Implications of Imposing Lee-Yang Theorem On The Partition Function of The Universe

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The contributions of this note are threefold: First, this note shows that it's possible to generally apply Lee-Yang Theorem to solutions of Einstein equation. Secondly, since Lee-Yang Theorem could be applied, then we can use this theorem to check whether there is a phase transition of the universe spacetime. If so, in a similar fashion that is done by A. Einsten [3] that the existence of phase transition might be the best indicator to show the existence of quantum gravity, since only if there is a smaller building blocks, sometimes it's called the spacetime atoms, could make phase transition happen. It follows that this note actually gives readers a handy tool to investigate the existence of quantum gravity. Thirdly, this existence of the applicability of Lee-Yang Theorem on partition function of spacetime manifold might also shed some light on the connection between the number theory and geometrical objects (the spacetime manifold). The connection to the Riemann Zeta function is quite interesting when one is also studying the distribution of non-trivial zeroes of the Riemann Zeta function, i.e. the Riemann Hypothesis[10].

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I. INTRODUCTION

For concreteness, consider the following gravitational field equation:

$$G^{\mu\nu} = \kappa\langle T^{\mu\nu}\rangle, \tag{1}$$

where $\langle T^{\mu\nu}\rangle$ denotes the expected value of the energy-momentum tensor of matter in the whole Universe, $\kappa = 8\pi G, c = \hbar = 1$, $G^{\mu\nu}$ is the Einstein tensor which is with the following expression: $G^{\mu\nu} = R^{\mu\nu} - \mathrm{g}^{\mu\nu}R/2$, and $R^{\mu\nu}$ is Riemann tensor, i.e., so-called the Ricci curvature of $\mathrm{g}^{\mu\nu}$ and R is Ricci scalar.

Now, let's introduce a bare CC, λ, required adding a term $-\lambda \mathrm{g}^{\mu\nu}$ to the original gravitational field equation:

$$G^{\mu\nu} = \kappa\langle T^{\mu\nu}\rangle - \lambda \mathrm{g}^{\mu\nu}. \tag{2}$$

The next step is taken quantum fluctuation in to account. Quantum fluctuation results in vacuum energy, ρ_{vac}, which will contribute to $\langle T^{\mu\nu}\rangle$.

$$\langle T^{\mu\nu}\rangle = T^{\mu\nu}_m - \Lambda \mathrm{g}^{\mu\nu}, \tag{3}$$

where $\Lambda = \lambda + \kappa\rho_{vac}$. To see more deeper, let's focus on cosmic time, which is corresponding to effective CC, t_Λ. t_Λ can be derived from effective CC as follows:

$$t_\Lambda \approx \Lambda^{-1/2} \approx 9.7\mathrm{Gyr}, \tag{4}$$

which is pretty close to the present age of the Universe $t_U \approx 13.7$Gyr.

For the *de facto* reason, let's define an action, $I_{tot}[\mathrm{g}_{\mu\nu}, \Psi^a, \Lambda; \mathcal{M}]$, of the whole Universe on a manifold $\mathcal{M}$, with boundary $\partial\mathcal{M}$, and effective CC, Λ, matter fields Ψ^a and metric $\mathrm{g}_{\mu\nu}$. Usually, bare CC, λ, is a fixed parameter and the wave (partition) function of the Universe, $Z[\lambda; \mathcal{M}] \equiv Z_\Lambda[\mathcal{M}]$, is proffered by

$$Z_\Lambda[\mathcal{M}] = \sum e^{iI_{tot}} \times [\text{gauge fixing terms}], \tag{5}$$

where $\{Q^a\}$ are some fixed boundary quantities which are generalized charges on $\partial\mathcal{M}$, and the sum is over all histories (i.e., configurations of the metric and matter, $\mathrm{g}_{\mu\nu}, \Psi^a$) is consistent with these fixed charges. The dominant contribution to $Z_\Lambda[\mathcal{M}]$is from the histories for which I_{tot} is stationary for $\mathrm{g}_{\mu\nu}$ and Ψ^a variations that preserve the $\{Q^a\}$. In these dominant histories, the matter and metric fields obey their classical field equations.

II. A FURTHER INVESTIGATION

Because Shaw-Barrow's theory can be tested in the near future, thus by that time works in this note can also be tested. In conformity with [1, 2], one can simply think that in an *ad hoc* solution which is a homogeneous and isotropic cosmological metric:

$$ds^2 = a^2(\tau)[-d\tau^2 + (1 + kx^2/4)^{-2}dx^i dx^i], \tag{6}$$

where k denotes the spatial curvature. The observer is at $(\tau, x) = (\tau_0, 0)$ and $\partial\mathcal{M}$ is at the surface $\tau = 0$ where $a = 0$. Take

$$T^{\mu\nu}_m = (\rho_m + P_m)U^\mu U^\nu + P_m \mathrm{g}^{\mu\nu}, \tag{7}$$

where $U^\mu = -a^{-1}(\tau)\nabla^\mu\tau$. With

$$H = \frac{a_{,\tau}}{a^2}, \tag{8}$$

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Einstein's equations tender

$$H^2 = \frac{\kappa\rho_m}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{9}$$

and

$$\frac{\rho_{m,\tau}}{a} = -3H(\rho_m + P_m). \tag{10}$$

In [1], Shaw and Barrow found that up to the linear order $O(kx^2)$, the action can be rewritten as follows:

$$I_{cl} = \frac{4\pi}{3}\int_0^{\tau_0} a^4(\tau)(\tau_0-\tau)^3\left[\frac{1}{\kappa}\Gamma - P_{eff}(a)\right]d\tau, \tag{11}$$

where $P_{eff}(a) = P_m - \mathcal{L}_m$ and $\Gamma = (k/a^2)[2/3 + \tau/(\tau_0 - \tau)]$. I_{cl} is defined to be I_{tot} evaluated with $g_{\mu\nu}$ and the matter fields obeying their classical field equations. Hence,

$$I_{cl} = \frac{4\pi}{3}\int_0^{\tau_0} a^4(\tau)(\tau_0-\tau)^3\left[\left(\frac{1}{\kappa}\frac{k}{a^2}\right)\left[\frac{2}{3} + \frac{\tau}{(\tau_0-\tau)}\right] - P_m - \mathcal{L}_m\right]d\tau, \tag{12}$$

where P_m can be contributed by radiation, dark matter and baryonic matter which are labeled by "rad", "dm", and "b" respectively. For the dark matter and baryonic matter $P_{rad} = \rho_{rad}/3$, $\mathcal{L}_{rad}/\rho_{rad} \approx 0$ $P_{dm}/\rho_{dm} \approx 0$ and $\mathcal{L}_{dm}/\rho_{dm} \approx 0$. For baryonic matter, $p_{b/\rho_b \approx 0}$, $\mathcal{L}_b \approx -\zeta_b\rho_b$, where for some $\zeta_b \sim O(1)$ is calculated by QCD. For the structures of baryonic matter, the chiral bag model gives the estimate $\zeta_b \sim 1/2$[1]. As a result that $\rho_b \gg \rho_{rad}$, the dominant contribution to P_{eff} comes from baryonic matter and $P_{eff} \approx \zeta_b\rho_b$. The terms in I_{cl} only depends on λ through the scale factor $a(\tau)$, $\Gamma \propto a^{-2}$ and $P_{eff} \approx \zeta_b\rho_b \propto a^{-3}$. Therefore, equation (11) as well as (12) can be rewritten again as follows:

$$I_{cl} = \frac{4\pi}{3}\int_0^{\tau_0} a^4(\tau)(\tau_0-\tau)^3\left[\left(\frac{1}{\kappa}\right)\frac{1}{a^2(\tau)} - \frac{1}{a^3(\tau)}\right]d\tau. \tag{13}$$

The above action can be more simplified:

$$I_{cl} = \frac{4\pi}{3\kappa}\int_0^{\tau_0} (\tau_0-\tau)^3\left[a^2(\tau) - a(\tau)\right]d\tau. \tag{14}$$

For the matter dominate era, suppose the scale factor $a(\tau) \propto \tau^{2/3}$, then one can derive that

$$I_{cl} = \frac{4\pi}{3\kappa}\int_0^{\tau_0} (\tau_0-\tau)^3\left[\tau^{4/3} - \tau^{2/3}\right]d\tau \tag{15}$$

$$= \left(\frac{4\pi}{3\kappa}\right)\left(\frac{243\tau_0^{14/3}(11\tau_0^{2/3} - 26)}{80080}\right), \tag{16}$$

and the classical action, I_{cl}, versus observer's cosmic time, τ_0, is plotted in FIG. 1.

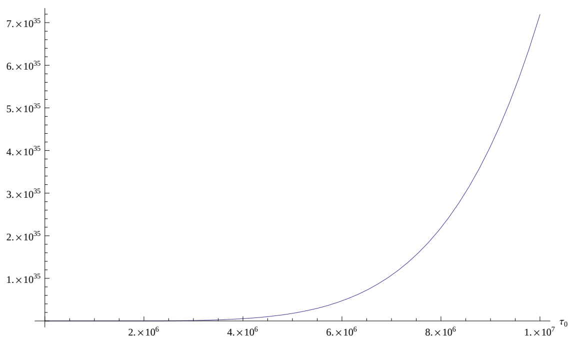

FIG. 1: A I_{cl}-τ_0 diagram which interprets that the monotonic increasing tendency corresponding to observer's cosmic time increasing.

In conformity with [1], section IIA, one has the following form in the classical limit:

$$Z[\mathcal{M}] \approx \sum_{\alpha=1}^{N}\sum_{\Lambda}\mu[\Lambda]\exp(iI_{cl}[\Lambda;\mathcal{M}]). \tag{17}$$

Motivated by simplicity and extracting the significant theoretical meaning, now assuming that the weight function can be fixed as unit function and only focus on the classical solution, i.e. the Universe we live then the first summation can temporarily neglect. Then one can easily derive the *sui generis* partition:

$$Z[\mathcal{M}] \approx \exp(iI_{cl}[\Lambda;\mathcal{M}]). \tag{18}$$

Now, substitute equation (16) into equation (18), then we could have

$$Z[\mathcal{M}] \approx \exp\left(\left(\frac{4\pi i}{3\kappa}\right)\left(\frac{243\tau_0^{14/3}(11\tau_0^{2/3} - 26)}{80080}\right)\right). \tag{19}$$

this partition function can be plotted by decomposed into real-part, $Re[Z]$, and imaginary part, $Im[Z]$. The parameter is observer's cosmic time, τ_0 as in FIG. 1.

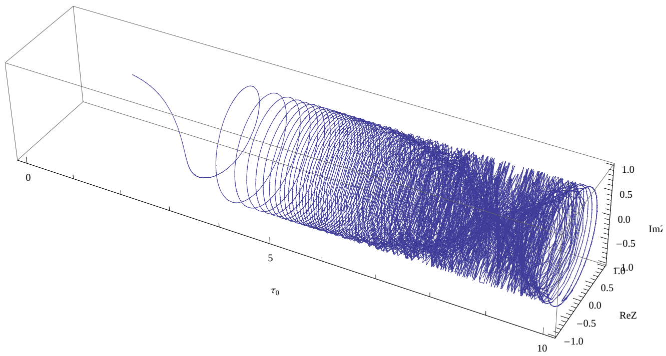

FIG. 2: A 3D diagram for illustrating the rate changing as the observer's cosmic time increasing.

According to FIG. 1; apparently, there is a tendency that as τ_0 increasing, the corresponding value of classical action proliferating as well. This tendency is also reflected in FIG. 2. In the finite temperature field theory perspective, the physical meaning of the observer's cosmic time, τ_0, is related to inverse temperature with respect to the partition function which is derived as follows. Applying wick rotation, then τ_0 replaced by $-it$. Furthermore, $\beta = t/\hbar = t$, where $\beta := 1/kT$, k is the Boltzmann constant and $\hbar = 1$. Thus we get a vital relation $t = 1/kT = i\tau_0$.

Consider that we can investigate critical phenomena, phase-transitions, of a system by investigating the distribution of Lee-Yang zeros. This notion is first proposed by Lee and Yang in 1952[4, 5], and generalized by Fisher and many other following researchers. Currently 3D quantum gravity(QG) still has many ambiguities[6, 7], especially on partition function[8]. Witten[9] applied the notion of Lee-Yang zeros to partition function of 3D quantum gravity and found that this procedure can be used to explain Hawking-Page phase transitions of black holes. All Lee-Yang zeros can be located by tessellation mappings. Because Witten's approach is concentrated on the partition function of the BTZ black holes with different geometries, it's natural to think that whether the partition function in Shaw-Barrow's can also have phase transition. However, the difference here is that the former is applied to a black hole, and the later, in this note, was applied to a universe. (Though from classical general relativity point of view, a black hole solution is equivalent to a universe solution of the Einstein equation.) Thus, the goal in the remaining of this section is concentrated to lay on foundation for investigating the distribution of Lee-Yang zeros of the partition function which is corresponds to the Universe.

Rewritten summation of effective CC, Λ, in equation (17) into observer's cosmic time, since we have already known that Λ depend on τ in the previous discussion (Λ is in the implicit expression of $a(\tau)$, and $a(\tau) \propto \tau^{2/3}$, thus Λ can be labeled by $\tau = \tau_0$). Change variable $\tau_0 = n, n \in \mathbf{N}$.

$$Z[\mathcal{M}] \approx \lim_{N\to\infty} \sum_{n=0}^{N} \exp\left(\left(\frac{4\pi i}{3\kappa}\right)\left(\frac{243 n^{14/3}(11n^{2/3} - 26)}{80080}\right)\right), \tag{20}$$

Simplify the above expression,

$$Z[\mathcal{M}] \approx \lim_{N\to\infty} \sum_{n=0}^{N} \exp\left(\left(\frac{4\pi i 243}{3\kappa 80080}\right)\left(n^{14/3}(11n^{2/3} - 26)\right)\right). \tag{21}$$

Assume

$$A_1 \equiv \left(\frac{4\pi 243}{3\kappa 80080}\right). \tag{22}$$

Therefore,

$$Z[\mathcal{M}] \approx \lim_{N\to\infty} \sum_{n=0}^{N} \exp\left(A_1 i \left(11n^{16/3} - 26n^{14/3}\right)\right). \tag{23}$$

$$\approx \lim_{N\to\infty} \sum_{n=0}^{N} \left\{\exp\left(A_1 i 11 n^{16/3}\right) \exp\left(-A_1 i 26 n^{14/3}\right)\right\}. \tag{24}$$

Define that $z \equiv (\exp(A_1 i))^{\frac{1}{3}}$. Thus, the equation (24) is taking the following form:

$$Z[\mathcal{M}] \approx \lim_{N\to\infty} \sum_{n=0}^{N} \left(z^{11n^{16}} z^{-26n^{14}}\right) \tag{25}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \left(z^{11n^{16}-26n^{14}}\right) \tag{26}$$

$$= \sum_{K=0}^{\infty} z^K, \tag{27}$$

where I let $(11n^{16} - 26n^{14}) := K, K \in \mathbf{Z}$.

Finally, according to equation (27) we derive:

$$Z[\mathcal{M}] = \sum_{K=0}^{\infty} z^K \tag{28}$$

$$= \prod_{i=1}^{\infty} \left(1 - \frac{z}{z_i}\right), \tag{29}$$

where the fundamental theorem of algebra have been applied, and z_i denotes the partition function zeros(after Lee-Yang theorem be applied, we can call them as Lee-Yang zeros).

The point is that the equation that was derived in this note: (28) and (29) imply that Lee-Yang theorem can be applied in this theory. Furthermore, this analysis method can be generalized to whole classes of Shaw-Barrow theory. Thus, phase-transitions can be numerically analyzed by using equation (28) and (29) with keeping track of the location and distribution of Lee-Yang zeros.

III. CONCLUSION

This note shows that we can apply Lee-Yang Theorem to the exponential class of the partition function of the universe implies the possibility of the phase transition of the spacetime of our universe. However, there are at least two important open questions should be focused on: First, to make a further investigation on this work to study whether phase transition really happens in the spacetime manifold of our universe. If this is the case, then as mentioned earlier in the Abstract, in a similar fashion in [3], we can see the implication of the phase transition–it implies the existence of quantum gravity. Secondly, a systematic investigation on the distribution of Lee-Yang zeros are necessary to be constructed. Since by studying this distribution, we can learn the information of the phase transition of the sapcetime manifold (if it exists). Furthermore, the connection to the Riemann Zeta function is quite interesting: the connection between quantum gravity and number theory, especially when one is studying the connections between geometrical objects (Riemann manifolds) and numbers. It's due to the example we have shown in this paper, this work may shed some light on using General Relativity (Einstein-Hilbert Action, and Einstein equation and solutions) to study number theory, especially on building the intuitions on the distirbution of partition functions. Since now we have a mapping between the distribution of zeroes and geometrical objects, hence this appraoch might also useful for physicists to investigate the distribution of non-trivial zeroes of the Riemann Zeta function, i.e. the Riemann Hypothesis[10], or in general, Dirichlet L-functions in a partition function perspective, and giving these math a physical meaning.

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