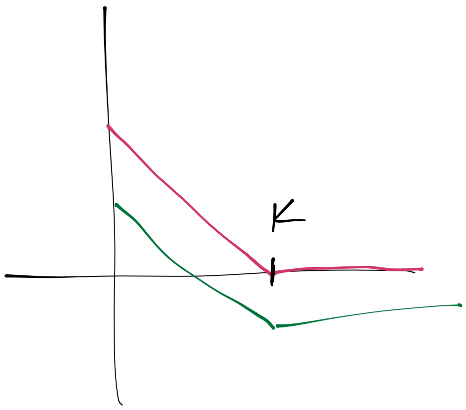
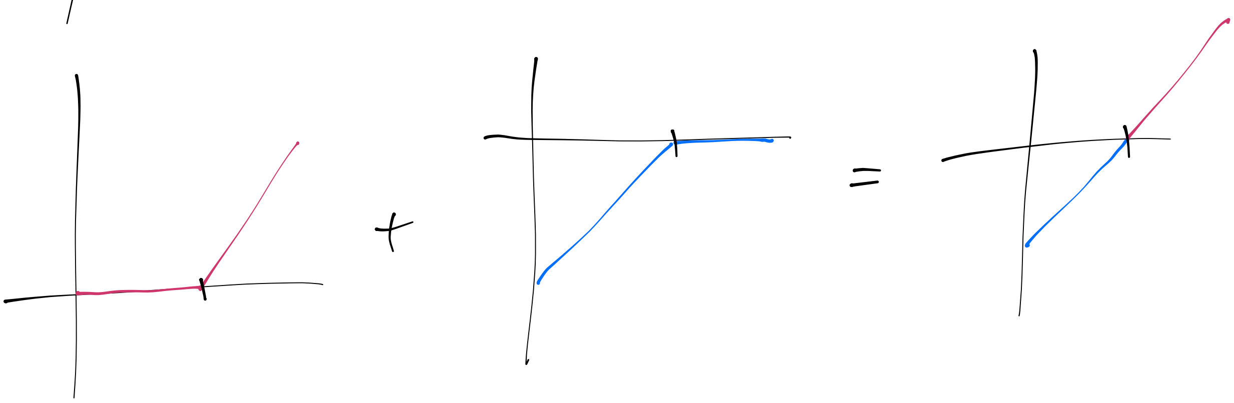
A put option is a contract where one party has the right to sell a specified asset for a specified price at a specified time. The other party has the obligation to buy the asset if the first party chooses to sell.



Synthetic Forward



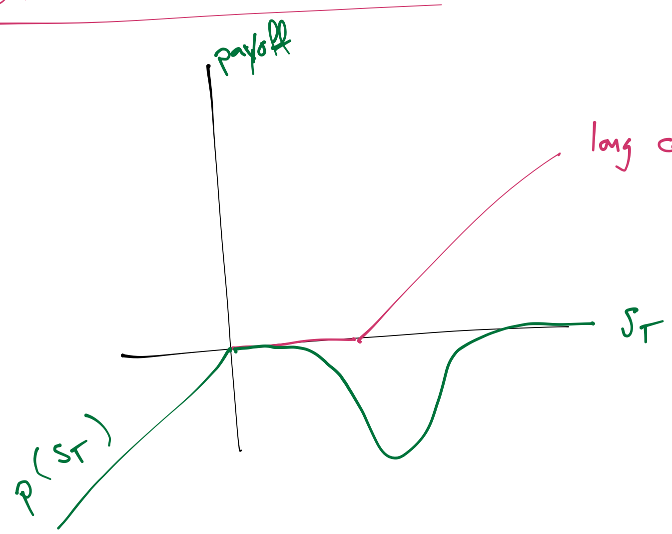
Put-Call Parity

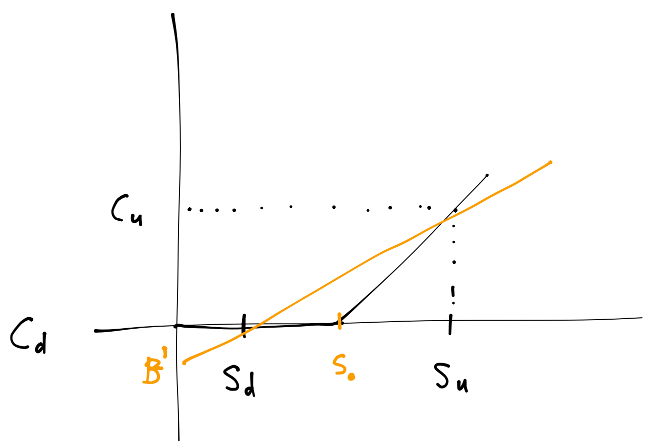
$$C - P = PV(F_{0,T} - K)$$

$$= \left(S_0 e^{(r-\delta)T} - K\right) e^{-rT}$$

$$C(K,T) - P(K,T) = S_0 e^{-\delta T} - K e^{-rT}$$

Binomial Trees





Suppose at $t=T$, the price can only be S_u or S_d

Slope < 1

e.g. $u = 1.2$, $d = 0.75$
if $S = 100$, $S_u = 120$, $S_d = 75$

$$\text{Slope} = \frac{C_u - C_d}{S_u - S_d} = \Delta, \quad \text{some fraction of the asset}$$

$$B' = \frac{C_u(0 - S_d) + C_d(S_u - 0)}{S_u - S_d} = \frac{C_d S_u - C_u S_d}{S_u - S_d}$$

↳ a Zero - Coupon Bond

$$\Delta' = \frac{C_u - C_d}{S(u-d)} \qquad B' = \frac{S(uC_d - dC_u)}{S(u-d)} = \frac{uC_d - dC_u}{u-d}$$

The Law of One Price

- if two portfolios have the same payoff in all possible cases, they must have the same price.

Δ is a function of the asset. assets grow via dividends.

$$\Delta = \Delta' e^{-\delta T} = e^{-\delta T}\left(\frac{C_u - C_d}{S(u-d)}\right)$$

$$B = B' e^{-rT} = e^{-rT}\left(\frac{uC_d - dC_u}{u-d}\right)$$

$$\Delta S_0 + B = S_0 e^{-\delta T}\left(\frac{C_u - C_d}{S(u-d)}\right) + e^{-rT}\left(\frac{uC_d - dC_u}{u-d}\right)$$

$$= \frac{1}{u-d}\left[e^{-\delta T}(C_u - C_d) + e^{-rT}(uC_d - dC_u)\right]$$

$$= \frac{1}{u-d}\left[e^{-\delta T}C_u - e^{-\delta T}C_d + e^{-rT}uC_d - e^{-rT}dC_u\right]$$

$$= \frac{1}{u-d}\left[e^{-\delta T}C_u - e^{-rT}dC_u - e^{-\delta T}C_d + e^{-rT}uC_d\right]$$

$$= \frac{1}{u-d}\left[C_u\left(e^{-\delta T} - d e^{-rT}\right) + C_d\left(u e^{-rT} - e^{-\delta T}\right)\right]$$

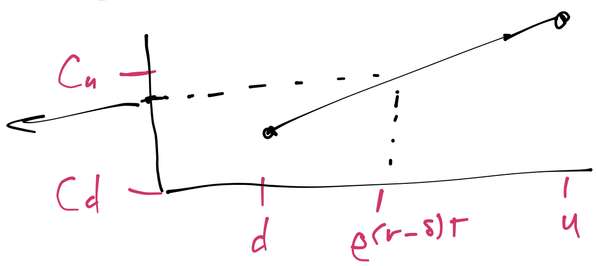
$$= \frac{e^{-rT}}{u-d}\left[C_u\left(e^{(r-\delta)T} - d\right) + C_d\left(u - e^{(r-\delta)T}\right)\right]$$

---

linear interpolation:

$$= e^{-rT}\left[\frac{C_u\left(e^{(r-\delta)T} - d\right) + C_d\left(u - e^{(r-\delta)T}\right)}{u-d}\right]$$

payoff if $S_T = F_{0,T}$



$F_{0,T} = S_0 e^{(r-\delta)T}$

Probability

$$= e^{-rT}\left[\left(\frac{e^{(r-\delta)T} - d}{u-d}\right)C_u + \left(\frac{u - e^{(r-\delta)T}}{u-d}\right)C_d\right]$$

$$p^* = \frac{e^{(r-\delta)T} - d}{u-d} \equiv \left[\text{probability that } S_T = S_u\right]$$

E.g. $S = 100$ $\quad r = 3\%$

$u = 1.2$ $\quad \delta = 1\%$

$d = 0.75$ $\quad T = .5$

$K = 105$

$$\Delta = e^{-\delta T}\left(\frac{C_u - C_d}{S(u-d)}\right) = e^{-.01(.5)}\left(\frac{15-0}{100(1.2-0.75)}\right)$$

$$= e^{-.005}\left(\frac{1}{3}\right)$$

$$= 0.3317$$

$$B = e^{-rT}\left(\frac{uC_d - dC_u}{u-d}\right)$$

$$= e^{-.03(.5)}\left(\frac{1.2(0) - (.75)15}{1.2 - .75}\right)$$

$$= e^{-0.015}(-25) = -24.6278$$

$$S\Delta + B = 100(.3317) - 24.6278$$

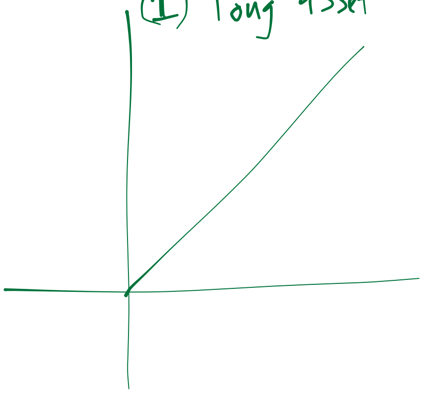
$$= 8.54$$

what are "matching portfolio" payoffs?

$$\underline{S_T = S_d}$$
$$\tfrac{1}{3}(120) - 25$$
$$40 - 25 = 15$$

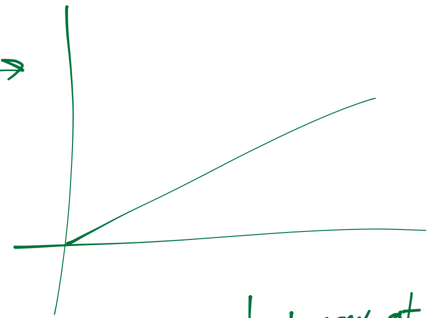
$$\underline{S_T = S_u}$$
$$\tfrac{1}{3}(75) - 25$$
$$25 - 25 = 0$$

(1) long asset

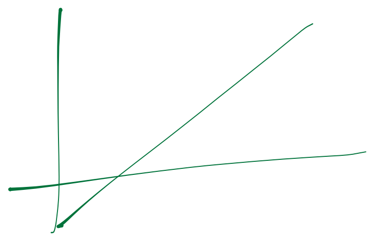


$\rightarrow$

(Δ) long asset



$\downarrow$ borrow at $t=0$



$$p^* = \frac{e^{(r-\delta)T} - d}{u-d} = \frac{e^{(.03-.01)(.5)} - .75}{1.2 - .75}$$

$$= 0.5779$$

$$E[\text{Payoff}] = (.5779)(15) + (1 - 0.5779)(0)$$

$$= 8.6685$$

discount to $t=0$ $\quad (8.6685)\, e^{-.03(.5)} = 8.54$

$$\begin{tikzcd}[row sep=small] & S_u = 120,\ C_u = 15 \\ S_0 = 100,\ C_0 = \arrow[ur, "u"] \arrow[dr, "d"'] & \\ & S_d = 75,\ C_d = 0 \end{tikzcd}$$

$$C_0 = \text{discounted expected value}$$
$$= e^{-rT}\left(p^* C_u + (1-p^*) C_d\right)$$

We're not going to use this type of tree where you select u and d very often.

We're going to prefer a Forward Tree,

where
$$u = e^{(r-\delta)T + \sigma\sqrt{T}}$$
$$d = e^{(r-\delta)T - \sigma\sqrt{T}}$$

σ is the std. dev. of the asset's price changes.

VaR