The Fundamental Valuation of a Convertible Bond: A Precise Framework for University Economics

Built from the Foundational Work of Calamos Investments & Solvency Analytics

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Contents

1 The Core Economic Principle: The "Either/Or" Nature

A convertible bond is a **hybrid security** that fuses the characteristics of corporate debt and equity options. A common but imprecise analogy is to view its value (V_{conv}) as the sum of a bond and a call option. This is incorrect because its two core components are **mutually exclusive**.

1.1 The Mathematical Expression of Mutual Exclusivity

The holder's action at the moment of conversion is mathematically absolute:

- **Gain:** The holder receives a fixed number of shares (C_r) of the underlying stock, valued at the current market price, S_t. The total value gained is $S_t \times C_r$.
- **Forfeit:** The holder surrenders the bond instrument. This means they irrevocably forfeit the right to **all future coupon payments** and the **final principal repayment** at maturity. They give up the entire straight bond component.

The convertible's true value arises from the holder's ability to optimally exercise this exchange. This American-style, path-dependent optimal exercise strategy is the primary source of the convertible's unique valuation dynamics, distinguishing it from a simple portfolio of a bond and a separate call option.

2 The Two Floors: The Dynamic Lower Bounds of Value

A convertible's price is anchored by two no-arbitrage floors. These are not static values but dynamic functions of market conditions.

2.1 The Bond Floor (BF) or Investment Value

This is the convertible's value as a pure debt instrument, representing the "downside protection" component. It is the Net Present Value (NPV) of all remaining cash flows, discounted at the issuer's cost of straight debt.

$$BF = \sum_{i=1}^{n} C_i \cdot e^{-r_b \cdot t_i} + N \cdot e^{-r_b \cdot T}$$

where:

- C_i: The coupon payment at time t_i.
- N: The principal (or par value) repaid at maturity.
- n: The total number of remaining coupon payments until maturity.
- T: The time to maturity of the bond, in years.
- r_b: The discount rate appropriate for the issuer's credit risk, defined as $r_b = r_f + CS$.
- r_f: The risk-free interest rate.
- CS: The issuer's credit spread over the risk-free rate.

Note: For simplicity, this formula assumes a constant credit spread 'CS'. Advanced models would incorporate a term structure of credit spreads that may vary stochastically.

2.2 The Conversion Value (Parity, P_a)

This is the convertible's immediate value if converted into stock. It is defined by the fixed **Conversion Ratio** (C_r).

$$P_a = S_t \times C_r$$

The absolute no-arbitrage lower bound for the convertible's market price (P) is therefore:

$$P \geq \max(BF, P_a)$$

3 The Credit-Equity Correlation Puzzle: Is Downside Protection Real?

The Central Question and the "Sinking Floor"

The Bond Floor is the theoretical foundation of a convertible's downside protection. However, a company's stock price and its creditworthiness are typically negatively correlated. If the stock price falls dramatically (signaling distress), the company's credit spread widens, increasing its cost of debt.

This creates the "sinking floor" phenomenon: the very event the floor is meant to protect against (a falling stock price) simultaneously causes the floor itself to fall. The critical question is: **Under what mathematical conditions does a convertible actually provide meaningful protection, and when is this protection an illusion?**

Modeling the Negative Correlation

To analyze this, we must model the relationship between the stock price (S_t) and the credit spread (CS). A common and intuitive way to model this empirical relationship is with a power law function:

$$CS(S_t) = CS_{\text{initial}} \cdot \left(\frac{S_{\text{initial}}}{S_t}\right)^{\beta} \quad (\text{for } \beta \geq 0)$$

Here:

- CS_{initial} and S_{initial} are the credit spread and stock price at the time of analysis (or issuance).
- β is the **credit elasticity parameter**. It measures how sensitive the credit spread is to changes in the stock price. A higher β signifies a stronger negative correlation. If $\beta = 0$, there is no relationship.

Mathematical Conditions for Meaningful Protection

Let's trace the causal chain. Assume the stock price drops from S_{initial} to a new, lower level S_{new}.

1. **The new Credit Spread (CS_{new}) increases.** Based on our model, the new spread is $CS_{\text{new}} = CS_{\text{initial}} \cdot (S_{\text{initial}}/S_{\text{new}})^{\beta}$. Since $S_{\text{initial}} > S_{\text{new}}$, this term is greater than 1, so $CS_{\text{new}} > CS_{\text{initial}}$.
2. **The discount rate (r_b) for the bond component increases.** The new discount rate is $r_{b,\text{new}} = r_f + CS_{\text{new}}$.
3. **The Bond Floor (BF_{new}) falls.** The Bond Floor is a present value calculation. A higher discount rate reduces the present value of future cash flows. Therefore, the new bond floor, BF_{new}, will be lower than the initial bond floor.

The critical insight is the trade-off between the Bond Floor's initial "height" (determined by CS_{initial}) and the credit elasticity β.

- **Protection is REAL when:** The issuer has high credit quality (CS_{initial} is low) and low credit elasticity (β is low). This is typical for investment-grade issuers.
- **Protection is an ILLUSION when:** The issuer is highly speculative (CS_{initial} is high) with high credit elasticity (β is high). The "floor" sinks in tandem with the stock price, offering little to no actual protection.

4 The Core Valuation Model and Option Premium

The strategic choice to hold or convert is valued using a framework like the **binomial tree model**. The value (V) at any node in the tree is the greater of converting immediately or waiting.

$$V(t, S_t) = \max(\underbrace{S_t \times C_r}_{\text{Convert Now (Parity)}}, \underbrace{H_t}_{\text{Hold / Continuation Value}})$$

where the **Continuation Value** (H_t) is the risk-neutral discounted expected value of the convertible in the next time step.

$$H_t = e^{-r_f \Delta t} \cdot [qV_u + (1-q)V_d]$$

Here, V_u and V_d are the convertible's values in the subsequent "up" and "down" states of the stock price, respectively.

Pedagogical Note 1: Risk-Neutral Valuation (r_f vs. r_b)

A key point is why we discount the Bond Floor at the risky rate r_b but the Continuation Value at the risk-free rate r_f. This is because the binomial model uses ***risk-neutral probabilities****. The probability q is mathematically constructed so that it already accounts for the asset's risk. By embedding risk into the probability, we can simplify the discounting to the risk-free rate. It is a cornerstone of modern option pricing.*

The risk-neutral probability q is derived from the no-arbitrage condition:

$$q = \frac{e^{(r_f - \delta)\Delta t} - d}{u - d}$$

where:

- u: The multiplicative "up" factor for the stock price in one time step ($S_u = S \cdot u$).
- d: The multiplicative "down" factor ($S_d = S \cdot d$). Often defined as $d = 1/u$.
- δ: The continuous dividend yield of the underlying stock. Forfeiting dividends is a cost of holding the stock, so it affects the optimal strategy.

Pedagogical Note 2: Integrating the Credit Puzzle into the Model

The Sinking Floor phenomenon from Section 3 is not just a theoretical concern; it can be directly integrated into the binomial model. A comprehensive model would not use a static Bond Floor. Instead, at each node (t, S_t) in the tree, the model would first calculate the corresponding credit spread $CS(S_t)$ and use it to compute a local, state-dependent Bond Floor. This dynamic floor would then serve as a more accurate lower bound on the Continuation Value, fully capturing the credit-equity correlation.

4.1 The Value of Optionality: The Premium

A convertible almost always trades at a premium to its floors. This is the market's price for the flexibility and potential upside that the conversion option provides.

$$\text{Option Premium} = V_{\text{model}} - \max(BF, P_a)$$

Here, V_{model} is the fair value derived from a valuation model (like the binomial tree), which captures the value of the future strategic choices. This premium is largest when uncertainty is highest (at-the-money) and shrinks as the optimal decision becomes obvious.

5 The Unified View: A Dynamic Framework

The convertible's value is a dynamic interplay between its floors and its option premium. This framework explains its "chameleon-like" behavior by tracking which term dominates the core valuation equation.

- **Out-of-the-Money (Bond-Like):** The Continuation Value (H_t), anchored by the Bond Floor, dominates. Value is primarily sensitive to credit spreads and interest rates.
- **In-the-Money (Equity-Like):** The Conversion Value (P_a) dominates. Value is primarily sensitive to stock price, and its Delta ($\partial V / \partial S$) approaches the Conversion Ratio (C_r).
- **At-the-Money (Hybrid):** $P_a \approx H_t$. The Option Premium is at its maximum, and sensitivity to volatility (Vega) is highest.

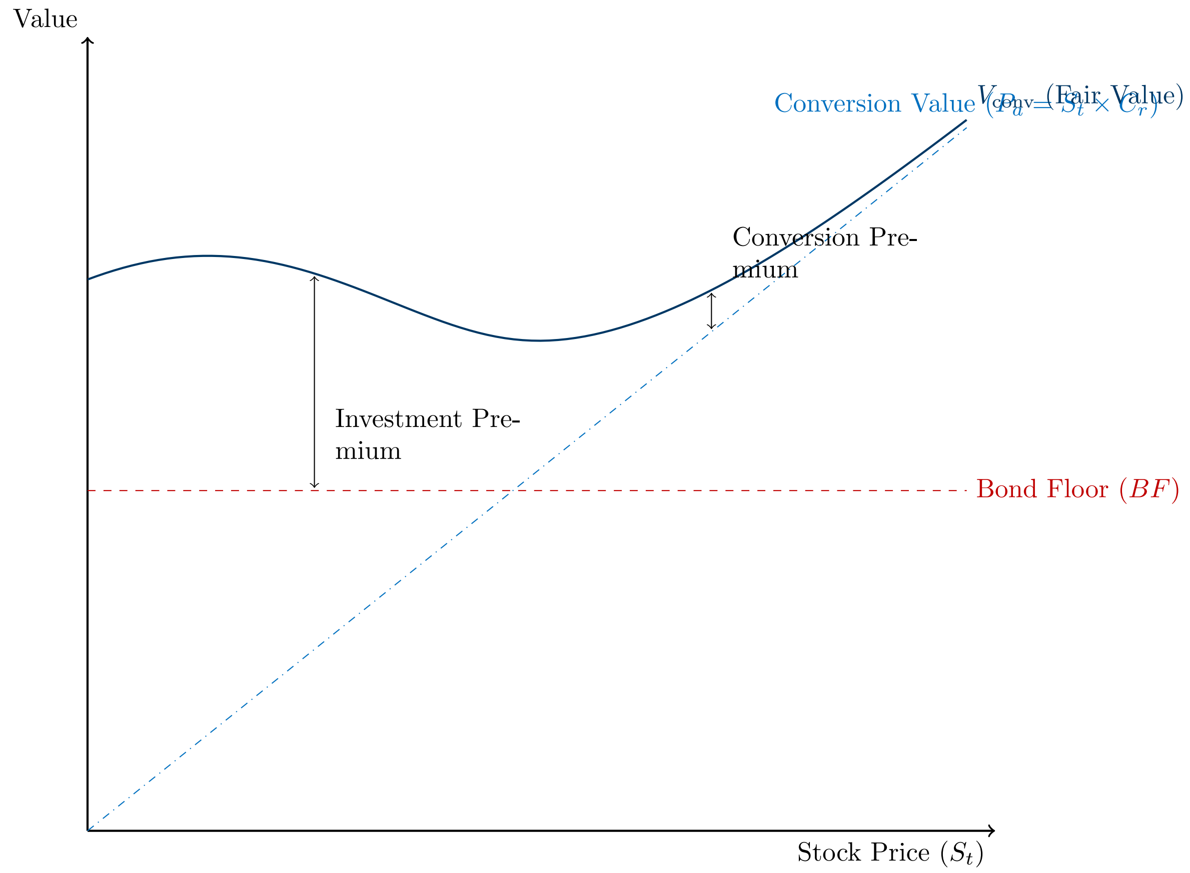

Figure 1: The Convertible Bond Value Profile