Applied Mathematics III: Homework 3 Model Solutions

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1. Define $f(z)$,

$$f(z) \equiv n^z \Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n}) \cdots \Gamma(\frac{z+n-1}{n})$$

$$\begin{aligned} f(x) &= n^z \cdot n \cdot \Gamma(\frac{z+1}{n}) \cdots \Gamma(\frac{z+n-1}{n})\Gamma(\frac{z}{n}+1) \\ &= zf(z) \end{aligned}$$

Here we intrduce a thorem,

If a function $f(x)$ satisfies the following three conditions, then it is identical in tis domain of definition with the gamma function:

- $f(x+1) = xf(z)$
- The domain of definition of $f(x)$ contains all $x > 0$, and is log convex for these x. (A function $f(x)$ defined and positive on a certain interval is called log convex if the function $\log[f(x)]$ is convex.
- $f(1) = 1$

The proof of the theorem can be found in the appendix.

Clearly $f(z)$ satisfies above conditons, so we can write,

$$f(x) = a_n \Gamma(x)$$

$$x = 1 \implies a_n = f(1) = n\Gamma(\frac{1}{n})\Gamma(\frac{2}{n}) \cdots \Gamma(\frac{n-1}{n})$$

Using the identity,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

We can write,

$$a_n = \frac{\pi^{\frac{n-1}{2}}}{\prod_{k=1}^{[n/2]} \sin \frac{k\pi}{n}} = \frac{\pi^{\frac{n-1}{2}}}{\sqrt{\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}}} = \sqrt{n} \cdot 2^{\frac{1-n}{2}}$$

Whrere the identity $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$ has been used.

$$\therefore (2\pi)^{\frac{n-1}{2}} \Gamma(z) = n^{z-\frac{1}{2}} \Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n}) \cdots \Gamma(\frac{z+n-1}{n})$$

Another way to do this problem:

Consider

$$\frac{d^2}{dz^2}(\ln\Gamma(z)) = \sum_{m=0}^{\infty}\frac{1}{(z+m)^2}$$

$$\begin{aligned}
\sum_{k=0}^{n-1}\frac{d^2}{dz^2}[ln\Gamma(\frac{z+k}{n})] &= \sum_{m=0}^{\infty}\sum_{k=0}^{n-1}\frac{1}{(\frac{z+k}{n}+m)^2}\\
&= n^2\sum_{m=0}^{\infty}\sum_{k=0}^{n-1}\frac{1}{(z+k+mn)^2}\\
&= n^2\sum_{m=0}^{\infty}\frac{1}{(z+m)^2}\\
&= n^2\frac{d^2}{dz^2}(\ln\Gamma(z))
\end{aligned}$$

Integrate both sides,

$$\sum_{k=0}^{n-1}\ln\Gamma(\frac{z+k}{n}) = \ln\Gamma(z)+az+b$$

$$\Longrightarrow \prod_{k=0}^{n-1}\Gamma(\frac{z+k}{n}) = e^{az+b}\Gamma(z)$$

To find a and b, set $z=1$.

$$\begin{aligned}
\Gamma(\frac{1}{n})\Gamma(\frac{2}{n})\cdots\Gamma(1-\frac{1}{n})\Gamma(1) &= \frac{\pi^{\frac{n-1}{2}}}{\sin(\frac{\pi}{n})\sin(\frac{2\pi}{n})\cdots\sin(\frac{\pi}{2})}\\
&= \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}\\
&= e^{a+b}\Gamma(1)\\
&= e^{a+b}
\end{aligned}$$

Now set $z=n$ =integer.

$$\prod_{k=1}^{n-1}\Gamma(\frac{n+k}{n}) = \frac{(n-1)!}{n^{n-1}}\prod_{k=1}^{n-1}\Gamma(\frac{k}{n}) = e^{an+b}\Gamma(n) = e^{an+b}(n-1)!$$

$$\Longrightarrow e^{a(n-1)} = \frac{1}{n^{n-1}} \quad\Longrightarrow e^a = \frac{1}{n},\ e^b = (2\pi)^{\frac{n-1}{2}}\sqrt{n}$$

$$\therefore (2\pi)^{\frac{n-1}{2}}\Gamma(z) = n^{z-\frac{1}{2}}\prod_{k=1}^{n-1}\Gamma(\frac{z+k}{n})$$

2.

$$\begin{cases}\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}\\ 2^{2z-1}\Gamma(z')\Gamma(z'+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z')\end{cases}$$

Choosing $z=1/3,\ z'=1/6$,

$$\Longrightarrow \begin{cases}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}\\ 2^{-\frac{2}{3}}\Gamma(\frac{1}{6})\Gamma(\frac{2}{3}) = \sqrt{\pi}\Gamma(\frac{1}{3})\end{cases}$$

$$\Longrightarrow \Gamma(\frac{1}{6}) = 2^{-\frac{1}{3}}(\frac{3}{\pi})^{\frac{1}{2}}\Gamma(\frac{1}{3})^2$$

3.

$$\begin{aligned}
& \int_0^\infty \frac{u^{x-1}}{e^u-1}du \\
&= \int_0^\infty \frac{e^{-u}u^{x-1}}{1-e^{-u}}du \\
&= \int_0^\infty e^{-u}u^{x-1}(1+e^{-u}+e^{-2u}+\cdots) \\
&= \sum_{k=1}^\infty \int_0^\infty e^{-ku}u^{x-1}du \\
(let\, t=ku) &= \sum_{k=1}^\infty \frac{1}{k^x}\int_0^\infty e^{-t}t^{x-1}dt \\
&= \sum_{k=1}^\infty \frac{1}{k^x}\Gamma(x)
\end{aligned}$$

$$\therefore \sum_{k=1}^\infty \frac{1}{k^x} = \frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u-1}du,\ \text{converges when}\ \Re(x)>1$$

4.

$$\begin{aligned}
\Gamma(x+iy) &= \lim_{n\to\infty}\frac{1\cdot2\cdot3\cdots n}{(x+iy)(x+iy+1)\cdots(x+iy+n)}n^{x+iy} \\
&= \lim_{n\to\infty}\frac{1\cdot2\cdot3\cdots n}{x(x+1)\cdots(x+n)}n^x\cdot\frac{n^{iy}}{(1+i\frac{y}{x})(1+i\frac{y}{x+1})(1+i\frac{y}{x+n})} \\
&= \Gamma(x)\lim_{n\to\infty}\frac{n^{iy}}{(1+i\frac{y}{x})(1+i\frac{y}{x+1})\cdots(1+i\frac{y}{x+n})}
\end{aligned}$$

$$\begin{aligned}
\therefore |\Gamma(x+iy)| &= |\Gamma(x)|\lim_{n\to\infty}\frac{|n^{iy}|}{|(1+i\frac{y}{x})||(1+i\frac{y}{x+1})|\cdots|(1+i\frac{y}{x+n})|} \\
&= |\Gamma(x)|\prod_{n=1}^\infty[1+(\frac{y}{x+n})^2]^{-\frac{1}{2}}
\end{aligned}$$

5.

$$\begin{aligned}
\Gamma(m)\Gamma(n) &= \lim_{a^2\to\infty}\int_0^{a^2}e^{-u}u^{m-1}du\int_0^{a^2}e^{-v}v^{n-1}dv \\
(\text{let}\ u=x^2,\ v=y^2) &= \lim_{a\to\infty}4\int_0^a e^{-x^2}x^{2m-1}dx\int_0^a e^{-y^2}y^{2n-1}dy \\
&= 4\lim_{a\to\infty}\int_0^a e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}rdrd\theta \\
&= 4\lim_{a\to\infty}\int_0^a e^{-r^2}r^{2m+2n-2}rdr\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta d\theta \\
(\text{let}\ u=r^2,\ t=\cos^2\theta) &= \lim_{a\to\infty}\int_0^a e^{-u}u^{m+n-1}du\int_0^1 t^{m-1}(1-t)^{n-1}dt \\
&= \Gamma(m+n)\cdot\int_0^1 t^{m-1}(1-t)^{n-1}dt
\end{aligned}$$

$$\begin{cases}\Gamma(x) & = \frac{e^{-\gamma x}}{x}\prod_{n=1}(1+\frac{x}{n})^{-1}e^{\frac{x}{n}} \\ \Gamma(y) & = \frac{e^{-\gamma y}}{y}\prod_{m=1}(1+\frac{y}{m})^{-1}e^{\frac{y}{m}} \\ \Gamma(x+y) & = \frac{e^{-\gamma x-\gamma y}}{x+y}\prod_{l}(1+\frac{x+y}{l})^{-1}e^{\frac{x+y}{l}}\end{cases}$$

$$\begin{aligned}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} &= \frac{x+y}{xy}\frac{(1+\frac{x}{1})^{-1}e^{\frac{x}{1}}(1+\frac{x}{2})^{-1}e^{\frac{x}{2}}\cdots(1+\frac{y}{1})^{-1}e^{\frac{y}{1}}(1+\frac{y}{2})^{-1}e^{\frac{y}{2}}\cdots}{(1+\frac{x+y}{1})^{-1}e^{\frac{x+y}{1}}(1+\frac{x+y}{2})^{-1}e^{\frac{x+y}{2}}\cdots} \\ &= \frac{x+y}{xy}\prod_{n=1}^{\infty}\frac{(1+\frac{x}{n})^{-1}(1+\frac{y}{n})^{-1}}{(1+\frac{x+y}{n})^{-1}} \\ &= \frac{x+y}{xy}\prod_{n=1}(1+\frac{xy/n}{n+x+y})^{-1}\end{aligned}$$

Using Stirling's approximation, at large x and y,

$$\begin{aligned}B(x,y) &\approx \frac{\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}\sqrt{2\pi}y^{y-\frac{1}{2}}e^{-y}}{\sqrt{2\pi}(x+y)^{x+y-\frac{1}{2}}e^{-x-y}} \\ &= \sqrt{2\pi}\frac{x^{x-\frac{1}{2}}y^{y-\frac{1}{2}}}{(x+y)^{x+y-\frac{1}{2}}}\end{aligned}$$

6.

$$\begin{aligned}\Gamma(\frac{1}{p}) &= \int_0^{\infty} du\, e^{-u}u^{\frac{1}{p}-1} \\ (\text{Let}\, u=t^p) &= p\cdot\int_0^{\infty} dt\, e^{-t^p}\end{aligned}$$

$$\therefore \int_0^{\infty} dte^{-t^p} = \frac{\Gamma(\frac{1}{p})}{p}$$

$$I(x) = \int_b^a dt\, f(t)e^{x\phi(t)}$$

$$\phi(t) = \phi(c) + \frac{1}{n!}(t-c)^{(n)}\phi^{(n)}(c) + \cdots$$

$$\begin{aligned}I(x) &\approx e^{x\phi(c)}f(c)\int_{-\infty}^{\infty}\exp[\frac{x}{n!}\phi^{(n)}(c)(t-c)^n]d(t-c) \\ &= e^{x\phi(c)}f(c)2\int_0^{\infty} exp[\frac{x}{n!}\phi^{(n)}(c)u^n]du \\ (\text{using the result above}) &= 2e^{x\phi(c)}f(c)\cdot\frac{\Gamma(\frac{1}{n})}{n}[\frac{x}{n!}|\phi^{(n)}(c)|]^{-\frac{1}{n}}\end{aligned}$$

7.

$$\begin{aligned}\text{erfc}(x) &= 1 + \frac{2}{\sqrt{\pi}}\int_x^0 dte^{-t^2} \\ &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} dte^{-t^2} + \frac{2}{\sqrt{\pi}}\int_x^0 dte^{-t^2} \\ &= \frac{2}{\sqrt{\pi}}\int_x^{\infty} dte^{-t^2}\end{aligned}$$

Now, $x \gg 1$,

$$\begin{aligned}
\int_x^\infty dt e^{-t^2} &= \int_x^\infty d(e^{-t^2})(-\frac{1}{2t}) \\
&\approx -\frac{1}{2}\frac{e^{-t^2}}{t}\Big|_x^\infty - \frac{1}{2}\int_x^\infty \frac{e^{-t^2}}{t^2}dt \\
&= \frac{1}{2}\frac{e^{-x^2}}{x} + \frac{1}{4}\int_x^\infty \frac{d(e^{-t^2})}{t^3}dt \\
&\approx \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \frac{1\cdot 3}{8x^5}e^{-t^2} - \cdots
\end{aligned}$$

$$\begin{aligned}
\mathrm{erf}(x) &= 1 - \mathrm{erfc}(x) \\
&\approx 1 - \frac{2}{\sqrt{\pi}}(\frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \frac{1\cdot 3}{8x^5}e^{-t^2}\cdots) \\
&= 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{4x^4} - \cdots) \\
&= 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{x^{2n}2^n}
\end{aligned}$$

8.

$$I(x) = \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dt\, \frac{e^{-t^2}}{t^{2x}}$$

$$t = \sqrt{x}u,\ dt = \sqrt{x}du$$

$$I(x) = \int \sqrt{x}du\frac{e^{-xu^2}}{(\sqrt{x}u)^{2x}} = \sqrt{x}x^{-x}\int du\frac{e^{-xu^2}}{u^{2x}} = \sqrt{x}x^{-x}\int due^{-x(-2\ln u - u^2)}$$

$$f(u) = -2\ln u - u^2$$

$$f'(u) = -2\frac{1}{u} - 2u = 0$$

Saddle pts: $\pm i$
At $+i$,

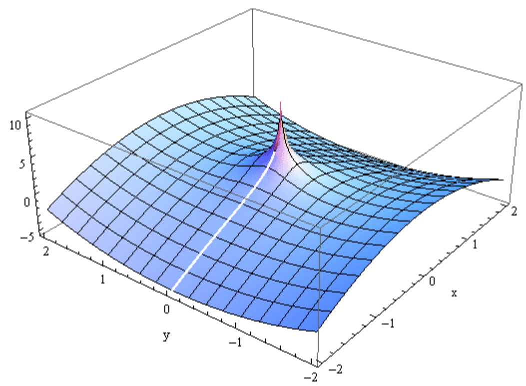
$$\begin{aligned}
f(u) &\approx f(i) + \frac{1}{2}f''(i)(z-i)^2 \\
&= (1-\pi i) - 2(z-i)^2
\end{aligned}$$

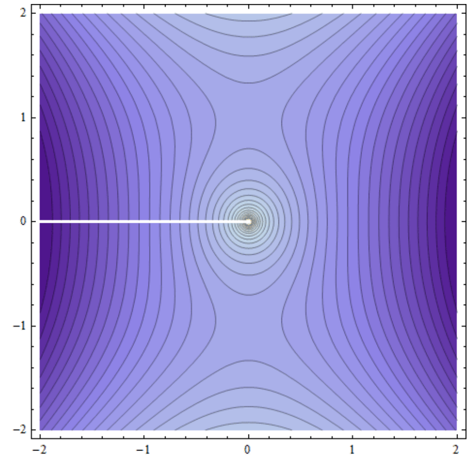
$$I(x) \approx \sqrt{x}x^{-x}e^{xf(i)}\int \exp[-2x(z-i)^2]dz$$

Let $z - i = re^{i\phi}$, $dz = dre^{i\phi}$. ϕ being the angle that the contour approaches the saddle pt. Choose the steepest descent angle $\phi = 0$.

$$\begin{aligned}
I(x) &\approx \sqrt{x}x^{-x}e^{i\phi}e^{xf(i)}\int \exp[-2xr^2]dr \\
&= \sqrt{x}x^{-x}e^{x(1-\pi i)}\sqrt{\frac{\pi}{2x}}
\end{aligned}$$

To convince ourselves why we only include the contribution from$+i$, we can inspect the 3D and contour plots of $\Re[f(z)]$.





It is clear that given the limits of integral $(\pm\infty + i\epsilon)$, the only reasonable contour is to go through$+i$ and follow the direction of steepset descents. Any contour attempting to go through both $\pm i$ would either cross the brach cut or deviate from the steepest descent direction.

9.

$$K_\nu(x) = \frac{1}{2}\int_0^\infty dt\, t^{\nu-1} e^{-\frac{x}{2}(t+\frac{1}{t})}$$

$$\phi(t) = -\frac{1}{2}(t+\frac{1}{t})$$

$$\phi'(t) = 0 \implies t = \pm 1$$

$$\phi''(1) = -1$$

We take the positive root since it's on the integration contour.(positive real axis)

$$\phi(t) \approx \phi(1) + \frac{\phi''(1)}{2}(t-1)^2$$

$$\begin{aligned} K_\nu(x) &\approx \frac{1}{2}\cdot 1^{\nu-1}\int_{-\infty}^\infty e^{x(-1-\frac{1}{2}u^2)}du \\ &= e^{-x}\sqrt{\frac{\pi}{2x}} \end{aligned}$$

10.

$$P_n(\cos\alpha) = \frac{1}{2^{n+1}\pi i}\oint_c \frac{dt(t^2-1)^n}{(t-cos\alpha)^{n+1}} = \frac{1}{2^{n+1}\pi i}\oint_c \frac{e^{f(t)}}{(t-\cos\alpha)}$$

$$f(t) = \log(t^2-1) - \log(t-\cos\alpha)$$

$$f'(t) = \frac{2t}{t^2-1} - \frac{1}{t-\cos\alpha} = 0 \implies t_\pm = \frac{-2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{-2} = e^{\pm i\alpha}$$

$$f''(t) = \frac{(t^2-1)\cdot 2 - 4t^2}{(t^2-1)^2} + \frac{1}{(t-\cos\alpha)^2}$$

$$f''(e^{\pm i\alpha}) = \frac{2(e^{\pm 2i\alpha}-1) - 4e^{\pm 2i\alpha}}{(e^{\pm 2i\alpha}-1)^2} + \frac{1}{(\pm i\sin\alpha)^2}$$
$$= \frac{\exp[\mp i(\alpha+\pi/2)]}{\sin\alpha} \text{ (after some simple calculation.)}$$

$$\int \frac{dt(t^2-1)^n}{(t-\cos\alpha)} \approx \int \frac{\exp\{x[f(t_\pm) + \frac{f''(t_\pm)}{2}(t-t_\pm)^2]\}}{(t_\pm - \cos\alpha)} dt$$

Set $t - t_\pm = re^{i\phi_\pm}$

$$\int \frac{dt(t^2-1)^n}{(t-\cos\alpha)} \approx \sum_\pm \frac{\exp[xf(t_\pm)]e^{i\phi_\pm}}{t_\pm - \cos\alpha} \int \exp[\frac{x}{2\sin\alpha} e^{\mp i(\alpha+\frac{\pi}{2})+2i\phi_\pm} r^2] dr$$

Direction of steepest descent contour:

$$\text{at } t_+: \ -(\alpha+\frac{\pi}{2}) + 2\phi_+ = \pi,\ \phi_+ = \frac{3}{4}\pi + \frac{\alpha}{2}$$
$$\text{at } t_-: \ -(\alpha+\frac{\pi}{2}) + 2\phi_- = \pi,\ \phi_- = \frac{1}{4}\pi - \frac{\alpha}{2}$$

$$-\frac{1}{4}\pi < \phi_+ < \frac{1}{4}\pi$$

$$\frac{3}{4}\pi < \phi_- < \frac{5}{4}\pi$$

It is obvious that the two saddle pts. lie to the "right" and "left" of the pole $\cos\alpha$. So the contour can be easily deformed to go through these two saddle pts. without crossing any poles or branch cuts.

$$\exp[nf(t_+)] = [\frac{t_+^2 - 1}{t_+ - \cos\alpha}]^n = 2^n e^{in\alpha}$$

$$\exp[nf(t_-)] = [\frac{t_-^2 - 1}{t_- - \cos\alpha}]^n = 2^n e^{-in\alpha}$$

Contribution at t_+ :

$$\frac{1}{2^{n+1}\pi i} \frac{2^n e^{in\alpha + i\frac{3}{4}\pi + i\frac{\alpha}{2}}}{i\sin\alpha} \sqrt{\frac{2\pi\sin\alpha}{n}} = -\frac{e^{in\alpha + i\frac{3}{4}\pi + i\frac{\alpha}{2}}}{\sqrt{2\pi n\sin\alpha}}$$

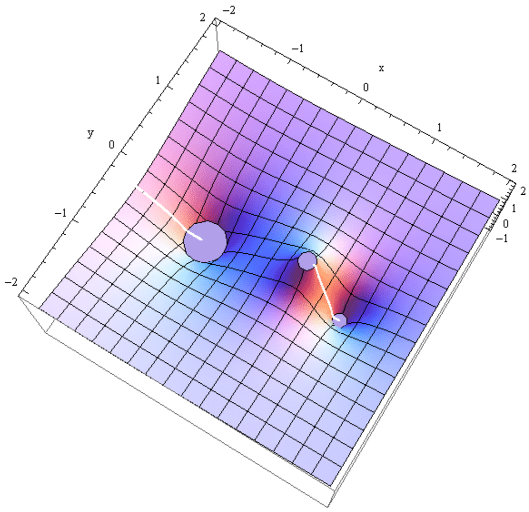
while at t_- :

$$\frac{1}{2^{n+1}\pi i} \frac{2^n e^{-in\alpha + i\frac{1}{4}\pi - i\frac{\alpha}{2}}}{-i\sin\alpha} \sqrt{\frac{2\pi\sin\alpha}{n}} = \frac{e^{-in\alpha + i\frac{1}{4}\pi - i\frac{\alpha}{2}}}{\sqrt{2\pi n\sin\alpha}}$$

Combing the two contributions,

$$-\frac{e^{in\alpha + i\frac{3}{4}\pi + i\frac{\alpha}{2}}}{\sqrt{2\pi n\sin\alpha}} + \frac{e^{-in\alpha + i\frac{1}{4}\pi - i\frac{\alpha}{2}}}{\sqrt{2\pi n\sin\alpha}} = \frac{-ie^{in\alpha + i\frac{\pi}{4} + i\frac{\alpha}{2}} + ie^{-in\alpha - i\frac{\pi}{2} - i\frac{\alpha}{2}}}{\sqrt{2\pi n\sin\alpha}} = \sqrt{\frac{2}{\pi\sin\alpha n}} \sin(n\alpha + \frac{\alpha}{2} + \frac{\pi}{4})$$

Again, inspecting the 3D and contour plots of $\Re[f(z)]$, we conclude that the contour of steepest descents must include both the saddle points $e^{\pm i\alpha}$. ($\cos\alpha = 0.5$)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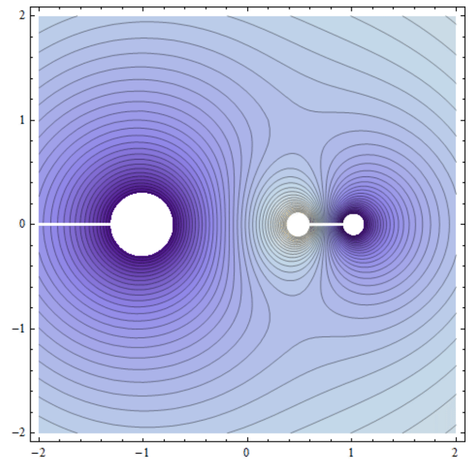

Proof of the theorem used in Problem 1.(From *The Gamma Function*, Emil Artin)

Theorem 2.1

If a function $f(x)$ satisfies the following three conditions, then it is identical in its domain of definition with the gamma function:

(1) $f(x+1) = xf(x)$.

(2) The domain of definition of $f(x)$ contains all $x > 0$, and is log convex for these x.

(3) $f(1) = 1$.

Proof

The existence of a function with these properties (the gamma function) has already been proved.

Suppose $f(x)$ is a function that satisfies our three conditions. Then Eq. (2.5) is valid because of condition (1), and $f(n) = (n-1)!$ for all integers $n > 0$ because of condition (3). It suffices to show that $f(x)$ agrees with $\Gamma(x)$ on the interval $0 < x \leqslant 1$. If this is the case, then $f(x)$ must agree with $\Gamma(x)$ everywhere because of condition (1). Let x be a real number, $0 < x \leqslant 1$, and n an integer $\geqslant 2$. The inequality

$$\frac{\log f(-1+n) - \log f(n)}{(-1+n)-n} \leqslant \frac{\log f(x+n) - \log f(n)}{(x+n)-n} \leqslant \frac{\log f(1+n) - \log f(n)}{(1+n)-n}$$

expresses the monotonic growth of the difference quotient for particular values, and is therefore valid because of condition (2). Since $f(n) = (n-1)!$, we have

$$\log (n-1) \leqslant \frac{\log f(x+n) - \log (n-1)!}{x} \leqslant \log n$$

or

$$\log (n-1)^x (n-1)! \leqslant \log f(x+n) \leqslant \log n^x (n-1)!.$$

But the logarithm is a monotonic function; hence

$$(n-1)^x (n-1)! \leqslant f(x+n) \leqslant n^x (n-1)!.$$

With the help of Eq. (2.5), we get the following inequality for $f(x)$ itself:

$$\frac{(n-1)^x (n-1)!}{x(x+1)\cdots(x+n-1)} \leqslant f(x) \leqslant \frac{n^x (n-1)!}{x(x+1)\cdots(x+n-1)}$$

$$= \frac{n^x n!}{x(x+1)\cdots(x+n)} \frac{x+n}{n}.$$

Since this inequality holds for all $n \geqslant 2$, we can replace n by $n+1$ on the left side. Thus

$$\frac{n^x n!}{x(x+1)\cdots(x+n)} \leqslant f(x) \leqslant \frac{n^x n!}{x(x+1)\cdots(x+n)} \frac{x+n}{n}.$$

An easy calculation gives the inequality

$$f(x)\frac{n}{x+n} \leqslant \frac{n^x n!}{x(x+1)\cdots(x+n)} \leqslant f(x).$$

As n approaches infinity, we get

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}.$$

But $\Gamma(x)$ is also a function that satisfies our three conditions. Hence the relation we have just derived is still valid if we put $\Gamma(x)$ instead of $f(x)$ on the left side. This completes the proof of the theorem.

As a corollary we have the formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}. \tag{2.7}$$

Actually Eq. (2.7) was only proved for the interval $0 < x \leqslant 1$. To show that it holds in general, we denote the function under the limit sign by $\Gamma_n(x)$. It is easy to see that

$$\Gamma_n(x+1) = x\Gamma_n(x)\frac{n}{x+n+1}, \qquad \Gamma_n(x) = \frac{1}{x}\frac{x+n+1}{n}\Gamma_n(x+1).$$

These two expressions help clarify the following fact: As n approaches infinity, if the limit in Eq. (2.7) exists for a number x, it also exists for $x+1$. Conversely, if it exists for $x+1$ and $x \neq 0$, it also exists for x. Hence the limit exists for exactly those values of x for which $\Gamma(x)$ is defined. If we denote the limit in Eq. (2.7) by $f(x)$, we get the equation $f(x+1) = xf(x)$. Since $f(x)$ already agrees with $\Gamma(x)$ on the interval $0 < x \leqslant 1$, it must also agree everywhere else. Equation (2.7) was derived by Gauss, and it is often used as the fundamental definition of the gamma function.

Another form of Eq. (2.7) which is important in the theory of functions was derived by Weierstrass. A simple manipulation shows that

$$\Gamma_n(x) = e^{x(\log n - 1/1 - 1/2 - \cdots - 1/n)}\frac{1}{x}\frac{e^{x/1}}{1+x/1}\frac{e^{x/2}}{1+x/2}\cdots\frac{e^{x/n}}{1+x/n}.$$

But the limit

$$C = \lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)$$