Applied Mathematics III : Homework 5

1. Given the definitions of Fourier Transform and its inverse:

$$\mathcal{F}[f(x)] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}, \quad \mathcal{F}^{-1}[\hat{f}(k)] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \hat{f}(k) e^{ikx},$$

we can also define the Fourier Cosine Transform and it inverse as:

$$\mathcal{F}_c[f_c(x)] = \hat{f}_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx f_c(x) \cos kx, \quad \mathcal{F}_c^{-1}[\hat{f}_c(k)] = f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx \hat{f}_c(k) \cos kx.$$

The Fourier Sine Transform $\mathcal{F}_s[f_s(x)]$ and its inverse $\mathcal{F}_s^{-1}[\hat{f}_s(k)]$ can also be defined analogously with $\cos kx \to \sin kx$. Show that for $a > 0$:

$$\mathcal{F}_c[e^{-ax}] = \sqrt{\frac{\pi}{2}} \frac{a}{k^2 + a^2}, \quad \mathcal{F}_s[e^{-ax}] = \sqrt{\frac{\pi}{2}} \frac{k}{k^2 + a^2}.$$

Show also that for $x > 0$:

$$\int_0^{\infty} dk \frac{k \sin kx}{k^2 + a^2} = \frac{\pi}{2} e^{-ax}, \quad \int_0^{\infty} dk \frac{\cos kx}{k^2 + a^2} = \frac{\pi}{2a} e^{-ax}.$$

2. By taking the Fourier Transform of the differential equation:

$$\frac{d^2 \phi(x)}{dx^2} - s^2 \phi(x) = f(x)$$

show that its solution is given by:

$$\phi(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{\hat{f}(k) e^{ikx}}{k^2 + s^2}.$$

3. Find the Fourier Transform of the function:

$$\begin{aligned} f(x) &= 1, \ |x| < 1 \\ &= 0, \ |x| \geq 1. \end{aligned}$$

Determine the convolution of $f(x)$ with itself, then without further calculation, determine its Fourier Transform. Finally deduce that:

$$\int_{-\infty}^{\infty} dk \frac{\sin^2 k}{k^2} = \pi, \quad \int_{-\infty}^{\infty} dk \frac{\sin^4 k}{k^4} = \frac{2\pi}{3}.$$

4. By finding the complex Fourier series for its LHS to show that either side of the equation:

$$\sum_{n=-\infty}^{\infty} \delta(x + nL) = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{-\frac{i2\pi nx}{L}}$$

can represent a periodic train of impulses. By representing $f(x+nL)$ where L is a constant, in terms of the Fourier Transform $\hat{f}(k)$ of $f(x)$, show that:

$$\sum_{n=-\infty}^{\infty} f(x+nL) = \frac{\sqrt{2\pi}}{L} \sum_{n=-\infty}^{\infty} \hat{f}\left(\frac{2n\pi}{L}\right) e^{\frac{i2\pi nx}{L}}.$$

This result is known as *Poisson Resummation Formula* in the literature.

5. Consider the Bessel function $J_0(x)$ with integral representation:

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp(ix\cos\theta).$$

Show that its Fourier Transform can be expressed as:

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} J_0(x) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta \delta(\cos\theta - k).$$

Notice that for $|k| > 1$, the delta function is never satisfied, and there are two values of θ which satisfy it for $|k| < 1$. Show that:

$$\begin{aligned} g(k) &= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1-k^2}}, \quad |k| < 1 \\ &= 0, \quad |k| > 1. \end{aligned}$$

6. By considering the Fourier Transform of $f(x) = e^{-a|x|}$ and $g(x) = e^{-b|x|}$ and the convolution product, show that:

$$\int_{-\infty}^{\infty} \frac{dk}{(k^2+a^2)(k^2+b^2)} = \frac{\pi}{ab(a+b)}.$$

7. Solve for $f(x)$ in the integral equation:

$$\int_{-\infty}^{\infty} dy f(x-y) f(y) = e^{-ax^2}, \quad a > 0.$$

8. The displacement $x(t)$ of a damped harmonic oscillator satisfies the differential equation:

$$\frac{d^2x(t)}{dt^2} + 2\gamma \frac{dx(t)}{dt} + q^2 x(t) = f(t), \quad \gamma > 0,$$

and assuming the Fourier Transform of $x(t)$ and $f(t)$ exist. Show that:

$$x(t) = \int_{-\infty}^{\infty} dt' G(t-t') f(t'), \quad G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \frac{e^{iwt}}{q^2 + 2i\gamma w - w^2}.$$

Verify by explicit differentiation under integration that $G(t)$ satisfies:

$$\frac{d^2G(t)}{dt^2} + 2\gamma \frac{dG(t)}{dt} + q^2 G(t) = \delta(t).$$