Let's get started.

# Part I

# Sequences and Summations

To make sense of Riemann sums, we will first study sequences and summations.

# Chapter 1

# Sequences

## 1.1 Introduction

Intuitively, a sequence is a list of numbers. More formally, a **sequence** $\{a_n\}_{n\geq m}$ is the list of numbers $a_m, a_{m+1}, a_{m+2}, \ldots$.

**Example 1.1.1 A Sequence.** The sequence $\{a_n\}_{n\geq 2}$ where $a_n = 3n-4$ is the sequnce below:

$$2, 5, 8, 11, 14, 17, \ldots$$

We might more often describe that sequence by declaring that $a_n = 3n-4$ for $n \geq 2$ □

**Activity 1.1.1** List the first five terms of the sequences below.

**(a)** $a_n = n$ for $n \geq 0$

**(b)** $b_n = -2n-1$ for $n \geq 2$

**(c)** $c_n = \frac{2n+1}{n}$ for $n \geq 1$

**(d)** $d_n = \left(\frac{2}{3}\right)^n$ for $n \geq 0$

**(e)** $e_n = (-1)^n$ for $n \geq 0$

A **closed form** for a sequence $a_m, a_{m+1}, \ldots$ is an explicit formula for the sequence.

**Example 1.1.2 Closed Form.** A closed form for the sequence $2, 4, 6, 8, \ldots$ is $a_n = 2n$ for $n \geq 1$.

Another closed form for the same sequence is $a_n = 2(n-1)$ for $n \geq 0$ □

**Activity 1.1.2** Determine a closed form expression for the sequences below. That is, write each as $a_n =$ something, $n \geq m$. (You get to choose $m$.) There are many (many!) correct expressions.

**(a)** $2, 5, 8, 11, 14, \ldots$

**(b)** $-3, 6, -12, 24, -48, \ldots$

**(c)** $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \ldots$

**(d)** $-1, 1, -1, 1, -1, \ldots$

**(e)** $1, 2, 1, 4, 1, 6, 1, 8, \ldots$

## 1.2 Convergence

For us, the most important feature of a sequence will be whether or not it converges, and, in the case of convergence, what it converges to. The definition of convergence should look familiar:

A sequence $a_n$ **converges** to the real number $L$ if given any $\varepsilon > 0$ there is an integer $N$ so that whenever $n \geq N$, $|a_n - L| < \varepsilon$. That is, if $a_n$ is always close to $L$ when $n$ is big, then $a_n$ converges to $L$. We write this in the usual way: $\lim\limits_{n\to\infty} a_n = L$. Otherwise, the sequence **diverges**.

If $a_n$ is a sequence and for any $M > 0$ there is an integer $N$ so that whenever $n \geq N$, $a_n > M$, then we say $a_n$ **diverges to infinity** and write $\lim\limits_{n\to\infty} a_n = \infty$. If $a_n$ is a sequence and for any $M < 0$ there is an integer $N$ so that whenever $n \geq N$, $a_n < M$, then we say $a_n$ **diverges to negative infinity** and write $\lim\limits_{n\to\infty} a_n = -\infty$.

**Example 1.2.1** $\lim\limits_{n\to\infty} 3n - 4 = \infty$ and $\lim\limits_{n\to\infty} \dfrac{3n-4}{2n+1} = \dfrac{3}{2}$. □

**Activity 1.2.1** For each of the sequences on the previous page, determine which converge and which do not. In case of convergence, determine the limit. In case of divergence, determine which diverge to infinity, which diverge to negative infinity, and which simply diverge.

**Activity 1.2.2** Compute the limit as $n$ goes to $\infty$ of the sequences below. Most of the usual rules for computing limits of functions apply in the context of sequences.

**(a)** $a_n = \frac{1}{n}$, $n \geq 1$

**(b)** $b_n = \frac{3n-4}{1-4n}$, $n \geq 0$

**(c)** $c_n = \left(\frac{3}{4}\right)^n$, $n \geq 0$

**(d)** $d_n = \frac{1}{n} - \frac{1}{n-1}$, $n \geq 2$

**(e)** $e_n = \sin(\pi n)$, $n \geq 0$

**Activity 1.2.3** Give an example of a sequence $a_n$ and a continuous, non-zero function $f$ so that the sequence $a_n$ diverges, but the sequence $f(a_n)$ converges. Show that both $a_n$ and $f(a_n)$ have the stated properties.

**Activity 1.2.4** Create a sequence $a_n$ so that $0 \leq a_n \leq 3$ for all $n$, and $a_n$ is **increasing** in the sense that $a_n \leq a_{n+1}$ for all $n$. Show that your sequence has the required properties. Does your sequence converge?

**Activity 1.2.5** Create a sequence $a_n$ so that $1 \leq a_n \leq 4$ for all $n$, and $a_n$ is **decreasing** in the sense that $a_n \geq a_{n+1}$ for all $n$. Show that your sequence has the required properties. Does your sequence converge?

**Activity 1.2.6** Create a sequence $a_n$ so that $2 \leq a_n \leq 3$ for all $n$, and $a_n$ diverges. Is $a_n$ increasing? Decreasing?

**Activity 1.2.7** Is it true that if $a_n \leq b_n$ for all $n$, and $b_n$ is convergent, then $a_n$ is also convergent? If so, explain why. If not, give an example to show this.

**Activity 1.2.8** Is it true that if $a_n \geq b_n$ for all $n$, and $b_n$ is divergent, then $a_n$ is also divergent? If so, explain why. If not, give an example to show this.

# Chapter 2

# Summations

## 2.1 Introduction

Given a sequence $a_k$, the **summation** of $a_k$ as $k$ goes from $m$ to $n$ is denoted

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

The variable $k$ used for the **index of summation** is arbitrary.

**Example 2.1.1 A Summation.** $\displaystyle\sum_{k=1}^{5} k^2 = 1 + 4 + 9 + 16 + 25$ □

**Activity 2.1.1** Determine the summations below.

**(a)** $\displaystyle\sum_{k=-2}^{5} k$

**(b)** $\displaystyle\sum_{j=0}^{3} 2j^2 + 5j - 1$

**(c)** $\displaystyle\sum_{k=0}^{5} 2$

**(d)** $\displaystyle\sum_{\ell=-3}^{3} \ell^3$

**Activity 2.1.2** Combine the summations below into a single summation. Do not compute the sum.

**(a)** $\displaystyle\sum_{k=0}^{25}(k+1)(k-1) + 3\sum_{k=0}^{25} k(k+1)$

**(b)** $\displaystyle\sum_{k=0}^{25}(k+1)(k-1) + 3\sum_{j=0}^{25} j(j+1)$*Hint*: The index of summation is arbitrary.

**(c)** $\displaystyle\sum_{k=0}^{25}(k+1)(k-1) + 3\sum_{k=2}^{27} k(k+1)$*Hint*: Shift the index of summation so that the two sums start at the same place, but don't change the value of the sum!

## 2.2 Telescoping Sums

Given a function $f$, the **finite difference** function $\Delta f$ is given by $\Delta f(k) = f(k) - f(k-1)$.

**Example 2.2.1 A Finite Difference.** If $f(x) = 2x + 1$, then $\Delta f(3) = f(3) - f(2) = 7 - 5$ □

**Activity 2.2.1** Let $f(x) = x^2 - x + 1$. Compute the following.

**(a)** $\Delta f(1)$

**(b)** $\Delta f(2)$

**(c)** $\Delta f(3)$

**(d)** $\Delta f(4)$

**(e)** $\displaystyle\sum_{k=1}^{4} \Delta f(k)$

Sometimes we specify the finite difference with simpler notation.

**Example 2.2.2 Another Way to Write Finite Differences.** If $f(x) = 3x-2$, then we can write $\Delta f(k)$ as simply $\Delta(3k-2)$. In this case, $\Delta(3k-2) = (3k-2) - (3(k-1)-2) = 3k - 2 - (3k - 3 - 2) = 3$. □

**Activity 2.2.2** Compute the following.

**(a)** $\Delta k$

**(b)** $\Delta k^2$

**(c)** $\Delta k^3$

**(d)** $\Delta 1$

**Activity 2.2.3** Let $f$ be a function.

**(a)** Expand the summation $\displaystyle\sum_{k=1}^{3} \Delta f(k)$. Is there any simplification to do? If so, simplify.

**(b)** Without expanding, simplify: $\displaystyle\sum_{k=1}^{10} \Delta f(k)$

**(c)** Without expanding, simplify: $\displaystyle\sum_{k=3}^{10} \Delta f(k)$

**(d)** Without expanding, simplify: $\displaystyle\sum_{k=m+1}^{n} \Delta f(k)$

Summations of the form $\displaystyle\sum_{k=m+1}^{n} \Delta f(k)$ are called **telescoping sums**, a reference to the collapsible telescopes of yore. These are some of the easiest summations to deal with, since we can determine the value of the sum computing all of the terms of the sequence being summed.

## 2.3 Some Useful Summation Formulas

**Activity 2.3.1** It will be convenient for us to have simple formulas for computing the sums $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$, and similar types of sums. Conveniently, we can view these as telescoping sums.

**(a)** Show that $k = \Delta\left[\frac{k(k+1)}{2}\right]$. That is, show that the right-hand side of the equation simplifies to $k$.

**(b)** Use the previous part to determine a formula for $\displaystyle\sum_{k=1}^{n} k$.

**(c)** Show that $k^2 = \Delta\left[\frac{k(k+1)(2k+1)}{6}\right]$.

**(d)** Use the previous part to determine a formula for $\displaystyle\sum_{k=1}^{n} k^2$.

## 2.4 Geometric Sequences and Sums

A sequence of the form $a_n = b \cdot r^n$ where $b$ and $r$ are fixed real numbers is called a **geometric sequence**.

A sum of the form $\displaystyle\sum_{k=m}^{n} b \cdot r^k$ is called a **geometric sum**. Notice that a geometric sum is a sum of terms of a geometric sequence.

**Activity 2.4.1** Let $r$ be any real number, but $r \neq 1$.

**(a)** Show that $r^k = \Delta\left[\dfrac{r^{k+1}}{r-1}\right]$.

**(b)** Let $b$ be any real number. Compute $\displaystyle\sum_{k=0}^{n} br^k$.

**(c)** Compute $\displaystyle\sum_{k=0}^{10} 3\left(\frac{4}{5}\right)^k$.

**(d)** Compute $\displaystyle\sum_{k=0}^{8} \left(-\frac{5}{4}\right)^k$.

# Part II

# Integration

This worksheet discusses material from section 1.1 of your book. We begin with some friendly review.

# Chapter 3

# Riemann Sums

Recall that the definite integral of $f$ on the closed interval $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, \ldots, n$. This definite integral represents the area under the curve $f$ from $a$ to $b$.

## Chapter 4

# One Tricky Integral

$$\int_0^\pi \sin(x)\, dx$$