A One-Curve Prism for Cross-Currency Pricing: $C(t,T) = F_t(T)/S_t$

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Abstract

We collapse cross-currency pricing to a single primitive: the forward–spot ratio

$$C(t,T) \;:=\; \frac{F_t(T)}{S_t}.$$

Treating C and the domestic discount curve $P_d(t,T)$ as primitives, all forwards, FX swaps, and cross-currency swaps (CCS) become linear functionals of C. Covered interest parity (CIP) is the constraint $C = P_f/P_d$; its violation is a scalar basis factor $B = C/(P_f/P_d)$ that "morphs" otherwise commuting pricing faces.

1 Setup and notation

Fix an observation time t and maturity $T > t$. Let d denote the domestic currency (e.g. USD/SOFR) and f the foreign currency (e.g. EUR/STR). We assume collateralised OIS discounting.

- $P_d(t,T)$ and $P_f(t,T)$: domestic and foreign discount factors.
- S_t: spot FX quoted as units of d per 1 unit of f.
- $F_t(T)$: outright FX forward for delivery at T.

Define the *cross-currency conversion curve*

$$\boxed{C(t,T) \;:=\; \frac{F_t(T)}{S_t}}. \tag{1}$$

For later use, define the *basis factor*

$$\boxed{B(t,T) \;:=\; \frac{C(t,T)}{P_f(t,T)/P_d(t,T)}} \qquad (B \equiv 1 \;\Leftrightarrow\; \text{CIP holds}). \tag{2}$$

Equivalently, the "log-defect" (curvature) is $\phi(t,T) := \log B(t,T)$; in continuous time one may write $B(t,T) = \exp\big(\int_t^T \beta(t,u)\,\mathrm{d}u\big)$ for an instantaneous basis curve β.

2 The prism as commuting faces (with a defect)

Consider the square formed by converting $f \to d$ today, then discounting in d to T, versus discounting in f to T and then converting using the forward. CIP asserts that this square *commutes*:

$$\boxed{S_t\,P_f(t,T) \;=\; P_d(t,T)\,F_t(T)}. \tag{3}$$

Using (1), this is $P_f/P_d = C$. Allowing for violation inserts the defect B:

$$S_t\, P_f(t,T) \;=\; P_d(t,T)\, F_t(T)\, B(t,T)^{-1}\,, \qquad \text{or} \qquad C(t,T) = \frac{P_f(t,T)}{P_d(t,T)}\, B(t,T). \tag{4}$$

Thus, B is the exact scalar by which the "CIP face" fails to commute.

3 Everything from C: forwards, FX swaps, and Arrow prices

Forwards. Trivially, $F_t(T) = S_t\, C(t,T)$; forwards *are* the vertical edge of the prism.

FX-swap points. The forward points are

$$F_t(T) - S_t \;=\; S_t\,[C(t,T) - 1]. \tag{5}$$

Domestic price of a foreign ZCB. The Arrow–Debreu price in d of paying 1_f at T is

$$\boxed{Z^{d\leftarrow f}(t,T) \;=\; P_d(t,T)\, S_t\, C(t,T)}\,, \tag{6}$$

which will span NDFs and the foreign leg of a CCS when valued in d.

4 Interest-rate swaps (IRS)

Let a coupon schedule $t < T_1 < \cdots < T_n = T$, with accrual fractions Δ_i (domestic) and δ_i (foreign). Define annuities

$$A_d(t;\boldsymbol{T}) = \sum_{i=1}^{n} \Delta_i\, P_d(t,T_i), \qquad A_f(t;\boldsymbol{T}) = \sum_{i=1}^{n} \delta_i\, P_f(t,T_i). \tag{7}$$

Then the spot par swap rates are

$$\boxed{K_d^{\text{spot}}(t;\boldsymbol{T}) = \frac{1 - P_d(t,T)}{A_d(t;\boldsymbol{T})}, \qquad K_f^{\text{spot}}(t;\boldsymbol{T}) = \frac{1 - P_f(t,T)}{A_f(t;\boldsymbol{T})}}\,, \tag{8}$$

and the forward-start par rate for $U \in \{T_1, \ldots, T_{n-1}\}$ is

$$\boxed{K_d^{\text{fwd}}(t;U \to T) = \frac{P_d(t,U) - P_d(t,T)}{\sum_{i:\,T_i>U} \Delta_i\, P_d(t,T_i)}}\,, \tag{9}$$

with the analogous expression for f. These are the rate–discount commutation identities on the IRS faces.

5 Valuing foreign cash flows over forwards

Any foreign cash flow stream $\{c_f(T_i)\}$ valued in domestic currency is a simple linear functional of C:

$$\boxed{\text{PV}_d(t) = \sum_i c_f(T_i)\, P_d(t,T_i)\, S_t\, C(t,T_i)}\,. \tag{10}$$

6 Cross-currency swap (CCS) as a linear projection of C

Consider an MTM/re-notionalising float–float CCS that pays domestic float $+x$ and receives foreign float. Let $L_{d,i}$ and $L_{f,i}$ be the forward floating rates over $[T_{i-1}, T_i]$. The domestic and foreign legs (valued in d) are

$$\mathrm{PV}_d^{\mathrm{pay}} = N_d \sum_{i=1}^{n} \Delta_i \, P_d(t,T_i) \, \big(L_{d,i} + x\big), \tag{11}$$

$$\mathrm{PV}_d^{\mathrm{recv}} = \frac{N_d}{S_t} \sum_{i=1}^{n} \delta_i \, P_d(t,T_i) \, F_t(T_i) \, L_{f,i} = N_d \sum_{i=1}^{n} \delta_i \, P_d(t,T_i) \, C(t,T_i) \, L_{f,i}. \tag{12}$$

The par spread x solves $\mathrm{PV}_d^{\mathrm{pay}} = \mathrm{PV}_d^{\mathrm{recv}}$:

$$\boxed{x(t;\boldsymbol{T}) = \frac{\displaystyle\sum_{i=1}^{n} \delta_i \, P_d(t,T_i) \, C(t,T_i) \, L_{f,i} - \sum_{i=1}^{n} \Delta_i \, P_d(t,T_i) \, L_{d,i}}{\displaystyle\sum_{i=1}^{n} \Delta_i \, P_d(t,T_i)}}. \tag{13}$$

Under exact CIP, $C = P_f/P_d$, and (13) reduces to the familiar symmetric form. With basis, insert $C = \frac{P_f}{P_d} B$ to isolate the "basis-only" contribution:

$$\boxed{x_{\mathrm{basis}}(t;\boldsymbol{T}) = \frac{\displaystyle\sum_{i=1}^{n} \delta_i \, P_f(t,T_i) \, [B(t,T_i) - 1] \, L_{f,i}}{A_d(t;\boldsymbol{T})}}. \tag{14}$$

7 Two prisms: today vs. in one year

Let t_0 be today and $t_1 = t_0 + 1\,\mathrm{y}$. The two prisms are fully described by the two curves $C_{t_0}(T)$ and $C_{t_1}(T)$ together with $P_d(t,\cdot)$ (and, if desired, P_f). If markets were shape-preserving and CIP exact, then

$$C(t,\cdot) \equiv \frac{P_f(t,\cdot)}{P_d(t,\cdot)} \quad \text{for } t \in \{t_0, t_1\}, \tag{15}$$

so the prisms are congruent up to the trivial time shift. In reality, the morphism is measured by the change in curvature

$$\Delta\phi(T) \;=\; \phi(t_1,T) - \phi(t_0,T) \;=\; \log \frac{B(t_1,T)}{B(t_0,T)}\,, \tag{16}$$

which pushes mechanically into forwards, FX-swap points, and CCS spreads via (1), (5), and (13).

8 Cheat sheet

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\begin{aligned}
&\text{(CIP with defect)} && S_t\,P_f = P_d\,F_t(T)\,B^{-1} && \iff && C = \frac{P_f}{P_d}\,B.\\
&\text{(Forward)} && F_t(T) = S_t\,C.\\
&\text{(Foreign ZCB in } d\text{)} && Z^{d\leftarrow f} = P_d\,S_t\,C.\\
&\text{(Spot IRS par)} && K_\alpha^{\text{spot}} = \frac{1 - P_\alpha(t,T)}{\sum_i \Delta_i^\alpha P_\alpha(t,T_i)}.\\
&\text{(Forward IRS par)} && K_d^{\text{fwd}} = \frac{P_d(t,U) - P_d(t,T)}{\sum_{i:T_i>U} \Delta_i\,P_d(t,T_i)}.\\
&\text{(CCS par)} && x = \frac{\sum_i \delta_i\,P_d\,C\,L_{f,i} - \sum_i \Delta_i\,P_d\,L_{d,i}}{\sum_i \Delta_i\,P_d}.\\
&\text{(Basis-only CCS)} && x_{\text{basis}} = \frac{\sum_i \delta_i\,P_f\,[B-1]\,L_{f,i}}{\sum_i \Delta_i\,P_d}.
\end{aligned}
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Sign conventions. Market CCS quotes vary by street (which leg carries the spread, pay/receive orientation). Equation (13) is written for "pay d float $+\,x$, receive f float" and values everything in domestic currency. Adjust signs consistently if using the opposite convention.