Chapter 1: Mortality Models

A. Future Lifetime Random Variable for a Newborn

We start by looking at how long a newborn will live. Unfortunately (for both the newborn and for us), we don't know how long that will be. So, let's consider the "future lifetime" of the newborn to be a **Random Variable** and let's define…

X is the Random Variable of the future lifetime of a newborn

We'll assume (and correctly) that X is continuous and non-negative (why?)

	$\overbrace{\text{Future Lifetime of 0-year old}}^{X}$			
Age	0	1	…	Death

Distribution Function of X

Remember that a Distribution Function is the probability that a Random Variable (for example, X) is less than a value (for example, x)

$$F_x = \Pr[X \le x],\ x \ge 0$$

The probability that the newborn dies on or before reaching age x

Note that since we're working with X, which is a continuous Random Variable, I'm not concerned about the difference between "<" and "≤" or ">" and "≥".

Survival Function of X

Remember that the Survival Function is the probability that a Random Variable (for example, X) is greater than a value (for example, x).

$$S_x = \Pr[X \ge x],\ x \ge 0$$

The probability that the newborn dies <u>after</u> reaching age x

Note that since we are looking at a binary system (life/death) it makes sense that the usual rules about Distribution and Survival Functions apply:

$$F_x = 1 - S_x \text{ and } F_x + S_x = 1$$

B. Future Lifetime Random Variable for Someone Age x: T_x

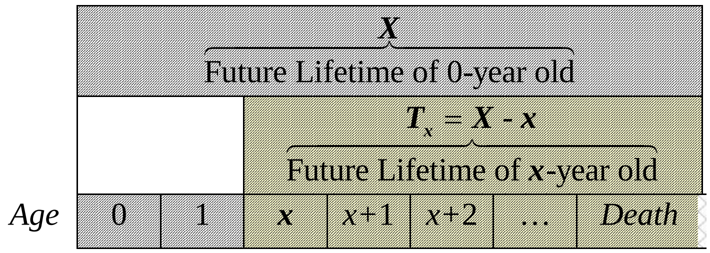
Since insurance is usually purchased at ages other than 0, we have to look at the same functions and approaches for any age x.

Define T_x [or $T(x)$] as the <u>continuous</u> Random Variable for the future lifetime of someone age x.

T_x takes on values from 0 to the number of years the x-year old lives

$T_x = T(x) = X - x_{|X>x}$(Notation of T differs by text.)

The future lifetime of someone currently age *x* is equal to the future lifetime Random Variable of a newborn [***X***] minus the age the purchaser is now (*x*), given that the 0-year old lives to at least age *x*.



↑ Note that the newborn has lived to age ***x***

We could also say that Random Variable $\boldsymbol{X}=\boldsymbol{T_0}$.

And, since we usually know that we're talking about someone who is age *x*, I'll often write ***T*** instead of T_x [or $T(x)$].

Distribution Function of T_x

Similar to the Distribution Function just discussed for Random Variable ***X***, we can think about the probability that Random Variable ***T*** is less than a number of years (***t***). We start with ***t***=0 at age ***x*** and continue until ***t***=∞ (in theory).

$$F_X(t)=\Pr[T_x \le t]=\Pr\left[\underbrace{T_0 \le x+t}_{\substack{\text{Future lifetime}\\ \text{of a newborn}\\ \text{is less than}\\ \text{current age } x\\ \text{plus } t}} \mid \underbrace{T_0 > x}_{\substack{\text{But, the}\\ \text{newborn}\\ \text{must be}\\ \text{alive at}\\ \text{age } x}}\right]=\frac{\Pr[x<T_0\le x+t]}{\Pr[T_0>x]}$$

Survival Function of T_x

The "survival function" represents the probability that the newborn dies after attaining age *x*, or – said another way - the probability that the newborn survives to at least age *x*. We'll see in a bit that $\boldsymbol{S_x}$ and $\boldsymbol{p_x}$ are related.

$$\boldsymbol{S_{x+t}=\Pr[T_x \ge t],\ t \ge 0}$$
$$\boldsymbol{S_{x+t}=1-F_{x+t}=1-\Pr[T_x \le t]}$$

Breaking down the symbol: S_{x+t}
where x is the Current Age
t is the number of years in the future

C. Rules for valid survival models

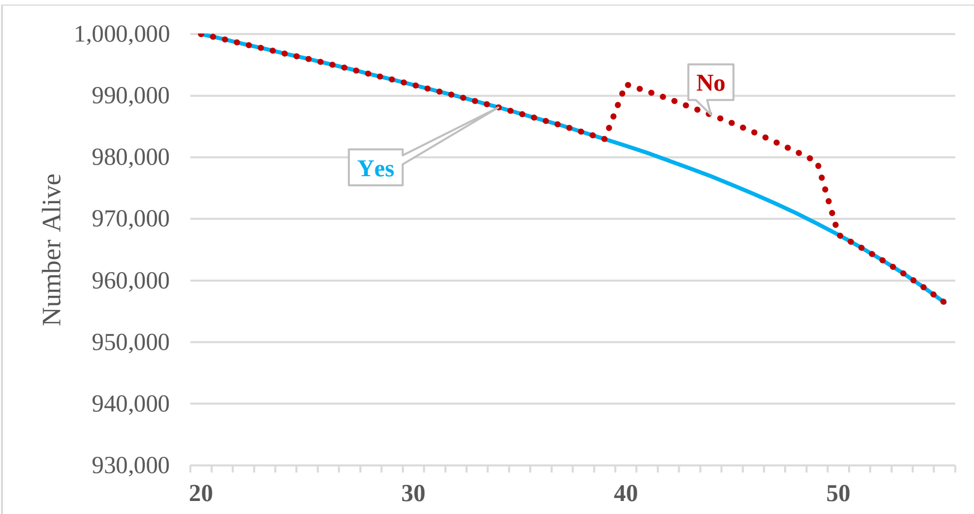
Can any function S_x represent survival/life? **No!** A few *important* rules have to apply.

Always check these, because if the survival model is not possible, then no result can be computed.

Condition 1: $S_0=1$ (Probability a life age 0 survives 0 years must be 1). This is the "if you're alive today, you're alive today" Rule.

Condition 2: $\lim_{t\to\infty} S_{0+t} \Rightarrow S_{0+\infty} \Rightarrow 0$. This is the "everyone dies" Rule.

Condition 3: Non-increasing! (Monotonically decreasing). Can't have a 60 year-old more likely to survive than a 59-year old. This is the "No zombies Rule!".



Assumptions (that will be behind all calculations)

Assumption 1: $S_0(t)$ is differentiable for all $t > 0$. Add to condition (3) and it means that $\frac{d}{dt}S_{0+t} \le 0$ for every $t > 0$

Assumption 2: $\lim_{t\to\infty}\left[t \cdot S_{x+t}\right]=0$

Assumption 3: $\lim_{t\to\infty}\left[t^2 \cdot S_{x+t}\right]=0$

Assumptions 2 and 3 allow us to be sure that the mean and variance of T_x exist

Problem 1-1

a) $S_x = \dfrac{1+\frac{2}{x+2}}{2}, x \ge 0$

b) $S_x = \left(1-\dfrac{x}{120}\right)^{\frac{1}{6}}, 0 \le x \le 120$

For each distribution, ***calculate the following probabilities:***

1) the probability a newborn lives to age 30

2) the probability a 30-year old dies before age 50

3) the probability a 40-year old survives to at least age 65

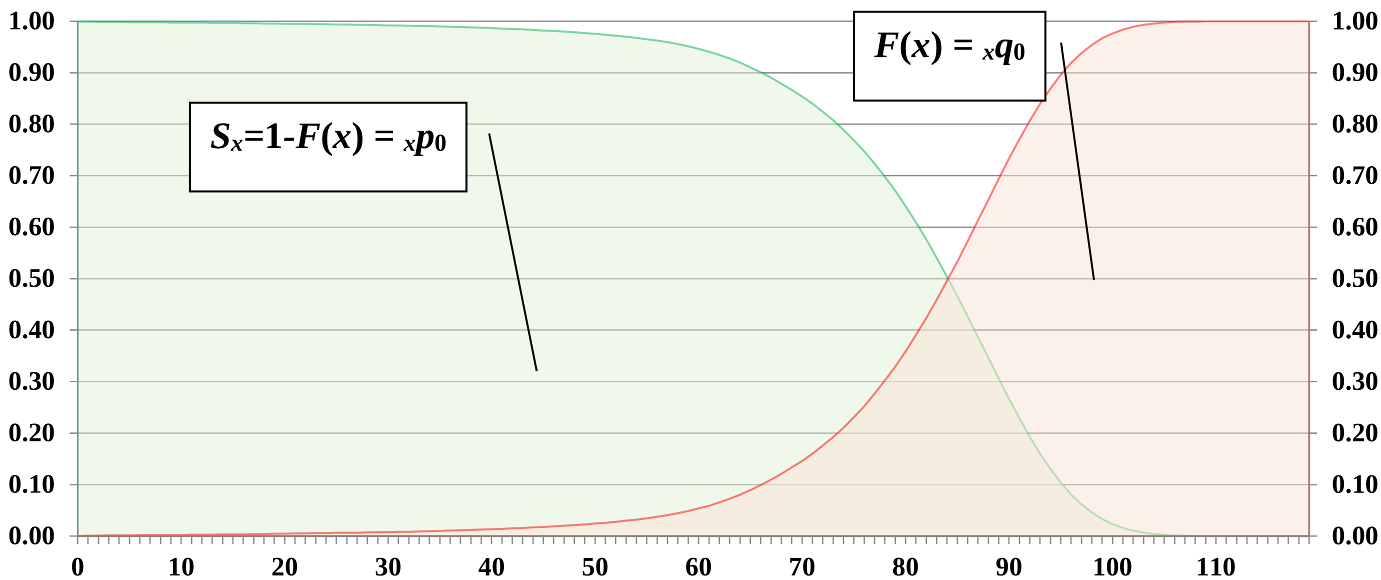
Problem 1-2

Given $S_x = 1 - \frac{x^2}{100}, 0 \leq x \leq 10$***, calculate F(x) and*** $S_4(t)$

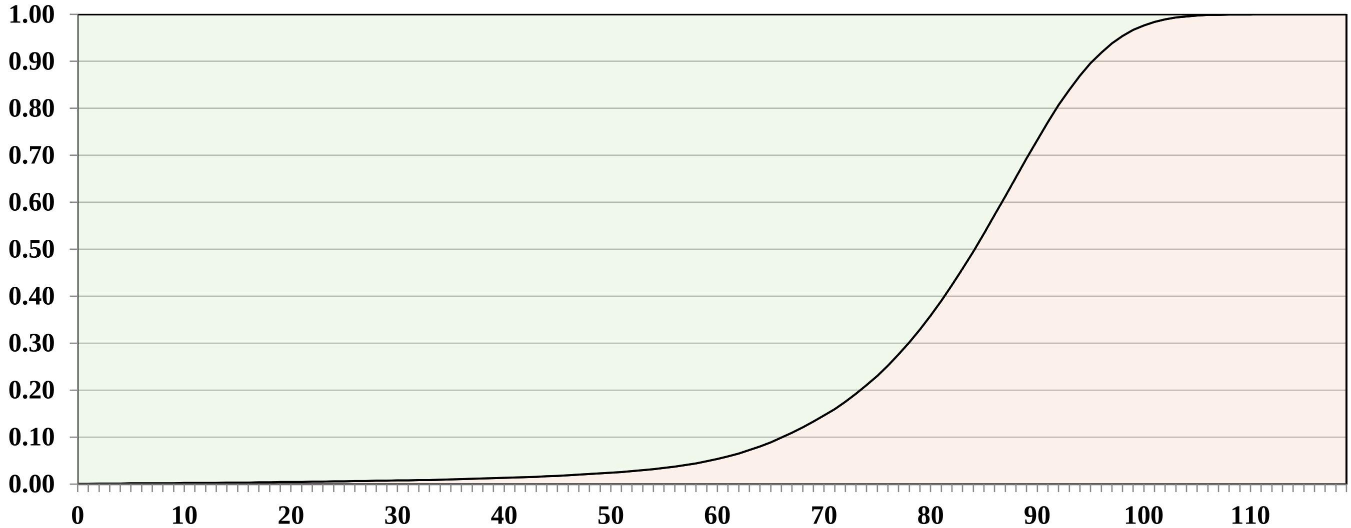
D. $F(x)$ and S_x [functions for a newborn] as a graph

We can construct a graph that shows $F(x)$ and S_x for a life table.

Since $F(x)$ represents the probability that a newborn will not live to age x, we can also use "actuarial notation" to describe the areas.



Note that if we flip the ${}_xp_0$ and put it on top of ${}_xq_0$, we have



Which shows what we know intuitively:

The chance that you die in a year PLUS the chance that you don't die in that year = 1

E. Probability Density Function

$f(x)$ is the probability that a newborn dies between x and $x+dt$

Define it as $f(x)=\frac{d}{dx}F(x)\left[=F'(x)\right]=\frac{-d}{dx}S_x=-S'_x$

Or, working backwards, $F(x)=\int_0^x f(z)dz \,\Big|\, S_x=\int_x^\infty f(z)dz$

This function doesn't have a "practical" description; it's more of a "tool" (as we will see).

F. All three functions

So, if we know any of F_x, $f(x)$, or S_x, we can find the other two functions.

		Want to have this…		
		F_x	$f(x)$	S_x
In terms of this…	F_x	---	F'_x	$1-F_x$
	$f(x)$	$\int_0^x f(z)dz$	---	$\int_x^\infty f(z)dz$
	S_x	$1-S_x$	$-S'_x$	---

Problem 1-3

$S_x=\left(1-\frac{x}{100}\right)^2, 0\le x\le 100$ ***Calculate F_x and $f(x)$***

G. p and q. And ℓ

- p is the symbol for **survival**. It means "living".
- q is the symbol for non-survival. It means non-living = "death".

- ${}_t p_0$ is the probability that a newborn (age 0) lives **at least** t **years**
- p_0 (a blank in front of the p) is the probability that a newborn lives **1 year or more** (no symbol before the p means 1). We could write this as ${}_1 p_0$, but no one does.

- ${}_t q_0$ is the probability that a newborn dies within t years.
- q_0 is the probability that a newborn dies within 1 year (a blank before the q means the same as with p). We could write as ${}_1 q_0$, but – again - no one does

$$F_0(x) = F_x = \Pr[\text{newborn } (0) \text{ dies before } x] = 1 - \frac{\overbrace{\Pr[T_0 > x]}^{\substack{\text{Probabilty a newborn's lifetime is} \\ \textit{at least } x \text{ years} \\ \text{(newborn lives to } \textit{at least} \text{ age } x\text{)}}}}{\underbrace{\Pr[T_0 > 0]}_{\substack{\text{Probabilty a newborn's lifetime is} \\ \textit{at least } 0 \text{ years} \\ \text{(newborn lives to } \textit{at least} \text{ age } 0\text{)}}}} = 1 - \frac{S_x}{\underbrace{S_0}_{=1}} = 1 - {}_x p_0 = {}_x q_0$$

Also, $S_0(x) = S_x = {}_x p_0$

ℓ_0 represents the number of people who are alive at age 0
ℓ_x, is the number of people alive at age x and $\ell_x = \ell_0 \cdot S_x$

The number of people alive at age x (ℓ_x) can be determined as the number who are alive at age 0 times the chance that a newborn is alive at age x.

Example 1-1 – Meaning of p and q in terms of ℓ

If we have ℓ_0 babies born, and we want to know the probability that a newborn baby will live long enough to enter BU's Actuarial Science program (assume that means age 25).

If we can fast-forward 25 years, we can divide the number of those babies who are still alive at age 25 (ℓ_{25}) by the number we started with (ℓ_0): ${}_{25} p_0 = \frac{\ell_{25}}{\ell_0}$

So, if we start with 1,000,000 newborns, and a mortality table shows there are 999,000 of those alive at age 25, then ${}_{25} p_0 = \frac{999{,}000}{1{,}000{,}000} = 0.999$

Similarly, the probability that a newborn <u>does not survive</u> to age 25 is

$$\underbrace{{}_{25}q_0}_{\substack{\text{Probability that}\\ \text{newborn dies}\\ \text{before age 25}}} = \frac{\overbrace{\ell_0 - \ell_{25}}^{\text{This is the number of newborns who don't survive}=\sum_0^{24} d_x}}{\ell_0} = 1 - \frac{\ell_{25}}{\ell_0} = 1 - {}_{25}p_0 = 1 - 0.999 = 0.001$$

Notice that we have not indicated (nor asked for) <u>when</u> the pre-age 25 death will occur; our focus is on "living to age 25" and "not living to age 25".

It should now be clear that since there are only two alternatives (living to age 25 or not living to age 25), we can generalize and say that:

$$\boldsymbol{{}_tp_x = 1 - {}_tq_x}$$

…the same relationship we had between $\boldsymbol{F(x)}$ and $\boldsymbol{S_x}$.

Chaining survival functions and probabilities

For some year in the future ($\boldsymbol{n}$), with $\boldsymbol{n<t}$, we can say that

$$\underbrace{{}_tp_x}_{\substack{\text{Probability}\\ x\text{ lives to}\\ \text{time }t}} = \underbrace{{}_np_x}_{\substack{\text{Probability}\\ x\text{ lives to}\\ \text{time }n}} \cdot \underbrace{{}_{t-n}p_{x+n}}_{\substack{\text{Probability}\\ x+n\text{ lives to}\\ \text{time }t-n}} = \frac{\overbrace{\ell_{x+n}}^{\substack{\text{Number alive}\\ \text{at age }x+n}}}{\underbrace{\ell_x}_{\substack{\text{Number alive}\\ \text{at age }x}}} \cdot \frac{\overbrace{\ell_{(x+n)+(t-n)}}^{\substack{\text{Number alive at}\\ \text{age }(x+\cancel{n})+(t-\cancel{n})\\ =\text{age }x+t}}}{\underbrace{\ell_{x+n}}_{\substack{\text{Number alive}\\ \text{at age }x+n}}} = \frac{\ell_{x+n}}{\ell_x} \cdot \frac{\ell_{x+t}}{\ell_{x+n}} = \frac{\ell_{x+t}}{\ell_x} = {}_tp_x$$

With $\boldsymbol{x}$=20, $\boldsymbol{n}$=5, $\boldsymbol{t}$=8:

$\boldsymbol{n}$=5					$\boldsymbol{t-n}$=3			
x					$\boldsymbol{x+n}$			$\boldsymbol{x+n+t}$
20	21	22	23	24	**25**	26	27	**28**

So, ${}_8\boldsymbol{p}_{20} = {}_5\boldsymbol{p}_{20} \cdot {}_3\boldsymbol{p}_{25}$

Relating a newborn to someone age *x* surviving to age *x+t*

Let's look at the relationship between the probability a newborn survives to an age in the future ($\boldsymbol{x+t}$) using this "chaining" approach:

$$\underbrace{{}_{x+t}p_0}_{\substack{\text{Newborn lives}\\ \text{to age }x+t}} = \underbrace{{}_xp_0}_{\substack{\text{Newborn}\\ \text{lives to age }x}} \cdot \underbrace{{}_tp_x}_{\substack{x\text{-year old lives}\\ \text{to age }x+t}}$$

We can (finally!) look at survival of older-than-newborn (you know, people who have money!)

The probability that a newborn survives to age $\boldsymbol{x+t}$ is equal to

- The probability a newborn (age 0) survives to age $\boldsymbol{x}$, <u>times</u>

- The probability this person, who is now age $\boldsymbol{x}$, survives to age $\boldsymbol{x+t}$

Which means $\boldsymbol{S_{x+t}} = \boldsymbol{S_x}$ times ${}_t\boldsymbol{p_x}$, a result that we used in earlier Problems.

$${}_t p_x = \frac{S_{x+t}}{S_x} = \frac{\text{Probability that a newborn survives to age } x+t}{\text{Probability that a newborn survives to age } x}$$

Make sure you understand that for every $\boldsymbol{z=x+t}$, if $\boldsymbol{x<z}$ (i.e., $\boldsymbol{t} > 0$), $\Pr[\boldsymbol{x} \le \boldsymbol{X} \le \boldsymbol{z}]$ is the probability that the newborn (age 0) dies between ages $\boldsymbol{x}$ and $\boldsymbol{z}$.

$$\overbrace{\qquad\qquad\qquad\qquad}^{\boldsymbol{T_x = X - x}}$$
Future Lifetime of $\boldsymbol{x}$-year old

Age	$\boldsymbol{x}$	$x+1$	…	$\boldsymbol{z}$	…	*Death*

So, if we consider every possible age $\boldsymbol{z}$ between age $\boldsymbol{x}$ and death, we can develop the following relationship between survival for a newborn and survival for someone who is age $\boldsymbol{x}$.

$${}_{z-x}p_x = \frac{S_{x+(z-x)}}{S_x} = \frac{S_z}{S_x} \Rightarrow \underbrace{S_z}_{\substack{\text{Newborn}\\\text{survives to}\\\text{age } z}} = \underbrace{S_x}_{\substack{\text{Newborn}\\\text{survives to}\\\text{age } x}} \cdot \underbrace{{}_{z-x}p_x}_{\substack{\text{Person}\\\text{age } x\\\text{survives to}\\\text{age } z}}$$

Since you've probably mastered this idea, let's complicate and analyze!

"Deferred Death"

The probability that $\boldsymbol{x}$ survives for $\boldsymbol{t}$ years and <u>dies within the next $\boldsymbol{u}$ years</u> is

$\boldsymbol{x}$ remains alive					$\boldsymbol{x}$ dies during this period				
$\boldsymbol{t}$ years					$\boldsymbol{u}$ years				
$\boldsymbol{x}$					$\boldsymbol{x+t}$				$\boldsymbol{x+t+u}$

We'll use the symbol: ${}_{t|u}\boldsymbol{q_x}$

$$\underbrace{t}_{\substack{x \text{ lives}\\\text{to age}\\x+t}} \,|\, \underbrace{u}_{\substack{x \text{ dies}\\\text{between}\\x+t \text{ and}\\x+t+u}} \boldsymbol{q_x}$$

$${}_{t|u}\boldsymbol{q_x} = \Pr[\boldsymbol{t} \le \boldsymbol{T_x} \le \boldsymbol{t} + \boldsymbol{u}]$$

Note the "|" in the symbol. To the *left* is the number of years that we're waiting; to the *right* is the number of years after that in which we are looking for death to happen

So, ${}_{t|u}\boldsymbol{q_x}$ is equal to <u>all</u> of the following and there are *three different ways* to evaluate:

1) The probability that $\boldsymbol{x}$ lives to $\boldsymbol{t}$, then dies between $\boldsymbol{x+t}$ and $\boldsymbol{x+t+u}$

$${}_{t|u}\boldsymbol{q_x} = {}_t p_x \cdot {}_u q_{x+t} = \frac{S_{x+t}}{S_x} \cdot \left(1 - \frac{S_{x+t+u}}{S_{x+t}}\right) = \frac{S_{x+t} - S_{x+t+u}}{S_x}$$

2) The probability that $\boldsymbol{x}$ <u>dies between</u> $\boldsymbol{x}$ and $\boldsymbol{x+t+u}$,
<u>remove</u> (i.e. subtract) the probability that $\boldsymbol{x}$ dies before $\boldsymbol{x+t}$

$$ {}_{t|u}\boldsymbol{q}_{\boldsymbol{x}} = {}_{t+u}q_x - {}_{t}q_x = \left(1 - \frac{S_{x+t+u}}{S_x}\right)\left(1 - \frac{S_{x+t}}{S_x}\right) = \frac{S_{x+t} - S_{x+t+u}}{S_x} $$

3) The probability that ***x*** <u>survives to</u> ***x+t,***
<u>remove</u> (i.e. subtract) the probability that ***x*** survives to ***x+t+u***

$$ {}_{t|u}\boldsymbol{q}_{\boldsymbol{x}} = {}_{t}p_x - {}_{t+u}p_x = \frac{S_{x+t}}{S_x} \cdot \frac{S_{x+t+u}}{S_x} = \frac{S_{x+t} - S_{x+t+u}}{S_x} $$

Note that these are all the same, as proven by the fact that they all $= \dfrac{S_{x+t} - S_{x+t+u}}{S_x}$

Example 1-2 – Death at Specific Future Year(s)

Let's look at 15|10q20, which means we want to look the probably a death happens ***between age 35 and age 45****. Any other time death occurs (before 35 or after 45) is not our focus.*

1) *Probability that a* 20*-year lives to* 35*, then dies before* 45
$_{15}q_{20} = ({}_{15}p_{20})({}_{10}q_{35})$

Survival					Death (35 to 45)				
20	21	22	…	34	35	36	…	44	45

2) *Probability that a* 20*-year dies between* 20 *and* 45*, minus the probability that the* 20*-year dies between* 20 *and* 35
$_{15}q_{20} = {}_{25}q_{20} - {}_{15}q_{20}$

Death (20 to 45)									
Death (20 to 35)									
					Death (35 to 45)				
20	21	22	…	34	35	36	…	44	45

3) *Probability that a* 20*-year lives to* 35 *minus the probability that the* 20*-year lives to* 45
$_{15}q_{20} = {}_{15}p_{20} - {}_{25}p_{20}$

Survival (20 to 35)									
Survival (20 to 45)									
					Death (35 to 45)				
20	21	22	…	34	35	36	…	44	45

To help with this analysis, let's use values of
${}_{15}p_{20}$=0.92195; ${}_{15}q_{20}$=0.07805; ${}_{25}p_{20}$=0.86603; ${}_{25}q_{20}$=0.13397; ${}_{10}p_{35}$=0.93934; ${}_{10}q_{35}$=0.06066

Then

(1) ${}_{15|}q_{20}=({}_{15}p_{20})\,({}_{10}q_{35})=(0.92195)(0.6066)= 0.05592$

(2) ${}_{15|}q_{20}={}_{25}q_{20} - {}_{15}q_{20}=0.13397-0.07805=0.05592$

(3) ${}_{15|}q_{20}={}_{15}p_{20} - {}_{25}p_{20}=0.92195-0.86603=0.5592$

Problem 1-2 (Continued)

Given $S_0(x)=1-\dfrac{x^2}{100}, 0 \le x \le 10$, ***Calculate*** ${}_{2|2}q_4$

H. Force of Mortality (μ)

One way to think about it:

More mathematically, define the "**force of mortality**" at age x (for a newborn) by taking smaller and smaller Δx around X:

$$\mu_x = \lim_{\Delta x \to 0} \frac{\Pr[x \le X \le x+\Delta x \mid X \ge x]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\overbrace{F_0(x+\Delta x)-F_0(x)}^{\Pr[x \le X \le x+\Delta x]}}{\Delta x \underbrace{\left[1-F_0(x)\right]}_{\Pr[X \ge x]}} = \lim_{\Delta x \to 0} \frac{F_{x+\Delta x}-F_x}{\Delta x\left[1-F_x\right]}$$

$$= \lim_{\Delta x \to 0} \frac{S_0(x)-S_0(x+\Delta x)}{\Delta x \cdot S_0(x)} = \frac{f_0(x)}{1-F_0(x)} = \frac{f_0(x)}{S_0(x)} = \frac{-d}{dx}\ln\left[S_0(x)\right] = \frac{-S'_x}{S_x}$$

Also, $\mu_{x+t} = \frac{-\partial}{\partial(x+t)} \ln\left[S_{x+t}\right] = \frac{-S'_{x+t}}{S_{x+t}}$

The following statements regarding μ_x are interesting (to me, anyway):

- $\boldsymbol{\mu_x}$ is **conditional, instantaneous death** at age ***x***
- For very small Δx, $\boldsymbol{\mu_x}\,\Delta x$ is the **probability that a newborn who has reached age *x* dies between age *x* and age *x*+Δ*x*:** $\mu_x \Delta x \approx \Pr\left[x \le X \le x + \Delta x \mid X \ge x\right]$
- Remembering statistics notation and language: "**force of mortality**" means the same thing as "**hazard rate**"

Another way to look at $\boldsymbol{\mu_x}$ (not mathematically correct, but maybe this helps with the idea of $\boldsymbol{\mu_x}$)

$$\mu_x \approx 2\left({}_{\frac{1}{2}}q_x\right) \approx 4\left({}_{\frac{1}{4}}q_x\right) \approx 12\left({}_{\frac{1}{12}}q_x\right) \approx 365\left({}_{\frac{1}{365}}q_x\right) \approx \ldots = \lim_{n\to 0}\left[n\left({}_{\frac{1}{n}}q_x\right)\right]$$

It's important to note that most mortality tables *do not* show fractional ages, so this formula is hard to see in the real world.

Relationship between Force of Mortality (μ_x) and Survival (S_x)

Remembering Calculus,

$$\mu_x = \frac{-d}{dx} \ln\left[S_0(x)\right] = \frac{-d}{dx} \ln\left[S_x\right]$$

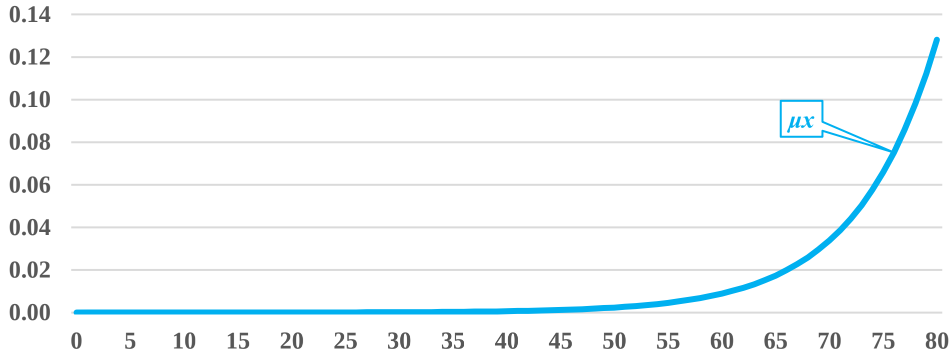
Which means

$$\Rightarrow \quad S_0(x) = S_x = e^{-\int_0^x \mu_z dz}$$

This will be used over and over again in Long Term Actuarial Math, so it's good to know/understand it: to get the survival function, integrate the force of mortality. This same relationship holds (as you probably know) between "survival" and "hazard" functions.

Important facts about μ_x

- It is non-negative: $\boldsymbol{\mu_x} \ge 0$ for all x
- It diverges: $\int_0^\infty (\mu_x)\, dx = \infty$.
- In the real world, $\boldsymbol{\mu_x}$ increases with x, especially at older ages (this graph uses the same underlying mortality as used in Chapter 1: C).



- But, in this class, our models won't always do that (hey, it's the classroom, not the real world!)

Using μ_x when we're looking at survival

We can look at the probability that *x* will survive to age *x*+*n* (*n* years from today) as being the same as the probability that *x* will die *after* *x*+*n* years (since life/death is binary):

x	***x*+1**	**…**	***x*+*n***	***x*+*n*+0.001**	***x*+*n*+0.002**	***x*+*n*+0.003**	**…**
Current			**Survive to age *x*+*n***	**Die the moment (!) after *x*+*n*…**	**…or the next moment…**	**…or the next moment…**	

$$
\begin{aligned}
{}_n p_x &= \text{Probability that } x \text{ lives to } x+n \\
&= \text{Probability that } x \text{ dies } \textit{after} \text{ age } x+n \\
&= \underbrace{\int_n^\infty \underbrace{{}_t p_x}_{\substack{x \text{ lives} \\ \text{from now} \\ \text{until } t \text{ years} \\ \text{from now}}} \cdot \underbrace{\mu_{x+t}}_{\substack{\text{Now (at} \\ \text{age } x+t\text{),} \\ \text{immedate death} \\ \text{occurs}}} \, dt}_{\substack{\text{Start } n \text{ years from now,} \\ \text{go to infinity (and beyond!)}}}
\end{aligned}
$$

The idea we've been working with so far has been that T_x is continuous.

This means that all the functions we've reviewed (F_{x+t}, S_{x+t}, $f_X(t)$) can be computed using the rules outlined and stuff from Calculus.

One relationship worth knowing is

$$f_X(t)dt \approx \Pr\left[t \le T_x \le t+dt\right]$$
$$= \Pr\left[x \text{ dies between } (x+t) \text{ and } (x+t+dt)\right]$$
$$= \Pr\left(x \text{ lives to } x+t\right) \cap \Pr\left(x \text{ dies before } x+t+dt\right)$$
$$= \Pr\left(x \text{ lives to } x+t\right) \cdot \Pr\left(x \text{ dies before } x+t+dt \mid x \text{ lives to } x+t\right)$$
$$f_X(t)dt \approx {}_t p_x \cdot \mu_{x+t} dt$$
$$f_X(t) = {}_t p_x \cdot \mu_{x+t}$$

I. All functions

Can expand from a newborn to someone age x

		Want to find this…			
		S_x	$f(t)$	F_{x+t}	μ_{x+t}
In terms of…	S_x		$\dfrac{-S'_{x+t}}{S_x}$	$1-\dfrac{S_{x+t}}{S_x}$	$\dfrac{-S'_{x+t}}{S_{x+t}}$
	$f(t)$	$\int_t^\infty f_x(z)dz$		$\int_0^t f_x(z)dz$	$\dfrac{f_x(t)}{1-\int_0^t f_x(z)dz}$
	F_{x+t}	$1-F_x$	F'_{x+t}		$\dfrac{F'_{x+t}}{1-F_{x+t}}$
	μ_x	$e^{-\int_0^x \mu_z dz}$	$\boxed{{}_t p_x \cdot \mu_{x+t}}$	$1-e^{-\int_0^x \mu_z dz}$	
	${}_t p_x$	${}_x p_0$		$1-({}_x p_0)({}_t p_x)$	$-\dfrac{d}{dt}\ln {}_t p_x$

Note that $f_x(t) = {}_t p_x\ \mu_{x+t}$

Example 1-3 – Using the table

If we want to calculate S_x *in terms of* μ_{x+t}*, we'd locate the* S_x *in across the top. Then, we'd find the row with* μ_{x+t}*, and where they intersect, we'd find that*

$$S_x = e^{-\int_0^x \mu_z dz}$$

Problem 1-4

$S_x = e^{-\frac{x^2}{500}}, x \geq 0$. ***Calculate*** ${}_t p_{20} \cdot \mu_{20+t}$ and ${}_{40} p_{20} \cdot \mu_{60}$

Problem 1-5

Assume the force of mortality μ_x *is constant at all ages* [$\mu_x = c$ *for all* x].
The probability that a 35*-year old dies within the next* 10 *years is* 0.10.
Calculate ${}_t p_x$

J. Discrete Case: K_x

Of course, when we're out of the classroom, we can't work with (truly) continuous functions. The world we live in requires a "discrete" approach.

So, let's change our approach and focus on living full years.

Define K_x to be Random Variable representing completed years

			Now				Death
x	0	1	**40**	41	42	65	**65y1m**
			$\underbrace{\text{25 Full Years}}_{\text{Now:40; Death at 65y1m}}$				

Example (like before) K_{40}=25 (25 whole years have been lived; ignore the extra month)

Now				Death	
x	$x+1$	$x+2$	$x+k$ (Integer)	$x+t$ (non-integer)	$x+k+1$ (Integer)

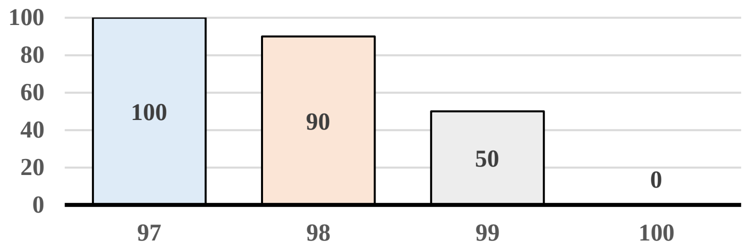
$$f_X(k) = \Pr(K = k) = \Pr(k \leq t < k+1) = {}_{k|}q_x = \frac{d_{x+k}}{\ell_x}$$

Problem 1-6

We find 100 *people who are* 97 *("97 year olds") who – for some reason - want to buy insurance from us. We'll be working with this group all semester.*

This group has the following mortality (the group's "life table"):

x	*Alive at age* (ℓ_{x})	Deaths (d_x)
97	100	10
98	90	40
99	50	50
100	0	--



Calculate the probability density function, $f_{97}(k)$, for this group.

K. Life Table Functions

In Chapter 0, I pointed out that the "real world" uses "mortality tables" for calculations. We'll look at some "theoretical" Laws in a bit, but our first focus will be on these "real world" tables.

A small segment of a Life Table follows:

x	ℓ_x
20	**1,000,000**
21	998,360
22	996,606
23	994,733
24	992,730
25	990,589

In this table, we started with 1,000,000 lives at age 20 (α=20) and created the other values.

We can bring this back to the prior discussion by looking at the probability of survival from age 20 to age 25 by the following formula:

$$ {}_t p_x = S_x(t) = \frac{S_{x+t}}{S_x} = \frac{\frac{\ell_{x+t}}{\ell_0}}{\frac{\ell_x}{\ell_0}} = \frac{\frac{\ell_{x+t}}{\cancel{\ell_0}}}{\frac{\ell_x}{\cancel{\ell_0}}} = \frac{\ell_{x+t}}{\ell_x} \overset{\text{Example}}{\Rightarrow} {}_5 p_{20} = \frac{\ell_{25}}{\ell_{20}} = \frac{990{,}589}{1{,}000{,}000} = 0.990589 $$

We can also calculate the probability of dying between age 20 and 25:

$$ {}_tq_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} \overset{\text{Example}}{\Rightarrow} {}_5q_{20} = \frac{\ell_{20} - \ell_{25}}{\ell_{20}} = \frac{1,000,000 - 990,589}{1,000,000} = 0.009411 $$

$$ {}_tq_x = 1 - {}_tp_x = 1 - \frac{\ell_{x+t}}{\ell_x} \overset{\text{Example}}{\Rightarrow} {}_5q_{20} = 1 - {}_5p_{20} = 1 - 0.990589 = 0.009411 $$

L. Analytic Laws

We can't use a "real" mortality table in our analysis, since the table does not have an easy-to-use formula, as we discussed the first week of class. For example, the table from earlier in the notes is based on real mortality, not a made up model.

I tried to find a formula that worked and it turns out (no surprise) to be a 120-degree polynomial! So, that won't work since we can't really dig in to it.

So, we use some "analytic laws" (i.e., made up mortality rules) to help us with mortality calculations and – later – insurance, annuities, premiums, and more.

Note that all of these laws can be *either* Continuous or Discrete.

These "analytic laws" are used because

(a) They are simpler mathematically than "real" mortality;
(b) They often highlight characteristics of the functions we're looking at;
(c) I can create some pretty cool problems with them.

DeMoivre's Law ("DML")

Developed by Abraham DeMoivre: "*the number of people alive is in constant decline*"

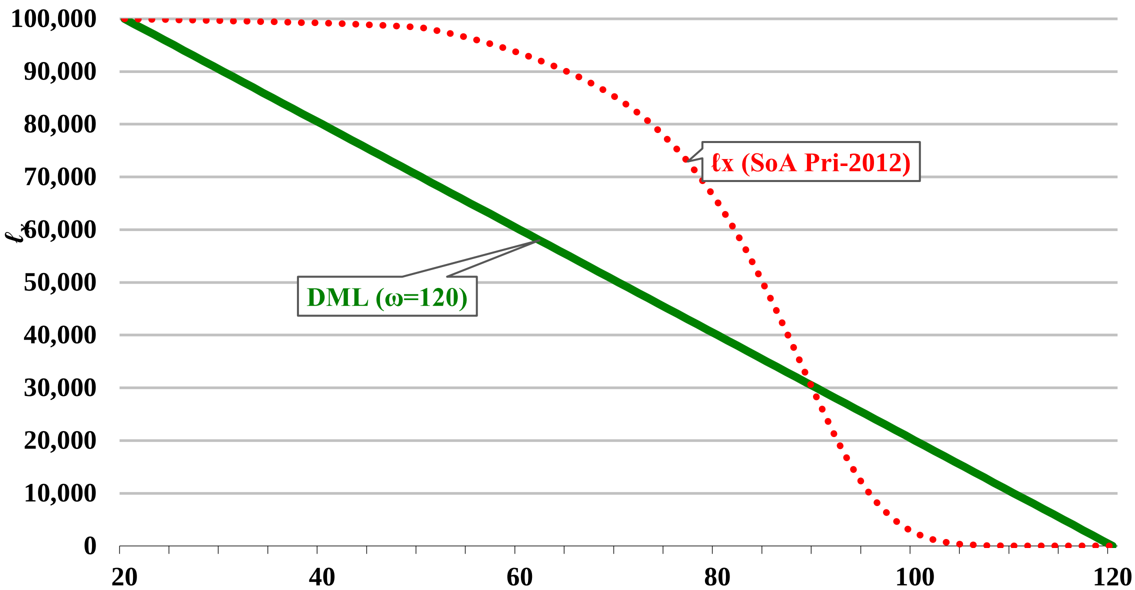
Basic rule: $\ell_x = c(\omega - x)$

Where x is the current age of the person we're looking at, ω is the last possible age someone can be alive; c is a constant

The number alive at any age is declines regularly and by a constant amount (a "straight line" decrease). Deaths are ***uniform*** throughout someone's lifetime.

In the following graph, I show ℓ_x for DeMoivre's Law (abbreviated as ***DML***) from age 20 to 120, with ω=120 and c=10,000 (to make the graph look better). For comparison to the real world, I also show a mortality table the SoA recently produced (the Pri-2012 table[2]).

[2] https://www.soa.org/resources/experience-studies/2019/P2012-private-mortality-tables/
Note that "Pri" is short for "Private retirement plan" and "2012" represents the central year of the final dataset from which mortality tables were developed.



We can develop the following functions from the DeMoivre's Law assumption:

$$S_x = \frac{\ell_x}{\ell_0} = \frac{c(\omega - x)}{c(\omega - 0)} = \frac{\omega - x}{\omega} = 1 - \frac{x}{\omega}, 0 \le x \le \omega$$

$${}_t p_x = \Pr[x \text{ survives to } x+t] = \frac{\text{Number alive at age } x+t}{\text{Number alive at age } x} = \frac{\ell_{x+t}}{\ell_x} = \frac{c[\omega - (x+t)]}{c[\omega - x]}$$

Under DeMoivre's Law, the probability that someone who is currently age *x* lives until at least age *x*+*t* is

$${}_t p_x = \frac{S_{x+t}}{S_x} = \frac{\omega - (x+t)}{\omega - x} = \boldsymbol{\frac{\omega - x - t}{\omega - x}}$$

Note that while this can be either continuous or discrete, we'll focus on the discrete version.

We can also look at the probabilities of death between time x and time $x+t$

$${}_t q_x = 1 - {}_t p_x = 1 - \left(\frac{\omega - x - t}{\omega - x}\right) = \boldsymbol{\frac{t}{\omega - x}}$$

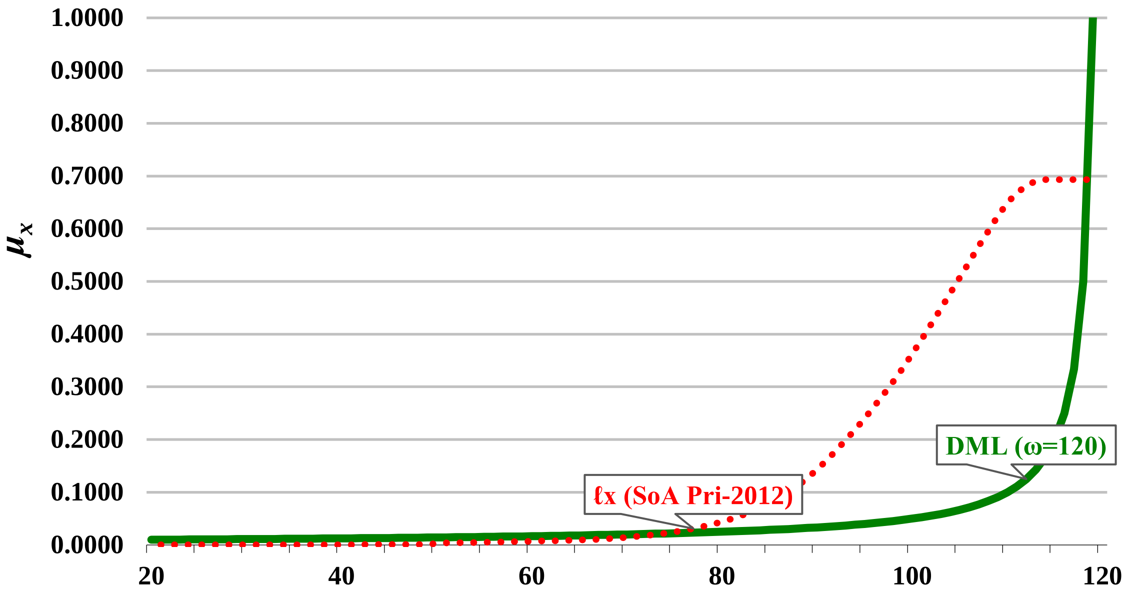
And, we can also (remembering earlier developments) determine the probability that ***x*** dies in some future year ***t***:

$$\underbrace{{}_{t|}q_x}_{\substack{\text{Probability}\\\text{someone age}\\x\text{ survives to}\\\text{age }x+t\text{, then}\\\text{dies within}\\1\text{ year}}} = \underbrace{{}_t p_x}_{\substack{\text{Live from}\\\text{age }x\text{ to}\\\text{age }x+t}} \cdot \underbrace{q_{x+t}}_{\substack{\text{Person now}\\\text{age }x+t\text{ dies}\\\text{within }1\text{ year}}} = \frac{\cancel{\omega - x - t}}{\omega - x} \cdot \frac{1}{\cancel{\omega - (x+t)}} = \boldsymbol{\frac{1}{\omega - x}}$$

We can find $\boldsymbol{\mu_x}$ from the definition of $\boldsymbol{S_x}$ using one of the earlier formulas:

$$\mu_{x+t} = \frac{-S'_{x+t}}{S_{x+t}} = -\frac{-\frac{\partial}{\partial(x+t)}\left[\frac{\omega-(x+t)}{\omega}\right]}{\left[\frac{\omega-(x+t)}{\omega}\right]} = \frac{-\left(-\frac{1}{\omega}\right)}{\frac{\omega-(x+t)}{\omega}} = \frac{\mathbf{1}}{\boldsymbol{\omega}\text{ - }(\boldsymbol{x}+\boldsymbol{t})}$$

$$\mu_x = \frac{1}{\omega - x}$$



Also from the formula for $\boldsymbol{f_X(t)}$

$$f_X(t) = {}_tp_x \cdot \mu_{x+t} = \underbrace{\frac{\cancel{\omega-(x+t)}}{\omega - x}}_{{}_tp_x} \cdot \underbrace{\frac{1}{\cancel{\omega-(x+t)}}}_{\mu_{x+t}} = \frac{\mathbf{1}}{\boldsymbol{\omega}\text{ - }\boldsymbol{x}}$$

Notice that this is <u>the same</u> formula as $\boldsymbol{\mu_x}$ (although it means something different).

Note that the FAM/ALTAM exam <u>do not</u> require knowledge of the name “DeMoivre”. But, we’ll use it in this class from time to time since it makes a few of the future topics clearer. Be sure you are ready to answer questions in which you are told that

$$S_x = 1 - \frac{x}{\omega} = \frac{\omega - x}{\omega}, 0 \le x \le \omega$$

Sx*=1-*x/ω, 0 ≤ x ≤ ω, since this *is* DeMoivre’s Law!

Problem 1-7

You have been told that the insurance company you work for uses

$S_x = 1 - x/\omega \quad 0 \le x \le \omega$

in valuing insurance, annuities, and so on. Your boss, an ASA, told you that you should use a value of $\omega = m$ *(an unknown constant). You do so and compute a value of* $_tp_{20} = 0.375$. *Your boss' boss (an FSA) yells at your boss (which you enjoy) and says that the value of* **m** *was wrong and that you should have used a value of* **1.2*m*** *(a 20% increase).*

You have to redo your calculations, and you determine that, using the new ω *produces a* $_tp_{20} = 0.50$, *a* 33.33% *increase.*

Calculate m

"Modified" (or "Generalized") DeMoivre's Law

Similar to DML, but we'll add a new parameter, α, and define μ_x as the DML version of μ_x raised to the power of α. We can then determine the functions for this Law:

$$\mu_x^{Mod} = \alpha \cdot \mu_x^{DML} \text{ for all } x$$

$$\text{Then, } S_x^{Mod} = e^{-\int_0^x \alpha \cdot \mu_z^{DML} dz} = e^{-\alpha \int_0^x \mu_z^{DML} dz} = \left(S_x^{DML}\right)^{\alpha}$$

$$_t p_x^{Mod} = \frac{S_{x+t}^{Mod}}{S_x^{Mod}} = \frac{\left(S_{x+t}^{DML}\right)^{\alpha}}{\left(S_x^{DML}\right)^{\alpha}} = \left(\frac{S_{x+t}^{DML}}{S_x^{DML}}\right)^{\alpha} = \left({}_t p_x^{DML}\right)^{\alpha} = \left(\frac{\boldsymbol{\omega} - (\boldsymbol{x} + \boldsymbol{t})}{\boldsymbol{\omega} - \boldsymbol{x}}\right)^{\boldsymbol{\alpha}}$$

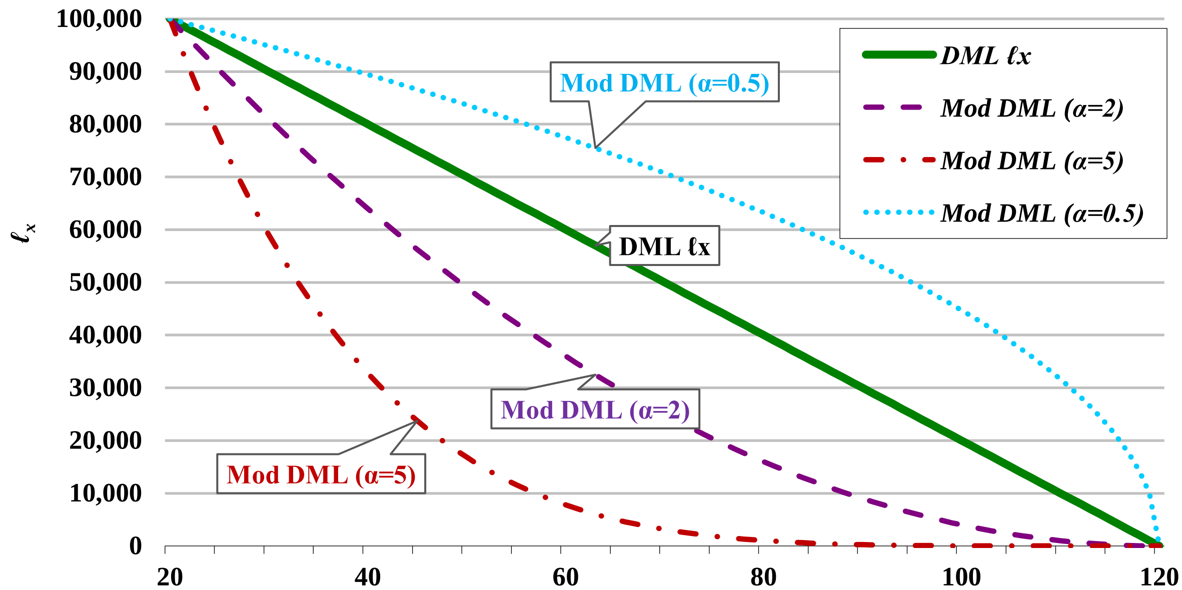
For Modified DML, multiply DML definition of μ_x by α.

$$\mu_x^{Mod} = \alpha \cdot \mu_x^{DML} = \alpha \cdot \frac{1}{\omega - x} = \frac{\boldsymbol{\alpha}}{\boldsymbol{\omega} - \boldsymbol{x}}$$

$$\ell_x = c(\omega - x)^{\alpha} = \ell_0 \left(\frac{\omega - x}{\omega}\right)^{\alpha}$$

$$S_x = \frac{\ell_x}{\ell_0} = \left(\frac{\omega - x}{\omega}\right)^{\alpha} = \left(1 - \frac{x}{\omega}\right)^{\alpha}$$

A graph of DML and Modified DML, with c=10,000 and ω=120:



Constant Force of Mortality ("CFM")

What is we said that the "force of mortality" remained constant for life (unlike the real world, where (see earlier) it should increase). So, let's use this idea and defined the force of mortality at all ages as follows:

μ_x=c (where c is a constant for all ages)

We often write $\mu_x = \mu$ (where the "μ" is a constant) to make this clearer.

We can develop the following functions:

$$S_x = e^{-\int_0^x \mu_z\, dz} = e^{-\int_0^x \mu\, dz} = e^{-\mu z|_0^x} = \boldsymbol{e^{-\mu x}}$$

$$_t p_x = \frac{S_{x+t}}{S_x} = \frac{e^{-\mu(x+t)}}{e^{-\mu x}} = \boldsymbol{e^{-\mu t}}$$

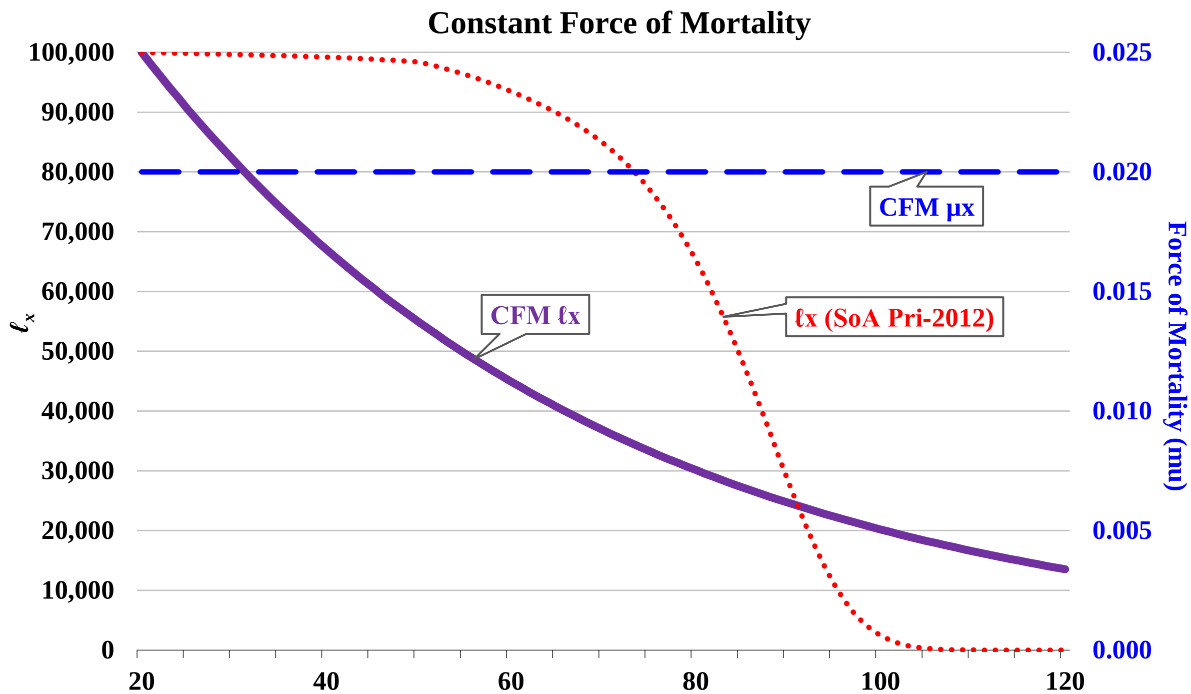
$$_t q_x = 1 - e^{-\mu t}$$

$$_{t|}q_x = {}_t p_x \cdot q_{x+t} = e^{-\mu t}\left(1 - e^{-\mu}\right)$$

$$f_x(t) = {}_t p_x \cdot \mu_{x+t} = \mu \cdot e^{-\mu t}$$

Notice that the "$\boldsymbol{x}$" doesn't matter in $\boldsymbol{{}_t p_x}$; only the "$\boldsymbol{t}$" is in the equation.

With the Constant Force of Mortality analytic law, we only care about t (future years), since x doesn't matter.



Notice that the equations <u>do not</u> refer to x, only to t (the future number of years being considered)

Problem 1-8

Mortality follows a Constant Force of Mortality model and $\boldsymbol{\mu} \leq 1$

We know that ${}_{3|3}q_{66}=0.003$.

Calculate $\boldsymbol{\mu}$

Makeham's Law

DML, Modified DML, and CFM are not good representations of real world mortality. We'll sometimes use them in class because they'll help highlight ideas, but maybe using a model that's more close the way mortality works in the world would be useful.

An actuary named Benjamin Gompertz came up with an approach in 1825 which was refined by William M. Makeham in 1860: https://archive.org/details/jstor-41134925

Makeham's defines the force of mortality as $\boldsymbol{\mu_x=A+Bc^x}$ where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{c}$ are constants.

So, we have

$$\mu_x = A + Bc^x$$

Then, $S_x = e^{-\int_0^x \mu_z dz} = e^{-\int_0^x \left(A+Bc^x\right)dz} = e^{-Ax-\frac{Bc^x-B}{\ln(c)}} = \exp\left(-Ax - \frac{Bc^x - B}{\ln(c)}\right)$

And ${}_t p_x = \frac{S_{x+t}}{S_x} = e^{\left\{-At-\frac{B}{\ln(c)}c^x\left(c^t-1\right)\right\}} = \exp\left(-At - \frac{B}{\ln(c)}c^x\left(c^t-1\right)\right)$

Dickson, Hardy and Watts uses it as their basis for Life Insurance (and later) calculations!

In particular, the text book defines $\boldsymbol{A}$=0.00022; $\boldsymbol{B}$=0.0000027; $\boldsymbol{c}$=1.124

Using these values, sample values of $\boldsymbol{p_x}$ and $\boldsymbol{q_x}$ that the text develops are:

x	p_x	q_x
20	0.99975	0.00025
21	0.99975	0.00025
22	0.99974	0.00026
23	0.99974	0.00026
24	0.99973	0.00027
25	0.99973	0.00027
26	0.99972	0.00028
27	0.99971	0.00029
28	0.99970	0.00030
29	0.99970	0.00030
30	0.99968	0.00032
⋯	⋯	⋯
40	0.99947	0.00053
⋯	⋯	⋯
50	0.99879	0.00121
⋯	⋯	⋯
60	0.99660	0.00340
⋯	⋯	⋯
70	0.98959	0.01041
⋯	⋯	⋯
80	0.96734	0.03266
⋯	⋯	⋯
90	0.89908	0.10092

Gompertz' Law

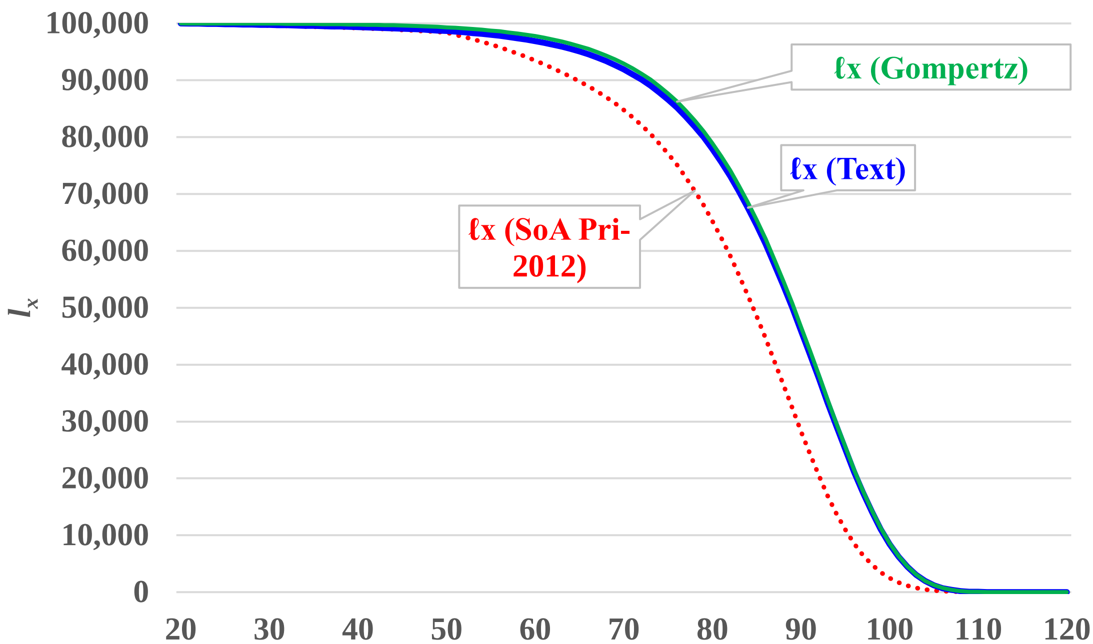
Gompertz' Law is similar to Makeham's Law, but the "A" from Makeham is removed:

$$\mu_x = Bc^x$$

$$\text{Then, } S_x = e^{-\int_0^x \mu_z dz} = e^{-\int_0^x Bc^x dz} = e^{-\frac{Bc^x - B}{\ln(c)}} = \exp\left(-\frac{Bc^x - B}{\ln(c)}\right)$$

$$\text{And } {}_t p_x = \frac{S_{x+t}}{S_x} = e^{\left[-\frac{B}{\ln(c)} c^x \left(c^t - 1\right)\right]} = \exp\left(-\frac{B}{\ln(c)} c^x \left(c^t - 1\right)\right)$$

Using the Textbook's values of *A*, *B*, and *c*, we can compare Makeham's Law and Gompertz' Law to the real world mortality:



Problem 1-9

*Mortality follows Gompertz' Law with **B**=0.00027 and **c**=1.1*

Calculate f_{10}(50)

M. Expected Value of T_x and K_x: "Life Expectancy"

The expected value of any "future lifetime" function (either discrete or continuous) represents how long someone who is age x is expected to live. This is known as **life expectancy**.

Many people (incorrectly) will say that the value of a life annuity (for example) is equal to $\ddot{a}_{\overline{\text{Life Expectancy}}|}$. It isn't. That approach assumes that once we know the *expected value* of someone's life, they'll live *exactly* that long. But, as actuaries, we look at everything year by year.

If someone dies in the next year, we no longer follow their life path.

But, if they do not, then maybe they'll die in the next year (with probability p_x).

But, if they do not, then maybe they'll die in the next year (with probability ${}_2p_x$).

…and we keep going until we reach the end of the table (or values that are close enough to zero so that we can just relaaaaaax).

"Complete" Life Expectancy (basis is T_x)

Remember those "expected value" formulas we looked at in Chapter 0, "Expected Values"? They're back!

$$E[T_x] = \int_{-\infty}^{\infty} t \cdot f(t)dt \xrightarrow{\text{Can't live backward}} \int_{0}^{\infty} t \cdot f(t)dt$$

$$= \int_{0}^{\infty} t \cdot \underbrace{{}_t p_x \cdot \mu_{x+t}}_{f_x(t)} dt \xrightarrow[\text{(Trust me!)}]{\text{By parts}} \int_{\mathbf{0}}^{\infty} \boldsymbol{{}_t p_x \, dt}$$

Note: that for some survival models, the final age (∞) is an age.

For example, for both DML and Modified DML, the end of the table is $\boldsymbol{\omega}$**-*x***.

The symbol we will use for this (remember the discussion from Chapter 0) is $\overset{\circ}{e}_x$ (note that the $\circ$ over the "***e***" means "complete")

"Curtate" Life Expectancy (basis is K_x)

$$E[K_x] = \sum_{k=0}^{\infty} k \cdot \Pr\underbrace{[K_x = k]}_{\substack{\text{Whole years} \\ \text{of future} \\ \text{lifetime } (K_x) \\ \text{equals } k}} = \sum_{k=0}^{\infty} k \cdot {}_{k|}q_x$$

$$= 0 \cdot {}_{0|}q_x + 1 \cdot \underbrace{{}_{1|}q_x}_{\substack{\text{Live } x \to x+1 \\ \text{Die } x+1 \to x+2 \\ \text{See discussion} \\ \text{of } {}_{t|u}q_x \Rightarrow p_x - {}_2p_x}} + 2 \cdot {}_{2|}q_x + 3 \cdot {}_{3|}q_x + \cdots$$

$$= 1(p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots$$

$$= p_x \underbrace{- {}_2p_x + 2({}_2p_x)}_{{}_2p_x} \underbrace{- 2({}_3p_x) + 3({}_3p_x)}_{{}_3p_x} - 3({}_4p_x) + \cdots$$

$$= p_x + {}_2p_x + {}_3p_x + \cdots$$

$$= \sum_{k=1}^{\infty} {}_kp_x = \frac{\ell_{x+1}}{\ell_x} + \frac{\ell_{x+2}}{\ell_x} + \frac{\ell_{x+3}}{\ell_x} + \cdots + \frac{\ell_\infty}{\ell_x}$$

$$e_x = \frac{\ell_{x+1} + \ell_{x+2} + \ell_{x+3} + \cdots + \ell_\infty}{\ell_x} = \frac{\sum_{k=1}^{\infty} \ell_{x+k}}{\ell_x}$$

Symbol is $\boldsymbol{e_x}$ (no circle)

Approximate relationship between curtate and complete life expectancy

$$\mathring{e}_x \approx e_x + \frac{1}{2} \Rightarrow e_x \approx \mathring{e}_x - \frac{1}{2}$$

$\boldsymbol{x}$'s curtate life expectancy is approximately equal to $\boldsymbol{x}$'s complete life expectancy minus ½ (year). In the world outside of the classroom, "life expectancy" is often calculated using this (incorrect, but simple) adjustment.

Also, note that $e_x = E[K_x] \le E[T_x] = \mathring{e}_x$

Example 1-4 – Real World Example of Life Expectancy

Using the mortality table developed for the US in 2021 by the CDC (you can find a copy on Blackboard), calculate the Life Expectancy using the "ℓ_x" approach for

a) A newborn

b) A 25-year old

c) A 65-year old

d) A 95-year old

Also, calculate (d) using the "p_x" approach.

We'll use Excel for this (on the screen in class)

"Temporary" Life Expectancy

A question that no one (who is not an actuary) ever asks:

"Over the next n years, how many years will someone age x live?"

This is sometimes used in accounting calculations, but it does not have the same common use as a full life expectancy,

For example, maybe we have a mortality model that allows us to determine that a 70-year old is expected to live 8.56 years over the next 10 years.

Put another way: if we have 1,000 80 year olds and we track each one for the next 10 years, the *average* number of years each person in the group would have lived is 8.56. Some will live 10 years; others shorter amounts of time. Note that the average has to be less than 10 years.

The symbols we will use are $e_{x:\overline{n}|}$ and $\overset{\circ}{e}_{x:\overline{n}|}$ where the "$\overline{n}|$" is the number of years that we're using for the "temporary" period (should look familiar from AT721).

Integration/Summation Formulas for Temporary Life Expectancy

$$\overset{\circ}{e}_{x:\overline{n}|} = \int_0^n {}_t p_x dt$$

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x = \frac{\ell_{x+1} + \ell_{x+2} + \cdots + \ell_{x+n}}{\ell_x}$$

As we did earlier, we can develop (outside of the scope of this class) an *approximation* of the relationship between the curtate and complete *temporary* life expectancy.

$$\overset{\circ}{e}_{x:\overline{n}|} \approx e_{x:\overline{n}|} + \frac{{}_n q_x}{2} \Rightarrow e_{x:\overline{n}|} \approx \overset{\circ}{e}_{x:\overline{n}|} - \frac{{}_n q_x}{2}$$

Example 1-5 – Real World Example of Temporary Life Expectancy

Using the mortality table developed 2021 for the US (CDC), calculate the 10-year Temporary Life Expectancy for a 25-year old.

Problem 1-10

Given $\mu_x = 0.06; \quad x \geq 0$, *compute* $\overset{\circ}{e}_{35:\overline{10}|} - e_{35:\overline{10}|}$ *exactly.*

Percentile, Median, and Mode of T_x

Percentile

If $0 \leq \boldsymbol{k} \leq 100$, we say the **$k^{th}$ percentile of T_x** is the point π_k for which

$$_{\pi_k}p_x = \Pr\left[T_x \geq \pi_k\right] = 1 - 0.01k$$

For example, the 95th percentile of T_x is $_{\pi}p_x = \Pr[T_x \geq \pi] = 0.05$. Which means there is (only!) a 5.0% chance that the future lifetime of x will be more than π.

And, if we were to use the "SoA" table ("real world"), the 95th percentile of T_{30} = 98 (a 30 year old has a 5% chance of living to age 98).

Median

Remember, the "Median" is the point in a function for which the probability of being less than that point is 50% (1/2).

For our survival models, this means that the **median of T_x** is the value of $\boldsymbol{t}$ (if such a value exists) for which

$$_tq_x = {_tp_x} = \Pr\left[T_x \geq t\right] = 0.5 \Rightarrow F_{x+t} = S_{x+t} = 0.5$$

The median age uses the symbol $\boldsymbol{m(x)}$ **[=*x*+*t*]**.

Put another way, it is the age at which survival from x is a "50/50" proposition.

It should be clear that the median is the 50th percentile of T_x ($\pi_{0.50}$)

Mode

The **mode** of T_x is the point at which the pdf of T_x is maximized (good news: we don't use this a lot…but that doesn't make it any less interesting).

Problem 1-11

Given $S_x = 1 - \dfrac{x^2}{100}, 0 \leq x \leq 10$, ***find the mean and median of*** $\boldsymbol{F_{4+t}}$

(*Continuation of Problem* 1-2)

N. Life Expectancy for Special Laws

DeMoivre's Law

DeMoivre's Law is a special case of the "Uniform Distribution of Deaths".

When we talk about the **"Uniform Distribution of Deaths (UDD)"** we mean that deaths occur *equally throughout the year*.

We covered this in Chapter 0 as "Uniform Distribution". Check that out to help you follow what comes next. Under UDD:

	$x+k$	$x+t$	$x+k+1$
	s = *time from begin year*	*Death at age x+t*	*End of year*

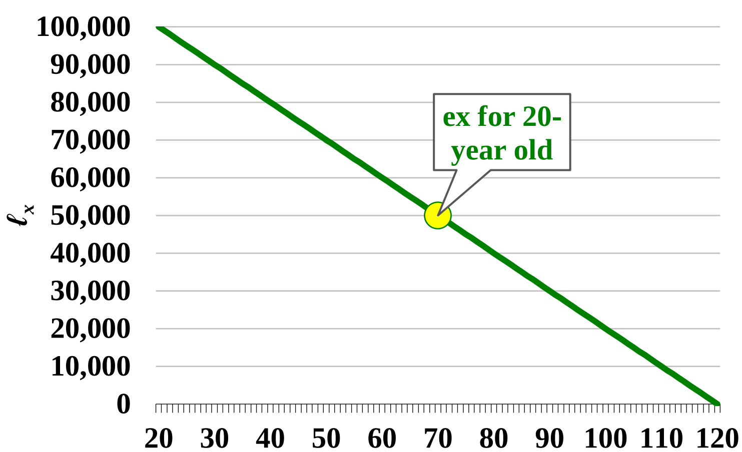
$T_x = K_x + s$, where K and s are independent.

Since we're using "Uniform" distribution, we will further know that s is uniform on $[k, k+1]$

Life Expectancies

We can determine this without integration/summation (although, that works, too)

T_x is uniform on $[0, \omega - x]$, so $\mathring{e}_x$ is the midpoint. Look at an example when $x=20$ and $\omega=120$



Develop using calculus

$$\mathring{e}_x = \int_0^\infty ({}_t p_x)dt = \int_0^{\boldsymbol{\omega\text{-}x}} ({}_t p_x)dt = \int_0^{\boldsymbol{\omega\text{-}x}} \left(\frac{\omega - x - t}{\omega - x}\right)dt$$

$$= \frac{1}{\omega - x}\int_0^{\boldsymbol{\omega\text{-}x}} (\omega - x - t)dt = \frac{1}{\omega - x}\left[(\omega - x)t - \frac{t^2}{2}\right]_0^{\boldsymbol{\omega\text{-}x}}$$

$$= \frac{1}{\omega - x}\left[(\boldsymbol{\omega - x})^2 - \frac{(\boldsymbol{\omega - x})^2}{2}\right] = \frac{1}{\omega - x}\,\frac{(\boldsymbol{\omega - x})^2}{2} = \frac{\boldsymbol{\omega - x}}{\boldsymbol{2}}$$

For the curtate life expectancy, remember that ***s*** is the point between *k* and *k*+1, and that it's uniform (since DML implies UDD). So, we can evaluate its expected value as:

$$E[s] = \int_0^1 s \cdot f(s)ds = \int_0^1 (s)(1)ds == \int_0^1 (s)ds = \frac{s^2}{2}\bigg|_0^1 = \frac{1}{2}$$

And, we can get the curtate life expectancy using this exact value.

$$\mathring{e}_x = E[T_x] = E[K_x + s] = E[K_x] + E[s] = e_x + \frac{1}{2} \Rightarrow e_x = \mathring{e}_x - \frac{1}{2}$$

Unlike the earlier discussion, this is the exact result. It's not an approximation, since we are using UDD as the rule for mortality. You can see why the "real world" will just use this; they (!) often simplify calculations by assuming deaths are uniform.

For the temporary life expectancy, it can be proved (but we won't do it here), that under DML

$$\mathring{e}_{x:\overline{n}|} = n \cdot {}_n p_x + \frac{n}{2}\,{}_n q_x$$

Also, $\mathring{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + \frac{n}{2}\,{}_n q_x$

Modified DeMoivre's Law

We'll follow similar mathematical logic as we did for the calculation of values for DML:

$$\mathring{e}_x = \int_0^\infty ({}_t p_x)dt = \int_0^{\boldsymbol{\omega\text{-}x}} \left(\frac{\omega - x - t}{\omega - x}\right)^\alpha dt$$

$$= \left(\frac{1}{\omega - x}\right)^\alpha \int_0^{\boldsymbol{\omega\text{-}x}} (\omega - x - t)^\alpha dt = \left(\frac{1}{\omega - x}\right)^\alpha \left[\frac{(\omega - x - t)^{\alpha+1}}{\alpha + 1}\right]_0^{\boldsymbol{\omega\text{-}x}}$$

$$= \left(\frac{1}{\omega - x}\right)^\alpha \left[\frac{-(\omega - x - [\boldsymbol{\omega - x}])}{\alpha + 1} - \frac{-(\omega - x)^{\alpha+1}}{\alpha + 1}\right] = \frac{\boldsymbol{\omega - x}}{\boldsymbol{\alpha + 1}}$$

Which is the prior (DML) result, but we're dividing the difference between last age (ω) and current age (x) by α+1 instead of "2". And, yet, if we remember that DML can be looked at as Modified DML with α=1, we have the same result as with DML

Constant Force of Mortality

$$\mathring{e}_x = \int_0^\infty \left({}_t p_x\right) dt = \int_0^\infty \left(e^{-\mu t}\right) dt == \left.\frac{-e^{-\mu t}}{\mu}\right|_0^\infty = \frac{\mathbf{1}}{\boldsymbol{\mu}}$$

For discrete case $\boldsymbol{K_x}$: is a geometric series (remember page 4) with the ratio between any two consecutive values (r) can be seen to be $\boldsymbol{r=e^{-\mu}}$

$$e_x = \sum_{k=1}^{\infty} {}_k p_x = p_x + {}_2p_x + {}_3p_x + {}_4p_x + \cdots$$
$$= \underbrace{e^{-\mu t} + e^{-2\mu t}}_{e^{-2\mu t} = \boldsymbol{e^{-\mu t}} e^{-\mu t}} \underbrace{+}_{e^{-3\mu t} = \boldsymbol{e^{-\mu t}} e^{-2\mu t}} \underbrace{e^{-3\mu t} + e^{-4\mu t}}_{e^{-4\mu t} = \boldsymbol{e^{-\mu t}} e^{-3\mu t}} + \cdots$$

So, using the geometric series approach we've reviewed, we can say:

$$e_x = \sum_{k=1}^{\infty} {}_k p_x = \frac{\text{First Term - Term after last term}}{1 - \text{ratio}} = \frac{p_x - {}_{(\infty+1)}p_x}{1-r}; r = p_x = e^{-\mu}$$
$$= \frac{e^{-\mu}}{1-e^{-\mu}} = \frac{p_x}{1-p_x} = \frac{p_x}{q_x}$$

Makeham's Law and Gompertz' Law

The math for these laws do not lend themselves to a simple mathematical formula. So, if we want Life Expetancy for either, the approach would be to calculate p_x or ℓ_x and use the formulas discussed earlier.

Problem 1-12

For an iPhone, you know that $S_x^{iPhone} = 1 - \frac{x}{\omega}, 0 \le x \le \omega$ and $\overset{o}{e}_0^{\,iPhone} = 2.5$

(Yes, machines can have "life expectancies").

An Android phone is similar to the iPhone in its $\boldsymbol{\omega}$*. The survival function for the Android phone is* $S_x^{Android} = \begin{cases} 1 & ,0 \le x \le 1 \\ \dfrac{\omega - x}{\omega - 1} & ,1 \le x \le \omega \end{cases}.$

Calculate the Complete Life Expectancy of the Android Phone.

Problem 1-13

$$S_x = \frac{\sqrt[3]{k^3 - x}}{k}, 0 \le x \le k^3.$$ You also know that $\overset{o}{e}_{40} = 2\overset{o}{e}_{80}$.

Calculate $\overset{\circ}{e}_{60}$

O. Recursion Formulas

General Form of Recursion formulas (will continually use in this class)

$$u(x) = c(x) + d(x)u(x+1) \text{ [Backward Recursion Formula]}$$

$$u(x+1) = -\frac{c(x)}{d(x)} + \frac{1}{d(x)}u(x) \text{ [Forward Recursion Formula]}$$

$$u(???) = ???? \ [u(0), u(\omega), \ldots]$$

Note that to be complete, a recursion formula <u>must</u> have an initial value (at 0 or ω)!

Recursion formula for Curtate Whole Life Expectancy

$$\begin{aligned} e_x &= \sum_{k=1}^{\infty} {}_k p_x = p_x + \sum_{k=2}^{\infty} {}_k p_x = p_x + p_x \sum_{k=1}^{\infty} {}_k p_{x+1} \\ &= p_x + p_x \cdot e_{x+1} = \mathbf{p_x(1 + e_{x+1})} \end{aligned}$$

Initial Value: $e_\omega = 0$

Recursion formula for Complete Whole Life Expectancy

$$\overset{\circ}{e}_x = \int_0^\infty {}_t p_x dt = \underbrace{\int_0^1 {}_t p_x dt}_{\overset{\circ}{e}_{x:\overline{1}|}} + \int_1^\infty {}_t p_x dt$$

$$\int_1^\infty {}_t p_x dt \xrightarrow{s=t-1, ds=dt, t=s+1} \int_0^{\infty-1} {}_{s+1} p_x ds = \int_0^\infty p_x \cdot {}_s p_{x+1} ds$$

$$= p_x \int_0^\infty {}_s p_{x+1} ds = p_x \cdot \overset{\circ}{e}_{x+1}$$

$$\boldsymbol{\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{1}|} + p_x \cdot \overset{\circ}{e}_{x+1}}$$

Initial Value: $e_\omega = 0$

Other Recursion Formulas

$$e_x = e_{x:\overline{n}|} + {}_n p_x \cdot e_{x+n} \Rightarrow e_{x:\overline{n}|} = e_x - {}_n p_x \cdot e_{x+n}$$

$$\overset{\circ}{e}_x = \overset{\circ}{e}_{x:\overline{n}|} + {}_n p_x \cdot \overset{\circ}{e}_{x+n} \Rightarrow \overset{\circ}{e}_{x:\overline{n}|} = \overset{\circ}{e}_x - {}_n p_x \cdot \overset{\circ}{e}_{x+n}$$

Problem 1-14

Given $e_{88} = \frac{3}{2}; e_{89} = \frac{2}{3}; e_{90} = \frac{1}{5}; e_{91} = 0$. ***Calculate*** ${}_{1|2}\mathbf{q}_{88}$

Problem 1-15

> *James is 29 years old and wants to buy an insurance policy.*
> *James likes to skydive but promises he will stop at age* 30.
> *The insurance company agrees to sell James a policy, but that they will use a value of* $\boldsymbol{p}_{29}$ *that reflects the additional risk at age* 29 *for James.*
> *The insurance company defines* $q'_{29}=q_{29}+0.05$, *where the "'" means "extra risk".*
> *For ages* 30 *and older,* $q_x=q'_x$ *(since James agreed not to skydive on/after age* 30*).*
> *On the standard table,* $e_{30}=20$
> *Calculate* $e_{29}-e'_{29}$

P. Variance of T_x and K_x

Once we know the *mean* of a Random Variable, we get curious about its Variance.

So, let's look at the variance of T_x and K_x.

We'll follow similar approach as we did for expected values, but using <u>second moment</u>.

Remember: $\boldsymbol{Var[X]=E[X^2]-E[X]^2}$

Continuous Case

$$E[T_x^2]=\int_0^\infty t^2\cdot f(t)\,dt=\int_0^\infty t^2\cdot {}_tp_x\cdot\mu_{x+t}\,dt\xrightarrow[\text{(again, trust me!)}]{\text{By parts}}2\int_0^\infty t\cdot {}_tp_x\,dt$$
$$\Rightarrow Var[T_x]=E[T_x^2]-\left(E[T_x]\right)^2$$

Discrete Case

$$Var[K_x]=E[K_x^2]-\left(E[K_x]\right)^2$$

Problem 1-16

> ${}_tq_{90}=0.04t^2;\ 0\le t\le 5$
> ***Calculate*** $\boldsymbol{Var[T_{90}]}$

Variances for Special Laws

DeMoivre

Basic/Easy Calculation:

$$E\left[T_x^2\right]=\int_0^{\infty} t^2\cdot f_x(t)\,dt=\int_0^{\omega-x} t^2\cdot {}_tp_x\cdot\mu_{x+t}\,dt=\int_0^{\omega-x} t^2\cdot\frac{\omega-x-t}{\omega-x}\cdot\frac{1}{\omega-x-t}\,dt=\frac{1}{\omega-x}\left.\frac{t^3}{3}\right|_0^{\omega-x}=\frac{(\omega-x)^2}{3}$$

By Parts:

$$=2\int_0^{\infty} t\cdot {}_tp_x\,dt=2\int_0^{\omega-x} t\,\frac{\omega-x-t}{\omega-x}\,dt=2\frac{1}{\omega-x}\int_0^{\omega-x}(\omega-x)t-t^2\,dt=\frac{2}{\omega-x}\left[\frac{(\omega-x)t^2}{2}-\frac{t^3}{3}\right]_0^{\omega-x}=\frac{(\omega-x)^2}{3}$$

$$Var[T_x]=E\left[T_x^2\right]-E\left[T_x\right]^2=\frac{(\omega-x)^2}{3}-\left(\frac{\omega-x}{2}\right)^2=\boldsymbol{\frac{(\omega-x)^2}{12}}$$

ModDML

$$E\left[T_x^2\right]=2\int_0^{\infty} t\cdot {}_tp_x\,dt=2\int_0^{\omega-x} t\left(\frac{\omega-x-t}{\omega-x}\right)^{\alpha}dt=\frac{2(\omega-x)^2}{(\alpha+1)(\alpha+2)}$$

$$Var\left[T_x\right]=E\left[T_x^2\right]-E\left[T_x\right]^2=\frac{2(\omega-x)^2}{(\alpha+1)(\alpha+2)}-\left(\frac{\omega-x}{\alpha+1}\right)^2=\boldsymbol{\frac{\alpha(\omega-x)^2}{(\alpha+1)^2(\alpha+2)}}$$

CFM

$$E\left[T^2\right]=2\int_0^{\infty} t\cdot {}_tp_x\,dt=2\int_0^{\infty} te^{-\mu t}dt=2\frac{-e^{-\mu t}(\mu t+1)}{\mu^2}=0-\left(-\frac{1}{\mu^2}\right)=\frac{2}{\mu^2}$$

$$Var[T]=E\left[T^2\right]-E\left[T\right]^2=\frac{2}{\mu^2}-\left(\frac{1}{\mu}\right)^2=\boldsymbol{\frac{1}{\mu^2}}$$

Problem 1-17

$S_x=1-x/\omega$, $0\le x\le\omega$. $\mathring{e}_{16}=42$.

Calculate $Var[T_{16}]$

Problem 1-18

A survival Random Variable $\mathbf{T_x}$ has a force of mortality at time t:

$$\mu_{0+t}=\mu_t=\frac{3t^2}{1000-t^3},\quad 0\le t<10$$

Calculate $Var[T_x]$

Q. Fractional Age Assumptions

How do we deal with values between integral ages? Interpolation!

Uniform Distribution of Death Assumption

We will assume that **<u>between integral ages, deaths are uniform</u>** (not necessarily the same as UDD, as we'll see). So, for any age x+t ($t \leq 1$), we can use *linear interpolation*:

$$\ell_{x+t} = (1-t)\cdot \ell_x + t \cdot \ell_{x+1}$$

$$\text{Since } \ell_{x+t} = \ell_x (p_x) = \ell_x (1-q_x)$$

$$\ell_{x+t} = (1-t)\cdot \ell_x + t\ell_x (1-q_x) = \ell_x - \cancel{t\ell_x} + \cancel{t\ell_x} - t\ell_x q_x = \ell_x (1-tq_x)$$

And we can also write a formula to represent survival from the *beginning* of the year to a point (t) between x and x+1:

$${}_t q_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} \overset{UDD}{=} \frac{\ell_x - \ell_x (1-tq_x)}{\ell_x} = 1-(1-tq_x) = \boldsymbol{tq_x}$$

$${}_t p_x = 1 - q_x = 1 - tq_x$$

Survival from mid-year to mid-year

When looking at the probability x+t survives to/dies before a non-integral age (x+s) less than or equal to x+1, we can look at it one of two (mathematically identical) ways (depends on text):

	x	***x*+*s***	***x*+*s*+*t***	***x*+1**
		Fraction of year defined as = *t*		

Death between $x+s$ and $x+s+t$ ("span"=t)

$${}_t q_{x+s} = 1 - {}_t p_{x+s} = 1 - \frac{{}_{t+s} p_x}{{}_s p_x} = 1 - \frac{1 - {}_{t+s} q_x}{1 - {}_s q_x} = 1 - \frac{1-(t+s)q_x}{1-sq_x} = \frac{tq_x}{1-sq_x}; (s+t \leq 1)$$

Survival from mid-year to end of year

Substitute values into the prior question. In this case, the age we're staring with is mid-year (x+t) and the age we're looking to get to is end of year (x+1).

$${}_t q_{x+s} = \frac{tq_x}{1-sq_x} \xrightarrow[\text{and end at age } x+1 \ (1-t \text{ year to get to year-end})]{\text{Apply to case where we start at age } x+s} {}_{1-t} q_{x+t} = \frac{(1-t)q_x}{1-tq_x}$$

Force of Mortality for UDD

$$q_x = {}_t p_x \cdot \mu_{x+t} \rightarrow \mu_{x+t} = \frac{q_x}{{}_t p_x} = \frac{q_x}{1 - {}_t q_x} = \frac{q_x}{1-tq_x}$$

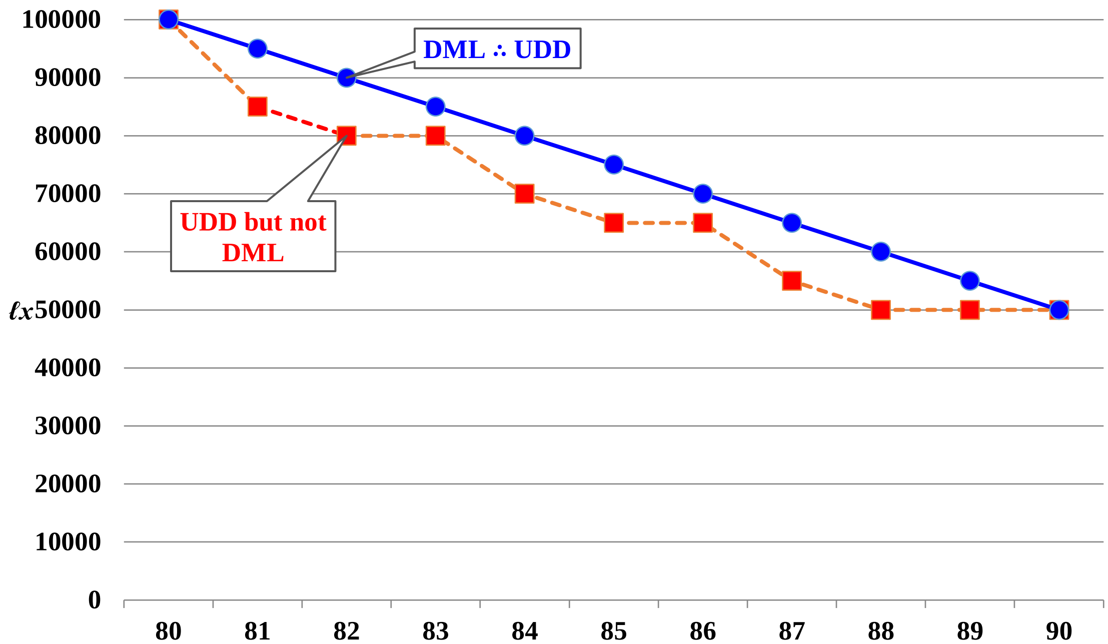
DML and UDD

Important to remember: DeMoivre's Law implies UDD, but UDD does not necessarily imply DML

DeMoivre's Law has UDD each year from ***x*** onward, and each year connects with the prior year. But UDD could be true for an individual year yet not connect with the prior/following year.

Note in this graph that the blue/solid (DML → UDD) line has the same slope each year, so that it implies UDD.

But, the red/dotted line (DML x→ UDD) has a different slope each year so that while it is UDD in any year, it is not the same as DML.



Constant Force of Mortality Assumption

Constant Force assumption has CFM each year from ***x*** onward, and ***μ*** does not change.

We'll interpolate (again) but not linearly…exponentially!

$$\ell_{x+t} = \ell_x^{(1-t)} \cdot \ell_{x+1}^t$$

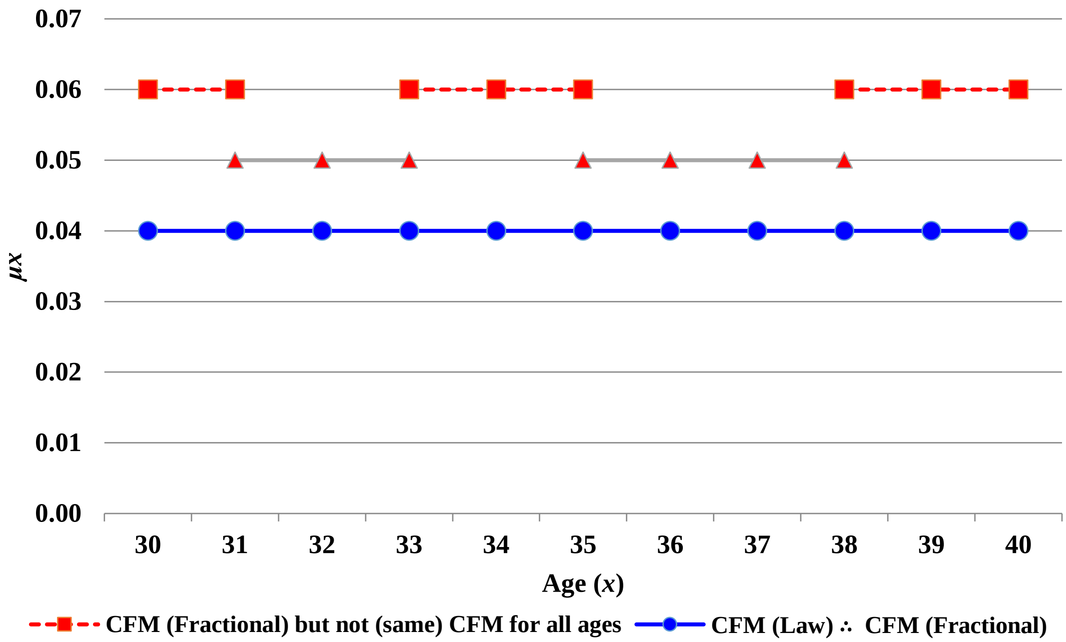
$${}_t p_x \xrightarrow{\text{For CFM, age } (x) \text{ doesn't matter; time } (t) \text{ matters}} = (p_x)^t = \left(e^{-\mu}\right)^t = e^{-\mu t}$$

Beginning of year to time t: ${}_t\boldsymbol{q}_x = \mathbf{1} - \boldsymbol{e}^{-\mu t}$

Middle of year $(x+s)$ to t years from then: ${}_t q_{x+s} = 1 - e^{-\mu t}$

Middle of the year $(x+t)$ to end of year $(1-t)$: ${}_{1-t}\, q_{x+t} = 1 - e^{-\mu(1-t)}$

Important to remember (sound familiar?): Constant Force of Mortality (for interpolation) implies CFM, but CFM does not necessarily imply CFM as mortality law.



The probability depends on **time after *x*, not *x*.**

Force of Mortality

Of course, it's $\boldsymbol{\mu_x=\mu}$

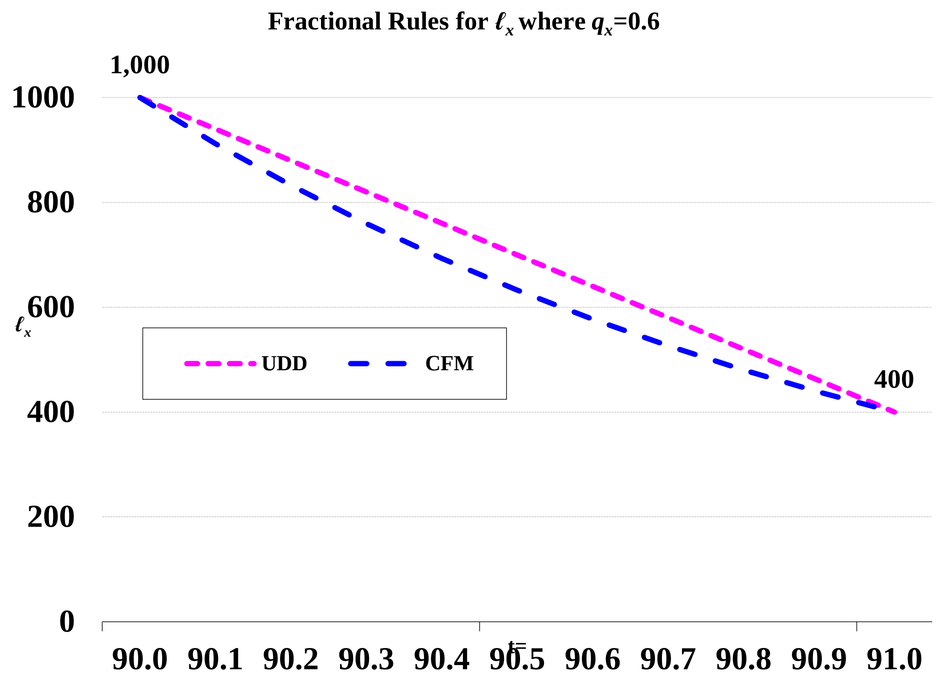
Example of Interpolations

Tabular

In the following table and graphs, q_x=0.60 and we start out with 1000 lives at age 90:

ℓ_{x+t}	ℓ_x^{UDD}	ℓ_x^{CF}
90.00	**1,000.0**	**1,000.0**
90.10	940.0	912.4
90.20	880.0	832.6
90.30	820.0	759.7
90.40	760.0	693.1
90.50	700.0	632.5
90.60	640.0	577.1
90.70	580.0	526.6
90.80	520.0	480.4
90.90	460.0	438.4
91.00	**400.0**	**400.0**

Graphs



Summary of fractional assumptions

Given $s + t \leq 1$

	Underlying Relationship	**Death between *x* and *t* (Death during first fraction of year)** ${}_tq_x$	**Death between *x+s* and *x+s+t*** ${}_tq_{x+s}$	**Death between *x+t* and *x+1* (death during last fraction of year)** ${}_{1-t}q_{x+t}$	**Force of Mortality** μ_{x+t}
Uniform Distribution	$\ell_{x+t} = (1-t)\ell_x + t \cdot \ell_{x+1}$	$t \cdot q_x$	$\dfrac{t \cdot q_x}{1-s \cdot q_x}$	$\dfrac{(1-t)q_x}{1-t \cdot q_x}$	$\dfrac{q_x}{1-t \cdot q_x}$
Constant Force	$\ell_{x+t} = \ell_x^{(1-t)} \cdot \ell_{x+1}^t$	$1-e^{-\mu t}$	$1-e^{-\mu t}$	$1-e^{-\mu(1-t)}$	μ

Note about Memorizing these rules

It's fine to memorize these formulas for ***q***, and there will be occasions throughout Actuarial Math where that might be useful.

But, it's possible to compute whatever we need without memorizing those formulas (we'll have to remember the basic "rules", though)

Problem 1-19

Given q_x=0.06. ***Calculate*** ${}_{\frac{1}{2}}q^{CFM}_{x+\frac{1}{3}}$ and ${}_{\frac{1}{2}}q^{UDD}_{x+\frac{1}{3}}$

Problem 1-20

Given Uniform Distribution of Deaths within year for fractional periods.
q_x=0.06; q_{x+1}=0.09.

What is the probability that someone age x+2/3 dies within next 2/3 of a year?

Problem 1-21

Same as prior problem, but use Constant Force of Mortality assumption?

R. Select & Ultimate Probabilities

Applications

- Life Insurance: better-than-"typical" mortality in first few years of policy issue [Why?]
- Pension: worse that "typical" withdrawal in first few years of employment [Why?]

Notation

Use brackets [] in the subscript: ${}_{10}p_{[x]}$

Remember that q_x is probability that x dies within next year; q_{x+1} is probability that x+1 dies within next year, …

$q_{[x]}$: probability x dies within next year, policy is issued at x;
$q_{[x]+1}$:probability x+1 dies within next year, policy is issued at x (not x+1)
Note that q_{x+1} is not the same as $q_{[x]+1}$

Relationship – Select & Ultimate

$$ {}_k p_{[x]} = \frac{\ell_{[x]+k}}{\ell_{[x]}} \left(\neq \frac{\ell_{[x+k]}}{\ell_{[x]}} \right) $$

$$ {}_k p_{[x]+j} = \frac{\ell_{[x]+k+j}}{\ell_{[x]+j}} $$

Example 1-6

Assume a 3-year "select" period. This means that the rates for the first 3 years (0-1, 1-2, 2-3) are different from the "normal" mortality

- *In the first policy year (0-1), mortality is 85% of the general mortality*
- *In the second policy year (1-2), mortality is 90% of the general mortality*
- *In the third policy year (2-3), mortality is 95% of the general mortality*
- *In the fourth policy year (3-4), mortality is 100% of the general mortality (select period has stopped)*

x	ℓ_x	q_x	$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
21	1,200	0.0417	18				1,200	21
22	1,150	0.0522	19				1,150	22
23	1,090	0.0642	20				1,090	23
24	1,020	0.0784	21	?	?	?	**1,020**	24
25	940	0.0532	22					
26	890							

We need to find the ? values such that this table remains valid.
That is, if we just pick <u>any</u> value for $\ell_{[21]}$, we would not end up with exactly **1,020** *lives at 24, which is the end of the select period for [21].*
We know that

$$q_{[21]} = 0.85q_{21}$$

$$q_{[21]+1} = 0.90q_{22}$$

$$q_{[21]+2} = 0.95q_{23}$$

1) *Determine <u>ultimate</u>, then work backwards to get <u>select</u>*
2) $\ell_{[x]+t+1} = \ell_{[x]+t} \cdot p_{[x]+t}$ *to get select rates*

In this example, we know that $\ell_{24}=\ell_{[21]+3}=1{,}020$. So,

$$\underbrace{\ell_{24}}_{\substack{\text{This is the same as } \ell_{[21]+3},\\ \text{but since we are outside of the}\\ \text{"select" period, we drop}\\ \text{the "[", "]", and "+"}}} = \ell_{[21]+2} \cdot p_{[21]+2} \longrightarrow$$

$$\ell_{[21]+2} = \frac{\ell_{24}}{p_{[21]+2}} = \frac{\ell_{24}}{1-q_{[21]+2}} = \frac{\ell_{24}}{1-0.95q_{23}} = \frac{1020}{1-0.95\cdot 0.0642} = 1086.2$$

$$\ell_{[21]+1} = \frac{\ell_{[21]+2}}{p_{[21]+1}} = \frac{\ell_{[21]+2}}{1-q_{[21]+1}} = \frac{\ell_{[21]+2}}{1-0.90q_{22}} = \frac{1086.2}{1-0.90\cdot 0.0522} = 1139.8$$

$$\ell_{[21]} = \frac{\ell_{[21]+1}}{p_{[21]}} = \frac{\ell_{[21]+1}}{1-q_{[21]}} = \frac{\ell_{[21]+1}}{1-0.85q_{21}} = \frac{1139.8}{1-0.85\cdot 0.0417} = 1181.6$$

So, this means that if we start out with 1,181.6 *lives who are issued insurance at select age* [21], 1,139.8 *will be alive one year later (select age* [21]+1, *real age* 22*), and* 1,086.2 *will be alive two years later (select age* [21]+2, *real age* 23, *and* 1,020 *will be alive at age* 24, *which is the end of the select period.*

Problem 1-22

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}
35	0.05	0.08	0.12
36	0.07	0.10	0.13
37	0.09	0.11	0.15

$\ell_{[35]}=1000$. ***Calculate*** $\boldsymbol{\ell_{[37]}}$

Problem 1-23

2 *year select and ultimate table.*

x	q_x
40	0.015
41	0.021
42	0.030
43	0.045
44	0.066

$q_{[x]}=\dfrac{q_x}{3};q_{[x]+1}=\dfrac{2q_{x+1}}{3};\ell_{[42]}=5000.\text{ Find }\ell_{[40]+1}$

Calculate $\boldsymbol{{}_3p_{[40]}}$**,** $\boldsymbol{{}_3p_{[41]}}$**,** $\boldsymbol{{}_3p_{40}}$**,** $\boldsymbol{{}_3p_{41}}$ ***and compare (consider 3p[40]+1 and 3p41)***

Problem 1-24

10-*year select and ultimate*

$\ell_{[30]+t}=\dfrac{\sqrt{60}}{9}\left(1-\dfrac{t}{100}\right);0\le t<10;\ \ell_{30+t}=\dfrac{\sqrt{70-t}}{10};10\le t\le 70$

Calculate $\mathring{e}_{[30]}$

S. Mortality Improvement

Human life expectancy is not a static item. It improves as improvements in medical care and disease prevention are developed and worsens when significant health challenges exist.

Outside of the classroom, there are many elements that are incorporated into the measurement of the expected life of a person (the "e_x" that the world is interested in hearing about).

The calculations in the prior sections used a mortality that was *static*. It did not change over time so that a person who turns 30 in 2021 should have a longer expected lifetime than a someone who turned 30 in 1981.

Here's proof of the change:

Table 1. Life expectancy at birth, by sex: United States, 2010–2018

Year	Total	Male	Female
2010	78.7	76.2	81.0
2011	78.7	76.3	81.1
2012	78.8	76.4	81.2
2013	78.8	76.4	81.2
2014	78.9	76.5	81.3
2015	78.7	76.3	81.1
2016	78.7	76.2	81.1
2017	78.6	76.1	81.1
2018	78.7	76.2	81.2

SOURCE: NCHS, National Vital Statistics System, Mortality.

I'll refer to this change as a "mortality improvement" but will allow for the possibility that the "improvement" is a negative (that is, a "dis-improvement"). Actuaries kind of expect that people will live longer in future generations and so prefer to talk about "improvement".

So, let's figure out how to adjust our calculations to reflect this real-world concept.

Notation

When we were not thinking about mortality changes, we used the symbol q_x for the probability of death between age x and age x+1.

Because we need to recognize *when* the q is being calculated, we modify the notation to be:

***q*(*x*,*t*) means the probability that someone age *x* dies before age *x*+1 <u>in year *t*</u>.**

We'll also use ***p*(*x*,*t*)** to be the probability that someone age ***x*** is still alive at age ***x***+1 in year ***t***.

For the actual *improvement*, we'll use the Greek letter phi and create the following formula, which is the ***percent change*** in mortality from year ***t*-1** to year ***t***:

$$\varphi(x,t)=1-\frac{q(x,t)}{q(x,t-1)}$$

Mortality "Scales"

There are two broad approaches to project future mortality.

If we create a table of all of the *improvements* in the prior topic, we have a **mortality improvement scale**.

Let ***t***=0 be the current year (think "2023").

Given

- The values of $\varphi(x,t)$ for t=1, 2, 3, …
- The value of $q(x,0)$ (the probability someone age x dies this year)

we can project all future probabilities for age x like this:

$$q(x,t)=q(x,0)\left\{[1-\varphi(x,1)]\left[1-\varphi(x,2)\right]\cdots\left[1-\varphi(x,t)\right]\right\}$$

Single Factor Mortality Improvement Scales

We can look at a mortality improvement that changes **only with age** (time is not important).

This is a "one dimensional" improvement scale.

So, $\varphi(x,1)=\varphi(x,2)=\varphi(x,3)=\ldots$ (Same mortality change every year; time doesn't matter).

Because there is no "time" component, we simplify the improvement notation to just $\boldsymbol{\varphi_x}$, and the formula can be simplified as: $q(x,t)=q(x,0)[1-\varphi_x]^t$

Example 1-7

You are given the following base mortality and improvement scale:

Age	$q(x,0)$	φ_x
80	0.20	1.0%
81	0.30	0.9%
82	0.40	0.8%

Calculate $_3p_{80}$*, both with and without a mortality improvement*

Without mortality improvement, the answer is $_3p_{80}$=(1-0.2)(1-0.3)(1-0.4)=**0.336**

Bringing in the mortality improvement

Someone age 80 *(this year) has a* **0.20** *chance of dying this year*

Someone age 80 *(this year) has a chance of dying between* 81 *and* 82 *of:*

> 0.30 * (1-0.009) [*the "base" probability of death between age* 81 *and* 82*, and* <u>*one year*</u> *of improvement* (*of* 0.9%)=**0.2973** (*people who are age* 80 *this year are slightly (by* 0.9%*)* <u>*less likely*</u> *to die between age* 81 *and* 82 *than someone who is age* 81 *this year.*

Someone age 80 *(this year) has a chance of dying between* 82 *and* 83 *of:*

> $0.40 * (1-0.008)^2$ [*the "base" probability of death between age* 82 *and* 83*, and* <u>*two years*</u> *of improvement* (*of* 0.8%)=**0.393626** (*people who are age* 80 *this year are slightly (by* 0.015936%*)* <u>*less likely*</u> *to die between age* 82 *and* 83 *than someone who is age* 82 *this year.*

Now, using these adjusted mortality values:

$_3p_{80}$=(1-0.2)(1-0.2973)(1-0.393626)=**0.340879**

Compare this to the "non-improved" value: it is <u>*larger*</u> *(by roughly* 1.4%*)*

Two Factor Mortality Improvement Scales

Getting back to our original approach, we will use a $\varphi(x,t)$ that <u>does</u> change depending on the year. This is more complex, but likely more accurate approach.

Example 1-8

You are given the following base mortality and improvement scale:

Age	$q(x,0)$	$\varphi(x,1)$	$\varphi(x,2)$	$\varphi(x,3)$
80	0.20	1.0%	0.85%	0.70%
81	0.30	0.9%	0.75%	0.60%
82	0.40	0.8%	0.65%	0.50%

Calculate ${}_3p_{80}$*, both with a mortality improvement*

Now, we have to recognize the *__when__* *of the improvement.*

Someone age 80 *(this year) has a* **0.20** *chance of dying this year*

Someone age 80 *(this year) has a chance of dying between* 81 *and* 82 *of:*

$0.30 * (1-\varphi(81,1)) = 0.30*(1-0.009)=\mathbf{0.2973}$

Someone age 80 *(this year) has a chance of dying between* 82 *and* 83 *of:*

$0.40 * (1-\varphi(82,1))(1-\varphi(82,2) = 0.40*(1-0.08)(1-0.065) =\mathbf{0.3942}$

${}_3p_{80}=(1-0.2)(1-0.2973)(1-0.3942)=\mathbf{0.340545}$ (*slightly worse than the single factor approach, since the improvement at age* 82 *in year* 2 *is not as large*)

Problem 1-25

An actuary for a pension plan recently performed an experience study. As a result of this experience study, the actuary revises the mortality table for the 2023 actuarial valuation. The new mortality assumption uses a generational mortality table. Selected base mortality rates for 2023 are shown below:

x	q_x
60	0.00508
61	0.00566
62	0.00631
63	0.00704

The actuary believes that every year mortality *__improves__* *by* 0.5% *(i.e., the chance of death* *__decreases__* *by* 0.5% *each year).*

Calculate the value of q_{63} *in* 2026.

Problem 1-26

Same as prior Example, but using Generational Mortality

Following is an excerpt of the mortality projection scale:

	Age			
Year	60	61	62	63
2023	(0.0007)	0.0006	0.0023	0.0043
2024	(0.0036)	(0.0029)	(0.0015)	0.0003
2025	(0.0058)	(0.0056)	(0.0046)	(0.0031)

Calculate the value of q_{63} *the actuary uses in* 2026.

Real World Example

The most recently-issued (by the SoA) "Mortality Improvement" (for pension plans, by the way) is the MP-2021 (male) improvement scale[3]. Here's what it looks like (graphically).

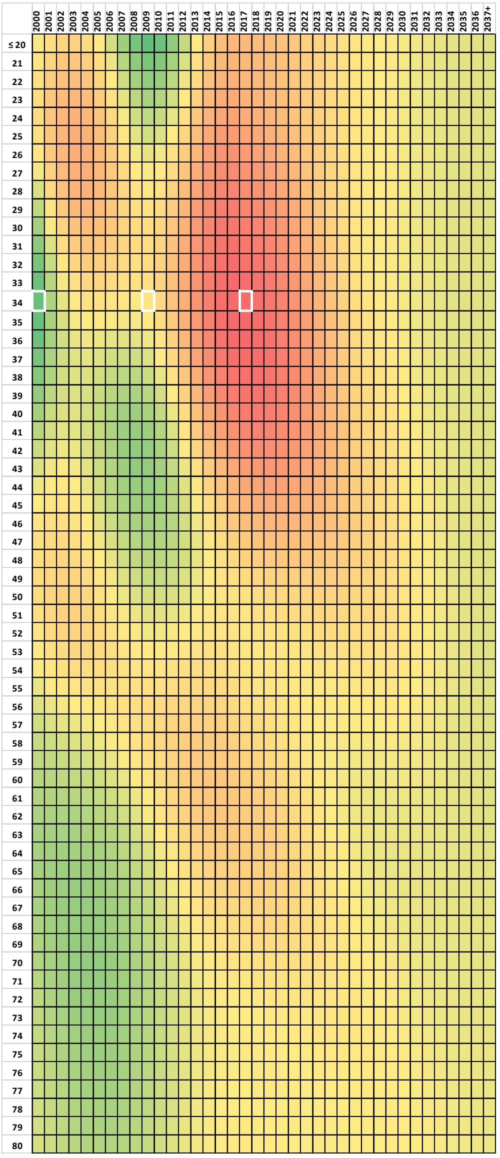
Age down the left side (20 - 80); year of improvement across the top (2000- 2027)

For example, someone who is 65 in 2021 is expected to have the improvement over the "base" Pri-2012 rate by the value shown in the column labelled "2021" (0.0016). So the mortality for a 65-year old in 2022 is (1-0.0016)=0.9984 times the "base" rate for a 65 year old.

	2021	2022	2023	2024
60	0.0030	0.0052	0.0071	0.0087
61	0.0019	0.0043	0.0065	0.0084
62	0.0010	0.0034	0.0058	0.0079
63	0.0002	0.0026	0.0050	0.0073
64	-0.0003	0.0020	0.0044	0.0066
65	-0.0005	**0.0016**	0.0038	0.0060
66	-0.0004	0.0014	0.0035	0.0055
67	0.0000	0.0015	0.0033	0.0052
68	0.0008	0.0019	0.0033	0.0049
69	0.0018	0.0025	0.0035	0.0047
70	0.0029	0.0032	0.0039	0.0048

[3] https://www.soa.org/resources/experience-studies/2021/mortality-improvement-scale-mp-2021/

Graphically, the improvement scale looks like this:



Red: little improvement or possibly “disimprovement” (Ex: age 34, 2017 = -4.73%)

Yellow: some improvement (Ex: age 34, 2009 = +0.47%)

Green: most improvement (Ex: age 34, 2000 = +3.59%)