The Multi-Currency No-Arbitrage Prism: A Complete Framework for Analysis and Trading

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Abstract

This paper introduces the Multi-Currency No-Arbitrage Prism, a graph-theoretic framework $G = (V, E, w)$ where vertices (c_i, t_j) represent currency-time assets and weighted edges represent financial transformations. In a theoretical, frictionless market, the geometry of the prism is constrained by no-arbitrage principles, such as Covered Interest Parity (CIP) and Triangular Arbitrage. These manifest as a zero-sum condition on all closed cycles in log-space ($\sum w = 0$), ensuring the prism's faces are perfectly planar. We leverage this ideal geometry as a benchmark to measure real-world market imperfections. By populating the graph with empirical market data, we quantify arbitrage opportunities as non-zero cycle sums ($\sum w_{\text{emp}} = b \neq 0$). The most significant of these, the cross-currency basis b, is visualized as a geometric "gap" or "warp" in the prism's faces. We formalize these deviations into a system-wide data structure, termed the Arbitrage Basis Tensor, and present a complete, step-by-step methodology for a historical backtest of a strategy designed to trade these geometric inconsistencies. The resulting analysis demonstrates the framework's utility not only for systematic trading but also as a powerful indicator of systemic risk and global funding stress.

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Notation and Conventions

This framework relies on a set of consistent definitions and assumptions.

- **Currencies and Time**: c_i denotes currency i. t_j denotes time period j.
- **Time Interval**: $\Delta t = t_m - t_j$ represents the time interval measured in years.
- **Interest Rates**: All interest rates r_i for currency c_i are continuously compounded annualized rates. Market quotes using simple rates must be converted.
- **Exchange Rates**: $S_{i\to k}$ is the spot exchange rate (units of c_k per c_i). $F_{i\to k}(t_j, t_m)$ is the forward rate agreed at t_j for delivery at t_m.
- **Log-Space**: Lowercase letters denote log-values: $s_{i\to k} = \log(S_{i\to k})$ and $f_{i\to k} = \log(F_{i\to k})$. Exchange rates are anti-symmetric in log-space: $s_{k\to i} = -s_{i\to k}$.
- **Market Observables**: S, F, and r are assumed to be market-observable quantities (e.g., mid-rates).

1 The Building Block of Arbitrage: The Two-Currency Rectangle

To construct our framework, we begin with the simplest case that can harbor a time- and currency-spanning arbitrage. This foundational example establishes the direct relationship between a financial arbitrage condition and its geometric representation. We define a minimal graph with two currencies, a base currency c_1 and a quote currency c_2, and two time periods, t_0 (today) and t_1 (a future date).

1.1 Vertices and Edges: The Currency-Time Grid

The vertices of this graph represent distinct assets: (c_1, t_0), (c_1, t_1), (c_2, t_0), and (c_2, t_1). The directed edges connecting them represent financial transactions, where the weight of each edge is its **log-multiplier**. A closed cycle, corresponding to a standard Covered Interest Parity arbitrage trade, involves four transactions with the following economic logic and weights:

1. **Spot Conversion**: An edge from $(c_1, t_0) \to (c_2, t_0)$ represents converting c_1 to c_2. The weight is $+s_{1\to 2}(t_0)$.
2. **Investing the Quote Currency**: An edge from $(c_2, t_0) \to (c_2, t_1)$ represents investing c_2. This generates a positive return, so its weight is $+r_2 \cdot \Delta t$.
3. **Forward Hedge**: An edge from $(c_2, t_1) \to (c_1, t_1)$ hedges the position back to c_1. Its weight is $-f_{1\to 2}(t_0, t_1)$.
4. **Borrowing the Base Currency**: An edge from $(c_1, t_1) \to (c_1, t_0)$ completes the cycle. This represents the funding cost of the initial unit of c_1, an expense with a negative weight of $-r_1 \cdot \Delta t$.

1.2 The No-Arbitrage Condition (Covered Interest Parity)

The foundational principle of our framework is the **No-Arbitrage Axiom**: *In a frictionless market, the sum of log-weights around any closed cycle must be exactly zero.* Applying this to the transaction path defined above yields the correct zero-sum equation:

$$s_{1\to 2}(t_0) + r_2 \cdot \Delta t - f_{1\to 2}(t_0, t_1) - r_1 \cdot \Delta t = 0 \tag{1}$$

Rearranging this equation to solve for the theoretical forward rate gives the standard CIP formula:

$$f_{1\to 2}(t_0, t_1) = s_{1\to 2}(t_0) + (r_2 - r_1) \cdot \Delta t \tag{2}$$

1.3 The Geometry of Arbitrage: The Basis "Gap"

The No-Arbitrage Axiom forces the four vertices of our cycle to be **coplanar**, forming a perfect, flat rectangle. In real markets, however, frictions cause this ideal relationship to break down. We quantify this deviation by defining the **log cross-currency basis**, b, as the non-zero sum of the empirical log-weights:

$$b_{1\to 2}(t_0, t_1) = s_{1\to 2}(t_0) + r_2 \cdot \Delta t - f_{1\to 2}(t_0, t_1) - r_1 \cdot \Delta t \tag{3}$$

This basis is the quantitative measure of the pre-cost arbitrage profit. A positive basis, $b_{1\to 2} > 0$, means that the return from the synthetic forward (created via spot and lending c_2) exceeds the cost (market forward and funding c_1). This implies the synthetic forward is **overpriced** relative to the market forward. Geometrically, the basis represents the "gap" or "warp" in the otherwise planar face of our prism.

2 Expanding the Model: From Rectangle to Prism

2.1 Adding Time: The Multi-Tenor Cylinder

Extending the model to three time periods, $T = \{t_0, t_1, t_2\}$, introduces the concept of consistency across the term structure of forward contracts. In an arbitrage-free market, the pricing of forwards with different maturities must be self-consistent. This relationship can be expressed in log-space through the following forward rate parity condition:

$$f_{1\to 2}(t_0, t_2) = f_{1\to 2}(t_0, t_1) + f_{1\to 2}(t_1, t_2) - s_{1\to 2}(t_1) \tag{4}$$

It is crucial to recognize a subtlety in this equation: the spot rate at the future time, $s_{1\to 2}(t_1)$, is unknown at time t_0. Therefore, this formula does not serve as a direct, executable trading rule at time t_0 in the same way that standard CIP does.

Instead, it functions as a fundamental **consistency condition** that must hold for the forward curve to be considered arbitrage-free. Any deviation from this parity, which can be observed through the market pricing of forward-starting instruments (like a forward-starting swap priced at t_0 for the period t_1 to t_2), represents a distinct and more complex arbitrage opportunity. Such opportunities are related to the term structure of funding costs and implied volatility, revealing deeper market dislocations beyond the scope of simple spot-starting CIP. The framework's ability to identify these cycles demonstrates its completeness.

2.2 Adding Currencies: The Triangular Base

Introducing a third currency, $C = \{c_1, c_2, c_3\}$, gives rise to a new, fundamental cycle that exists at each fixed time slice: **Triangular Arbitrage**. The No-Arbitrage Axiom requires that the product of the spot exchange rates for a three-legged cycle must be one ($S_{1\to2} \cdot S_{2\to3} \cdot S_{3\to1} = 1$). In our log-space framework, this translates to:

$$s_{1\to2}(t_0) + s_{2\to3}(t_0) + s_{3\to1}(t_0) = 0 \tag{5}$$

Geometrically, this condition ensures that the triangular "bases" of our prism are perfectly flat. For $n = 3$ currencies, there is only one independent triangular cycle per time period.

3 Organizing Deviations: The CIP Basis Structure

The fully generalized framework is a weighted directed graph $G = (V, E, w)$. We formally define the **Arbitrage Potential $\mathcal{A}$(Cyc)** of any cycle as the sum of its observed log-weights:

$$\mathcal{A}(\text{Cyc}) = \sum_{e\in\text{Cyc}} w_{\text{emp}}(e) \tag{6}$$

To capture all CIP deviations simultaneously and systematically, we introduce a multi-indexed data structure we term the **CIP Basis Tensor**, **B**. It is important to note that we use the term "tensor" in the context of data science and computation—to describe a multi-indexed array—rather than in its strict physical or mathematical sense, which would imply specific transformation properties under a change of coordinates.

This structure, **B**, is a rank-4 array where each element represents the basis for a specific currency pair and time interval: $\mathbf{B}_{ik}^{jm} = b_{i\to k}(t_j, t_m)$. Other deviations, such as those from triangular or forward-forward parity, form separate mathematical objects and are components of the overall system-wide arbitrage picture.

We can aggregate all market-wide CIP dislocations into a single metric by taking the Frobenius norm of this data structure, populated with values observed at a single point in time, t_{obs}. This observation time is distinct from the structure's time indices (j, m), which refer to the contract tenors. To avoid redundancy from symmetry ($b_{ki} = -b_{ik}$), the norm is calculated over the set of unique cycles:

$$\|\mathbf{B}(t_{obs})\| = \sqrt{\sum_{i<k,j<m} (b_{i\to k}(t_j, t_m)|_{t_{obs}})^2} \tag{7}$$

This norm acts as a powerful systemic risk indicator. A generalized risk measure would also incorporate the norms of other arbitrage types.

4 Computational Implementation and Backtesting Strategy

4.1 The Vectorized Approach: Incidence Matrix `A` and Weight Vector `W`

The framework's elegance translates directly into an efficient computational strategy through a **vectorized approach**. This approach uses a static **Incidence Matrix** `A`, where each row defines a unique arbitrage cycle and each column corresponds to a unique financial instrument (e.g., $s_{i \to k}$ for $i < k$). A second component, the **Weight Vector** `W_emp`, holds the observed log-prices of all instruments at a given time. The **Arbitrage Potential Vector**, containing the basis values for all defined cycle types, is then calculated in a single matrix-vector multiplication:

$$\mathbf{B}_{\text{vector}} = \mathbf{A} \cdot \mathbf{W}_{\text{emp}} \tag{8}$$

4.2 Backtesting Methodology

A full historical backtest follows a clear, systematic process:

1. **Data Acquisition**: Collect historical daily time series for all required market-observable instruments. Specify the source of risk-free rates (e.g., OIS, SOFR).
2. **Daily Calculation**: For each day, compute the Arbitrage Potential Vector.
3. **Strategy Simulation**: Initiate a trade if $|\mathcal{A}(\text{Cyc})| > \text{Cost}$, where the transaction cost for each leg e is rigorously defined, typically as $\text{Cost}_e = \frac{1}{2}\log(\text{ask}_e/\text{bid}_e)$. The realized log-profit, $|b| - \text{Cost}$, is locked in at initiation and realized on the maturity date.
4. **Performance Analysis**: Analyze the P&L stream using standard metrics. Plot the systemic risk indicator $\|\mathbf{B}(t_{obs})\|$ against a known market stress index (e.g., VIX or the SOFR-OIS spread).

A Full Derivation of Fundamental No-Arbitrage Equations

This appendix provides the rigorous derivations for the fundamental no-arbitrage conditions within a 3-currency, 3-period prism. All interest rates r are continuously compounded.

Part 1: Triangular Arbitrage Cycles

For each time slice $t_j \in \{t_0, t_1, t_2\}$, the no-arbitrage condition for the cycle $(c_1, t_j) \to (c_2, t_j) \to (c_3, t_j) \to (c_1, t_j)$ is:

$$s_{1\to 2}(t_j) + s_{2\to 3}(t_j) + s_{3\to 1}(t_j) = 0 \tag{9}$$

Part 2: Covered Interest Parity (CIP) Cycles

For each currency pair (c_i, c_k) and time interval (t_j, t_m), with $\Delta t = t_m - t_j$, the zero-sum condition is:

$$s_{i\to k}(t_j) + r_k \cdot \Delta t - f_{i\to k}(t_j, t_m) - r_i \cdot \Delta t = 0 \tag{10}$$

This general formula is applied to all 9 CIP cycles in the 3x3 prism:

- **Pair** c_1/c_2:

$$s_{1\to 2}(t_0) + r_2 \cdot (t_1 - t_0) - f_{1\to 2}(t_0, t_1) - r_1 \cdot (t_1 - t_0) = 0 \tag{11}$$
$$s_{1\to 2}(t_0) + r_2 \cdot (t_2 - t_0) - f_{1\to 2}(t_0, t_2) - r_1 \cdot (t_2 - t_0) = 0 \tag{12}$$
$$s_{1\to 2}(t_1) + r_2 \cdot (t_2 - t_1) - f_{1\to 2}(t_1, t_2) - r_1 \cdot (t_2 - t_1) = 0 \tag{13}$$

- **Pair** c_1/c_3:

$$s_{1\to 3}(t_0) + r_3 \cdot (t_1 - t_0) - f_{1\to 3}(t_0, t_1) - r_1 \cdot (t_1 - t_0) = 0 \tag{14}$$
$$s_{1\to 3}(t_0) + r_3 \cdot (t_2 - t_0) - f_{1\to 3}(t_0, t_2) - r_1 \cdot (t_2 - t_0) = 0 \tag{15}$$
$$s_{1\to 3}(t_1) + r_3 \cdot (t_2 - t_1) - f_{1\to 3}(t_1, t_2) - r_1 \cdot (t_2 - t_1) = 0 \tag{16}$$

- **Pair** c_2/c_3:

$$s_{2\to 3}(t_0) + r_3 \cdot (t_1 - t_0) - f_{2\to 3}(t_0, t_1) - r_2 \cdot (t_1 - t_0) = 0 \tag{17}$$
$$s_{2\to 3}(t_0) + r_3 \cdot (t_2 - t_0) - f_{2\to 3}(t_0, t_2) - r_2 \cdot (t_2 - t_0) = 0 \tag{18}$$
$$s_{2\to 3}(t_1) + r_3 \cdot (t_2 - t_1) - f_{2\to 3}(t_1, t_2) - r_2 \cdot (t_2 - t_1) = 0 \tag{19}$$

While not all of these cycles are mathematically independent, they represent all the fundamental arbitrage relationships that must be checked.