NUMBER THEORY HOMEWORK 2: A CONJECTURE ON REPUNIT NUMBERS

WILL CHUANG

1. Data collecting, and Brutal force computing

Data collecting

- {2,3,4,5,6,7}
- {3,4,5,6,7,8}
- {4,5,6,7,8,9}
- {5,6,7,8,9,10}
- ....

Brutal force computing

If we assume the six consecutive number as:

$$\{6k, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5\}$$

then according the possible number of choices are the Stirling number of second kind:

$$\left\{ \begin{matrix} 6 \\ 2 \end{matrix} \right\} = 31.$$

Since we know that there are six cases can be excluded in the beginning, because it's not possible to single out one number, and equals to a product that is formed by the other five numbers.
Hence, we only need to consider the following 25 cases:

- $\{6k, 6k+1\}$, $\{6k+2, 6k+3, 6k+4, 6k+5\}$
  $\Rightarrow (6k)(6k+1) = (6k+2)(6k+3)(6k+4)(6k+5)$
  No rational solution.

- $\{6k, 6k+2\}$, $\{6k+1, 6k+3, 6k+4, 6k+5\} \Rightarrow (6k)(6k+2) = (6k+1)(6k+3)(6k+4)(6k+5)$
  No rational solution.

- $\{6k, 6k+3\}$, $\{6k+1, 6k+2, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k, 6k+4\}$, $\{6k+1, 6k+2, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k, 6k+5\}$, $\{6k+1, 6k+2, 6k+3, 6k+4\}$
  No rational solution.

- $\{6k+1, 6k+2\}$, $\{6k, 6k+3, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+3\}$, $\{6k, 6k+2, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+4\}$, $\{6k, 6k+2, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+5\}$, $\{6k, 6k+2, 6k+3, 6k+4\}$
  No rational solution.

- $\{6k+2, 6k+3\}$, $\{6k, 6k+1, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+2, 6k+4\}$, $\{6k, 6k+1, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k+2, 6k+5\}$, $\{6k, 6k+1, 6k+3, 6k+4\}$
  No rational solution.

- $\{6k+3, 6k+4\}$, $\{6k, 6k+1, 6k+2, 6k+5\}$
  No rational solution.

- $\{6k+3, 6k+5\}$, $\{6k, 6k+1, 6k+2, 6k+4\}$
  No rational solution.

- $\{6k+4, 6k+5\}$, $\{6k, 6k+1, 6k+2, 6k+3\}$
  No rational solution.

- $\{6k, 6k+1, 6k+2\}$, $\{6k+3, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+2, 6k+3\}$, $\{6k, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+2, 6k+3, 6k+4\}$, $\{6k, 6k+1, 6k+5\}$
  No rational solution.
- $\{6k, 6k+2, 6k+4\}$, $\{6k+1, 6k+3, 6k+5\}$
  No rational solution.
- $\{6k, 6k+1, 6k+4\}$, $\{6k+2, 6k+3, 6k+5\}$
  No rational solution.
- $\{6k+1, 6k+2, 6k+4\}$, $\{6k+3, 6k+5, 6k\}$
  No rational solution.
- $\{6k+1, 6k+2, 6k+5\}$, $\{6k+3, 6k+4, 6k\}$
  No rational solution.
- $\{6k+1, 6k+3, 6k+4\}$, $\{6k+2, 6k+5, 6k\}$
  No rational solution.
- $\{6k+1, 6k+4, 6k+5\}$, $\{6k+2, 6k+3, 6k\}$
  No rational solution.
- $\{6k+2, 6k+4, 6k+5\}$, $\{6k+1, 6k+3, 6k\}$
  No rational solution.

2. Conjectures and Proofs

Conjecture 1

Claim:
For the case $(6k+a)(6k+b)(6k+c) = (6k+d)(6k)(6k+e)$, there is no rational solutions for this equation, if $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$.

Proof. Let's expand the equation, and simplify it:

$$(6k+a)(6k+b)(6k+c) = (6k+d)(6k)(6k+e)$$

$$\Rightarrow abc + (6ab + 6ac + 6bc - 6de)k + (36a + 36b - 36e)k^2 = 0$$

$$\Rightarrow \frac{abc}{(36a + 36b - 36e)} + \frac{(6ab + 6ac + 6bc - 6de)}{(36a + 36b - 36e)}k + k^2 = 0$$

Since $(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta$, hence

$$\alpha\beta = \frac{abc}{36(a+b-e)} \in \mathbb{Z}$$

$$\alpha+\beta = \frac{-(ab+ac+bc-de)}{6\cdot(a+b-e)} = \frac{de-a(b+c)-bc}{6(a+b-e)} \in \mathbb{Z}$$

where $k=\alpha,$ or β, and $\alpha,\beta\in\mathbb{Z}$. Therefore,

$$6 \mid de-a(b+c)-bc, (\exists \text{ symmetry:} b \longleftrightarrow c, d \longleftrightarrow e)$$

$$36 \mid abc$$

That is to form abc, there are two cases: either $(3,4,5)$ or $(2,4,5)$. Thus, there are six sub-cases to test:

(1)
$$(a,b,c,d,e)=(3,4,5,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-45 \Rightarrow 6 \nmid -45 \Rightarrow$$

(2)
$$(a,b,c,d,e)=(4,3,5,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-45 \Rightarrow 6 \nmid -45 \Rightarrow$$

(3)
$$(a,b,c,d,e)=(5,3,4,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-45 \Rightarrow 6 \nmid -45 \Rightarrow$$

(4)
$$(a,b,c,d,e)=(2,4,5,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-35 \Rightarrow 6 \nmid -35 \Rightarrow$$

(5)
$$(a,b,c,d,e)=(4,2,5,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-34 \Rightarrow 6 \nmid -34 \Rightarrow$$

(6)
$$(a,b,c,d,e)=(5,2,4,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-35 \Rightarrow 6 \nmid -35 \Rightarrow$$
$$(a,b,c,d,e)=(3,4,5,1,2)$$
$$\Rightarrow de-a(b+c)-bc=-45 \Rightarrow 6 \nmid -45 \Rightarrow$$

$\square$

Conjecture 2

Claim:
For the case $(6k + a)(6k + b) = (6k + c)(6k + d)(6k)(6k + e)$, there is no rational solutions for this equation, if $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$.

Proof. Consider

$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma$

where $\alpha, \beta, \gamma\delta \in \mathbb{Z}$. Compare to our equation (after expanding, and simplifying):

$$k^4 \frac{(216)(e+d+c)k^3 + 36(de+ce+cd-1)k^2 + 6(cde-b-a)k - ab}{1296} = 0$$

Since $k \in \mathbb{Z}$, hence

$$\alpha\beta\gamma\delta = \frac{-ab}{1296}$$

That is

$$1296 \mid ab \Rightarrow 1296 \leq ab,$$

Because $a, b \in \{1, 2, 3, 4, 5\}$, it follows that this is not possible. This completes the proof! $\square$