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The Value of Price Discrimination in Large Social Networks

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Abstract. We study the value of price discrimination in large social networks. Recent trends in industry suggest that, increasingly, firms are using information about social network to offer personalized prices to individuals based upon their positions in the social network. In the presence of positive network externalities, firms aim to increase their profits by offering discounts to influential individuals that can stimulate consumption by other individuals at a higher price. However, the lack of transparency in discriminative pricing may reduce consumer satisfaction and create mistrust. Recent research focuses on the computation of optimal prices in deterministic networks under positive externalities. We want to answer the question of how valuable such discriminative pricing is. We find, surprisingly, that the value of such pricing policies (increase in profits resulting from price discrimination) in very large random networks are often not significant. Particularly, for Erdös–Renyi random networks, we provide the exact rates at which this value decays in the size of the networks for different ranges of network densities. Our results show that there is a nonnegligible value of price discrimination for a small class of moderate-sized Erdös–Renyi random networks. We also present a framework to obtain bounds on the value of price discrimination for random networks with general degree distributions and apply the framework to obtain bounds on the value of price discrimination in power-law networks. Our numerical experiments demonstrate our results and suggest that our results are robust to changes in the model of network externalities.

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Keywords: personalized pricing in networks • value of price discrimination • social networks • centrality

1. Introduction

The use of social network information in operations and marketing has become increasingly prevalent in recent years. The increasing popularity of social platforms, such as Facebook, Twitter, Wechat, and Instagram, has enabled convenient online social interactions among people. Those social interactions make individuals more likely to influence or to be influenced by peers in the social network. In the meantime, the rapid development of information technology has enabled firms to utilize the social network information of their consumers in their marketing and operations decisions. Indeed, it is reported that firms such as Gilt Groupe and PALMS resort have been using such information for targeted advertising, steering, and identifying influencers in the social network for product promotion and viral marketing.[1] Other internet companies also find using social network information to be a potential way to improve their operations and marketing efficiencies.[2] Fainmesser and Galeotti (2020) provide a summary of the industrial practice of influencer marketing, including the use of discriminative pricing strategies.

Among the different uses of social network information in operations and marketing, one important use is in pricing. In many situations, firms want to sell products to consumers on the social platform (either their own products or to serve as a platform to sell products of a third party). In such cases, firms may use the information about a social network to inform their pricing decisions. For example, firms want to know which individual should be targeted for a particular promotion or might be willing to pay a higher price. Apparently, such information is very helpful for firms to wisely use their resources (e.g., campaign budget or limited web page space) to maximal benefit.

However, despite the potential benefit, the use of social network information in pricing has raised serious concerns. For government agencies, such practices inherently cause inequity and may violate certain regulations (see, e.g., Obama White House Report 2015). Consumers are concerned about their privacy as well as the prospect that they might be offered a higher price than their peers.[3] Given such concerns, firms face the dilemma of whether to use social network information in their pricing decisions. To solve the dilemma, it is important for the firm to understand how much potential gain there is from the use of social network information in pricing; if the gain is small, then there may not be much motivation for such practices in the first place.

The paper aims to shed light on this question. Specifically, we investigate how much value can be added if the firm utilizes social network information to inform its pricing decisions (to offer personalized prices based on the positions of the individuals in a social network). In particular, we try to answer the following questions.

1. Given a specific social network, what is the value for the firm to offer personalized pricing to each individual consumer compared with offering a uniform price to all consumers?
2. What is the value of information about the network structure if used for pricing?
3. Which type of network (structure) generate more additional value if the firm uses discriminative pricing?

To answer these questions, we consider a model that is widely used in operations management to study pricing decisions under network effects. In the model, there is a network of consumers. Each consumer may influence other consumers with different weights. The firm sells a divisible product to all consumers, and it has the ability to offer personalized prices to each individual consumer (e.g., by showing personalized coupons or promotions). Each consumer has a utility function that depends on the purchased amount of the product as well as the purchase amount by the consumer's neighbors in the social network.

Then, we consider two pricing strategies for the firm. In the first strategy, the firm knows the social network structure and utilizes this information to offer personalized prices to maximize its profit. In the second strategy, the firm does not apply personalized offerings and simply offers a uniform price to all consumers. Note that not offering personalized prices may be due to ignorance of the network structure or simply because the firm decides not to offer personalized prices. Under our model assumptions, the optimal uniform price does not depend on the network information, so if the firm decides not to offer personalized prices, then it can set prices ignoring the network effects. In either case, the consumers decide the purchase amount based on their utility function (with network influences). Then, we compare the profits obtained by the firm using these two strategies (namely optimal personalized prices versus optimal uniform prices). The difference of the two profits is the value of price discrimination. It can also be viewed as the value of network information for the corresponding network.

The value of price discrimination is always nonnegative. Nevertheless, the magnitude of the value depends largely on the network structure. First, we find that, for certain networks, there is no value of price discrimination. That is, for those networks, even if the firm knows the network structure and can offer discriminative prices, the optimal action is to offer a uniform price for all consumers. We identify a critical property for this class of networks: the number of walks of any given length starting from each node must be equal to the number of walks of the same length ending at that node. This result gives a precise characterization of when the value of price discrimination is zero for a given network.

Having understood the value of price discrimination for a given network, we further consider the value of price discrimination for random networks. In particular, we focus on a class of random networks: the Erdös–Renyi networks. In an Erdös–Renyi network, a consumer has a unit influence on another consumer with a certain probability (we call it the *influence probability*). We consider the asymptotic value of price discrimination for large Erdös–Renyi networks (the relative improvement in profit of discriminative over uniform pricing). We have the following findings for Erdös–Renyi networks:

1. When the influence probability is below a certain threshold, the asymptotic value of price discrimination increases with the influence probability.
2. When the influence probability is above a certain threshold, the asymptotic value of price discrimination decreases with the influence probability.
3. We identify the range of the influence probability in which the asymptotic value of price discrimination reaches the maximum.
4. In all cases, the value of price discrimination is asymptotically vanishing as the network size becomes large.

To explain our results, we note that, when the network is very sparse (the influence probability is below a threshold), the network is very fragmented, and the social influence of any individual is contained within the individual's closest neighbors, leading to small gains from price discrimination (we show that the gain of discriminative pricing depends largely on the number of long paths in the network). When the network is very dense (the influence probability is above a threshold), the network becomes very balanced, also leading to a small value of price discrimination (we show that the existence of cycles reduces the gain of discriminative pricing). When the density of the network is intermediate, there exist long paths but not as many cycles, and the value of price discrimination reaches its maximum. However, even at the maximum, the value of discriminative pricing still asymptotically decays to zero, indicating that, for large Erdös–Renyi networks with high chance, there is little value of applying discriminative prices. Meanwhile, complementary to the asymptotic results, we find that, for Erdös–Renyi networks with a certain range of influence probability, the rate at which the value of price discrimination decays is slow, suggesting that the value of price discrimination may be nonnegligible for small- or moderate-size networks.

We also extend our analysis to random networks with general degree distributions. Specifically, we provide a general framework to obtain upper bounds for the value of price discrimination based on the maximum degrees and the second moments of the degree distributions of the random networks. As an application of this general framework, we investigate the asymptotic value of price discrimination for random networks with power-law degree distributions. We show that, for power-law networks, the value of price discrimination asymptotically vanishes as the size of the network increases. More specifically, we provide the rates of decay for power-law networks with different ranges of parameters. These results suggest that the value of price discrimination may not be significant for a broader class of large random networks.

In addition to our theoretical results, we perform numerical experiments to demonstrate our results. We first use synthetic data and show that the numerical results match our theoretical results. Then we use social networks from real data sets (that are often not Erdös–Renyi networks) to compute the value of price discrimination. We show that, for large networks, the value of price discrimination is often quite small. However, for moderate-size networks, there could be a nonnegligible value of price discrimination. To show the robustness of our results, we also numerically test variants of our model and observe similar results.

The remainder of the paper is organized as follows. In Section 2, we review related literature to this work. In Section 3, we introduce the basic model considered in this paper. In Section 4, we study the conditions under which there is no value of price discrimination. Then in Section 5, we consider a class of random networks—the Erdös–Renyi networks—and study the asymptotic value of price discrimination. We present the main results in Section 5.1 and provide the key proof concepts in Section 5.2. In Section 6, we provide a general framework to study the value of price discrimination in general random networks and apply the framework to power-law networks. We conduct some numerical experiments in Section 7. We conclude the paper in Section 8.

2. Literature Review

The study of network effects has been an active research topic in recent years. Through cascades of influence, network effects can shape critical outcomes in a social network, such as the spread of information, ideas, and disease (see, e.g., Pastor-Satorras and Vespignani 2001, Chamley 2004, Banerjee et al. 2013, Muchnik et al. 2013); choice or adoption of products by consumers (see, e.g., Rogers 1976, Bapna and Umyarov 2015, Wang and Wang 2017); and so on.

Our work is related to the literature considering positive network externalities as introduced in Farrell and Saloner (1985) and Katz and Shapiro (1985). Specifically, our work is related to the work on optimal marketing strategies with positive network externalities. There are often two objectives in such studies, one aims at *influence maximization* across the network, and the other aims at *revenue maximization*. In the following, we discuss the literature on both streams.

Influence maximization problems consider diffusion of influence and aim at identifying the best *seeds* to maximize the spread of social influence. For example, Domingos and Richardson (2001) introduce the concept of influence maximization in virtual marketing by initially targeting the seeds and then triggering the influence cascade among consumers. Such problems have been widely studied subsequently in various settings; see, for example, Richardson and Domingos (2002) and Banerjee et al. (2013). Particularly, Kempe et al. (2003) show that this problem is computationally complex and provide a provable approximation algorithm for the problem. Recently, Akbarpour et al. (2017) consider the value of network information for diffusion problems and suggest that a random seeding strategy with a few more seeds can prompt a larger cascade than optimally targeting, the result of which is similar in spirit to ours but in a different setting.

Our work mainly belongs to the revenue maximization stream. Revenue maximization problems not only consider the diffusion of influence, but also aim at maximizing the revenue. In this stream, there has been much research on efficient marketing strategies using network effects, especially on the influence, exploit, or pricing strategy in the setting of sequential purchase decisions (see, e.g., Hartline et al. 2008, Arthur et al. 2009, Haghpanah et al. 2013, Crapis et al. 2016, Zhou and Chen 2018). Such problems are often computationally complex, and most literature focuses on the computational approaches for such problems. There is also a significant amount of work on pricing in the presence of a network effect and simultaneous purchase decisions (see, e.g., Campbell 2013, Du et al. 2016, Chen et al. 2018, Cohen and Harsha 2020). Again, most such research focuses on the computational aspects of the problem.

Our work is closely related to the static pricing problem of selling a divisible product to a group of consumers with positive network externalities, in particular, the works of Candogan et al. (2012), Bloch and Quérou (2013), and Fainmesser and Galeotti (2015). In particular, we build upon the model in Candogan et al. (2012), which is a deterministic model, and we introduce structural randomness in the model. These works consider a two-stage game in which the monopolist first chooses the prices, and then the consumers, embedded in a social network, make purchasing decisions simultaneously. Candogan et al. (2012) and Bloch and Quérou (2013) assume full network information, and Fainmesser and Galeotti (2015) assume incomplete network information and consider a configuration network model. Candogan et al. (2012) and Fainmesser and Galeotti (2015) consider the amount of consumption of the consumers and assume quadratic utility in the consumption quantity, and Bloch and Quérou (2013) primarily consider a binary purchase decision and assume a linear utility function. In these works, prices can be set differently based upon individuals' positions in the social network. The question of interest is to characterize and identify optimal prices and profits in the networks and the complexity of computing optimal prices. In contrast, our main goal is to quantify the value of price discrimination, that is, to understand how much potential value can be added by discriminative pricing. We notice a few recent papers addressing similar problems but in different settings and approaches. For example, Momot et al. (2020) compare the value of degree information and conspicuity information in the setting of selective selling of exclusive products with negative network externalities. Alizamir et al. (2018) analyze a firm's optimal pricing problem for a new service with a local network effect in a nonstationary dynamic setting and study the impact of the network structure on the firm's revenue and pricing decisions. In contrast, our work studies the value of discriminative pricing in random social networks.

Our work is also connected to some literature in marketing and economics, such as network games (see, e.g., Ballester et al. 2006, Sundararajan 2007, Galeotti et al. 2010) and personalized pricing with heterogeneous customer valuations (see, e.g., Choudhary et al. 2005, Elmachtoub et al. 2021). Moreover, in developing the asymptotic value of price discrimination in random networks, we establish connections with random graph theory, in particular, the literature on counting the number of walks of different lengths in a network (see, e.g., Fiol and Garriga 2009, Levin and Peres 2017), on spectral graph theory (see, e.g., Krivelevich and Sudakov 2003, Chung et al. 2004, Preciado and Rahimian 2017), and on network centrality (see, e.g., Bonacich 1987). We illustrate more detailed connections to these literatures when discussing the proofs of our theoretical results.

3. Model

In this section, we introduce our monopolist-consumer model with network externalities. We characterize the consumption equilibrium, optimal discriminative pricing, and optimal uniform pricing under our model. We also define the value of price discrimination based on the increase in the profit under optimal discriminative pricing over that under optimal uniform pricing, namely the regret and the fractional regret.

3.1. Basic Model

We consider the pricing problem of a monopolist who sells a divisible product to consumers in a social network. Our basic model is motivated by Candogan et al. (2012), Bloch and Quérou (2013), and Fainmesser and Galeotti (2015). In our model, there are n consumers in a directed social network with nonempty adjacency matrix $G \in \mathbb{R}_{\geq 0}^{n\times n}$. The element G_{ij} represents the influence of consumer j on consumer i. The monopolist chooses a price p_i for each consumer i, and each consumer i chooses the consumption level x_i. The preferences of the consumers are represented by the following utility function:

$$u_i(x_i, \mathbf{x}_{-i}, p_i) = ax_i - x_i^2 + 4\rho x_i \sum_{j\neq i} \frac{G_{ij}}{\|G+G^T\|} x_j - p_i x_i,$$

where $\mathbf{x}_{-i}$ is a vector representing the consumption levels of all consumers other than i, $\|G+G^T\|$ is the spectral norm of $(G+G^T)$, $a > 0$ is a constant representing the strength of stand-alone utility, and $\rho \in (0,1)$ is a positive network externality coefficient representing the strength of network effect.

Our model is built upon the model in Candogan et al. (2012) except that we assume homogeneous consumer preferences and include the spectral norm of $(G+G^T)$ in the utility function to normalize the network effect (this is necessary as we later extend the model into a sequence of pricing problems over growing networks), and we introduce ρ as the network externality coefficient. The introduction of ρ is consistent with the model in Bloch and Quérou (2013) and Fainmesser and Galeotti (2015). As for the network information, our model is accordant with Candogan et al. (2012) and Bloch and Quérou (2013) in assuming full network information, that is, the adjacency matrix G.

We focus on the discriminative pricing strategy based upon network positions. To block other unnecessary idiosyncrasies, we assume that all consumers have identical preferences, and their decisions differ only because of their positions in the network; that is, a and ρ are homogeneous across consumers. With the homogeneous assumption, we can isolate the impact of network structure and avoid confounding effects resulting from heterogeneity among consumers. The consumers choose their consumption levels that maximize their utilities given the prices offered to them and the consumption levels of their peers in the network. Thus, the consumers are in a consumption game that is completely represented by $\{\mathcal{N}, [0,\infty)\; i \in \mathcal{N}, \{u_i\}_{i\in\mathcal{N}}\}$, where $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of consumers, $[0,\infty)$ is the set of possible consumption levels for each consumer, and u_i is the utility function for consumer i. The utility function $\{u_i\}_{i\in\mathcal{N}}$ for each consumer i is completely identified given the parameters a, ρ, price vector $\mathbf{p} \in \mathbb{R}^n$, and the network adjacency matrix G. According to Candogan et al. (2012), given a, ρ, and G, there is a unique consumption equilibrium given by

$$\mathbf{x}^*(\mathbf{p}) = \frac{1}{2}\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}(a\mathbf{1} - \mathbf{p}). \tag{3.1}$$

This equilibrium is well defined because $\left(I - \frac{2\rho}{\|G+G^T\|}G\right)$ is a positive definite matrix. This is because $\rho < 1$, and the spectral norm of $2G$ is at most the spectral norm of $(G + G^T)$ (see theorem 2 in Fan 1950). The assumption $\rho < 1$ is weaker than assumption 1 in Candogan et al. (2012), which requires that the row sums of the adjacency matrix G be uniformly bounded by twice the quadratic coefficient in the utility function (which is one in our model). If $\rho < 1$ is violated, then $\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}$ may not be well defined, and it would lead to an equilibrium in which consumers consume an infinite quantity.

Next, we consider the seller's problem. We assume that the monopolistic seller can observe the network structure and produce an arbitrary quantity at a constant marginal cost $c < a$. Then, the monopolist's pricing problem is defined as

$$\max_{\mathbf{p}}\, (\mathbf{p} - c\mathbf{1})^T\mathbf{x}^*(\mathbf{p}).$$

Again, as shown in Candogan et al. (2012), the optimal price vector is

$$\begin{aligned}\mathbf{p}^* = &\left(\frac{a+c}{2}\right)\mathbf{1} + \left(\frac{a-c}{8}\right)\frac{4\rho}{\|G+G^T\|}G\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right)\\ &-\left(\frac{a-c}{8}\right)\frac{4\rho}{\|G+G^T\|}G^T\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right)\\ =&\left(\frac{a+c}{2}\right)\mathbf{1} + \left(\frac{a-c}{2}\right)\frac{\rho}{\|G+G^T\|}(G-G^T)\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right),\end{aligned} \tag{3.2}$$

where $\mathcal{K}(G,\alpha) = (I - \alpha G)^{-1}\mathbf{1} = \sum_{i=0}^{\infty}(\alpha G)^i\mathbf{1}$ is the Bonacich centrality vector proposed in Bonacich (1987). According to the definition, the Bonacich centrality of consumer i is the discounted sum of weighted walks of all lengths ending at consumer i. The discount factor exponentially decreases in the length of the walk (for a walk of length k, the discount factor is α^k, where α is the discount rate), and the weight of a walk is the product of the weights of the edges in the walk. In particular, if the adjacency matrix G is binary, then all walks have weight one, and the Bonacich centrality of consumer i is the discounted sum of the number of walks of all lengths ending at consumer i.

We point out that the optimal price vector (3.2) consists of three parts: a common charge for all consumers (the first term), a markup term proportional to the influence received by the consumers (the second term), and a discount term proportional to the influence a consumer exerts on other consumers (the third term). A consumer influencing more central peers gets a higher discount than a consumer influencing less central peers, and a consumer influenced by more central peers gets a higher markup than a consumer influenced by less central peers. This structure has also been observed in Candogan et al. (2012), Bloch and Quérou (2013), and Fainmesser and Galeotti (2015). Plugging the optimal price into the objective function, we have that the optimal profit of the monopolist is

$$\begin{aligned}\pi^* &= \frac{1}{2}\left(\frac{a-c}{2}\right)^2\mathbf{1}^T\left(I - \frac{\rho}{\|G+G^T\|}(G+G^T)\right)^{-1}\mathbf{1}\\ &= \frac{1}{2}\left(\frac{a-c}{2}\right)^2\sum_{k=0}^{\infty}\left(\frac{\rho}{\|G+G^T\|}\right)^k\mathbf{1}^T(G+G^T)^k\mathbf{1}.\end{aligned} \tag{3.3}$$

We note that the optimal profit is proportional to the discounted sum of weighted walks of all lengths in the network with the adjacency matrix $(G + G^T)$.

In practice, the monopolist may not be able to provide discriminative pricing for various reasons. In such cases, under the assumption that $c < a$, according to Candogan et al. (2012), the optimal uniform price vector is

$$\mathbf{p}_0 = \frac{a+c}{2}\mathbf{1}.$$

We note that the optimal uniform price vector does not depend on the network information.[4] This implies that there is no value of network information if the monopolist cannot use discriminative pricing. However, this does not imply that the monopolist's profit under the optimal uniform price vector is independent of the network structure. Actually, under the optimal uniform price vector $\mathbf{p}_0$, the monopolist's profit is

$$\begin{aligned}\pi_0 &= \frac{1}{2}\left(\frac{a-c}{2}\right)^2\mathbf{1}^T\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}\mathbf{1}\\ &= \frac{1}{2}\left(\frac{a-c}{2}\right)^2\sum_{k=0}^{\infty}\left(\frac{2\rho}{\|G+G^T\|}\right)^k\mathbf{1}^TG^k\mathbf{1}.\end{aligned} \tag{3.4}$$

Thus, the monopolist's profit using the optimal uniform price is proportional to the discounted sum of weighted walks of all lengths in the directed network with adjacency matrix G. We next introduce the performance metrics used to evaluate pricing policies.

3.2. Regret and Fractional Regret

We define the monopolist's regret for a price vector $\mathbf{p}$ as the difference between the optimal profit and the profit under the price vector $\mathbf{p}$. That is, the regret of the monopolist under price vector $\mathbf{p}$ is given by

$$R(\mathbf{p}) = \pi^* - \pi(\mathbf{p}),$$

where $\pi(\mathbf{p}) = (\mathbf{p} - c\mathbf{1})^T \mathbf{x}^*(\mathbf{p})$, and $\mathbf{x}^*(\mathbf{p})$ is defined in (3.1).

We also consider another performance metric: the fractional regret. The fractional regret of the monopolist under price vector $\mathbf{p}$ is

$$R_F(\mathbf{p}) = 1 - \frac{\pi(\mathbf{p})}{\pi^*}.$$

The value of price discrimination can now be evaluated using the two performance metrics. We consider the following two metrics to quantify the value of price discrimination:

1. Monopolist's regret under optimal uniform pricing, that is,

$$R(\mathbf{p}_0) = \frac{1}{2}\left(\frac{a-c}{2}\right)^2\Bigg[\mathbf{1}^T\left(I - \frac{\rho}{\|G+G^T\|}(G+G^T)\right)^{-1}\mathbf{1} - \mathbf{1}^T\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}\mathbf{1}\Bigg]. \quad (3.5)$$

2. Monopolist's fractional regret under optimal uniform pricing, that is,

$$R_F(\mathbf{p}_0) = 1 - \frac{\mathbf{1}^T\left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}\mathbf{1}}{\mathbf{1}^T\left(I - \frac{\rho}{\|G+G^T\|}(G+G^T)\right)^{-1}\mathbf{1}}. \quad (3.6)$$

Because the monopolist does not use the network information for optimal uniform pricing, the value of price discrimination is also the value of network information. This equivalence is important because collecting and using network information may be operationally expensive for the monopolist and may create consumer privacy concerns. With the value of price discrimination, one can answer whether the cost of collecting network information is justified.

4. Value of Price Discrimination for Deterministic Networks

In this section, we study how the network structure plays a role in determining the value of price discrimination. In particular, we provide a characterization of networks for which there is no value of price discrimination. This does not imply that the network effect is absent, but that price discrimination does not generate additional profit in those networks. In Candogan et al. (2012), the authors show that, when the network G is symmetric, the optimal discriminative price vector equals the optimal uniform price vector, suggesting that there is no value of price discrimination if the network is symmetric. Therefore, symmetry is a sufficient condition under which there is no value of price discrimination. However, we find that there is a larger class of networks with no value of price discrimination, with symmetric networks forming a subclass.

In the following, we use the monopolist's regret under uniform pricing $R(\mathbf{p}_0)$ as the measure of value of price discrimination. We make the following definition.

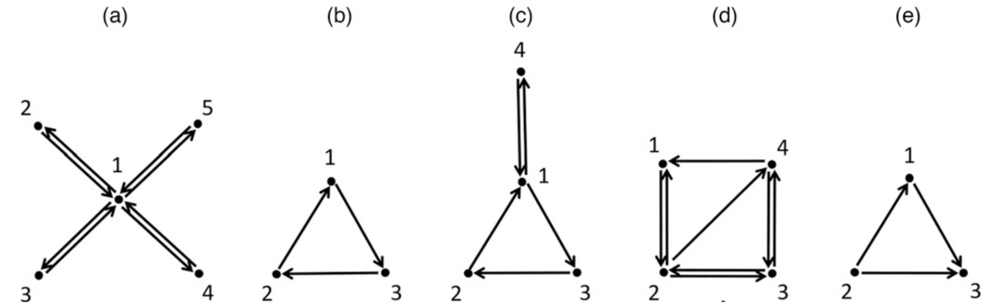
Definition 1 (Price Discrimination–Free Network). A network is a price discrimination–free network if, for any $\rho \in (0,1)$, $a > c > 0$, the value of price discrimination is zero, that is, $R(\mathbf{p}_0) = 0$.

In the following, we study necessary and sufficient conditions for a network to be price discrimination free. At first sight, the price discrimination–free property should have some connection with certain "symmetric" properties of networks, such as symmetry, identical centrality for nodes or balanced in-degrees and out-degrees for nodes. However, as the following five examples show, none of these properties is equivalent to the price discrimination–free property. In all the examples, we assume that all edges have weight one and that $a = 5$, $c = 4$, and $\rho = 0.9$.

Example 1 (Identical Centrality Is Not a Necessary Condition). First, we consider an example to show that identical Bonacich centrality for each node is not a necessary condition for a network to be price discrimination free. We consider the example in Figure 1(a). In Table 1, we present the adjacency matrix, the centrality vector (based on G and $(G+G^T)$), the optimal prices, and the regret under $\mathbf{p}_0$ for this network. As can be seen, in this network, the centrality of the consumer in the center is different from (larger than) the centrality of other consumers. However, the optimal prices are same for all consumers (this can also be seen by noting that the network is symmetric, and according to Candogan et al. (2012), a symmetric network leads to identical prices). Therefore, identical Bonacich centrality is not a necessary condition for a network to be price discrimination free.

Example 2 (Symmetry Is Not a Necessary Condition). Second, we consider an example to show that symmetry is not a necessary condition for a network to be price discrimination free. We consider the example in Figure 1(b), and we summarize the results for this network in Table 2. It is easily seen that the network is a directed cycle and is not symmetric. However, the optimal prices are the same. Therefore, symmetry is not

Figure 1. Example Networks



a necessary condition for a network to be price discrimination free; however, it is a sufficient condition as shown in Candogan et al. (2012).

Example 3 (Same In-Degree and Out-Degree Is Not a Sufficient Condition). Next, we consider an example to show that the in-degree of each node being equal to its out-degree is not a sufficient condition for a network to be price discrimination free (even for binary networks). We consider the example in Figure 1(c). Table 3 presents a summary of results for this network. It is easy to see that for each node in this network, its in-degree and out-degree are the same. However, the optimal prices are different, and there is a positive value of price discrimination for this network. Therefore, having same in-degree and out-degree for each node in a network is not a sufficient condition for a network to be price discrimination free. (Later, we show that same in-degree and out-degree for each node is a necessary condition.)

Example 4 (Identical G-Based Centrality Is Not a Sufficient Condition). Then, we consider an example to show that all nodes have identical G-based Bonacich centrality (meaning the Bonacich centrality of G) is not a sufficient condition for a network to be price discrimination free. We consider the example in Figure 1(d), and we summarize the results for this network in Table 4. As shown in Table 4, each node in this network has the same Bonacich centrality under adjacency matrix G, but the optimal prices are different, and there is a value of price discrimination for this network. Essentially, having the same G-based Bonacich centrality is equivalent to

$$\mathcal{K}\left(G, \frac{2\rho}{\|G+G^T\|}\right) = \left(I - \frac{2\rho}{\|G+G^T\|}G\right)^{-1}\mathbf{1} = \beta\mathbf{1}$$

for some constant β. This is further equivalent to

$$\left(\frac{2\rho\beta}{\|G+G^T\|}G\right)\mathbf{1} = (\beta-1)\mathbf{1},$$

which implies that all nodes in the network G have the same in-degree. Therefore, having identical G-based Bonacich centrality or having the same in-degree for each node is not a sufficient condition for a network to be price discrimination free.

Example 5 (Identical $(G+G^T)$-Based Centrality Is Not a Sufficient Condition). Finally, we consider an example to show that every node having the same $(G+G^T)$-based Bonacich centrality (meaning the Bonacich centrality of $(G+G^T)$) is not a sufficient condition for a network to be price discrimination free. We consider the example in Figure 1(e). Table 5 shows the results for this network. From Table 5, we can see that each node in this network has the same Bonacich centrality based on $(G+G^T)$, but the optimal prices are not the same, and there is a value of price discrimination for this network. In fact, following similar arguments in Example 4, having the same $(G+G^T)$-based Bonacich centrality is equivalent to having the same degrees for all nodes in the multigraph $(G+G^T)$. Therefore, having identical $(G+G^T)$-based Bonacich centrality or, equivalently, having the same degree for

Table 1. Centrality and Optimal Prices for the Network in Figure 1(a)

Adjacency matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|G+G^T\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right)$	Optimal prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&1&1&1\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}14.7368\\7.6316\\7.6316\\7.6316\\7.6316\end{pmatrix}$	$\begin{pmatrix}14.7368\\7.6316\\7.6316\\7.6316\\7.6316\end{pmatrix}$	$\begin{pmatrix}4.5000\\4.5000\\4.5000\\4.5000\\4.5000\end{pmatrix}$	0

Table 2. Centrality and Optimal Prices for the Network in Figure 1(b)

Adjacency matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|G+G^T\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right)$	Optimal prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}10.0000\\10.0000\\10.0000\end{pmatrix}$	$\begin{pmatrix}10.0000\\10.0000\\10.0000\end{pmatrix}$	$\begin{pmatrix}4.5000\\4.5000\\4.5000\end{pmatrix}$	0

all nodes in the multigraph $(G+G^T)$ is not a sufficient condition for a network to be price discrimination free.

We also point out that identical centrality based on both G and $(G+G^T)$ is a sufficient condition for a network to be price discrimination free. To see this, we note that having the same G-based Bonacich centrality is equivalent to $G\mathbf{1}=\alpha\mathbf{1}$ for some constant α, and having the same $(G+G^T)$-based Bonacich centrality is equivalent to $(G+G^T)\mathbf{1}=\beta\mathbf{1}$ for some constant β. Combining the two conditions yields $G^T\mathbf{1}=(\beta-\alpha)\mathbf{1}$. Because $\mathbf{1}^TG\mathbf{1}=\mathbf{1}^TG^T\mathbf{1}$, therefore, $G\mathbf{1}=G^T\mathbf{1}=\alpha\mathbf{1}$ and $\beta=2\alpha$. From Equation (3.2), having the same $(G+G^T)$-based Bonacich centrality and $G\mathbf{1}=G^T\mathbf{1}=\alpha\mathbf{1}$ together give the same optimal price for all the nodes in the network. Therefore, identical centrality based on both G and $(G+G^T)$ is a sufficient condition for a network to be price discrimination free.

These examples show that some intuitive condition is not enough to determine whether a network is price discrimination free or not. In the following, we present our main result, which shows that the price discrimination–free property is closely related to the balance of walks from and to each node in the network. We have the following theorem.

Theorem 1. *G is a price discrimination–free network if and only if $G^k\mathbf{1}=(G^T)^k\mathbf{1}$ for each positive integer k.*

The detailed proof of Theorem 1 is provided in Online Appendix A. Theorem 1 suggests that G is a price discrimination–free network if and only if, for any consumer in the network G and any k, the total weight of incoming walks of length k equals the total weight of outgoing walks of the same length. This shows that the price discrimination–free property is not a local property (restricted to the characteristics of the immediate neighborhood as the symmetry or the same in-degree and out-degree properties are), but a global property. It also shows that the same in-degree and out-degree for each node is a necessary but not a sufficient condition for a network to be a price discrimination–free network. However, when the network $(G+G^T)$ is regular, this condition is also sufficient as shown in the following corollary.

Corollary 1. *If $(G+G^T)$ is a regular network, then G is a price discrimination–free network if and only if $G\mathbf{1}=G^T\mathbf{1}$ or the in-degree equals the out-degree for each node.*

The corollary immediately follows from Theorem 1. When $(G+G^T)$ is regular, for each node in G, if the in-degree and out-degree are equal, then the total weight of incoming walks of any arbitrary length k equals the total weight of outgoing walks of the same length (thus, satisfying the condition for Theorem 1). Corollary 1 provides a simple test for regular networks. However, it may be difficult to obtain such simple tests for general networks. Using a test based upon evaluating the weights of smaller walks, a bound on the fractional regret may be obtained as a function of the positive network externality coefficient ρ. For example, if for each node and each $k\le K$ the total weight of incoming walks of length k is equal to the total weight of outgoing walks of the same length for the node, that is, $G^k\mathbf{1}=(G^T)^k\mathbf{1}$ for each $k\le K$, then the fractional regret $R_F(\mathbf{p}_0)$ is bounded by ρ^{2K+1}. The statement follows from the proof of Theorem 1 in Online Appendix A. A family of tests could help characterize networks with a "small" value of price discrimination. Such a family of tests is beyond the scope of this paper and is an interesting future direction of research.

5. Value of Price Discrimination for Erdös–Renyi Random Networks

In Section 4, we study conditions under which there exists a value of price discrimination for a deterministic network. In this section, we study the value

Table 3. Centrality and Optimal Prices for the Network in Figure 1(c)

Adjacency matrix	$\mathcal{K}\left(G, \frac{2\rho}{\|G+G^T\|}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\|G+G^T\|}\right)$	Optimal prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0&1\\0&0&1&0\\1&0&0&0\\1&0&0&0\end{pmatrix}$	$\begin{pmatrix}11.3172\\6.7715\\8.5973\\8.5973\end{pmatrix}$	$\begin{pmatrix}12.7599\\7.9521\\7.9521\\9.5658\end{pmatrix}$	$\begin{pmatrix}4.5000\\3.6931\\5.3069\\4.5000\end{pmatrix}$	0.3683

Table 4. Centrality and Optimal Prices for the Network in Figure 1(d)

Adjacency matrix	$\mathcal{K}\left(G, \frac{2\rho}{\lVert G+G^T\rVert}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\lVert G+G^T\rVert}\right)$	Optimal prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\0&1&1&0\end{pmatrix}$	$\begin{pmatrix}8.4042\\8.4042\\8.4042\\8.4042\end{pmatrix}$	$\begin{pmatrix}8.1142\\11.2550\\10.2706\\9.7904\end{pmatrix}$	$\begin{pmatrix}5.5782\\3.4218\\4.5000\\4.8459\end{pmatrix}$	0.7267

of price discrimination in random networks. We conduct an asymptotic analysis. That is, we consider a sequence of networks indexed by n. In the nth network, there are n consumers embedded in a random network $G(n)$, and for each network $G(n)$, we consider the same pricing problem as described in Section 3. We assume a, c, and ρ are the same for each pricing problem in the sequence. Specifically, the utility for consumer i in the nth network is

$$u_i(x_i(n), \mathbf{x}_{-i}(n), p_i(n)) = ax_i(n) - x_i(n)^2 + 4\rho x_i(n)\sum_{j\neq i}\frac{G_{ij}(n)}{\lVert G(n)+G(n)^T\rVert}x_j(n) - p_i(n)x_i(n), \tag{5.1}$$

where $\mathbf{x}(n)$ is the vector representing the consumption profile and $\mathbf{p}(n)$ is the price vector in the nth pricing problem. We denote $\mathbf{x}^*(n, \mathbf{p}(n))$ as the consumption equilibrium, $\mathbf{p}^*(n)$ as the optimal price vector, $\pi^*(n)$ as the optimal profit, $\mathbf{p}_0(n)$ as the optimal uniform price vector,[5] $\pi_0(n)$ as the profit under optimal uniform pricing, $R(\mathbf{p}_0(n))$ as the monopolist's regret under optimal uniform pricing, and $R_F(\mathbf{p}_0(n))$ as the monopolist's fractional regret under optimal uniform pricing in the nth pricing problem.

We focus on a special class of random networks: the directed Erdös–Renyi network. A directed Erdös–Renyi network $G(n, p(n))$ is a directed, binary random network with n nodes, in which links between ordered pairs of nodes exist independently with a probability $p(n)$ (we allow $p(n)$ to be a function of n). More precisely, the adjacency matrix of a directed Erdös–Renyi network $G(n, p(n))$ is a random binary matrix satisfying

$$G_{ij}(n, p(n)) = \begin{cases}1, & \text{with probability } p(n)\\ 0, & \text{with probability } 1-p(n)\end{cases}$$

for $i \neq j \in \{1, 2, \ldots, n\}$, and $G_{ii}(n, p(n)) = 0$ for $i \in \{1, 2, \ldots, n\}$. We note that the sample networks $G(n, p(n))$ are usually not symmetric. We call $p(n)$ the influence probability among consumers; it also represents the expected density of the random network.

Given that the networks are random, the performance metrics, including the monopolist's regret and fractional regret under optimal uniform pricing, are random. We are interested in the asymptotic properties of the performance metrics. In particular, we are interested in the expected value of these random performance metrics as n grows large.

5.1. Main Results

We now introduce our main results. A sketch of the proofs is presented in Section 5.2 with the complete proofs provided in the online appendix.

Our first result shows that, for any network density sequence $(p(n))_{n\in\mathbf{N}}$, the monopolist's expected fractional regret vanishes asymptotically. For simplicity of representation, we represent the sequence $(p(n))_{n\in\mathbf{N}}$ as $p(n)$ as needed. We have the following theorem.

Theorem 2. *Given any sequence of network densities $p(n)$, for the sequence of Erdös–Renyi random networks, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = o(1)$.*

Theorem 2 shows that, for Erdös–Renyi networks, the expected regret grows sublinearly in the size of the network, and the expected fractional regret vanishes asymptotically. This implies that the value of price discrimination is negligible asymptotically, and the optimal uniform pricing is good enough to guarantee almost all of the monopolist's profit when the size of the network grows large enough irrespective of the network density. We note that the optimal uniform price $\mathbf{p}_0(n) = \frac{a+c}{2}\mathbf{1}$ is independent of the network structure. This implies that the monopolist does not even need to invest effort to learn the underlying social networks in the asymptotic regime. We provide the detailed proof of Theorem 2 in Online Appendix D.

Table 5. Centrality and Optimal Prices for the Network in Figure 1(e)

Adjacency matrix	$\mathcal{K}\left(G, \frac{2\rho}{\lVert G+G^T\rVert}\right)$	$\mathcal{K}\left(G+G^T, \frac{\rho}{\lVert G+G^T\rVert}\right)$	Optimal prices	Regret $R(\mathbf{p}_0)$
$G=\begin{pmatrix}0&1&0\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1.9000\\1.0000\\3.6100\end{pmatrix}$	$\begin{pmatrix}10.0000\\10.0000\\10.0000\end{pmatrix}$	$\begin{pmatrix}4.5000\\0.0000\\9.0000\end{pmatrix}$	2.9363

Theorem 2 is powerful because it does not put any constraints on the sequence $p(n)$, but as a result, the uniform upper bound on the expected regret and expected fractional regret is loose. In the following, we present tighter asymptotic bounds of expected regret and expected fractional regret for specific ranges of $p(n)$. Note that asymptotically, Erdös–Renyi networks are empty almost surely if $p(n) = O(n^{-2})$ (see Erdos and Rényi 1960). In this range of network densities, there is no network effect and, thus, no value of price discrimination. Therefore, in the following discussions, we are only interested in the range in which $p(n) = \omega(n^{-2})$. Specifically, we consider three different cases: (i) expected in/out degree vanishes asymptotically, that is, $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ (in the random graph literature, such networks are often called *very sparse* networks); (ii) expected in/out degree stays asymptotically bounded and positive, that is, $p(n) = \Theta(n^{-1})$ (in the random graph literature, such networks are often called *critically sparse* networks); and (iii) expected in/out degree asymptotically grows faster than $\log n$, that is, $p(n) = \omega\left(\frac{\log n}{n}\right)$ (in the random graph literature, such networks are often called *dense* networks).

We first consider the case of very sparse networks in which $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$. We have the following results.

Theorem 3. *When $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$ and $p(n) = \omega(n^{-2})$, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta(n^2 p(n))$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = \Theta(np(n))$.*

Now we provide some explanations for the results in Theorem 3. Probabilistically, the networks in this case are acyclic, extremely sparse, and fragmented. In such cases, the effect of any consumer's purchase is restricted to the consumers in the same component in the network. With increasing network density, the size of the components grows and a central consumer is able to influence a larger set of consumers without being influenced by them. Because of this growing imbalance, with increasing network density in this range, the expected regret and expected fractional regret increase. The detailed proof of this case is given in Online Appendix D.

Next, we consider the case of critically sparse networks, in which $p(n) = \Theta(n^{-1})$.

Theorem 4. *When $p(n) = \Theta(n^{-1})$, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta\left(\frac{\log\log n}{\log n} n\right)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = \Theta\left(\frac{\log\log n}{\log n}\right)$.*

In this range of $p(n)$, a sharp phase transition happens. According to Janson et al. (1993), as $p(n)$ increases from $\frac{1}{n} - O\left(n^{-\frac{4}{3}}\right)$ to $\frac{1}{n} + O\left(n^{-\frac{4}{3}}\right)$, small components merge to form a giant component containing a positive fraction of consumers in the network and cycles emerge. Although the emergence of a giant component increases the value of price discrimination, the emergence of cycles reduces this value. However, there are not enough cycles in the network to balance the influence, and the effects of giant components and longer paths dominate. Therefore, in this range of $p(n)$, the value of price discrimination is higher than the case when $p(n) = O\left(n^{-(1+\epsilon)}\right)$ for some $\epsilon > 0$. Again, the detailed proof is given in Online Appendix D.

Finally, we consider the case of denser networks, in which $p(n) = \omega\left(\frac{\log n}{n}\right)$. We have the following results.

Theorem 5. *When $p(n) = \omega\left(\frac{\log n}{n}\right)$, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = \Theta\left(p(n)^{-1}\right)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = \Theta\left(n^{-1}p(n)^{-1}\right)$.*

In this range of $p(n)$, the network is connected with high probability, contains many cycles, and is asymptotically regular and balanced. The in/out degree distribution is tightly concentrated around the average degree and converges asymptotically to a normal distribution. Therefore, networks in this range of densities have a very small value of price discrimination. Furthermore, as the network gets denser, the value of price discrimination decreases. This is because the coefficient of variation of the in/out degree distribution becomes smaller, leading to a more balanced network. The detailed proof is given in Online Appendix D.

In summary, we find that, under different ranges of $p(n)$, the expected regret and expected fractional regret under the optimal uniform pricing strategy follow different rates. When $p(n)$ is relatively small or relatively large, there is not much value of price discrimination. For sparser networks, the value of price discrimination increases in density, and for denser networks, the value of price discrimination decreases in density. The value of price discrimination reaches its maximum in the range in which the average degree $np(n)$ is asymptotically bounded away from zero and is growing slower than the logarithm of n. We summarize our main results in Table 6.

5.2. Proof Concepts

Although the proofs of the main results are quite complicated and are presented in the online appendix, we provide a sketch of those proofs and intermediate results in this section. Random networks demonstrate very different properties for different densities, and therefore, the techniques to quantify the rates of regret/fractional regret are different for different ranges of network densities even though an overarching theme emerges in the proofs. The proof of Theorem 2 uses a combination of techniques used to prove Theorems 3–5 with some additional complexity, so we focus on the sketch of the proofs of Theorems 3–5 in this section.

Table 6. The Value of Price Discrimination for Different Ranges of Network Densities

Network density $p(n)$	Expected regret $\mathbf{E}_G[R(\mathbf{p}_0)]$	Expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)]$
$O(n^{-(1+\epsilon)})$	$\Theta(n^2 p(n))$	$\Theta(np(n))$
$\Theta(n^{-1})$	$\Theta\left(\frac{\log\log n}{\log n} n\right)$	$\Theta\left(\frac{\log\log n}{\log n}\right)$
$\omega\left(\frac{\log n}{n}\right)$	$\Theta\left(p(n)^{-1}\right)$	$\Theta\left(n^{-1}p(n)^{-1}\right)$

Overall, our proof technique relies on decomposing the profit and regret into components corresponding to walks of different lengths and then estimating each component. The following is the outline for the rest of this section.

1. In Section 5.2.1, we introduce two important quantities used in the proofs, namely the profit contribution from walks of different lengths and the regret contribution from walks of different lengths. We also demonstrate how these quantities behave for different network densities. Then, based on these concepts, we give a high-level summary of the proof ideas.

2. In Section 5.2.2, we introduce the techniques for obtaining the upper bounds on the expected regrets. To quantify the profit contributions and regret contributions from walks of different lengths, we need to characterize the asymptotic behavior of the largest eigenvalue of the multigraph $(G(n)+G(n)^T)$ and the number of walks of different lengths in the multigraph $(G(n)+G(n)^T)$ and network $G(n)$. We build upon the literature on the spectra of Erdös–Renyi random networks to characterize the asymptotic behavior of the largest eigenvalue of $(G(n)+G(n)^T)$. We then introduce novel techniques to quantify the number of walks of different lengths in random networks.

3. In Section 5.2.3, we introduce our approach for obtaining matching lower bounds. Our approach relies on counting short walks and small components in random networks.

5.2.1. Profit Contribution and Regret Contribution.

Before introducing the definitions of profit contribution and regret contribution, we first define the concept of value of network effect.

Definition 2 (Value of Network Effect). The value of network effect for the monopolist is the additional expected profit the monopolist can generate under optimal pricing resulting from the presence of network externalities. More specifically,

$$\begin{aligned}\mathrm{VoN}(n) &= \mathbf{E}_G[\pi^*(n)] - \frac{n}{2}\left(\frac{a-c}{2}\right)^2 \\ &= \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=1}^{\infty} \mathbf{E}_G\left[\left(\frac{\rho}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1}^T\left(G(n)+G(n)^T\right)^k \mathbf{1}\right]. \end{aligned} \tag{5.2}$$

In the definition, $\frac{n}{2}\left(\frac{a-c}{2}\right)^2$ is the optimal profit when there is no network effect, which is obtained by substituting $G(n)$ with an all-zero matrix in the optimal profit Equation (3.3). The value of network effect is a natural upper bound on the expected regret. It turns out, as we show in the proof of Theorem 3, that this upper bound is tight for sparse networks as it matches the lower bound. According to Equation (5.2), the value of network effect can be decomposed into the profit contributions from walks of different lengths. The *profit contribution from walks of length k* is the kth term in the series represented in Equation (5.2), which is proportional to the number of walks of length k when controlling for the spectral norm of the multigraph $(G(n)+G(n)^T)$. We note that, by the definition of spectral norm, for any nonempty network $G(n)$ and any $k>0$,

$$0 < \mathbf{1}^T\left(\frac{G(n)+G(n)^T}{\|G(n)+G(n)^T\|}\right)^k \mathbf{1} \le n, \tag{5.3}$$

and the highest value of $\mathbf{1}^T\left(\frac{G(n)+G(n)^T}{\|G(n)+G(n)^T\|}\right)^k\mathbf{1}$, for any given k, is attained when $(G(n)+G(n)^T)$ is regular (all row sums are equal).[6] Therefore, the value of network effect defined in Equation (5.2) satisfies the following inequality:

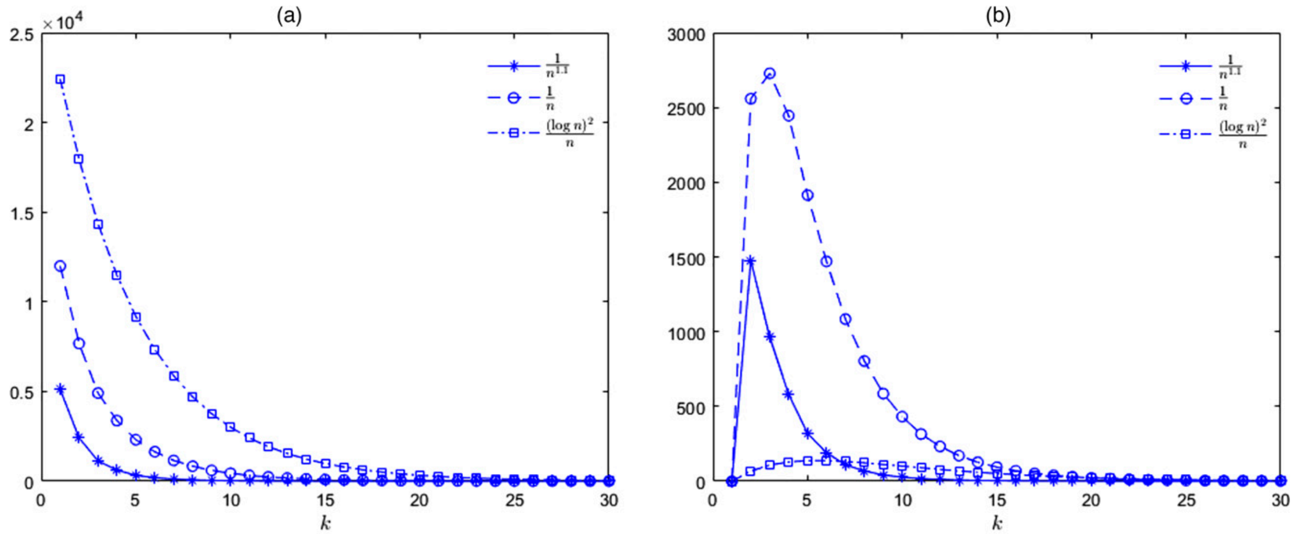
$$0 \le \mathrm{VoN}(n) \le \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \sum_{k=1}^{\infty} \rho^k n = \frac{1}{2}\left(\frac{a-c}{2}\right)^2 \frac{n\rho}{1-\rho}. \tag{5.4}$$

Correspondingly, *the regret contribution from walks of length k* is the difference between the profit contribution from walks of length k and the expected value of the kth term in the series represented in Equation (3.4) or

$$\frac{1}{2}\left(\frac{a-c}{2}\right)^2 \mathbf{E}_G\left[\left(\frac{\rho}{\|G(n)+G(n)^T\|}\right)^k \left(\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1} - 2^k\mathbf{1}^T G(n)^k \mathbf{1}\right)\right]. \tag{5.5}$$

The profit contribution and the regret contribution from longer walks asymptotically decay exponentially because $\rho<1$. Most of the value of network effect and the regret come from contributions from shorter walks. For better illustration of the profit contribution and regret contribution from walks of different lengths, we provide some numerical results in Figure 2. These results are obtained by generating $n = 100$ independent

Figure 2. (Color online) Average Profit Contribution and Regret Contribution from Walks of Different Lengths in Erdös–Renyi Networks (n = 100,000, ρ = 0.8)



Notes. (a) Average profit contribution. (b) Average regret contribution.

directed Erdös–Renyi networks with n = 100,000 nodes under each $p(n) \in \left\{\frac{1}{n^{1.1}}, \frac{1}{n}, \frac{(\log n)^2}{n}\right\}$. We choose a = 2, c = 0.5, and $\rho = 0.8$ in the numerical experiments. For each realization of $G(n, p(n))$ and each k, we compute the corresponding profit contribution and regret contribution from walks of length k and plot the average values in Figure 2. From Figure 2, the profit contribution from walks of length k decays as k increases for all network densities, and for any given k, the profit contribution from walks of length k increases with network density. The regret contribution from walks of length k is unimodular. For k = 0 and 1, the regret contribution is zero. As k increases, the regret contribution from walks of length k initially increases. This is because the expected number of walks of length k in the multigraph $(G(n) + G(n)^T)$ increases faster than the 2^k times the expected number of walks in the network $G(n)$ because of imbalance in the network. However, as k passes a certain threshold, the regret contribution from walks of length k starts decreasing in k. Furthermore, the location of the peak increases with the density of the network, suggesting that, when quantifying (approximating) the expected regret of denser networks, regret contributions from longer walks should be considered. We also observe that, when we move from very sparse networks to critically sparse networks, the peak value increases (because there are increasing numbers of longer walks and there are not many cycles), but when we move further to denser networks, the peak value decreases sharply (because there are increasing numbers of cycles). The observations are consistent with the results in Theorems 3–5 and are formalized in the proofs.

With these definitions, we provide some high-level proof ideas. First, we note that the profit contribution from walks of any length k dominates the regret contribution from walks of length k. Therefore, the value of network effect is a natural upper bound on the expected regret. For very sparse networks discussed in Theorem 3, the value of network effect turns out to be of the same order as the expected regret, and therefore, the bound is tight. For critically sparse networks discussed in Theorem 4, the regret contribution from walks of length one is zero. Therefore, we use the total profit contribution from walks of length greater than one as the upper bound on the expected regret. It turns out that the bound is also tight in this range because we are able to find a matching lower bound.

For denser networks in Theorem 5, the profit contributions from longer walks could be much larger than the regret contributions. Therefore, we quantify the expected regret directly. In particular, we decompose the expected regret into a finite sum (of regret contributions from walks of length up to $\sqrt{\log(np(n))}$ or the square root of the logarithm of the average degree) and a tail term. We obtain upper bounds on both the finite sum and the tail term. Then, we provide a matching lower bound on the expected regret and prove the bound is tight.

5.2.2. Deriving Upper Bounds on Expected Regrets. Despite the preceding observations, the value of network effect and the expected regret expressions are still very complicated. In particular, the kth term in Equations (5.2) and (5.5) are expectations of the ratio of two random variables (number of walks of length k and the spectral norm of the multigraph $(G(n) +$

$G(n)^T$) raised to the power k). To overcome this difficulty, we first obtain asymptotic properties for the spectral norm.

Then, we count the expected number of walks of any length k in multigraph $(G(n)+G(n)^T)$ to quantify the profit contribution from walks of length k. To prove Theorem 5, we also need to obtain the expected difference in the number of walks of length k in multigraph $(G(n)+G(n)^T)$ and 2^k times the number of walks of length k in network $G(n)$ to quantify the regret contribution from walks of length k. We present the results of these two parts in the following.

5.2.2.1. Asymptotic Spectral Norm of the Multigraph $(G(n)+G(n)^T)$. The asymptotic spectral norm of $(G(n)+G(n)^T)$ can be derived from the spectral property of undirected Erdös–Renyi networks. For undirected Erdös–Renyi networks, Krivelevich and Sudakov (2003) show that the largest eigenvalues are highly concentrated. Moreover, the largest eigenvalue demonstrates phase transitions for different ranges of $p(n)$. Specifically, Krivelevich and Sudakov (2003) show the following theorem and two following corollaries. In the following, we say a sequence of events E_n hold *asymptotically almost surely* if the probabilities of E_n converge to one.

Theorem 6 (Krivelevich and Sudakov 2003). *Let $G(n)$ be an undirected Erdös–Renyi network. If Δ is the maximum degree of $G(n)$, then almost surely the largest eigenvalue of the adjacency matrix of $G(n)$ satisfies $\|G(n)\| = (1+o(1))\max\{\sqrt{\Delta}, np(n)\}$, where the $o(1)$ term tends to zero as $\max\{\sqrt{\Delta}, np(n)\}$ tends to infinity.*

Therefore, for large, undirected Erdös–Renyi networks, the spectral norm is almost surely of the same order as the maximum between the square root of the largest degree and the average degree. Although Δ is a random variable given that $G(n)$ is a random network, one can use convergence results on Δ to obtain the asymptotic property for $\|G(n)\|$. The following corollary provided in Krivelevich and Sudakov (2003) is a result about the convergence of the largest degree Δ when $p = \Theta(n^{-1})$.

Corollary 2 (Krivelevich and Sudakov 2003). *Let $G(n)$ be an undirected Erdös–Renyi network. When $p(n) = \Theta(n^{-1})$, almost surely $\|G(n)\| = (1+o(1))\sqrt{\frac{\log n}{\log\log n}}$.*

Krivelevich and Sudakov (2003) omit the technical details of the proof of the Corollary 2. The proof uses the fact that, when $p(n) = \Theta(n^{-1})$, the binomial degree distribution of a node can be approximated by a Poisson degree distribution. Then the largest degree Δ is determined by the maximum of a set of independent and identically distributed (i.i.d.) Poisson random variables, which converges to $\frac{\log n}{\log\log n}$ as shown in Kimber (1983).

The following corollary shows that when $p(n)$ grows larger, the average degree dominates the square root of the largest degree, and the spectral norm is of the same order as the average degree.

Corollary 3 (Krivelevich and Sudakov 2003). *Let $G(n)$ be an undirected Erdös–Renyi network. When $p(n) \geq \frac{\sqrt{\log n}}{n}$, almost surely $\|G(n)\| = (1+o(1))np(n)$.*

Krivelevich and Sudakov (2003) do not provide the spectral norm of $G(n)$ for very sparse networks. In the following, we derive a bound on the spectral norm for this setting. In particular, we consider $p(n) = O(n^{-(1+\epsilon)})$ for some $\epsilon > 0$. The following lemma shows that, in this case, the spectral norm is asymptotically almost surely bounded by some constant.

Lemma 1. *Let $G(n)$ be an undirected Erdös–Renyi network. When $p(n) = O(n^{-(1+\epsilon)})$ for some $\epsilon > 0$ and $p(n) = \omega(n^{-2})$, asymptotically almost surely $1 \leq \|G(n)\| \leq m(\epsilon)$, where $m(\epsilon)$ is a constant that only depends on E.*

Lemma 1 shows that, when the network is very sparse, that is, $p(n) = O(n^{-(1+\epsilon)})$ for some $\epsilon > 0$ and $p(n) = \omega(n^{-2})$, the spectral norm does not grow in the size of the networks. This is different from the spectral property of denser networks (for example, the networks specified in Corollaries 2 and 3), in which the spectral norms grow in the size of the networks.

Now, we have the rate at which the spectral norm of the undirected Erdös–Renyi networks grows for different network densities. Remember that $(G(n)+G(n)^T)$ represents an undirected network. However, the elements in the matrix $(G(n)+G(n)^T)$ can take values from {0, 1, 2}, and it is not a binary network. Therefore, we still need some extra efforts to obtain the bound we need.

To establish the relation between the spectral norm of $G(n)$ and $(G(n)+G(n)^T)$, we decompose the directed Erdös–Renyi network adjacency matrix $G(n)$ into the sum of an upper triangle matrix $G_1(n)$ and a lower triangle matrix $G_2(n)$, that is, $G(n) = G_1(n) + G_2(n)$. Then, $(G_1(n)+G_1(n)^T)$ and $(G_2(n)+G_2(n)^T)$ are two independent undirected Erdös–Renyi networks with probability $p(n)$, and $(G(n)+G(n)^T)$ can be viewed as the sum of two independent undirected Erdös–Renyi networks with probability $p(n)$. By the property of spectral norm, we have

$$\|G(n)+G(n)^T\| \leq \|G_1(n)+G_1(n)^T\| + \|G_2(n)+G_2(n)^T\|.$$

Because $(G_1(n)+G_1(n)^T)$ is a nonnegative matrix, by the Perron–Frobenius theorem, there is a nonnegative eigenvector associated with its largest eigenvalue.

Using the property of the largest eigenvalue, we can show that

$$\|G_1(n)+G_1(n)^T\| \le \|G(n)+G(n)^T\|.$$

By Theorem 6 and the preceding analysis, $\|G(n)+G(n)^T\| \in [1+o(1), 2+o(1)]\max\{\sqrt{\Delta}, np(n)\}$ almost surely, in which Δ is the maximum degree of an undirected Erdös–Renyi network with influence probability $p(n)$. Thus, asymptotically $\|G(n)+G(n)^T\|$ is of the same order of the largest eigenvalue of the undirected Erdös–Renyi networks with the same probability $p(n)$.

Now, we derive an alternative lower bound of $\|G(n)+G(n)^T\|$ to obtain a sharper characterization of $\|G(n)+G(n)^T\|$ when $p(n)=\omega\left(\frac{\log n}{n}\right)$. We use $|E(G(n)+G(n)^T)|$ to denote the number of edges in multigraph $(G(n)+G(n)^T)$. When $p(n)=\omega(n^{-2})$, the number of edges in $(G_1(n)+G_1(n)^T)$ (or $(G_2(n)+G_2(n)^T)$) is $(1+o(1))\frac{n^2p(n)}{2}$ almost surely (Krivelevich and Sudakov 2003); therefore, $|E(G(n)+G(n)^T)|=(2+o(1))\frac{n^2p(n)}{2}$ almost surely. Also, because

$$\|G(n)+G(n)^T\| = \max_{\xi}\frac{\xi^T(G(n)+G(n)^T)\xi}{\xi^T\xi} \ge \frac{\mathbf{1}^T(G(n)+G(n)^T)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2|E(G(n)+G(n)^T)|}{n},$$

therefore, almost surely,

$$\|G(n)+G(n)^T\| \ge (2+o(1))np(n).$$

To summarize, although $\|G(n)+G(n)^T\|$ is a random variable, we have the following asymptotic characterization of $\|G(n)+G(n)^T\|$ for different ranges of values of $p(n)$:

$$\|G(n)+G(n)^T\| = \begin{cases} \Theta(1) & \text{, if } p(n)=O(n^{-(1+\epsilon)}) \text{ for } \epsilon>0 \\ \Theta\left(\sqrt{\frac{\log n}{\log\log n}}\right) & \text{, if } p(n)=\Theta(n^{-1}) \\ (2+o(1))np(n) & \text{, if } p(n)=\omega\left(\frac{\log n}{n}\right) \end{cases} \text{almost surely.} \tag{5.6}$$

5.2.2.2. Expected Number of Walks of Different Lengths. We now quantify the expected number of walks of length k in $(G(n)+G(n)^T)$. We start with the case $k=2$. We have the following lemma.

Lemma 2. *For directed Erdös–Renyi network $G(n)$,*

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^2\mathbf{1}\right] = 2n(n-1)p(n)(1-3p(n) + 2np(n)).$$

Note that Lemma 2 provides an exact calculation of the expected number of walks of length two in the multigraph $(G(n)+G(n)^T)$. The proof of Lemma 2 is based on considering the degree distribution of a neighboring node and is given in Online Appendix C. For walks with longer lengths, that is, $k\ge 3$, it is difficult to consider all possible repeated links, and thus, an exact calculation of walks is difficult. However, we observe that the number of walks of length k in the multigraph $(G(n)+G(n)^T)$ can be upper bounded by the product of the number of walks of length t for any $0\le t\le k$ and the spectral norm of $(G(n)+G(n)^T)$ raised to the power $k-t$. The result is given in the following lemma.

Lemma 3. *For any adjacency matrix $G(n)$, given a positive integer t, for any integer $k\ge t$,*

$$\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1} \le \|G(n)+G(n)^T\|^{k-t}\mathbf{1}^T\left(G(n)+G(n)^T\right)^t\mathbf{1}.$$

Lemmas 2 and 3 help obtain an upper bound on the expected number of walks of any length k asymptotically in the multigraph $(G(n)+G(n)^T)$ because the spectral norm converges asymptotically almost surely according to Equation (5.6). This bound helps us obtain a bound on the profit contribution from walks of different lengths and, in turn, the value of network effect in very sparse networks (in Theorem 3) and the total profit contribution from all walks of length greater than one in critically sparse networks (in Theorem 4).

For denser networks, the profit contribution from walks of length greater than one is significant, and the value of network effect is large. Therefore, we need to quantify the regret contribution from walks of different lengths. To quantify the regret contribution from walks of different lengths, we need the expected difference in the number of walks of length k in the multigraph $(G(n)+G(n)^T)$ and 2^k times the number of walks of length k in network $G(n)$. Unfortunately, an exact computation of the number of walks of any given length is difficult. However, we can provide a lower bound for the expected number of walks of different lengths in a directed Erdös–Renyi network $G(n)$ by ignoring the possibly repeated links. We have the following lemma.

Lemma 4. *For directed Erdös–Renyi network $G(n)$, for any integer $k\ge 2$,*

$$\mathbf{E}_G(\mathbf{1}^TG(n)^k\mathbf{1}) \ge n^2(n-1)^{k-2}(n-2)p(n)^k.$$

The following proposition provides bounds on the differences between $\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right]$ and $2^k\mathbf{E}_G\left[\mathbf{1}^TG(n)^k\mathbf{1}\right]$ for different k.

Proposition 1. *For any directed Erdös–Renyi network $G(n)$, the following statements hold.*

i. *For any integer* $k \geq 0$,

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right] \geq 2^k\mathbf{E}_G\left[\mathbf{1}^T G(n)^k\mathbf{1}\right].$$

ii. *For* $0 \leq k < n$ *and* $\frac{1}{n-k+1} \leq p(n) < 1$,

$$\mathbf{E}_G\left[\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right] - 2^k\mathbf{E}_G\left[\mathbf{1}^T G(n)^k\mathbf{1}\right] \leq (k-1)2^{k-1}P(n,k)p(n)^{k-1} + (k-1)^k 2^k P(n,k-1)p(n)^{k-2},$$

where $P(n,k) = \frac{n!}{(n-k)!}$ *represents the number of ways of permuting k out of n objects.*

The proof of Proposition 1 is quite involved. In the proof, we develop novel counting techniques using the concepts of *graph motifs*. We refer the readers to Online Appendix C for the detailed proof. We note that Proposition 1(i) only holds in expectation. In fact, there exists G for which $\mathbf{1}^T(G+G^T)^k\mathbf{1} < 2^k\mathbf{1}^T G^k\mathbf{1}$. An example of such G is

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

In this case, $\mathbf{1}^T(G+G^T)^3\mathbf{1} = 134$ and $2^3\mathbf{1}^T G^3\mathbf{1} = 136$, and therefore, $\mathbf{1}^T(G+G^T)^k\mathbf{1} < 2^k\mathbf{1}^T G^k\mathbf{1}$. Also, in Proposition 1(ii), we only bound the expected difference in the number of walks of length k in multigraph $(G(n)+G(n)^T)$ and 2^k times the number of walks of length k in network $G(n)$ for $k \leq n+1-\frac{1}{p(n)}$. For dense networks and larger k, we use a concentration inequality on the in/out degree of the consumers in the network to obtain a bound on the tail profit contribution. In particular, we show that, when the network density is large enough, that is, when $p(n) = \omega(\frac{\log n}{n})$, the in/out degree of every node is highly concentrated around the average degree. We have the following proposition.

Proposition 2. *Let* $c(n)$ *be a function of* n *such that* $\lim_{n\to\infty} c(n) = +\infty$ *and* $c(n)\log n < n$ *for all* n. *Let* $\delta(n)$ *be another function of* n *such that* $\delta(n) = \Theta(\frac{1}{\sqrt{c(n)}})$ *and* $\sqrt{\frac{12}{c(n)}} < \delta(n) < 1$. *If* $p(n) = \frac{c(n)\log n}{n}$, *then almost surely every node of the directed Erdös–Renyi network* $G(n)$ *has in/out degree in the range of* $[1-\delta(n), 1+\delta(n)]c(n)\log n$.

5.2.3. Deriving Lower Bounds on Expected Regrets and Expected Fractional Regrets. With these results, we are able to obtain upper bounds on the regrets needed for Theorems 3–5. To obtain matching lower bounds, we use different methods for sparse and dense networks. For very sparse and critically sparse networks in the range in Theorems 3 and 4, matching lower bounds of the regret can be obtained by counting the expected number of components in the network that consist of two nodes with exactly one directed link (see Online Appendix B for details). For denser networks in the range in Theorem 5, a lower bound on the expected regret can be obtained by the regret contribution from walks of length two. We refer the detailed analysis of this part to Online Appendix D.

Finally, in order to obtain bounds on the expected fractional regret, we also need to show that the optimal profit is of order $\Theta(n)$. By Equation (3.3),

$$\pi^*(n) = \frac{1}{2}\left(\frac{a-c}{2}\right)^2\sum_{k=0}^{\infty}\mathbf{E}_G\left[\left(\frac{\rho}{\|G(n)+G(n)^T\|}\right)^k\mathbf{1}^T\left(G(n)+G(n)^T\right)^k\mathbf{1}\right].$$

Considering only $k=0$ in this summation, we get the inequality $\pi^*(n) \geq \frac{n}{2}\left(\frac{a-c}{2}\right)^2$. Using Equation (5.3), we get that $\pi^*(n) \leq \frac{1}{2}\left(\frac{a-c}{2}\right)^2\sum_{k=0}^{\infty}\rho^k n = \frac{1}{2}\left(\frac{a-c}{2}\right)^2\frac{n}{1-\rho}$. Combining the lower and upper bounds of the optimal profit, we have

$$\pi^*(n) = \Theta(n). \tag{5.7}$$

The results on the fractional regret are, thus, established.

6. Value of Price Discrimination for Random Networks with General Degree Distributions

In this section, we extend our analysis to more general random networks. We first establish a result to obtain upper bounds on the value of price discrimination in general networks. Then, we apply the result to obtain upper bounds on the value of price discrimination for power-law networks.

The following theorem provides a general framework to obtain an upper bound on the expected regret and expected fractional regret in random networks with general degree distributions. The proof is provided in Online Appendix E.

Theorem 7. *For any sequence of directed, integer-valued random networks* $G(n)$, *if* $(G(n)+G(n)^T)$ *has degree distribution* $d(n) \sim F(n)$ *and the network satisfies*

1. $\max_{1\leq i\leq n}\sum_{j=1}^{n}\left(G(n)_{ij}+G(n)_{ji}\right) \geq \xi(n)$ *asymptotically almost surely for some sequence* $\xi(n)$ *and*
2. $\mathbf{E}[d(n)^2] \leq \gamma(n)$ *for some sequence* $\gamma(n)$,

then the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = O\left(\frac{n\gamma(n)}{\xi(n)}\right)$, *and the expected fractional regret* $\mathbf{E}_G[R_F(\mathbf{p}_0)] = O\left(\frac{\gamma(n)}{\xi(n)}\right)$.

Theorem 7 provides a general framework to evaluate the value of price discrimination in random networks by using the asymptotic behavior of the ratio of the second moment of the degree distribution and the maximum degree. Thus, if we can obtain a lower bound on the maximum degree of $(G(n)+G(n)^T)$ and an upper bound on the second moment of the degree distribution of $(G(n)+G(n)^T)$, then we can obtain an upper bound on the expected regret and expected fractional regret.

In the following, we apply Theorem 7 to obtain bounds on the regret and fractional regret for an important class of random networks: the power-law networks. The power-law networks, also called the scale-free networks, are a class of networks whose degree distribution follows a power law. For comprehensive discussions of the power-law networks, we refer the readers to Aiello et al. (2001).

We consider the continuous version of the power-law distributions with probability density function (p.d.f.) $f(x) \propto x^{-\alpha}$ and cumulative distribution function (c.d.f.) $F(x)$ such that $1-F(x) \propto x^{1-\alpha}$, where α is called the exponent of the power-law distribution. The exponent α of the power-law distribution has a significant impact on the properties of the distribution. We are interested in the range $\alpha > 2$ because the expected degree diverges for $\alpha \le 2$. The range of α of interest also covers the $\alpha = 3$ case obtained from the preferential attachment process originally shown in Barabási and Albert (1999). Power-law networks also demonstrate a structural cutoff because the maximum degree in a finite network is not unbounded. In Newman (2003), the authors show that, for networks of size n with power-law distributions following $p_d \sim d^{-\alpha}$, the expected maximum degree (or the structural cutoff) follows $d_{max} \sim n^{\frac{1}{\alpha-1}}$, which is of order $o(n)$ given that $\alpha > 2$. We use a more conservative cutoff and impose an upper bound n for the support of the degree distribution. This upper bound on the degree is always satisfied for binary networks because the in/out degree of any node in $G(n)$ is at most n. To obtain a valid probability distribution, we further impose a lower bound $x_{min} \ge 1$ on the support of the distribution. A lower bound on the support is also commonly used in power-law degree distributions to obtain valid distributions. In the preferential attachment process, the number of edges for every newborn node provides the lower bound on the degree (Barabási and Albert 1999). With these considerations, the p.d.f. of the power-law distribution we consider follows

$$f(x)=\frac{\alpha-1}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha}\Bigg/\left(1-\left(\frac{n}{x_{min}}\right)^{1-\alpha}\right) \text{ for } x_{min} \le x \le n, \tag{6.1}$$

and the c.d.f. of the distribution follows

$$F(x)=P(X\le x)=\left(1-\left(\frac{x}{x_{min}}\right)^{1-\alpha}\right)\Bigg/\left(1-\left(\frac{n}{x_{min}}\right)^{1-\alpha}\right) \text{ for } x_{min} \le x \le n. \tag{6.2}$$

When we consider a sequence of power-law distributions in terms of n, we use $F(n)$ to denote the sequence of c.d.f. of the distributions. In the following, we consider random networks $G(n)$ whose in-degrees $a_1,\dots,a_n$ are i.i.d. with distribution $F(n)$ and out-degrees $b_1,\dots,b_n$ are also i.i.d. with distribution $F(n)$. We allow pairwise correlation between the in-degree and out-degree of a node, that is, $corr(a_i,b_i)=\rho_{a,b}$ and $\rho_{a,b} \in [-1,1]$.[7] A valid degree sequence must ensure $\sum_{i=1}^n a_i = \sum_{j=1}^n b_j$. In Section 7.1.2, we provide a generative process for sampling valid in- and out-degrees. Particularly, we can build upon the configuration model (Molloy and Reed 1995, Newman et al. 2001, Chung and Lu 2002) to construct such directed random networks with power-law degree distributions.

The following Theorem 8 gives an upper bound on the asymptotic value of price discrimination for such random networks with power-law degree distributions.

Theorem 8. *For any exponent $\alpha > 2$, consider the sequence of power-law distributions with c.d.f. $F(n)$. For the sequence of random networks $G(n)$ with in- and out-degrees i.i.d. with distribution $F(n)$ and any degree correlation $\rho_{a,b} \in [-1,1]$, the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)] = o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)] = o(1)$.*

Moreover, for different values of α, we have the following bounds: the expected regret

$$\mathbf{E}_G[R(\mathbf{p}_0)] = \begin{cases} O(n^{4-\alpha-\frac{1}{\alpha-1}+\epsilon}), & \text{if } 2<\alpha<3 \\ O(n^{\frac{1}{2}+\epsilon}), & \text{if } \alpha = 3 \\ O\left(n^{\frac{\alpha-2}{\alpha-1}+\epsilon}\right), & \text{if } \alpha > 3, \end{cases}$$

for any $\epsilon > 0$, and the expected fractional regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)] = \begin{cases} O(n^{3-\alpha-\frac{1}{\alpha-1}+\epsilon}), & \text{if } 2<\alpha<3 \\ O(n^{-\frac{1}{2}+\epsilon}), & \text{if } \alpha = 3 \\ O\left(n^{-\frac{1}{\alpha-1}+\epsilon}\right), & \text{if } \alpha > 3, \end{cases}$$

for any $\epsilon > 0$.

Theorem 8 shows that, for random networks with power-law degree distributions, the expected regret grows sublinearly in n, and the expected fractional regret vanishes asymptotically. This yields the same conclusion as in Theorem 2 for Erdös–Renyi networks, suggesting that the value of price discrimination is small for a broader class of random networks. Furthermore, the expected fractional regret decays at least polynomially in the size of the network. We also point out that the degree correlation $\rho_{a,b}$ does not affect the rate of decay of the expected regret or the expected fractional regret; however, as we see in the numerical experiments in Section 7.1.2, the degree correlation $\rho_{a,b}$ does play a role in determining the magnitude of the value of price discrimination.

We now provide a proof sketch for Theorem 8. To apply Theorem 7, we need to obtain a bound on the second moment of the degree distribution in multigraph $(G(n)+G(n)^T)$ and a lower bound on the maximum degree of $(G(n)+G(n)^T)$. In particular, the degree of node i in $(G(n)+G(n)^T)$ is the sum of the in-degree a_i and out-degree b_i for node i. Thus, we can calculate the second moment of the degree distribution based on the p.d.f. of the power-law distribution in Equation (6.1) and the specified degree correlation $\rho_{a,b}$. Different values of α lead to different (exact) orders of the second moment in terms of n. To find the lower bound of the maximum degree in the multigraph $(G(n)+G(n)^T)$, we first obtain a bound on the maximum in-/out-degree in network $G(n)$, which is also a natural lower bound for the maximum degree in multigraph $(G(n)+G(n)^T)$. We then show that, with high probability, the maximum in- or out-degree is lower bounded by $x_{min}n^{\frac{1-\delta}{\alpha-1}}$ for any small enough $\delta>0$. This result is consistent with the structural cutoff result in Newman (2003) that provides the expected maximum degree in power-law networks. Finally, the (exact) rate of the second moment and the lower bound of the in-/out-degree jointly determine the upper bounds we obtain in Theorem 8. The detailed proof of Theorem 8 is provided in Online Appendix E.

To conclude this section, we provide an example of a sequence of random networks $G(n)$ in which the expected regret $\mathbf{E}_G[R(\mathbf{p}_0)]\neq o(n)$, and the expected fractional regret $\mathbf{E}_G[R_F(\mathbf{p}_0)]\neq o(1)$. We consider a sequence of directed, binary star networks $G(n)$ with growing size n. In the nth network, one of the n consumers is randomly picked as the sink, and the remaining $n-1$ consumers are the sources. In other words, exactly one random consumer is being influenced by everyone else in network $G(n)$. The optimal pricing strategy should charge the sink consumer the highest price, and charge everyone else in the network the same (lower) price. It can be verified that, for $G(n)$, the spectral norm $\|G(n)+G(n)^T\|=\sqrt{n-1}$, the expected profit under optimal uniform pricing

$$\mathbf{E}_G[\pi_0(n)]=\frac{1}{2}\left(\frac{a-c}{2}\right)^2(n+2\rho\sqrt{n-1}),$$

and the expected optimal profit

$$\mathbf{E}_G[\pi^*(n)]=\frac{1}{2}\left(\frac{a-c}{2}\right)^2\frac{n+2\rho\sqrt{n-1}}{1-\rho^2}.$$

So the expected regret

$$\mathbf{E}_G[R(\mathbf{p}_0)]=\frac{1}{2}\left(\frac{a-c}{2}\right)^2\frac{\rho^2}{1-\rho^2}(n+2\rho\sqrt{n-1})=\Theta(n),$$

and the expected fraction regret

$$\mathbf{E}_G[R_F(\mathbf{p}_0)]=\rho^2=\Theta(1).$$

7. Numerical Experiments

In this section, we conduct numerical experiments to validate our theoretical results. Our numerical experiments consist of four parts. In the first part, we show how the value of price discrimination changes as the network size n increases under different network densities in Erdös–Renyi networks and under different exponents and degree correlations in power-law networks. In the second part, we show how $p(n)$ affects the value of price discrimination under a given network size n in Erdös–Renyi networks. In the third part, we test a variant of our model to demonstrate the robustness of our results. Finally, we investigate the value of price discrimination on some real-world networks.

7.1. Impact of Network Size on the Value of Price Discrimination

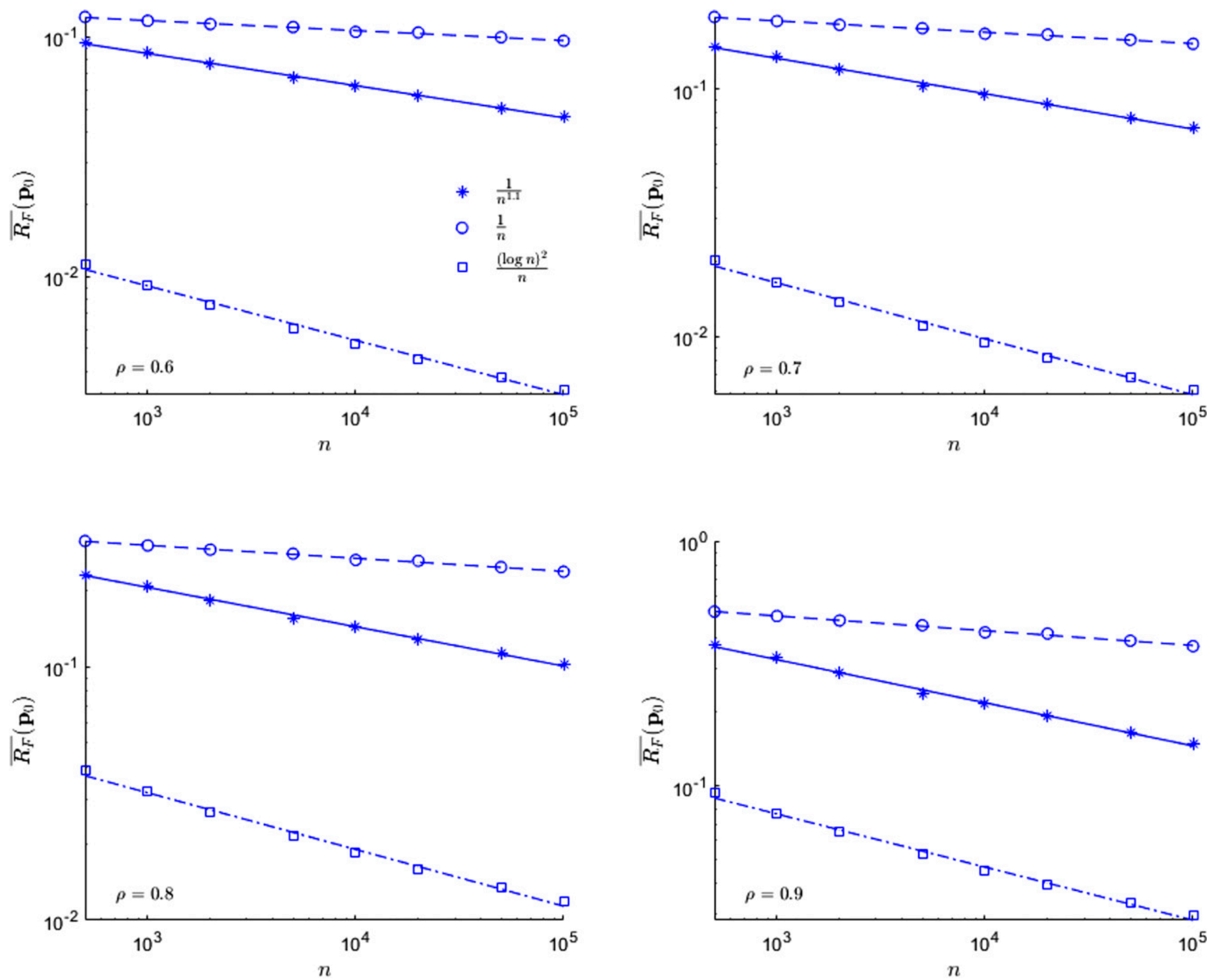
In our first set of experiments, we investigate the impact of network size on the value of price discrimination for both Erdös–Renyi and power-law networks.

7.1.1. Erdös–Renyi Networks.

For Erdös–Renyi networks, we numerically study the decreasing rates of average fractional regret for different $p(n)$ as n increases. We consider $\rho\in\{0.6,0.7,0.8,0.9\}$. For each $n\in\{500,1000,2000,5000,10000,20000,50000,100000\}$, we randomly generate $n=100$ independent directed Erdös–Renyi networks with $p(n)\in\left\{\frac{1}{n^{1.1}},\frac{1}{n},\frac{(\log n)^2}{n}\right\}$. For each realization of $G(n,p(n))$, we compute the corresponding fractional regret according to Equation (3.6). We take the average of the fractional regrets under the same combination of n and $p(n)$ and obtain the average fractional regret $\overline{R_F}(\mathbf{p}_0)$ corresponding to that combination.

To better compare the decreasing rates of the average fractional regret for different $p(n)$, we present the simulation results as scatterplots in log–log scale in Figure 3. The scatterplots for different values of $p(n)$ appear to follow straight lines in the log–log scale for the values of n of our choice, but they are not straight lines essentially. For comparison, we add lines that are the best-fit regression lines in the log–log scale. The slopes of lines in the log–log scale plot reflect the decreasing rates under different circumstances. From Figure 3, we know that all lines have negative slopes. This implies that, in general, the average fractional regret decays as n increases with different decay rates for different ranges of $p(n)$. This is consistent with our result in Theorem 2. In particular, the results show that, when $p(n)$ is relatively small or relatively large, the slope is steeper and the decreasing rate is faster. When $p(n)$ is moderately large, the slope is flatter and the decay rate is slower. This is consistent with results in Section 5. As ρ increases, the value of price discrimination increases uniformly across various $p(n)$. However, we can see from Figure 3 that the decay rates for different $p(n)$ are roughly the same under different values of ρ. This implies that the decay rates of the expected fractional regret do not depend on the choices of ρ.

Figure 3. (Color online) Log–Log Plot of Average Fractional Regrets Under Uniform Pricing Across Different n in Erdös-Renyi Networks



7.1.2. Power-Law Networks. For random networks with power-law degree distributions, we numerically study the decreasing rates of the average fractional regret for different power-law exponents and different pairwise correlations between the in- and out-degree of nodes as n increases.

We first provide an approach to construct a valid correlated in-degree sequence $a_1, \ldots, a_n$ and out-degree sequence $b_1, \ldots, b_n$. We first sample i.i.d. in-degrees $a_1, \ldots, a_n$ from the power-law distribution F. Without loss of generality, we assume the sequence $a_1, \ldots, a_n$ is sorted in descending order. Next, we sample i.i.d. random variables $Z_1, \ldots, Z_n$ as follows: for each i, $Z_i = 1$ with probability $|\theta|$, and $Z_i = 0$ with probability $1 - |\theta|$, where $\theta \in [-1, 1]$ is a parameter we use to control the correlation between in- and out-degrees (it is not necessarily the correlation $\rho_{a,b}$ between the in- and out-degree sequence). We define sets of nodes $I_0 = \{i : Z_i = 0, 1 \le i \le n\}$ and $I_1 = \{i : Z_i = 1, 1 \le i \le n\}$. The out-degrees $b_1, \ldots, b_n$ are constructed by a permutation of $a_1, \ldots, a_n$ as follows. If $\theta \ge 0$, then we set $b_i = a_i$ for $i \in I_1$ and set $\{b_i : i \in I_0\}$ by a random permutation of $\{a_i : i \in I_0\}$. If $\theta < 0$, then we set $b_i = a_{n-i+1}$ and set $\{b_i : i \in I_1\}$ by a random permutation of $\{a_{n-i+1} : i \in I_1\}$. By this construction, both the in- and out-degrees follow the power-law distribution F, and their sum of in- and out-degrees are equal. When $\theta > 0$, the in- and out-degree sequences have a positive correlation; when $\theta < 0$, the two degree sequences have a negative correlation.

Moreover, when $\theta \ge 0$, the correlation between in- and out-degree is $\theta + O(n^{-1})$.[8] So, asymptotically, the correlation is $\rho_{a,b} = \theta$. This is because when n is large (as in our simulation), an approximately equivalent representation of b_i is $b_i = Z_i a_i + (1 - Z_i)X$, where X follows the same power-law distribution F and is independent of a_i. We can then verify that $\mathbf{Var}(b_i) = \mathbf{Var}(a_i)$

and $\mathbf{Cov}(a_i, b_i) = \theta\mathbf{Var}(a_i)$. Therefore, for large networks, asymptotically, $\mathbf{c}orr(a_i, b_i) = \theta$ for any i and $\rho_{a,b} = \theta$.

We now describe the details of our numerical experiments. We set $x_{min} = 2$ and $\rho = 0.8$ for all cases. For each $\alpha \in \{2.5, 2.75, 3.0, 3.25, 3.5\}$, we generate a sequence of networks with different sizes n according to the power-law distribution specified in Equations (6.1) and (6.2). For each α and n, we generate the in- and out-degree sequences with different values of the parameter $\theta \in \{-1, -0.5, 0, 0.5, 1\}$ according to the approach described earlier in this section. For the in- and out-degree sequences associated with each θ, we use a configuration model (Molloy and Reed 1995, Newman et al. 2001) to construct the directed random network $G(n, \alpha, \theta)$ with the specified degrees and compute the corresponding fractional regret according to Equation (3.6). For each combination of parameters, we repeat this process for $n = 100$ times independently and compute the average fractional regret $\bar{R}_F(\mathbf{p}_0)$.

We present our results as scatterplots in the log–log scale as shown in Figure 4. We add the best-fit regression lines to better illustrate the decreasing trends of the fractional regret. We also list the average sample Pearson correlation between the in- and out-degree sequence (denoted as $\bar{r}_{a,b}$) corresponding to each θ. We observe that, for the simulated power-law networks, the average fractional regret decreases as the network size n increases. This observation is consistent with our theoretical results about the value of price discrimination on power-law networks in Theorem 8. Moreover, we observe that the slopes of the regression lines in Figure 4 are quite steep (as compared with the cases in which $p(n) = \frac{1}{n}$ in Figure 3), and thus, the decreasing rates of the average fractional regret in these power-law networks are relatively fast. This observation is again consistent with our theoretical results in Theorem 8 as we show that the rates of decreasing for power-law networks are at least as fast as polynomial decay.

Moreover, we observe that the average fractional regret generally increases as we decrease θ from one to zero, and further decreasing θ from 0 to -1 does not have much impact on the magnitude of the average fractional regret. On one hand, this observation suggests that a higher positive correlation leads to a smaller value of price discrimination. This can be explained by the level of imbalance in the network. As we increase the degree correlation to a higher positive value, the incoming and outgoing influence of each consumer in the network become more balanced, leading to a smaller value of price discrimination. On the other hand, when the correlation is zero or negative, the network is very unbalanced and, thus, demonstrates a relatively high value of price discrimination. When $\theta \leq 0$, the value of price discrimination remains similar across different values of θ. We point out that θ is not equivalent to the degree correlation $\rho_{a,b}$ when $\theta < 0$. Because the degrees of nodes are discrete and the degree distribution is highly asymmetric, a big mass of nodes would have in- or out-degrees concentrated around the lower bound of the support of the power-law distribution. Thus, it is actually difficult to create power-law networks with significantly anticorrelated in- and out-degrees. Therefore, networks with different nonpositive values of θ yield to similar degree sequences and, hence, similar value of price discrimination. We find that even choosing $\theta = -1$ leads to an average sample correlation $\bar{r}_{a,b} = -0.09$. Therefore, the results corresponding to negative θ almost coincide with the results corresponding to $\theta = 0$.

7.2. Impact of Network Densities on the Value of Price Discrimination

In our second set of experiments, we consider the case when the underlying networks are directed Erdös–Renyi networks with size n = 100,000. In our numerical experiments, we choose $\rho \in \{0.6, 0.7, 0.8, 0.9\}$. For each ρ, we consider $p(n) \in \left\{\frac{1}{n^{1.3}}, \frac{1}{n^{1.1}}, \frac{1}{n}, \frac{\log\log n}{n}, \frac{\sqrt{\log n}}{n}, \frac{\log n}{n}, \frac{(\log n)^2}{n}\right\}$. Then, for each $p(n)$, we randomly generate n = 100 independent directed Erdös–Renyi networks with 100,000 nodes. For each realization of $G(n, p(n))$, we compute the corresponding fractional regret according to Equation (3.6).

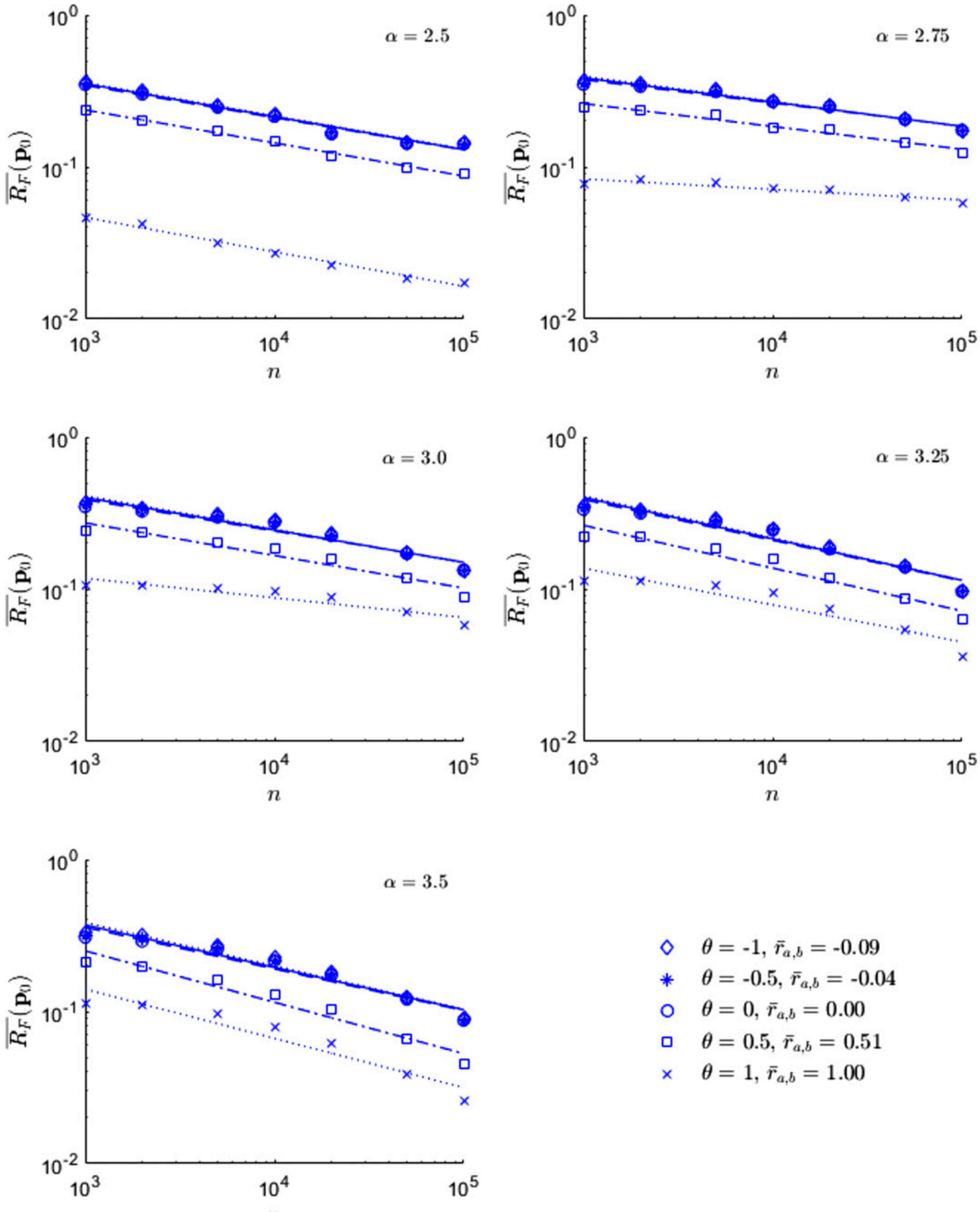
The values of fractional regrets are presented in the form of boxplots in Figure 5. We can see that for each ρ, the values of fractional regret first increase and then decrease as $p(n)$ increases. The peak is reached when $p(n) = \frac{\log\log n}{n}$. When $p(n)$ is very small ($p(n) = \frac{1}{n^{1.3}}$) or very large ($p(n) = \frac{(\log n)^2}{n}$), the fractional regret is less than 5%. In addition, as the value of ρ increases, the value of price discrimination increases uniformly across all values of $p(n)$. This implies that, larger ρ leads to larger value of price discrimination under the same n and $p(n)$.

Recall that in Theorems 3–5, our theoretical bounds of expected fractional regret under different ranges of $p(n)$ also yield similar trends for a fixed network size: when the network density is relatively small or relatively large, the expected fractional regret decreases very fast and has a small value of price discrimination for large n; when the network density is moderately large, the expected fractional regret decreases slowly such that, for decently large network size n, the value of price discrimination is nonnegligible.

7.3. The Value of Price Discrimination with Degree Normalization

Next, we investigate the robustness of our results. We perform similar analysis as in Sections 7.1 and 7.2 but with a variant of our model. In particular, instead of normalizing a consumer's local network effect by the

Figure 4. (Color online) Log-Log Plot of Average Fractional Regrets Under Uniform Pricing Across Different n in Power-Law Networks



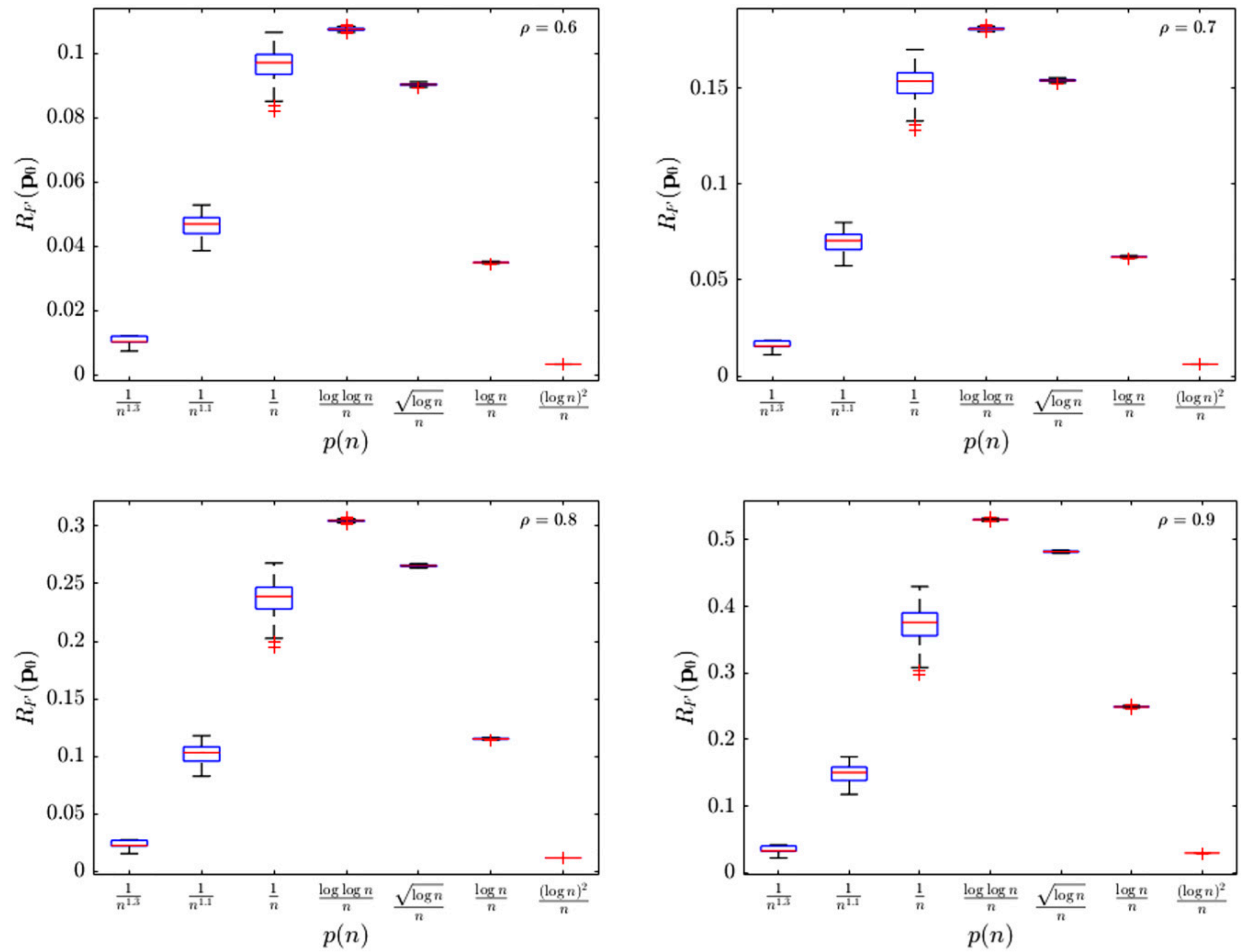
spectral norm of $(G + G^T)$, we normalize it by the total amount of influence (or the total in-degree of G) received by the consumer. Specifically, we consider the setting in which the preferences of the consumers are represented by the following utility function:

$$u_i(x_i, \mathbf{x}_{-i}, p_i) = ax_i - x_i^2 + x_i \sum_{j \neq i} \frac{G_{ij}}{\sum_{j \neq i} G_{ij}} x_j - p_i x_i.$$

For any i, if $\sum_{j \neq i} G_{ij} = 0$, then $G_{ij} = 0$ for all $j \neq i$, and we consider $0/0$ to be 0. This utility function implies that the more total influence consumer i receives, the less important the consumer's single neighbor's purchasing decision would be to the consumer. In this setting, the normalization factor could be different for different consumers.

In the simulation, we study the impact of network sizes and densities on the value of price discrimination under this model variant for Erdös–Renyi networks. The simulation process to study the impact of network sizes (network densities) is exactly the same as that described in Section 7.1.1 (Section 7.2) except that we modify the normalization factor of each node as the total number of in-degrees of the node. The

Figure 5. (Color online) Boxplot of Fractional Regrets Under Uniform Pricing in Erdös–Renyi Networks (n = 100,000)



simulation results are presented in Figure 6. As we can see from Figure 6, the observed patterns/trends in this model variant are similar to our results in the base model. Particularly, the average fractional regret still decreases as we increase the network size n, and the decay rates for different $p(n)$ are different. For a given network size, the values of fractional regret also first increase and then decrease as we increase the network density. The peak is reached when the network density is moderately large. This set of simulation results suggests that our results may be applicable to a larger set of models, and the theoretical results we obtained are robust with respect to this model variant.

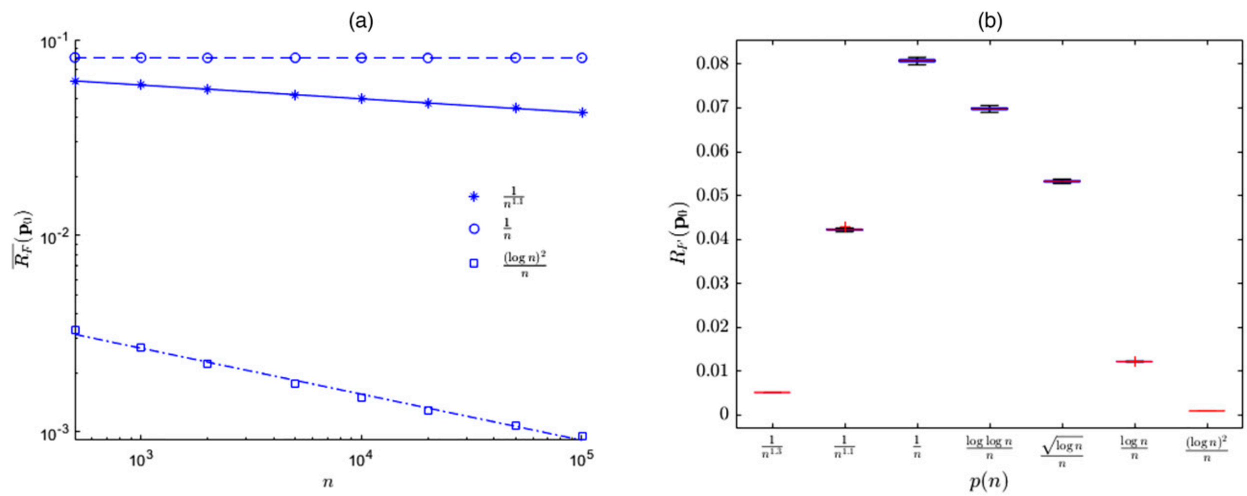
7.4. The Value of Price Discrimination on Real Networks

We select five social network data sets, namely "Epinions1", "LiveJournal1", "Slashdot0811", "Slashdot0902", and "Pokec" from the Stanford large network data set collection (Leskovec and Krevl 2014) and use the converted version of the data from the University of Florida sparse matrix collection (Davis and Hu 2011). All the selected data sets are directed binary social networks. A brief description of each network is summarized in Table 7.

For each data set, we extract the adjacency matrix G and calculate the corresponding fractional regret according to Equation (3.6) with given value of ρ. The basic statistics of the selected social networks and the fractional regrets under different values of ρ are summarized in Table 8. The degree correlation coefficient is the sample Pearson correlation coefficient of the in- and out-degree of the network. The level of symmetry is defined as the ratio between the number of links whose reverse is also in the network and the total number of links in the network.

The results are presented in Table 8. In Table 8, we can see that the value of price discrimination is very small for the largest network LiveJournal with the fractional regret of uniform pricing being less than 1% for all values of ρ. For the other networks with medium size, the value of price discrimination is generally

Figure 6. (Color online) Fractional Regrets Under Uniform Pricing in Erdös–Renyi Networks with Degree Normalization



Notes. (a) Regrets for different n and $p(n)$ in log–log scale. (b) Regrets for different $p(n)$ (n = 100,000).

small. However, it could already be significant in certain contexts, especially when ρ is closer to one. Moreover, we note that in Table 8, for the networks of similar sizes (Epinions1, Slashdot0811, and Slashdot0902), usually the higher the level of symmetry of the network, the smaller the fractional regret. Overall, although these real-world networks do not necessarily follow the theoretical models we investigate, the results can provide part support for our main results.

8. Conclusions and Future Work

In this paper, we study the asymptotic value of price discrimination in large random social networks. We find that, for Erdös–Renyi random networks, the value of price discrimination is not a positive fraction of the profit asymptotically for all network densities. Yet, when the average degree of the network stays as a constant or grows slower than the logarithm of the size of the network, the rate of expected fractional regret decays slowly, and therefore, for decently large networks, there could still be a nonnegligible value of price discrimination. The results for very sparse networks (average degree decreasing in the size of the network) are driven by the fact that the network is fragmented enough for the influence of any individual to be small although, for dense networks (average degree increasing at least as the logarithm of the size of the network) the results are driven by relatively equal positions of individuals in the network.

We also extend our analysis to random networks with general degree distributions. We propose a general framework based on degree information to provide

Table 7. Brief Descriptions of Selected Real-World Networks

Name of network	Description
Epinions1	A who-trusts-whom online social network of a general consumer review site Epinions.com. Members of the site can decide whether to “trust” each other. All the trust relationships interact and form the web of trust, which is then combined with review ratings to determine which reviews are shown to the users (Richardson et al. 2003).
LiveJournal1	LiveJournal is a free online community with almost 10 million members, allowing members to maintain journals and individual and group blogs and declare which other members are their friends (Backstrom et al. 2006, Leskovec et al. 2009).
Slashdot0811	Slashdot is a technology-related news website known for its specific user community with a feature allowing users to tag each other as friends or foes. The network contains friend/foe links between the users. The links don’t distinguish the friend or foe relationship, so the network is binary and nonnegative. The network was obtained in November 2008 (Leskovec et al. 2009).
Slashdot0902	Same as Slashdot0811. The network was obtained in February 2009 (Leskovec et al. 2009).
Pokec	Pokec is the most popular online social network in Slovakia. The popularity of the network has not changed even after the coming of Facebook. The network contains oriented user friendship data (Takac and Zabovsky 2012).

Table 8. Results from Real-World Data

Name of network	Epinions1	LiveJournal1	Slashdot0811	Slashdot0902	Pokec
Number of Nodes	75,888	4,847,571	77,360	82,168	1,632,803
Average in/out degree	6.7051	14.2326	11.7046	11.5430	18.7546
Degree correlation coefficient	0.5491	0.6490	0.9547	0.9343	0.7150
Level of symmetry, %	41	75	87	84	54
Fractional regret ($\rho = 0.9$)	0.1627	0.0048	0.0265	0.0417	0.0423
Fractional regret ($\rho = 0.8$)	0.0611	0.0029	0.0088	0.0131	0.0190
Fractional regret ($\rho = 0.7$)	0.0289	0.0019	0.0040	0.0058	0.0109
Fractional regret ($\rho = 0.6$)	0.0150	0.0012	0.0021	0.0030	0.0066

bounds for the expected value of price discrimination in random networks. We apply the general framework to random networks with power-law degree distributions and show that the value of price discrimination (in terms of expected fractional regret) vanishes as the size of the network increases for any exponent $\alpha > 2$.

Given our results, it would appear that the firms need to be more careful about using price discrimination because the value of such discriminative pricing policies under many cases may not be substantial, and the inequity and the lack of transparency in pricing can lead to lower consumer satisfaction and mistrust. Moreover, our analysis and results can serve as a first step in addressing the value of other marketing strategies on social networks, such as product promotions or referral programs.

Our analysis is purely structural and ignores the heterogeneity in consumer preferences. Although consumer preferences play a role in generating value from price discrimination, the interaction between heterogeneous consumer preferences and network structure and its role in generating value from price discrimination is an important direction of future work. Another possible future direction of work could focus on the moderate-sized networks in which there may be some value of price discrimination. Network information is often noisy, and using optimal pricing assuming perfect network information may be suboptimal. The design of robust pricing policies under noisy network information can also be a future research direction.

Endnotes

[1] See https://www.wired.com/2012/04/ff-klout/.

[2] See https://www.forbes.com/sites/neilhowe/2016/03/31/brands-are-under-the-social-influence/?sh=346832245f49.

[3] See https://hbr.org/2018/05/when-customers-are-and-arent-ok-with-personalized-prices for results from a large-scale experiment.

[4] Without the assumption that all consumers have identical preferences, the optimal uniform price vector may depend on the network structure G.

[5] The optimal uniform price for each consumer is $\frac{a+c}{2}$. The optimal uniform price vectors for different n differ only in the dimension of the vectors.

[6] To see this result, we note that $\|G(n) + G(n)^T\|^k = \|(G(n) + G(n)^T)^k\| = \max_\xi \frac{\xi^T(G(n)+G(n)^T)^k\xi}{\xi^T\xi} \geq \frac{\mathbf{1}^T(G(n)+G(n)^T)^k\mathbf{1}}{\mathbf{1}^T\mathbf{1}}$. Therefore, $\mathbf{1}^T\left(\frac{G(n)+G(n)^T}{\|G(n)+G(n)^T\|}\right)^k\mathbf{1} \leq \mathbf{1}^T\mathbf{1} = n$, and the equality holds only when $\mathbf{1}$ is the largest eigenvector of $(G(n) + G(n)^T)^k$. This is equivalent to that $(G(n) + G(n)^T)$ is a regular graph.

[7] Although we do not restrict the possible values of $\rho_{a,b}$, the correlation between in- and out-degree in such networks are usually much higher than –1 because of the highly asymmetric nature of the degree distribution and the fact that the sum of in-degrees of all nodes and the sum of out-degrees of all nodes are equal to the number of links in the network.

[8] Because there is a small probability $\frac{1}{|I_0|}$ (which in expectation is $\frac{1}{n(1-\theta)}$) that $b_i = a_i$ for any $i \in I_0$, therefore, b_i and a_i have a small correlation, which vanishes as $n \to \infty$.

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