The Multi-Currency No-Arbitrage Prism: A Complete Framework for Analysis and Trading

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Abstract

This paper introduces the Multi-Currency No-Arbitrage Prism, a graph-theoretic framework $G = (V, E, w)$ where vertices (c_i, t_j) represent currency-time assets and weighted edges represent financial transformations. In a theoretical, frictionless market, the geometry of the prism is constrained by no-arbitrage principles, such as Covered Interest Parity (CIP) and Triangular Arbitrage. These manifest as a zero-sum condition on all closed cycles in log-space ($\sum w = 0$), ensuring the prism's faces are perfectly planar. We leverage this ideal geometry as a benchmark to measure real-world market imperfections. By populating the graph with empirical market data, we quantify arbitrage opportunities as non-zero cycle sums ($\sum w_{\text{emp}} = b \neq 0$). The most significant of these, the cross-currency basis b, is visualized as a geometric "gap" or "warp" in the prism's faces. We formalize these deviations into a system-wide Arbitrage Basis Tensor $\mathbf{B}$ and present a complete, step-by-step methodology for a historical backtest of a strategy designed to trade these geometric inconsistencies. The resulting analysis demonstrates the framework's utility not only for systematic trading but also as a powerful indicator of systemic risk and global funding stress.

Contents

1 Multi-Currency No-Arbitrage Prism: A Complete Mathematical Formalism

This section establishes the foundational mathematics of the prism in an idealized, arbitrage-free world. We define its components as a weighted directed graph and derive its geometric properties as a direct consequence of fundamental no-arbitrage principles.

1.1 Core Components: The Graph G=(V,E,w)

The model is constructed as a weighted directed graph $G = (V, E, w)$ with the following components:

- **Vertices** V: The set of vertices is the Cartesian product $V = C \times T$, where $C = \{c_1, c_2, \ldots, c_n\}$ is a finite set of currencies and $T = \{t_0, t_1, \ldots, t_m\}$ is an ordered set of settlement dates. Each vertex (c_i, t_j) represents the asset "one unit of currency c_i deliverable at time t_j."
- **Edges** E: A directed edge $e : v_1 \to v_2$ represents a financial instrument or transaction that transforms asset v_1 into asset v_2.
- **Weights** w: The function $w : E \to \mathbb{R}$ assigns a real-valued weight to each edge. If a transaction transforms 1 unit of a starting asset into K units of an ending asset, the weight is its **log-multiplier**, $w(e) = \log(K)$.
- **The No-Arbitrage Axiom**: In a frictionless, arbitrage-free market, any sequence of transactions that forms a closed cycle must result in zero net profit. In our log-space framework, this imposes the fundamental constraint that for any closed cycle Cyc in the graph, the sum of the log-weights must be zero:
$$\sum_{e \in \mathrm{Cyc}} w(e) = 0 \tag{1}$$

1.2 Edge Definitions and Financial Meaning

- **Temporal Edges (Interest Rates)**: These vertical edges model the time value of money. An edge from (c_i, t_j) to (c_i, t_k) for $k > j$ represents investing in currency c_i. Its weight is the **log-accrual factor**, $\log(1/D_i(t_j, t_k)) = r_i(t_k - t_j)$, where D is the discount factor and r is the continuous interest rate.
- **FX Spot Edges (Spot Rates)**: These horizontal edges model the exchange of currencies at the same maturity. An edge from (c_i, t_j) to (c_k, t_j) for $i \neq k$ represents a spot FX transaction. Its weight is the **log-spot rate**, $s_{i\to k}(t_j) = \log(S_{i\to k}(t_j))$.
- **FX Forward Edges (Forward Rates)**: These diagonal edges model the exchange of currencies at different future dates. An edge from (c_i, t_j) to (c_k, t_m) for $i \neq k, m > j$ represents an FX forward contract. Its weight is the **log-forward rate**, $f_{i\to k}(t_j, t_m) = \log(F_{i\to k}(t_j, t_m))$.

1.3 Geometric Derivation from No-Arbitrage Principles

1.3.1 Triangular No-Arbitrage (The Prism's Bases)

A risk-free arbitrage attempt at time t_j involving three currencies implies the condition $S_{i\to k}(t_j) \times S_{k\to l}(t_j) \times S_{l\to i}(t_j) = 1$. Taking the natural logarithm yields the zero-sum rule for the triangular cycle, which enforces that the bases of the prism are geometrically "flat":

$$s_{i\to k}(t_j) + s_{k\to l}(t_j) + s_{l\to i}(t_j) = 0 \tag{2}$$

1.3.2 Covered Interest Parity (The Prism's Faces)

A zero-cost, risk-free portfolio involving borrowing, a spot conversion, investing, and a forward hedge leads to the standard CIP equation: $F_{i\to k}(t_j, t_m) = S_{i\to k}(t_j) \cdot D_i(t_j, t_m)/D_k(t_j, t_m)$. Rearranging and taking the logarithm reveals the zero-sum condition for the four-sided faces of the prism:

$$f_{i\to k}(t_j, t_m) - s_{i\to k}(t_j) - \log\left(\frac{D_i(t_j, t_m)}{D_k(t_j, t_m)}\right) = 0 \tag{3}$$

This proves that the quadrilateral faces of the theoretical prism must be **planar**.

1.4 The Geometric Construction and Visualization of the Prism

1.4.1 Defining the Coordinate System

We map the abstract graph to a 3D coordinate space (x, y, z):

- **Time Axis (Z)**: Maturities $t_j \in T$ are mapped to discrete coordinates z_j on the Z-axis.
- **Currency Plane (X-Y)**: At each time slice z_j, the n currencies are vertices of a regular n-gon. A vertex (c_i, t_j) is assigned the unique coordinate:

$$(c_i, t_j) \mapsto (R\cos(2\pi i/n), R\sin(2\pi i/n), z_j)$$

1.4.2 The Theoretical ("Perfect") Prism

The ideal prism consists of parallel currency polygons (bases) connected by vertical edges (interest rates) and planar quadrilateral faces (CIP), forming a perfect geometric structure.

1.4.3 The Empirical ("Warped") Prism and the Basis

In real markets, the CIP cycle sum is generally non-zero:

$$s_{\text{emp}} + (\text{accrual}_{k,\text{emp}}) - f_{\text{emp}} - (\text{accrual}_{i,\text{emp}}) = b_{i\to k}(t_j, t_m) \neq 0$$

This non-zero **basis vector** b means the path of the four-leg portfolio fails to close. Geometrically, the four vertices of the empirical CIP quadrilateral are **not coplanar**, and the face is "warped."

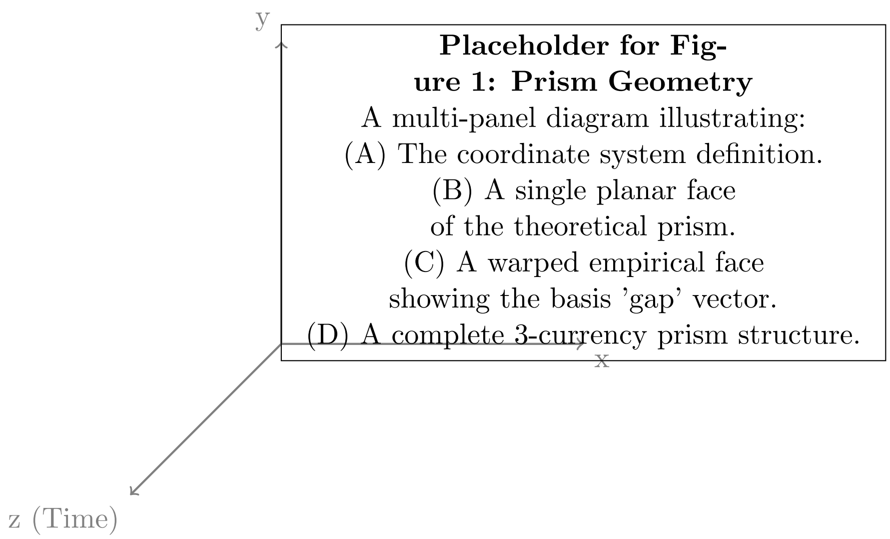

Figure 1: The Geometric Structure of the Theoretical and Empirical Prisms.

1.5 Advanced No-Arbitrage Conditions and Derivations

The framework's internal consistency allows for the derivation of advanced relationships, such as multi-tenor and multi-currency forward consistency, from the fundamental conditions.

1.6 The Complete Model and Extensions

The theoretical no-arbitrage state defines a potential field on the graph, making it ideal for arbitrage detection (e.g., using the Bellman-Ford algorithm) and for extension into a stochastic setting.

2 Quantifying Deviations: The Empirical Prism and the Arbitrage Field

2.1 The Theoretical vs. Empirical Graphs

- $G_{\textbf{theory}}$: A prescriptive model where weights are calculated to enforce $\sum w = 0$.
- $G_{\textbf{emp}}$: A descriptive model where all weights are populated from observable market quotes, representing the real world.

2.2 The Arbitrage Potential of a Cycle

We define the **arbitrage potential** $\mathcal{A}(\mathbf{Cyc})$ of any cycle in G_{emp} as the sum of its observed log-weights. This value represents the pre-cost, risk-free log-return.

$$\mathcal{A}(\text{Cyc}) = \sum_{e \in \text{Cyc}} w_{\text{emp}}(e) \tag{4}$$

2.3 The Fundamental Mispricing: The Cross-Currency Basis

The **log cross-currency basis** b is the arbitrage potential of the CIP cycle, representing the algebraic measure of the "gap vector".

$$b_{i\to k}(t_j, t_m) = f_{\text{emp}_{i\to k}}(t_j, t_m) - [s_{\text{emp}_{i\to k}}(t_j) - r_{i,\text{emp}}(t_m - t_j) + r_{k,\text{emp}}(t_m - t_j)] \tag{5}$$

3 Geometric Visualization of Arbitrage

3.1 The Basis Surface

To understand the dynamics of market dislocations, we visualize the basis as a surface. The **Basis Surface** $B_{i\to k}(T, t)$ is a 3D plot where the x-axis is historical time t, the y-axis is forward tenor T, and the z-axis represents the magnitude of the log-basis $|b_{i\to k}(T)|$ on that day. This surface is a dynamic map of the prism's "warping."

Placeholder for Figure 2: Basis Surface
A 3D surface plot of a major cross-currency basis (e.g., EURUSD) over historical time and forward tenor, showing significant spikes during periods of market crisis.

Figure 2: Visualization of the Cross-Currency Basis as a Dynamic Surface.

3.2 The Arbitrage Tensor: A System-Wide View

We define the **Arbitrage Basis Tensor**, a rank-4 tensor $\mathbf{B}$, to capture all mispricings simultaneously: $\mathbf{B}_{ik}^{jm} = b_{i\to k}(t_j, t_m)$. The Frobenius norm of this tensor, $\|\mathbf{B}(t)\| = \sqrt{\sum_{i,k,j,m}(b_{i\to k}(t_j, t_m))^2}$, aggregates all market dislocations into a single, powerful systemic risk indicator.

4 A Comprehensive Analysis of No-Arbitrage Principles

- **Fundamental Theorem of Asset Pricing (FTAP)**: The prism's $\sum w = 0$ axiom is the discrete-time, deterministic analogue of the FTAP, which links the absence of arbitrage to the existence of a risk-neutral measure $\mathcal{Q}$.
- **The Stochastic Extension**: In a continuous-time setting, the CIP condition becomes a no-arbitrage drift restriction on the spot exchange rate process under $\mathcal{Q}$:

$$\frac{dS_t}{S_t} = (r_d - r_f)dt + \sigma dW_t^{\mathcal{Q}}$$

- **The HJM and Multi-Curve Frameworks**: The prism is compatible with advanced term structure models and correctly captures the post-2008 multi-curve reality, where the basis b measures the spread between OIS discounting rates and term projection rates.

5 Backtesting the Prism: A Step-by-Step Mathematical and Computational Plan

5.1 Phase 1: Data Acquisition and Structuring

- **Universe**: G7 currencies; Tenors $T = \{\text{SPOT, 1W, 1M, 3M, 6M, 1Y, 2Y}\}$.
- **Required Data Series**: For each historical day, collect Spot FX rates, FX Swap Points, and OIS rates for each currency.

5.2 Phase 2: The Daily Calculation Engine

For each day t in the historical dataset, calculate the empirical weights w_{emp} and compute the full Arbitrage Basis Tensor $\mathbf{B}(t)$.

5.3 Phase 3: Strategy Simulation

- **Transaction Costs** *Cost*: Estimate a realistic bid-ask spread for each of the four legs of the CIP portfolio to get a total log-cost.
- **Entry Rule**: A trade is initiated if a measured arbitrage potential exceeds its cost: $|\mathcal{A}(\text{Cyc})| > \text{Cost}$.
- **Portfolio Execution**: Execute a four-legged portfolio to exploit the mispricing (e.g., for $b > 0$, sell the overpriced empirical forward and create a cheaper synthetic long forward).
- **Profit and Loss (P&L)**: The realized log-profit is $|b| - \text{Cost}$, logged at the maturity date of the contract.

5.4 Phase 4: Performance Analysis and Visualization

Analyze strategy performance using standard metrics (Sharpe Ratio, Max Drawdown). Plot the time series of the aggregate arbitrage tensor norm $\|\mathbf{B}(t)\|$ against a market stress index (e.g., VIX) to visually confirm its power as a systemic risk gauge.

Placeholder for Figure 3: Systemic Risk Plot
A chart showing two time series:
1. The Arbitrage Tensor Norm $\|\mathbf{B}(t)\|$.
2. A market stress index (e.g., VIX).
The plot should highlight their strong correlation during financial crises.

Figure 3: Systemic Risk Analysis: Correlation of Arbitrage Tensor Norm with VIX.