

# Spectrum of a Compact Operator

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# Overview

Spectrum of a  
Compact  
Operator

William  
Chuang

Set-up.

Fredholm  
Theorem

Spectrum of a  
compact  
operator

Bounds on the  
spectrum of a  
symmetric  
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References

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# Notations

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$H$  represents Hilbert space.

$\Lambda$  is for bounded linear operators in  $H$ .

$A$  stands for linear operator in  $\mathbb{R}^n$ .

$(, )$  inner products in Hilbert space  $H$ .

$\doteq$  means equal by definition.

$:=$  means the left hand side is defined as right hand side.

# Definitions.

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- $\Lambda$  is **symmetric** if  $(\Lambda x, y) = (x, \Lambda y)$ , for all  $x, y \in H$ , so  $\Lambda$  is **self-adjoint**
- The resolvent set of  $\Lambda$ , denoted as  $\rho(\Lambda)$ , is the set of numbers  $\eta \in \mathbb{R}$  such that  $\eta I - \Lambda$  is a bijection<sup>1</sup>
- The complement of the resolvent set:  $\sigma(\Lambda) \doteq \mathbb{R} \setminus \rho(\Lambda)$  is called the **spectrum**.
- The **point spectrum** of  $\Lambda$ , denoted as  $\sigma_p(\Lambda)$ , is the set of numbers  $\eta \in \mathbb{R}$  such that  $\eta I - \Lambda$  is not injective. In other words, If there exists a nonzero vector  $w \in H$  such that

$$\Lambda w = \eta w$$

then  $\eta \in \sigma_p(\Lambda)$  where  $\eta$  is an eigenvalue of  $\Lambda$ , and  $w$  is an associated **eigenvector**.

- The **essential spectrum** of  $\Lambda$ , denoted as  $\sigma_e(\Lambda) = \sigma(\Lambda) \setminus \sigma_p(\Lambda)$ , is the set of numbers  $\eta \in \mathbb{R}$  such that  $(\eta I - \Lambda)$  is injective, not surjective.

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<sup>1</sup>By open mapping theorem,  $(\eta I - \Lambda)^{-1}$  is continuous.

The chapter 6 is initiated from classical linear algebra. For a linear operator  $A : \mathbb{R}^n \mapsto \mathbb{R}^n$  in a finite-dimensional space, there are two results are what we would like to generalize to infinite-dimensional Hilbert space  $H$ :

- $A$  is one-to-one if and only if  $A$  is onto, since  $\dim(Ker(A)) = (Range(A))^\perp$ .
- If  $A$  is symmetric, then its eigenvalues are real, and the space  $\mathbb{R}^n$

The first result is still valid for operators with the form:

$$\Lambda = I - K,$$

where  $I$  is the identity, and  $K$  is a compact operator.

The second statement can be extended to any compact, and self-adjoint operator  $\Lambda : H \mapsto H$ .

# Fredholm Theorem.

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Let  $H$  be a Hilbert space over the reals and let  $K : H \mapsto H$  be a compact linear operator. Then

- $\text{Ker}(I - K)$  is finite-dimensional
- $\text{Range}(I - K)$  is closed
- $\text{Range}(I - K) = \text{Ker}(I - K^*)^\perp$
- $\text{Ker}(I - K) = \{0\} \Leftrightarrow \text{Range}(I - K) = H$
- $\text{Ker}(I - K)$  and  $\text{Ker}(I - K^*)$  have the same dimension

This theorem tells us whether a linear equation:

$$u - Ku = f$$

has solutions, and if so, whether those solutions are unique.

# Tow cases

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- Case 1:  $\text{Ker}(I - K) = \{0\}$ . The operator  $I - K$  is one-to-one and onto. For every  $f \in H$  the above linear equation has a unique solution.
- Case 2:  $\text{Ker}(I - K) \neq \{0\}$  Hence the homogeneous equation  $u - Ku = 0$  has a nontrivial solution. The above linear equation has solutions, if and only if  $f \in \text{Ker}(I - K^*)^\perp$ . That is, if and only if,  $(f, u) = 0, \forall u \in H$  such that  $u - Ku = 0$ .

# Spectrum of a compact operator

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## Theorem

*Let  $H$  be an infinite-dimensional Hilbert space, and let  $K : H \mapsto H$  be a compact linear operator.*

*Then*

- $0 \in \sigma(K)$
- $\sigma(K) = \sigma_p(K) \cup \{0\}$
- *Either  $\sigma_p(K)$  is finite, or else  $\sigma_p(K) = \{\lambda_k : k \geq 1\}$ , where the eigenvalues satisfy  $\lim_{k \mapsto \infty} \lambda_k = 0$*



# Bounds on the spectrum of a symmetric operator

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## Lemma

*Let  $\Lambda : H \mapsto H$  be a bounded linear selfadjoint operator on a real Hilbert space  $H$ . Define the upper and lower bounds*

$$m \doteq \inf_{u \in H, \|u\|=1} (\Lambda u, u), \quad M \doteq \sup_{u \in H, \|u\|=1} (\Lambda u, u). \quad (1)$$

*Then*

- *The spectrum  $\sigma(\Lambda)$  is contained in the interval  $[m, M]$*
- *$m, M \in \sigma(\Lambda)$*
- *$\|\Lambda\| = \max \{-m, M\}$*

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Thank you!