

Let's get started.

Part I

Sequences and Series

To make sense of Riemann sums, we will first study sequences and series.

Chapter 1

Sequences

1.1 Introduction

Intuitively, a sequence is a list of numbers. More formally, a **sequence** $\{a_n\}_{n \geq m}$ is the list of numbers $a_m, a_{m+1}, a_{m+2}, \dots$

Example 1.1.1 A Sequence. The sequence $\{a_n\}_{n \geq 2}$ where $a_n = 3n - 4$ is the sequence below:

$$2, 5, 8, 11, 14, 17, \dots$$

We might more often describe that sequence by declaring that $a_n = 3n - 4$ for $n \geq 2$ \square

Activity 1.1.1 List the first five terms of the sequences below.

- (a) $a_n = n$ for $n \geq 0$
- (b) $b_n = -2n - 1$ for $n \geq 2$
- (c) $c_n = \frac{2n+1}{n}$ for $n \geq 1$
- (d) $d_n = \left(\frac{2}{3}\right)^n$ for $n \geq 0$
- (e) $e_n = (-1)^n$ for $n \geq 0$

A **closed form** for a sequence a_m, a_{m+1}, \dots is an explicit formula for the sequence.

Example 1.1.2 Closed Form. A closed form for the sequence $2, 4, 6, 8, \dots$ is $a_n = 2n$ for $n \geq 1$.

Another closed form for the same sequence is $a_n = 2(n - 1)$ for $n \geq 0$ \square

Activity 1.1.2 Determine a closed form expression for the sequences below. That is, write each as $a_n = \text{something}$, $n \geq m$. (You get to choose m .) There are many (many!) correct expressions.

- (a) $2, 5, 8, 11, 14, \dots$
- (b) $-3, 6, -12, 24, -48, \dots$
- (c) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \dots$
- (d) $-1, 1, -1, 1, -1, \dots$
- (e) $1, 2, 1, 4, 1, 6, 1, 8, \dots$

1.2 Convergence

For us, the most important feature of a sequence will be whether or not it converges, and, in the case of convergence, what it converges to. The definition of convergence should look familiar:

A sequence a_n **converges** to the real number L if given any $\varepsilon > 0$ there is an integer N so that whenever $n \geq N$, $|a_n - L| < \varepsilon$. That is, if a_n is always close to L when n is big, then a_n converges to L . We write this in the usual way: $\lim_{n \rightarrow \infty} a_n = L$. Otherwise, the sequence **diverges**.

If a_n is a sequence and for any $M > 0$ there is an integer N so that whenever $n \geq N$, $a_n > M$, then we say a_n **diverges to infinity** and write $\lim_{n \rightarrow \infty} a_n = \infty$.

If a_n is a sequence and for any $M < 0$ there is an integer N so that whenever $n \geq N$, $a_n < M$, then we say a_n **diverges to negative infinity** and write $\lim_{n \rightarrow \infty} a_n = -\infty$.

Example 1.2.1 $\lim_{n \rightarrow \infty} 3n - 4 = \infty$ and $\lim_{n \rightarrow \infty} \frac{3n - 4}{2n + 1} = \frac{3}{2}$. □

Activity 1.2.1 For each of the sequences on the previous page, determine which converge and which do not. In case of convergence, determine the limit. In case of divergence, determine which diverge to infinity, which diverge to negative infinity, and which simply diverge.

Activity 1.2.2 Compute the limit as n goes to ∞ of the sequences below. Most of the usual rules for computing limits of functions apply in the context of sequences.

(a) $a_n = \frac{1}{n}, n \geq 1$

(b) $b_n = \frac{3n-4}{1-4n}, n \geq 0$

(c) $c_n = \left(\frac{3}{4}\right)^n, n \geq 0$

(d) $d_n = \frac{1}{n} - \frac{1}{n-1}, n \geq 2$

(e) $e_n = \sin(\pi n), n \geq 0$

Activity 1.2.3 Give an example of a sequence a_n and a continuous, non-zero function f so that the sequence a_n diverges, but the sequence $f(a_n)$ converges. Show that both a_n and $f(a_n)$ have the stated properties.

Activity 1.2.4 Create a sequence a_n so that $0 \leq a_n \leq 3$ for all n , and a_n is **increasing** in the sense that $a_n \leq a_{n+1}$ for all n . Show that your sequence has the required properties. Does your sequence converge?

Activity 1.2.5 Create a sequence a_n so that $1 \leq a_n \leq 4$ for all n , and a_n is **decreasing** in the sense that $a_n \geq a_{n+1}$ for all n . Show that your sequence has the required properties. Does your sequence converge?

Activity 1.2.6 Create a sequence a_n so that $2 \leq a_n \leq 3$ for all n , and a_n diverges. Is a_n increasing? Decreasing?

Activity 1.2.7 Is it true that if $a_n \leq b_n$ for all n , and b_n is convergent, then a_n is also convergent? If so, explain why. If not, give an example to show this.

Activity 1.2.8 Is it true that if $a_n \geq b_n$ for all n , and b_n is divergent, then a_n is also divergent? If so, explain why. If not, give an example to show this.

Part II

Integration

This worksheet discusses material from section 1.1 of your book. We begin with some friendly review.

Chapter 2

Riemann Sums

Recall that the definite integral of f on the closed interval $[a, b]$ is

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, \dots, n$. This definite integral represents the area under the curve f from a to b .

Chapter 3

One Tricky Integral

$$\int_0^\pi \sin(x) \, dx$$