mid solution 1 to 4

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Problem 1

(3%)Since

$$\int_{-\infty}^{\infty} dx R(x) = 2\pi i \sum_k \mathrm{Res} R(z_k)$$

Where z_j is a pole on upper plane.
(5%)The poles of the function are

$$z_k = e^{i(\frac{\pi}{2n}+\frac{k\pi}{n})} \quad k = 0 \cdots n-1$$

For simple pole, the residue of rational function $\frac{P(z)}{Q(z)}$ is

$$\mathrm{Res}(f, z_0) = \frac{P(z_0)}{Q'(z_0)}$$

(12%)Set $R(x) = \frac{1}{1+x^{2n}}$ the integral becomes

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}} = 2\pi i \sum_{k=0}^{n-1} \frac{1}{2nz_k^{2n-1}} = -2\pi i \frac{e^{i\frac{\pi}{2n}}}{2n} \frac{1-e^{i\pi}}{1-e^{i\frac{\pi}{n}}} = \frac{\pi/n}{\sin(\frac{\pi}{2n})}$$

Because $(1+x^{2n})^{-1}$ is even function, we have

$$\int_0^{\infty} \frac{dx}{1+x^{2n}} = \frac{\pi/2n}{\sin(\frac{\pi}{2n})}$$

Problem 2

(6%)Replecing $z = e^{i\phi}$

$$\int_{|z|=1} \frac{dz}{z}(z-z^{-1})^{2n} = 2^{2n} i^{2n+1} \int_0^{2\pi} d\phi \sin^{2n}\phi$$

(14%)To find residue, we can expand the function around $z = 0$

$$z^{-1}(z - z^{-1})^{2n} = z^{-1}(\cdots + C_n^{2n}(-1)^n(z)^n(\frac{1}{z})^n + \cdots)$$

So we have

$$2^{2n}i^{2n+1}\int_0^{2\pi} d\phi \sin^{2n}\phi = 2\pi i \text{Res}\{z^{-1}(z - z^{-1})^{2n}\} = 2\pi i(-1)^n C_n^{2n}$$

We get

$$\int_0^{2\pi} d\phi \sin^{2n}\phi = \frac{\pi}{2^{2n-1}}\frac{(2n)!}{(n!)^2}$$

Problem 3

(3%)Consider

$$I = \oint_{|z|\to\infty} dz\left(\frac{\sin(\alpha z)}{\alpha z}\right)^2 \frac{\pi}{\sin(\pi z)} = 0$$

(5%)The poles of integral is located at $z = k$ where $k \in \pm\mathbb{N}$, and $z = 0$.
(12%) Since $Res\{f(z = 0)\} = 1$. For $k \in \pm\mathbb{N}$, we may set $z = \xi + k$, and expand ξ around $\xi = 0$. i.e.

$$\frac{\pi}{\sin(\pi(\xi + k))} = \frac{\pi}{\sin(\pi\xi)\cos(\pi k) + \cos(\pi\xi)\sin(\pi k)} = \frac{(-1)^k\pi}{\sin(\pi\xi)}$$

We have

$$I = 0 = \sum_{k=-\infty}^{\infty}(-1)^k\pi\left(\frac{\sin(\alpha k)}{\alpha k}\right)^2 = 1 + 2\sum_{k=0}^{\infty}(-1)^k\pi\left(\frac{\sin(\alpha k)}{\alpha k}\right)^2$$

It implies

$$\sum_{k=0}^{\infty}(-1)^{k-1}\pi\left(\frac{\sin(\alpha k)}{\alpha k}\right)^2 = \frac{1}{2}$$

Problem 4

(3%)Replacing $\omega = e^{i\phi}$, by definition

$$J_k(z) = \frac{1}{2\pi i}\oint_c d\omega\frac{1}{\omega^{k+1}}e^{z(w-w^{-1})/2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\frac{e^{iz\sin\phi}}{e^{ik\phi}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{-i[k\phi - z\sin\phi]}$$

(3%)By Euler formula, we have

$$e^{-i[k\phi - z\sin\phi]} = \cos(k\phi - z\sin\phi) - i\sin(k\phi - z\sin\phi)$$

The 1st term is even-function, and 2nd term is odd-function, the integral can be written as

$$J_k(z) = \frac{1}{\pi}\int_0^\pi d\phi \cos(k\phi - z\sin\phi)$$

(14%)We have

$$e^{z(\omega-\omega^{-1})/2} = \sum_{k=-\infty}^{\infty} J_k(z)\omega^k$$

Expanding L.H.S.

$$e^{z(\omega-\omega^{-1})/2} = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} C_r^n \omega^{n-r}(-\omega)^{-r}\right)$$

We may set $k = n - 2r$, since the maximum value of r is $n = \infty$, and we have $(n \to \infty) - 2(r \to \infty) < k < (n \to \infty) - 2(r = 0)$ i.e. $-\infty < k < \infty$

$$\begin{aligned}
e^{z(\omega-\omega^{-1})/2} &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n} C_r^n \omega^{n-r}(-\omega)^{-r}\right) \\
&= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{\infty} C_r^{k+2r}\left(\frac{z}{2}\right)^{k+2r}\frac{(-1)^r}{(k+2r)!}\omega^k \\
&= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{\infty} \frac{(k+2r)!}{r!(k+2r-r)!}\left(\frac{z}{2}\right)^{k+2r}\frac{(-1)^r}{(k+2r)!}\omega^k \\
&= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(k+r)!}\left(\frac{z}{2}\right)^{k+2r}\omega^k = \sum_{k=-\infty}^{\infty} J_k(z)\omega^k
\end{aligned}$$

so we have

$$J_k(z) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(k+r)!}\left(\frac{z}{2}\right)^{k+2r}$$