{iz}}{z} = 0 = \int_A \frac{e^{iz}}{z} + \int_B \frac{e^{iz}}{z} + \int_C \frac{e^{iz}}{z} + \int_D \frac{e^{iz}}{z}$$

Two of contours integral $A$ and $C$ can be written as

$$\int_A \frac{e^{iz}}{z} + \int_C \frac{e^{iz}}{z} = \int_\rho^R dr \frac{e^{ir}}{r} + \int_{-R}^{-\rho} dr \frac{e^{ir}}{r} = I$$

By Jordan's lemma

$$\lim_{R\to\infty} \int_D \frac{e^{iz}}{z} = 0$$

So we get

$$I = -\int_B \frac{e^{iz}}{z} = \lim_{\rho\to 0} \int_0^\pi id\phi e^{i\rho e^{i\phi}} = i \int_0^\pi d\phi = i\pi$$

We get

$$
\begin{aligned}
\Re e\{I\} &= \int_{-\infty}^{\infty} dx \frac{\cos x}{x} = 0 \\
\Im m\{I\} &= \int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \pi
\end{aligned}
$$

$\blacksquare$

## Bonus Question

We have showed any polynormial of degree $n > 0$ exist at most one zero, i.e.

$$P_n(z_0) = 0$$

Where $z_0$ is a zero of $P_n(z)$. As we know

$$
\begin{aligned}
P_n(z) = P_n(z) - P_n(z_0) &= \sum_{k=0}^{n} a_n(z^k - z_0^k) \\
&= \sum_{k=0}^{n} a_n(z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + zz_0^{k-2} + z_0^{k-1}) \\
&= (z - z_0)P_{n-1}(z)
\end{aligned}
$$

By the argument, we have

$$
\begin{aligned}
P_n(z) &= (z - z_0)P_{n-1}(z) \\
&= (z - z_0)(z - z_1)P_{n-2}(z) = \cdots \\
&= (z - z_0)(z - z_1)\cdots(z - z_n)P_0
\end{aligned}
$$

Where $P_0$ is a constant. $\blacksquare$