Introduction to Quantum Gravity in (2+1)-Dimensions

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In this thesis, I introduce two kinds of Chern-Simons invariants, one is in classical form another is in quantum form, and set up the classical gravity in (2+1)-dimensions with Chern-Simons theory and then quantize it in torus universe as a concrete example. Finally I conclude that this work can continue to develop by coupling matter field to gravitational field.

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I. INTRODUCTION

There is a number of good textbooks in quantum gravity(QG, see [1–8]). And some good textbooks and review papers for lower dimensional quantum gravity or called quantum gravity in (2+1)-dimensions and planer quantum gravity (see [9–12, 25][62]). However, they are all designed for the graduated-level study. Therefore, a undergraduate student can only read some easy parts of them. The goal of this paper is try to fill in this gap, to give the undergraduates the very first glimpse of quantum gravity(QG) in (2+1)-dimensions with Chern-Simons theory which is a branch theory in topological quantum field theory(TQFT).

The past 35 years have witnessed remarkable growth in our understanding of fundamental physics. For instance, the Weinberg-Salam model has successfully unified the weak interactions and the eletromagnetism, and quantum chromodynamics has proven to be an remarkably accurate model for the strong interactions. Furthermore, the combination of the Weinberg-Salam model and quantum chromodynamics which is called the Standard Model has been success to explain experiment results such like particle decay rate and scattering cross-sections. These successes mostly have a common starting point, that is by perturbating the quantum field theory. However, general theory of relativity[13–22] is not belong in this framework.

Gravitational field theory still in the same state like Einstein's time, that is a beautiful model from the point of view of mathematics that agrees well with many experimental facts. However, from the modern point of view of physics, neither quantization of gravity nor unification with other three elementary forces. A quantum theory of gravity is not a well-defined concept, if we stand at the physics point of view. There are two limits, $G \to 0$ for particle physics and $\hbar \to 0$ for general relativity. Except these limits, there are few constraints can leave us. Over the last few decades, many advances have been made in the area of seeing quantum gravity(QG) as topological quantum field theory(TQFT). The motivation why using topological methods is that general relativity is a highly nonlinear theory, and if it be viewed as an ordinary field theory, then general relativity has a coupling constant, $G^{\frac{1}{2}}$. And, the dimensions of this coupling constant is an inverse mass. By the power-counting arguments, we know that the theory is nonrenormalizable. In general relativity, there are few problems still unsolved. In the classical point of view, we have cosmic censorship [26, 27], the nature of singularities and the conditions for formation of closed timelike curves. In the quantum point of view, the situation is even worse that we still cannot understand the fundamental physics of gravity in quantum level. Furthermore, if one would like to quantized general relativity in four dimensions, then one might face to several problems[12]. Faced with those difficult problems, it is helpful to look a simpler model but share the important features of general relativity to avoid computational difficulties. Therefore, beginning with Einstein's contemporaries, there are some efforts such like to increase the number of dimensions of space-time larger than four, for example, a well-known theory that string theory[23, 24] is a candidate theory in this developing direction. A growing number of research studies are now available to shed some light on the (2+1)-dimensions theory. In the begining if someone look at this theory not so carefully, one might think that its a trivial theory. However, with a second thought, one might find some interesting points of it. That is, in the opposite direction, a small minority has found inspiration in dimension less than four. General relativity in (2+1)-dimensions is one of such model. From a mathematical point of view one would like to keep the general features of quantum theory in one hand and the topological framework of space-time on the other hand. In other words, Both of the features are contained by In spite of the fact that a few disadvantages of this model are come from physically non-equivalent with the (3+1)-dimensional general relativity. However, (2+1)-dimensional model has proven highly instructive in the analysis of conceptual problems, for instance, the nature of time, the construction of states and observables, the role of topology and topology change[25], the relationships between different approaches to quantization and so on and so forth.

The structure of this paper is as follows. In the next section, I will set up Chern-Simons theory at first, since

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it's a powerful tool when analyzing planer classical gravity and quantization of classical gravity. In section III, I will set up gravity in (2+1)-dimensions with Chern-Simons approach, and sketch other approach, for instance, by using Einstein equation form. In section IV, I will quantize the classical gravity with Chern-Simons approach, and discuss the relationships between quantum gravity and topological quantum field theory, and finally list other approach in the end. Finally, some concluding remarks are presented in section V.

II. CHERN-SIMONS THEORY

A beautiful line of development in Riemannian geometry is the relationship between curvature and topology. In one of Chern's first major works in 1946, Chern proves a generalized Gauss-Bonnet theorem by producing a method what now be called a transgressing from on the unit sphere bundle of the manifold[65]. 25 years later, together with Simons, Chern took up transgression in the context of the theory of connections on arbitrary principle bundles[65]. Chern-Simons invariants of a connection are secondary geometric invariants. After these works, Chern and Weil developed the theory of primary topological invariants of connections. Both of these two kind of invariants are local in the sense that they are computed by integral of differential forms. The relationship between them is that the differential of the Chern-Simons form is the Chern-Weil form. The integral of the Chern-Weil form over a closed manifold is independent of the connection, so is a topological invariant. In the late 1980s Edward Witten proposed a new topological invariant of 3-manifolds from these same ingredients. Witten integrates the exponentiated Chern-Simons invariant over the infinite-dimensional space of all connections. Because the connection is integrated out, the result depends only on the underlying manifold. The result is called the quantum Chern-Simons invariant. The classical Chern-Simons invariant is closely related to the Atiyah-Patodi-Singer invariant, and was refined in the Cheeger-Simons theory of differential characters. It appeared in physics before Wittens work, for example in the theory of anomalies. The quantum Chern-Simons invariant is closely related to the Jones invariants[65] of link which has been widely used in knot theory[2].

A. Classical Chern-Simons functional

Let $G = SU(n)$ be the Lie group of unitary n×n matrices of determinant one for n≥ 2. Its Lie algebra **g** consists of n×n skew-Hermitian matrices of traceless. Note that G here is different from coupling constant in general relativity. Fix a closed oriented 3-manifold M. A connection A on the trivial G-bundle over M is a skew-Hermitian matrix of 1-forms with trace zero. In this case the classical Chern-Simons invariant(functional) of A is given by the explicit formula $CS(A) : \mathcal{A}_{\mathcal{M}} \to \mathbf{R}$,

$$A \longmapsto \frac{1}{8\pi^2}\int_M Tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \qquad (1)$$

where the wedge products are combined with matrix multiplication. The integrand in (1) is a 3-form, and the integral depends on the orientation of M. Chern and Simons were interested in the Levi-Civita connection on the tagent bundle of a Riemannian manifold M where the Chern-Simons invariant is an obstruction to certain conformal immersions and to derive combinatorial formulas for the first Pontrjagin number of a compact oriented 4-manifold. E. Witten[34] used the classical Chern-Simons invariant to derive a topological invariant of 3-manifolds which is not an invariant of Riemannian 3-manifolds. The classical approach to remove the dependence of (1) on the connection A is to treat $CS(A)$ as defining a variational problem and to find its critical points. However, the Euler-Lagrange equation asserts that the connection A is flat. There is not in general a unique flat connection, so there is no particular critical value to choose as a topological invariant, though many interesting topological may be formed from the space of flat connections. There is a driving engine of quantum field theory that is Feynman path integral. Very briefly, there are some basic forms of the path integral as following.

i.Sum-over-histories, developed in Feynman's version of quantum mechanics(QM)[53].

ii.Sum-over-field, started in Feynman's version of quantum electrodynamics(QED)[54] and latter improved by Fadeev-Popov[55].

iii.Sum-over-geometries/topologies, in topological field theory(TQFT) and quantum gravity(QG).

B. Witten's Chern-Simons invariant

Witten's intelligent approach is using this powerful method, he integrates out the variable A to obtain a topological invariant. The precise process will be sketch as follows.

Definition 1 *The gauge group $\mathcal{G}_{\mathcal{M}}$ to be the set of maps $g : M \to G$. Here g acts on P by right multiplication. If $g \in \mathcal{G}_{\mathcal{M}}$, then*

$$g^*A = g^{-1}Ag + g^{-1}dg. \qquad (2)$$

Observe that this is similar to the gauge change for affine connections. For more information on principal G-bundles and their connections, see[64, 65].

Lemma 1 *Suppose $\partial M = \emptyset$. Then the critical points of CS are flat connections.*

Lemma 2 *Suppose ∂M is not necessarily empty. Then*

$$CS(g^*A) = CS(A) + \frac{1}{8\pi^2}\int_{\partial M} Tr(A\wedge dgg^{-1}) - \int_M g^*\sigma. \qquad (3)$$

Here σ is the 3-form on $G = SU(2)$ given by $\frac{1}{24\pi^2}Tr(\mu \wedge \mu \wedge \mu)$, and μ is the Maurer-Cartan form on G.

Note that the last term is the Wess-Zumino term. If $\partial M = \emptyset, thentheboundarytermdropsout$, and

$$CS(g^*A) = CS(A) + \int_M g^*\sigma. \tag{4}$$

Notice $\int_M g^*\sigma \in \mathbf{Z}$ since it's the pullback of an integral class of G. Hence CS is a function

$$CS : A_M/\mathcal{G}_M \to \mathbf{R}/\mathbf{Z}. \tag{5}$$

It also makes sense to write $e^{2\pi i CS([A])}$, where $[A] \in A_M/\mathcal{G}_M$. Just like the WZW model[56, 57], consider the Feynman oath integral:

$$Z_k(M) = \int_{A_M/\mathcal{G}_M} e^{2\pi i k CS(A)} d\mu. \tag{6}$$

where k is the level. Cut a closed oriented 3-manifold M along an oriented surface Σ so that $M = M_1 \cup M_2, \partial M_1 = \Sigma, \partial M_2 = -\Sigma$. Finally, we can define topological invariant which is quite important in topology and TQFT.

Definition 2 *If the $L = L_1 \cup \ldots \cup L_m$ is a link in a 3-manifold M, then assign a representation V_j of G to each component L_j. Let $W_{L_j,R_j}(A)$ be the trace of the holonomy of A around L_j. Then the following topological invariant is obtained which is called Witten's invariant:*

$$Z_k(M; L_1, \ldots, L_m) = \int e^{2\pi i k CS(A)} \prod_{j=1}^{m} W_{L_j,R_j}(A) d\mu. \tag{7}$$

C. Application of Chern-Simons theory

Chern-Simons theory which is a powerful tool not only for mathematics but also for the modern mathematical physics. Beacau this theory can be widely used in many different field. There is hundred of papers using these invariant, thus there are rich stories to tell about both the classical and quantum Chern-Simons invariants in geometry, topology[61], mathematical physics, condensed physics, biophysics and so on and so forth. Chern-Simons theory can be made interesting and nontrivial in a number of ways:

i. coupling to dynamical matter fields
ii. coupling to a Maxwell term
iii. taking the space-time to have nontrivial topology
iv. nonabelian gauge fields
v. quantum gravity
vi. modified gravity

There are some good review papers for aspects of Chern-Simons theory[33, 58].

III. GRAVITY IN (2+1)-DIMENSIONS WITH CHERN-SIMONS APPROACH

(2+1)-dimensional gravity is a rich topic to analysis, I have therefore chosen to focus on a few key aspects with Chern-Simons approach, and it's necessarily leaving out many interesting parts: classical point sources and closed timelike curves[45], the classical and quantum behavior of point particles[46], matter couplings, supergravity[67], asymptotic behavior and global charges, the radial gauge, topologically massive gravity, Poincaré gauge theory, the construction of observables from topological field theories[28, 30, 31, 34] and so on and so forth.

A. Chern-Simons Approach

In this approach I will use the unit that $16\pi G = 1$ and start from the first-order form of the Einstein action. (For a review of the first- and second-order formalism, see[59]) The fundamental variables are the bundle of orthonormal frames,e^a_μ, and a spin connection $\omega_\mu{}^a{}_b$. The Einstein-Hilbert action are written as following

$$I_g = 2\int_M e^a(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c), \tag{8}$$

where $e^a = e_\mu{}^a dx^\mu$ and $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu$. The action is invariant under local SO(2,1) transformations,

$$\delta e^a = \epsilon^{abc} e_b \tau_c \tag{9}$$

$$\delta\omega^a = d\tau^a + \epsilon^{abc}\omega_b\tau_c, \tag{10}$$

as well as "local translations,"

$$\delta e^a = d\sigma^a + \epsilon^{abc}\omega_b\sigma_c \tag{11}$$

$$\delta\omega^a = 0. \tag{12}$$

I_g is invariant under diffeomorphism of space-time, M. However, this is not an independent symmetry: Witten has shown that when the triad $e_\mu{}^a$ is invertible, diffeomorphisms in the connected component of the identity are equivalent to transformations of the form(9)-(12)[66]. A distinct construction of the generators of diffeomorphisms in terms of generators of gauge transformations has been carried out by Bañado [59]. Equations of motion derive from the action (8) are as following:

$$T^a[e, \omega] = de^a + \epsilon^{abc}\omega_b \wedge e_c = 0 \tag{13}$$

and

$$R^a[\omega] = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c = 0. \tag{14}$$

The first equation is the standard torsion-free condition that determines ω in terms of e. The second equation implies the connection ω is flat, or equivalently that the curvature of the metric $g_{\mu\nu} = e_{\eta}{}^{a} e_{\nu}{}^{b} \eta_{ab}$ vanishes. In this formulation, the important point is that the global geometry is clear, in other words, if M is topologically nontrivial, a flat connection can still give rise to nonvanishing Aharonov-Bohm phases around noncontractible curves.

There are several ways to understand the solutions of equations(13)-(14). For example, we can learn about the field equations(13)-(14) by observing that the one-forms e^a and ω^a can be combined to form a single ISO(2,1) connection[66, 67]. The Lie algebra of ISO(2,1) has generators $\mathcal{J}^a$ and $\mathcal{P}^b$, with communication relations

$$[\mathcal{J}^a, \mathcal{J}^b] = \epsilon^{abc}\mathcal{J}_c, \tag{15}$$

$$[\mathcal{J}^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c, \tag{16}$$

$$[\mathcal{P}^a, \mathcal{P}^b] = 0. \tag{17}$$

If we write a single connection one-form

$$A = e^a \mathcal{P}_a + \omega^a \mathcal{J}_a \tag{18}$$

and define a "trace," an invariant inner product on the Lie algebra, by

$$Tr(\mathcal{J}^a \mathcal{P}^b) = \eta^{ab}, \tag{19}$$

$$Tr(\mathcal{J}^a \mathcal{J}^b) = Tr(\mathcal{P}^a \mathcal{P}^b) = 0, \tag{20}$$

it's easy to check that the first-order action(8) is simply the Chern-Simons action[34] for A,

$$I_{CS} = \frac{k}{4\pi}\int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{21}$$

where $k = -4\pi$ with the chosen units. Furthermore, the gauge transformations (9)-(12) now can be reinterpreted as standard ISO(2,1) gauge transformations of A. The field equations of s Chern-Simons theory simply require that A be flat[34]. Thus, one might expect solutions to the field equations to be labeled by ISO(2,1) holonomies, i.e., homomorphism $\rho \in M$, where M is precisely the space as follows.

It seems like one may be shown that the holonomy of a curve γ depends only on its homotopy class. Actually, the holonomy defines a group homomorphism

$$\rho : \pi_1(M) \to G. \tag{22}$$

The homomorphism ρ is not uniquely determined by the geometric structure. However, it is unique when conjugating by a constant element $h \in G$, i.e.,$\rho \longmapsto h \cdot \rho \cdot h^{-1}$. In the case of (2+1)-dimensional gravity, G is the Poincaré group, therefore, deriving a space of holonomies of the form

$$\mathcal{M} = Hom(\pi_1(M), ISO(2,1))/\sim, \tag{23}$$

$$\rho_1 \sim \rho_2 \;\; if \;\; \rho_2 = h \cdot \rho_1 \cdot h^{-1}, h \in ISO(2,1). \tag{24}$$

Therefore, I obtain the space M as above, the equivalence relation in (23)-(24) is easy to understand if we take Chern-Simons action into account, that is, a gauge transformation $g : M \to G$ acts on Wilson loops based at x_0 by conjugation by $g(x_0)$, thus, the quotient in (23)-(24) is simply an expression of gauge invariance.

Note that the traces $\mathrm{Tr}\rho[\gamma]$ are automatically invariant under conjugation, and thus provide a set of gauge-invariant observables. In general, these observables form an overcomplete set; Nelson and Regge[68, 69] and Martin[70] have investigated the identities among them.

It's possible to construct a model with cosmological constant, $\Lambda \neq 0$[66, 67, 71]. For $\Lambda = -1/\ell^2 < 0$, the two SO(2,1) connections

$$\Lambda^{(\pm)a} = \omega^a \pm \frac{1}{\ell} e^a \tag{25}$$

can be treated as independent variables, and the Einstein-Hilbert action becomes

$$I_g = I_{CS}[A^{(+)}] - I_{CS}[A^{(-)}], \tag{26}$$

where now $k = \ell\sqrt{2}/8G$ in the conventions of[72]. Note that the numerical value of k depends on the choice of representation and the definition of the trace in (21). For the case of $\Lambda > 0$, the Einstein-Hilbert action is equivalent to the Chern-Simons action for the SL(2,$\mathbf{C}$) connection

$$\tilde{A}^a = \omega^a + i\sqrt{\Lambda} e^a. \tag{27}$$

For the sign of Λ, the holonomies of the gauge field reproduce the holonomies of the corresponding geometric structure had already discussed by Witten[73].

B. Other Approaches

I briefly summarize several other approaches for classical gravity in (2+1)-dimensions as following:
i. analysis with Einstein's field equations[62],
ii. a direct analysis of the geometry[10]
iii. the ADM formalism(metric formalism)[10, 16, 74]

IV. QUANTUM GRAVITY IN THREE DIMENSIONS WITH CHERN-SIMONS APPROACH

A. Torus Universe

In order to obtain a concrete example, considering the torus universe, $M \approx \mathbf{R} \times T^2$, with a negative cosmological constant $\Lambda = -1/\ell^2$[38]. Therefore, one can use this example to understand the relationship between the four approach which are shows in last section. However, in this thesis I will focus on how to quantize the classical gravity in (2+1)-dimensions with Chern-Simons approach. For a concrete example, I will using torus universe which is described by Chern-Simons gravity. Thus, I only summarize the results of the torus universe here, for further details see[10, 38, 39].

It's useful to exhibit the Poisson brackets among the traces of the holonomies, which serve as aset of gauge-invariant observables.

Let the fundamental group of $\mathbf{R}\times T^2$ have two generators,$[\gamma_1]$and$[\gamma_2]$ and $r_a^{\pm}$ be four arbitrary parameters, then we have

$$R_1^{\pm} = \frac{1}{2}Tr\rho^{\pm}[\gamma_1] = \mathsf{cosh}\frac{r_1^{\pm}}{2}, \tag{28}$$

$$R_2^{\pm} = \frac{1}{2}Tr\rho^{\pm}[\gamma_2] = \mathsf{cosh}\frac{r_2^{\pm}}{2}, \tag{29}$$

$$R_{12}^{\pm} = \frac{1}{2}Tr\rho^{\pm}[\gamma_1\cdot\gamma_2] = \mathsf{cosh}\frac{r_1^{\pm}+r_2^{\pm}}{2}. \tag{30}$$

Therefore, we can obtain that

$$\{R_1^{\pm}, R_2^{\pm}\} = \mp\frac{1}{4\ell}(R_{12}^{\pm} - R_1^{\pm}R_2^{\pm}) \tag{31}$$

and cyclical permutations, reproducing the Poisson algebra of Nelson, Regge, and Zertuche[40].

B. Chern-Simons Quantum Gravity with torus universe

In section III, we can know that the fundamental gauge-invariant observables of quantum gravity are the traces $Tr\rho[\gamma]$ of the holonomies. These provide an overcomplete set of coordinates for the space of classical solutions, and it is not obvious that the entire set of Poisson brackets can be made into commutators of operators. This problem has been studied systematically by Nelson and Regge[40, 68], who demonstrate the existence of a well-behaved subalgebra of traces for which commutators can be consistently defined. In the case of the torus universe, this approach is straightforward. To quantize the algebra (31), the strategy is as following:
i. Replacing the classical Poisson brackets{ , } by commutators [,], with the following rule

$$[x, y] = xy - yx = i\hbar\{x, y\}; \tag{32}$$

ii. On the right hand side of (31), we replace the product by the symmetrized product,

$$xy \to \frac{1}{2}(xy + yx). \tag{33}$$

Then, the algebra we need is defined by the relations

$$\hat{R}_1^{\pm}\hat{R}_2^{\pm}e^{\pm i\theta} - \hat{R}_2^{\pm}\hat{R}_1^{\pm}e^{\mp i\theta} = \pm 2i\mathsf{sin}\theta\hat{R}_{12}^{\pm} \tag{34}$$

and cyclical permutations, with

$$\mathsf{tan}\theta = \frac{-\hbar}{8\ell}. \tag{35}$$

The algebra (34) is not a Lie algebra, but it is related to the Lie algebra of the quantum group $\mathsf{SU}_q(2)$[40], with $q = \mathsf{exp}(4i\theta)$. Substitute the parameters $r_a^{\pm}$ in the last subsection, this algebra can be represented by

$$\hat{R}_1^{\pm} = \mathsf{sec}\theta\mathsf{cosh}\frac{r_1^{\pm}}{2} \tag{36}$$

$$\hat{R}_2^{\pm} = \mathsf{sec}\theta\mathsf{cosh}\frac{r_2^{\pm}}{2} \tag{37}$$

$$\hat{R}_{12}^{\pm} = \mathsf{sec}\theta\mathsf{cosh}\frac{r_1^{\pm}+r_2^{\pm}}{2} \tag{38}$$

with

$$[r_1^{\pm}, r_2^{\pm}] = \pm 8i\theta, \tag{39}$$

$$[r_a^{+}, r_b^{-}] = 0. \tag{40}$$

We know that in classical (2+1)-dimensional gravity in torus universe we have the classical brackets:

$$\{r_1^{\pm}, r_2^{\pm}\} = \mp\frac{1}{\ell} and \{r_a^{+}, r_b^{-}\} = 0. \tag{41}$$

When $\Lambda \to 0$, these commutators differ from the naive quantization of the classical brackets(41),

$$[r_1^{\pm}, r_2^{\pm}] = \mp\frac{i\hbar}{\ell}, \tag{42}$$

by terms of order $\hbar^3$.

C. Other Approaches

There are many other approaches such like
i. covariant canonical quantization[12]
ii. the Wheeler-DeWitt equation[12]
iii. loop variables[12]
iv. lattice methods[44]
v. path integrals[43]. Since take into account the time and space constraint here, I neglect their details, and just list them together.

V. CONCLUSION

Chern-Simons quantization approach is self-consistent, but like phase space quantization, it suffers from an important deficiency. In this case, the observables $\hat{R}_a$ characterize the entire spacetime at once, and are therefore time-independent constants of motion. But the classical solutions to (2+1)-dimensional gravity, even for the simple $\mathbf{R} \times T^2$ topology, are most certainly not static. Those operators have somehow lost track of the dynamics of space-time. In realistic world Space-time is not 3-dimensional, and (2+1)-dimensional gravity is obviously not a physically realistic model of our universe. However, there remains a great deal to be learned from thess model. A special interest problem is coupling matter to quantum gravity. Another interesting point is that in the presence of gravity in (2+1)-dimensions, unbroken supersymmetry may lead to a vanishing cosmological constant without requiring the equality of boson and fermion masses. Finally, there are still many interesting works remain, since to quantize gravity until today is a challenge problem in physics.

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