Minimal Definitions and Notation

- **Time and dates.** Fix $t < T$. Coupon dates satisfy $t = T_0 < T_1 < \cdots < T_n = T$.
- **Currencies and units.** d = domestic currency, f = foreign currency. FX is quoted as S_t *units of d per 1 unit of f (d/f)*. Symbols 1_d and 1_f denote one unit of d or f.
- **Discount factors.** $P_d(t,T)$ and $P_f(t,T)$ are the time-t discount factors (present values) of 1_d or 1_f payable at T under the chosen discounting convention.
- **Spot and forward FX.** S_t is the spot FX at t (d/f). $F_t(T)$ is the outright forward FX for delivery at T (d/f).
- **Conversion factor.** $C(t,T) := \dfrac{F_t(T)}{S_t}$; it scales a T-dated foreign cash flow into domestic units at time t.
- **Covered interest parity (CIP).** CIP holds iff $C(t,T) = \dfrac{P_f(t,T)}{P_d(t,T)}$.
- **Basis.** $B(t,T) := \dfrac{C(t,T)}{P_f(t,T)/P_d(t,T)}$; $\phi(t,T) := \log B(t,T)$, with small-basis approximation $B - 1 \approx \phi$.
- **Domestic PV of foreign flows.** $\mathrm{PV}_d(t)$ denotes domestic present value at t; translating foreign flows uses S_t and $C(t,T_i)$.
- **Accrual fractions.** Δ_i (domestic) and δ_i (foreign) are day-count accruals over $[T_{i-1}, T_i]$.
- **Forward-floating rates.** $L_{d,i} = \dfrac{P_d(t,T_{i-1}) - P_d(t,T_i)}{\Delta_i \, P_d(t,T_i)}, \quad L_{f,i} = \dfrac{P_f(t,T_{i-1}) - P_f(t,T_i)}{\delta_i \, P_f(t,T_i)}$.
- **Domestic annuity.** $A_d(t;T) := \sum_{i=1}^{n} \Delta_i \, P_d(t,T_i)$.
- **CCS notionals.** For a (re-notionalising) float–float CCS, N_d is the domestic notional and $N_f := N_d/S_t$ is the foreign notional at inception.

The Commuting Square $\Rightarrow$ C as Primitive

We want the value today (in domestic currency, USD) of receiving 1 unit of foreign currency (EUR) at time T. There are two routes:

Example: To get €1 in one year, I can either (Path 1) lock in a forward today at, say, 1.12 \$/€, or (Path 2) buy discounted euros today and hold them.

$$
\begin{tikzcd}
1_f \text{ at } T \arrow[r, "\text{Path 1: Lock forward } F_t(T)"] \arrow[d, "\text{Path 2: Discount in } f\text{:} P_f(t,T)"'] & F_t(T) \cdot 1_d \text{ at } T \arrow[d, "\text{Discount in } d\text{:} P_d(t,T)"] \\
P_f(t,T) \cdot 1_f \text{ at } t \arrow[r, "\text{Convert at spot } S_t"'] & S_t \, P_f(t,T) \cdot 1_d \text{ at } t
\end{tikzcd}
$$

Following the top–then–right path gives $P_d(t,T)F_t(T)$. Following left–then–bottom gives $S_t P_f(t,T)$. Both must be equal, otherwise arbitrage exists:

$$S_t P_f(t,T) = P_d(t,T)F_t(T).$$

This equality matches discount–then–convert with convert–then–discount for the same €1 at T, eliminating a riskless arbitrage loop.

Define the conversion factor

$$\boxed{C(t,T) := \frac{F_t(T)}{S_t}}.$$

This is the time-t scaling that turns a T-dated foreign unit into domestic units via the forward over spot.

Example: if forward is 1.12 and spot is 1.10, then $C = 1.12/1.10 = 1.018$, meaning forwards trade at a 1.8% premium to spot.

When CIP holds:

$$C(t,T) = \frac{P_f(t,T)}{P_d(t,T)}.$$

Here the forward–spot ratio equals the relative discount factors, i.e. forwards are exactly consistent with covered interest parity.

When it fails, define the basis factor

$$B(t,T) := \frac{C(t,T)}{P_f(t,T)/P_d(t,T)}.$$

This measures the wedge between observed forwards and the CIP-implied level; $B = 1$ means no basis.

The power of C. Any foreign cash flow $\{c_f(T_i)\}$ has domestic PV

$$\boxed{\mathrm{PV}_d(t) = S_t \sum_i c_f(T_i)\, P_d(t,T_i)\, C(t,T_i).} \tag{1}$$

Each foreign payment is translated at its delivery tenor via C and discounted on the domestic curve to obtain value today.

Proof: $c_f(T_i) \to F_t(T_i)c_f(T_i)$ USD at $T_i \to P_d(t,T_i)F_t(T_i)c_f(T_i) = P_d(t,T_i)S_t C(t,T_i)c_f(T_i)$ today. □

Notice: we never needed P_f explicitly—just C from market forwards.

Direct Application: CCS in One Line from C

We now price the CCS by translating the foreign floating leg into d using C, then equating legs at par.

Setup. Float–float CCS with domestic notional N_d and foreign $N_f = N_d/S_t$. Pay: domestic float $+x$. Receive: foreign float. What spread x makes this fair?

Domestic leg (pay).

$$\mathrm{PV}_d^{\mathrm{pay}} = N_d \sum_i \Delta_i \, P_d(t,T_i)\big(L_{d,i} + x\big), \qquad L_{d,i} = \frac{P_d(t,T_{i-1}) - P_d(t,T_i)}{\Delta_i \, P_d(t,T_i)}.$$

This is the PV of domestic floating coupons plus spread x on notional N_d, with $L_{d,i}$ the period forward rate.

Foreign leg (receive). Each coupon is $\delta_i L_{f,i} N_f$ in EUR. Using the theorem:

$$\mathrm{PV}_d^{\mathrm{recv}} = S_t \sum_i (N_f \delta_i L_{f,i}) P_d(t,T_i) C(t,T_i) = N_d \sum_i \delta_i \, P_d(t,T_i) \, C(t,T_i) \, L_{f,i}.$$

This values the foreign float in d, using C at each T_i and the re-notionalisation $N_f = N_d/S_t$.

Par condition. At inception:

$$\sum_i \Delta_i P_d L_{d,i} + x \sum_i \Delta_i P_d = \sum_i \delta_i P_d C L_{f,i}.$$

At par, domestic float plus the spread annuity equals the foreign float translated into d.

$$x \sum_i \Delta_i P_d(t,T_i) = \sum_i \delta_i P_d(t,T_i) C(t,T_i) L_{f,i} - \sum_i \Delta_i P_d(t,T_i) L_{d,i}.$$

Rearrange to isolate the spread term on the left before dividing by the domestic annuity.

Spread formula. Let $A_d(t;T) = \sum_{i=1}^n \Delta_i \, P_d(t,T_i)$. Then

$$\boxed{x(t;T) = \frac{\sum_i \delta_i \, P_d(t,T_i) \, C(t,T_i) \, L_{f,i} - \sum_i \Delta_i \, P_d(t,T_i) \, L_{d,i}}{A_d(t;T)}.} \tag{2}$$

The spread is the annuity-weighted mismatch between foreign float (converted via C) and domestic float.

Basis isolation. With $C = (P_f/P_d)B$:

$$\sum_i \delta_i P_d C L_{f,i} = \sum_i \delta_i P_f B L_{f,i} = \sum_i \delta_i P_f L_{f,i} + \sum_i \delta_i P_f (B-1) L_{f,i}.$$

Substituting $C = (P_f/P_d)B$ splits the translated foreign float into the CIP piece and a basis-only adjustment.

Thus

$$\boxed{x_{\mathrm{basis}}(t;T) = \frac{\sum_i \delta_i P_f(t,T_i)(B(t,T_i) - 1) L_{f,i}}{A_d(t;T)}.} \tag{3}$$

This gives the additional spread attributable solely to basis.

For small deviations $B - 1 \approx \phi = \log B$:

$$x_{\mathrm{basis}}(t;T) \approx \frac{\sum_i \delta_i P_f(t,T_i) L_{f,i} \, \phi(t,T_i)}{A_d(t;T)}.$$

For small basis, $\log B \approx B - 1$ makes the correction a simple PV-weighted average of ϕ.

Final Result. We priced the CCS using only P_d and observables $C = F/S$. The foreign curve P_f never entered. The cross-currency world is encoded entirely in forward–spot ratios.