Calculus & Analytical Geometry II

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Preface

This is a collection of definitions, examples, and activities intended for use in a typical second semester calculus course. The activities are designed to introduce new ideas to students in class as they work in groups.

The author is in the process of recoding this material from its original LaTeX source code to PreTeXt code. The materials may appear incomplete at times, as the process is ongoing.

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Part I

Sequences and Summations

To make sense of Riemann sums, we will first study sequences and summations.

Chapter 1

Sequences

1.1 Introduction

Intuitively, a sequence is a list of numbers. More formally, a **sequence** $\{a_n\}_{n\geq m}$ is the list of numbers $a_m, a_{m+1}, a_{m+2}, \ldots$.

Example 1.1.1 A Sequence. The sequence $\{a_n\}_{n\geq 2}$ where $a_n = 3n - 4$ is the sequnce below:

$$2, 5, 8, 11, 14, 17, \ldots$$

We might more often describe that sequence by declaring that $a_n = 3n - 4$ for $n \geq 2$ □

Activity 1.1.1 List the first five terms of the sequences below.

(a) $a_n = n$ for $n \geq 0$

(b) $b_n = -2n - 1$ for $n \geq 2$

(c) $c_n = \frac{2n+1}{n}$ for $n \geq 1$

(d) $d_n = \left(\frac{2}{3}\right)^n$ for $n \geq 0$

(e) $e_n = (-1)^n$ for $n \geq 0$

A **closed form** for a sequence $a_m, a_{m+1}, \ldots$ is an explicit formula for the sequence.

Example 1.1.2 Closed Form. A closed form for the sequence $2, 4, 6, 8, \ldots$ is $a_n = 2n$ for $n \geq 1$.

Another closed form for the same sequence is $a_n = 2(n - 1)$ for $n \geq 0$ □

Activity 1.1.2 Determine a closed form expression for the sequences below. That is, write each as $a_n =$ something, $n \geq m$. (You get to choose m.) There are many (many!) correct expressions.

(a) $2, 5, 8, 11, 14, \ldots$

(b) $-3, 6, -12, 24, -48, \ldots$

(c) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \ldots$

(d) $-1, 1, -1, 1, -1, \ldots$

(e) $1, 2, 1, 4, 1, 6, 1, 8, \ldots$

1.2 Convergence

For us, the most important feature of a sequence will be whether or not it converges, and, in the case of convergence, what it converges to. The definition of convergence should look familiar:

A sequence a_n **converges** to the real number L if given any $\varepsilon > 0$ there is an integer N so that whenever $n \geq N$, $|a_n - L| < \varepsilon$. That is, if a_n is always close to L when n is big, then a_n converges to L. We write this in the usual way: $\lim_{n\to\infty} a_n = L$. Otherwise, the sequence **diverges**.

If a_n is a sequence and for any $M > 0$ there is an integer N so that whenever $n \geq N$, $a_n > M$, then we say a_n **diverges to infinity** and write $\lim_{n\to\infty} a_n = \infty$. If a_n is a sequence and for any $M < 0$ there is an integer N so that whenever $n \geq N$, $a_n < M$, then we say a_n **diverges to negative infinity** and write $\lim_{n\to\infty} a_n = -\infty$.

Example 1.2.1 $\lim_{n\to\infty} 3n - 4 = \infty$ and $\lim_{n\to\infty} \dfrac{3n-4}{2n+1} = \dfrac{3}{2}$. □

Activity 1.2.1 For each of the sequences on the previous page, determine which converge and which do not. In case of convergence, determine the limit. In case of divergence, determine which diverge to infinity, which diverge to negative infinity, and which simply diverge.

Activity 1.2.2 Compute the limit as n goes to ∞ of the sequences below. Most of the usual rules for computing limits of functions apply in the context of sequences.

(a) $a_n = \frac{1}{n}$, $n \geq 1$

(b) $b_n = \frac{3n-4}{1-4n}$, $n \geq 0$

(c) $c_n = \left(\frac{3}{4}\right)^n$, $n \geq 0$

(d) $d_n = \frac{1}{n} - \frac{1}{n-1}$, $n \geq 2$

(e) $e_n = \sin(\pi n)$, $n \geq 0$

Activity 1.2.3 Give an example of a sequence a_n and a continuous, non-zero function f so that the sequence a_n diverges, but the sequence $f(a_n)$ converges. Show that both a_n and $f(a_n)$ have the stated properties.

Activity 1.2.4 Create a sequence a_n so that $0 \leq a_n \leq 3$ for all n, and a_n is **increasing** in the sense that $a_n \leq a_{n+1}$ for all n. Show that your sequence has the required properties. Does your sequence converge?

Activity 1.2.5 Create a sequence a_n so that $1 \leq a_n \leq 4$ for all n, and a_n is **decreasing** in the sense that $a_n \geq a_{n+1}$ for all n. Show that your sequence has the required properties. Does your sequence converge?

Activity 1.2.6 Create a sequence a_n so that $2 \leq a_n \leq 3$ for all n, and a_n diverges. Is a_n increasing? Decreasing?

Activity 1.2.7 Is it true that if $a_n \leq b_n$ for all n, and b_n is convergent, then a_n is also convergent? If so, explain why. If not, give an example to show this.

Activity 1.2.8 Is it true that if $a_n \geq b_n$ for all n, and b_n is divergent, then a_n is also divergent? If so, explain why. If not, give an example to show this.

Chapter 2

Summations

2.1 Introduction

Given a sequence a_k, the **summation** of a_k as k goes from m to n is denoted

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

The variable k used for the **index of summation** is arbitrary.

Example 2.1.1 A Summation. $\displaystyle\sum_{k=1}^{5} k^2 = 1 + 4 + 9 + 16 + 25$ □

Activity 2.1.1 Determine the summations below.

(a) $\displaystyle\sum_{k=-2}^{5} k$

(b) $\displaystyle\sum_{j=0}^{3} 2j^2 + 5j - 1$

(c) $\displaystyle\sum_{k=0}^{5} 2$

(d) $\displaystyle\sum_{\ell=-3}^{3} \ell^3$

Activity 2.1.2 Combine the summations below into a single summation. Do not compute the sum.

(a) $\displaystyle\sum_{k=0}^{25}(k+1)(k-1) + 3\sum_{k=0}^{25} k(k+1)$

(b) $\displaystyle\sum_{k=0}^{25}(k+1)(k-1) + 3\sum_{j=0}^{25} j(j+1)$

Hint: The index of summation is arbitrary.

(c) $\displaystyle\sum_{k=0}^{25}(k+1)(k-1) + 3\sum_{k=2}^{27} k(k+1)$

Hint: Shift the index of summation so that the two sums start at the same place, but don't change the value of the sum!

2.2 Telescoping Sums

Given a function f, the **finite difference** function Δf is given by $\Delta f(k) = f(k) - f(k-1)$.

Example 2.2.1 A Finite Difference. If $f(x) = 2x + 1$, then $\Delta f(3) = f(3) - f(2) = 7 - 5$ □

Activity 2.2.1 Let $f(x) = x^2 - x + 1$. Compute the following.

(a) $\Delta f(1)$

(b) $\Delta f(2)$

(c) $\Delta f(3)$

(d) $\Delta f(4)$

(e) $\displaystyle\sum_{k=1}^{4} \Delta f(k)$

Sometimes we specify the finite difference with simpler notation.

Example 2.2.2 Another Way to Write Finite Differences. If $f(x) = 3x-2$, then we can write $\Delta f(k)$ as simply $\Delta(3k-2)$. In this case, $\Delta(3k-2) = (3k-2) - (3(k-1)-2) = 3k-2-(3k-3-2) = 3$. □

Activity 2.2.2 Compute the following.

(a) Δk

(b) Δk^2

(c) Δk^3

(d) $\Delta 1$

Activity 2.2.3 Let f be a function.

(a) Expand the summation $\displaystyle\sum_{k=1}^{3} \Delta f(k)$. Is there any simplification to do? If so, simplify.

(b) Without expanding, simplify: $\displaystyle\sum_{k=1}^{10} \Delta f(k)$

(c) Without expanding, simplify: $\displaystyle\sum_{k=3}^{10} \Delta f(k)$

(d) Without expanding, simplify: $\displaystyle\sum_{k=m+1}^{n} \Delta f(k)$

Summations of the form $\displaystyle\sum_{k=m+1}^{n} \Delta f(k)$ are called **telescoping sums**, a reference to the collapsible telescopes of yore. These are some of the easiest summations to deal with, since we can determine the value of the sum *without* computing all of the terms of the sequence being summed.

2.3 Some Useful Summation Formulas

Activity 2.3.1 It will be convenient for us to have simple formulas for computing the sums $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$, and similar types of sums. Conveniently, we can view these as telescoping sums.

(a) Show that $k = \Delta\left[\frac{k(k+1)}{2}\right]$. That is, show that the right-hand side of the equation simplifies to k.

(b) Use the previous part to determine a formula for $\displaystyle\sum_{k=1}^{n} k$.

(c) Show that $k^2 = \Delta\left[\frac{k(k+1)(2k+1)}{6}\right]$.

(d) Use the previous part to determine a formula for $\displaystyle\sum_{k=1}^{n} k^2$.

2.4 Geometric Sequences and Sums

A sequence of the form $a_n = b \cdot r^n$ where b and r are fixed real numbers is called a **geometric sequence**.

A sum of the form $\displaystyle\sum_{k=m}^{n} b \cdot r^k$ is called a **geometric sum**. Notice that a geometric sum is a sum of terms of a geometric sequence.

Activity 2.4.1 Let r be any real number, but $r \neq 1$.

(a) Show that $r^k = \Delta\left[\dfrac{r^{k+1}}{r-1}\right]$.

(b) Let b be any real number. Compute $\displaystyle\sum_{k=0}^{n} br^k$.

(c) Compute $\displaystyle\sum_{k=0}^{10} 3\left(\frac{4}{5}\right)^k$.

(d) Compute $\displaystyle\sum_{k=0}^{8} \left(-\frac{5}{4}\right)^k$.

Part II

Integration

Chapter 3

Riemann Sums

3.1 Introduction

Recall that the **definite integral of f on the interval** $[a, b]$ is defined to be

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, \ldots, n$. This definite integral represents the area under the curve f from a to b.

Activity 3.1.1 Let $f(x) = x + 1$ and $[a, b] = [-1, 2]$.

(a) Make a sketch which indicates what $\displaystyle\int_{-1}^{2} f(x)\, dx$ represents.

(b) Determine Δx

(c) Determine x_i.

(d) Determine the Riemann sum $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$. Simplify.

(e) If you didn't already, make use of the fact that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} 1 = n$ to further simplify your Riemann sum.

(f) Compute $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.

(g) Compute $\displaystyle\int_{-1}^{2} x + 1\, dx$.

(h) Compare your results with your sketch. Does it make sense? Explain.

3.2 More Activities

Activity 3.2.1 Repeat the previous activity for the function $g(x) = x^2$ on the interval $[0, 2]$. That is, compute $\displaystyle\int_0^2 x^2\,dx$.

You may need to make use of the fact that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Activity 3.2.2 Compute $\displaystyle\int_{-2}^{2} x^2\,dx$.

Hint: Sketch a graph, look for symmetry, and use a previous result.

Activity 3.2.3 Let $f(x) = x^2 - 1$.

(a) Compute $\displaystyle\int_{-1}^{1} f(x)\,dx$.

(b) Is your result positive or negative? Draw a picture, and use it to explain why this makes sense.

Activity 3.2.4 Suppose that $\displaystyle\int_0^3 f(x)\,dx = 0$.

(a) Explain what this means in terms of the area under f from 0 to 3.

(b) Draw at least three different graphs of functions where $\displaystyle\int_0^3 f(x)\,dx = 0$. You do not need to come up with an explicit formula for your functions.

Activity 3.2.5 Let $f(x) = 1 - 3x^2$.

(a) Find all values of b so that $\displaystyle\int_0^b 1 - 3x^2\,dx = 0$.

(b) Sketch a graph to illustrate each case.

Chapter 4

One Tricky Integral

4.1 $\int_0^{\pi} \sin(x)\, dx$

Here you will compute $\int_0^{\pi} \sin(x)\, dx$ using the definition of the Riemann integral.

Activity 4.1.1 Consider $\int_0^{\pi} \sin(x)\, dx$.

(a) Check that in this case $x_i = i\Delta x$. (For what we are doing, we want everything in terms of Δx.)

(b) Write the Riemann sum $\sum_{i=1}^{n} \sin(x_i)\Delta x$.

(c) Write down the sum of angles formula for the cosine function. (Look this up if you have to.)

$$\cos(A + B) =$$

(d) Write down the difference of angles formula for the cosine function. (Look this up if you have to.)

$$\cos(A - B) =$$

(e) Use the above to simplify $\cos(A - B) - \cos(A + B)$.

(f) Replace $A - B$ with a and $A + B$ with b and simplify.

(g) Check that you have $\cos(b) - \cos(a) = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.

(h) Plug in $a = \frac{2i+1}{2}\Delta x$ and $b = \frac{2i-1}{2}\Delta x$ and simplify.

(i) You should have an equation involving $\sin(i\Delta x)$. Solve your equation for this term.

(j) Use your last result to rewrite the Riemann sum under investigation: $\sum_{i=1}^{n} \sin(x_i)\Delta x$.

(k) Factor out all terms which do not contain i. You should be left with a difference of two cosine terms in the summation.

(l) Evaluate that sum. (*Hint*: It is a telescoping sum. Write out some of the terms to see this. The whole point of all of the previous work was to get our sum to look like a telescoping sum.)

(m) Recall that $\displaystyle\int_0^\pi \sin(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\sin(x_i)\Delta x$.

(n) If $n \to \infty$, what happens to Δx?

(o) Express $\displaystyle\int_0^\pi \sin(x)\,dx$ as a limit involving Δx. That is, replace $\displaystyle\lim_{n\to\infty}$ with $\displaystyle\lim_{\Delta x\to c}$ for the appropriate value of c.

(p) Evaluate that limit.

(q) What is $\displaystyle\int_0^\pi \sin(x)\,dx$?

(r) Draw a picture illustrating what this definite integral represents.

Chapter 5

Integration by Substitution

5.1 Introduction

Every rule for differentiation has a corresponding rule for antidifferentiation. Derivatives have the Chain Rule; antiderivatives have Substitution.

5.2 Chain Rule

Recall that if F and g are functions, the Chain Rule for derivatives tells us how to determine the derivative of the composition function $F(g(x))$:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

If $f(x) = F'(x)$, we can rewrite this as

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x) \tag{5.2.1}$$

Activity 5.2.1 For the given $F(x)$ and $g(x)$ given below, use the Chain Rule to compute $\dfrac{d}{dx}F(g(x))$.

(a) $F(x) = x^4$, $g(x) = 3x^2 + 1$

(b) $F(x) = 3x^2 + 1$, $g(x) = x^4$

(c) $F(x) = \cos(x)$, $g(x) = x^2$

(d) $F(x) = x^2$, $g(x) = \cos(x)$

5.3 Antiderivatives by Substitution

Activity 5.3.1 Follow the short steps below to turn the Chain Rule for derivatives into the Substitution Rule for antiderivatives.

(a) Write (5.2.1) as a rule for antiderivatives. Your equation should be of the form $\displaystyle\int (\text{something})\, dx = (\text{something else}) + C$.

(b) Let $u = g(x)$. Compute $\dfrac{du}{dx}$.

(c) Solve that last equation for du.

(d) Plug in your work from the last two tasks into Task 5.3.1.a.

(e) Check that you have the following:

$$\int f(u)\, du = F(u) + C$$

where $F'(u) = f(u)$.

Substitution for antiderivatives allows us to compute antiderivatives for some functions.

Activity 5.3.2 Consider the antiderivative $\displaystyle\int \cos(5x)\, dx$

(a) Check that if $u = 5x$ and $f(x) = \cos(x)$, then the integrand is $f(u)$

(b) With $u = 5x$, determine du, and solve that equation for dx.

(c) Use your work to rewrite the original antiderivative in terms of u. Take care that there are not any x's at all in your expression.

(d) The new antiderivative in terms of u's should be simpler. Compute it.

(e) Take your result, and rewrite it in terms of x's again.

(f)

$$\int \cos(5x)\, dx =$$

Activity 5.3.3 Using a process similar to the last activity, determine all of the following antiderivatives.

(a) $\displaystyle\int \sin(7x)\, dx$

(b) $\displaystyle\int \sec^2(2x)\, dx$

(c) $\displaystyle\int \exp(10x)\, dx$

(d) $\displaystyle\int x\exp(x^2)\, dx$

(e) $\displaystyle\int \sqrt{3x+5}\, dx$

(f) $\displaystyle\int x\sqrt{x^2+1}\, dx$

(g) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\, dx$

(h) $\displaystyle\int \frac{\ln(x)}{x}\, dx$

(i) $\displaystyle\int \frac{1}{x(\ln(x))^2}\, dx$

(j) $\displaystyle\int \frac{x}{x-50}\, dx$ (Try $u = x - 50$.)

5.4 Definite Integrals by Substitution

Example 5.4.1 We'll compute $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} x\sin(x^2)\ dx$.

$$\begin{aligned} u = x^2\ ,\ du &= 2\ dx \\ x = \sqrt{\pi/2} &\Rightarrow u = \pi/2 \\ x = \sqrt{\pi} &\Rightarrow u = \pi \\ \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} x\sin(x^2)\ dx &= \int_{\pi/2}^{\pi} \frac{1}{2}\sin(u)\ du \\ &= (-\cos(u))_{u=\pi/2}^{\pi} \\ &= -\cos(\pi) + \cos(\pi/2) \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

Notice that when we made the substitution, we also substituted the limits of integration. That is crucial, since the original limits were for x, not u. □

Activity 5.4.1 Use substitution to compute the following definite integrals.

(a) $\displaystyle\int_0^{\pi/2} \sin(x)\cos(x)\ dx$

(b) $\displaystyle\int_9^{15} \frac{2x}{\sqrt{x^2-49}}\ dx$

(c) $\displaystyle\int_e^{e^2} \frac{\ln(x)}{x}\ dx$

(d) $\displaystyle\int_0^{4} \frac{3x}{\sqrt{1+6x^2}}\ dx$

(e) $\displaystyle\int_{\pi/3}^{\pi/2} \sec(2x)\tan(2x)\ dx$

Chapter 6

Inverse Trigonometric Functions

6.1 Introduction

Here we'll recall the differentiation formulas for inverse trigonometric functions, and use them to get the related antiderivative formulas.

Activity 6.1.1 Recall the following:

$$\begin{aligned}\frac{d}{dx}\arcsin(x) &= \frac{1}{\sqrt{1-x^2}}\\ \frac{d}{dx}\arctan(x) &= \frac{1}{1+x^2}\\ \frac{d}{dx}arcsec(x) &= \frac{1}{x\sqrt{x^2-1}}\end{aligned}$$

(a) Determine $\frac{d}{dx}\arcsin\left(\frac{x}{a}\right)$ where a is a constant. Simplify your answer to make it look nice.

(b) Determine $\frac{d}{dx}\arctan\left(\frac{x}{a}\right)$ where a is a constant. Simplify your answer to make it look nice.

(c) Determine $\frac{d}{dx}arcsec\left(\frac{x}{a}\right)$ where a is a constant. Simplify your answer to make it look nice.

(d) Rewrite each of the previous three statements about derivatives as their equivalent statement about antiderivatives.

6.2 Some Practice

Activity 6.2.1 Determine each of the antiderivatives below.

(a) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$

(c) $\displaystyle\int \frac{dx}{4+x^2}$

(d) $\displaystyle\int \frac{dx}{4+9x^2}$

(e) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

(f) $\displaystyle\int \frac{\tan(x)}{\sqrt{9-\cos^2(x)}}\,dx$

Activity 6.2.2 Determine each of the definite integrals below.

(a) $\displaystyle\int_0^5 \frac{dx}{25+x^2}$

(b) $\displaystyle\int_{-9/2}^{9/2} \frac{dx}{\sqrt{81-x^2}}$

Activity 6.2.3 Verify the following antiderivative formulas by taking an appropriate derivative. In each, a is a constant.

(a) $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$

(b) $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$