Atiyah-Singer Index Theorem

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1/10 Why we need Atiyah-Singer index theorem?

Reason 1/3

Given by Nakaharah, as we consider a differential operator defined on a manifold M, say, the d'Alembertian, the Dirac operator, or the Laplacian. These operators are regarded as maps of sections

$$D : \Gamma(M, E) \to \Gamma(M, F)$$

where E and F are vector bundles[a] over M. Because it is a differential operator, D carries analytic information on the spectrum and its degeneracy. That is, people usually interested in the zero eigenvectors of D and $D^\dagger$

[a] A vector bundle $E \mapsto M$ is a fibre bundle whose fibre is a vector space. A fibre bundle is a topological space which looks locally like a direct product of two topological spaces.

1/10 Why we need Atiyah-Singer index theorem?

Reason 2/3

We hope to find an invariant that on the RHS of the following equation:

$$\mathrm{index} D = \textbf{?}$$

To find/bulid the RHS, we may need to use the characteristic classes of E, F, and TM, and some quantities relate to D (but should not involve information about solutions of PDEs).

1/10 Why we need Atiyah-Singer index theorem?

Reason 3/3

What if we define

$$\ker D = \{s \in \Gamma(M,E) \mid Ds = 0\}$$

$$\ker D^{\dagger} = \{s \in \Gamma(M,F) \mid D^{\dagger}s = 0\}$$

Then we can define the RHS as

$$\dim \ker D - \dim \ker D^{\dagger}.$$

This analytic quantity is a topological invariant expressed in terms of an integral of an appropriate characteristic class over the manifold M.

2/10 What is Atiyah-Singer index theorem?

Scientists describe the world by measuring quantities and forces that vary over time and space. The rules of nature are often expressed by formulas, called differential equations, involving their rates of change. Such formulas may have an ”index,” the number of solutions of the formulas minus the number of restrictions that they impose on the values of the quantities being computed. i.e. **The analytical index is equal to its topological index.** The Atiyah-Singer index theorem calculated this number in terms of the geometry of the surrounding space.

2/10 What is Atiyah-Singer index theorem?

Formal Description [Nakahara]

Let (E, D) be an elliptic complex over an m-dimensional compact manifold M without a boundary. The index of this complex is given by

$$ind(E, D) = (-1)^{m(m+1)/2} \int_M ch(\bigoplus_r (-1)^r E_r) \left(\frac{Td(TM^{\mathbb{C}})}{e(TM)}\right)_{vol}. \tag{1}$$

In the integrand of the RHS, only m-forms are picked up, so that the integration makes sense.

Remarks: The division by e(TM) can really be carried out at the formal level. If m is an odd integer, the index vanishes identically, see below. Original references are Atiyah and Singer (1968a, b), Atiyah and Segal (1968).

2/10 How to derive it? (Example on the board!)

Basic Requiremets

smooth manifolds,and algebraic topology(especially cohomology). Some familiarity with basic notions of functional analysis: Hilbert spaces, bounded linear operators, L^2-spaces

2/10 How to derive it? (Derivations on the board!)

Method 1 Pseudodifferential operators

Method 2 Cobordism

Method 3 K theory

Method 4 Heat equation

Heat equations give a local formula for the index of any elliptic complex.

Method 5 Using Supersymmetry for Dirac Operator form

Method 6 For a Spin Complex form