Algorithms for System Identification Tool

1 Least Squares Method

Let us consider a system

$$y[\mathrm{k}+1] = Ax[\mathrm{k}] + Bu[\mathrm{k}], \tag{1}$$

where A and B are the system matrix and input matrix, $u, x, y \in \mathbb{R}$ stand for the input, state, and output, and a prediction system

$$\hat{y}[\mathrm{k}+1] = \theta^{\mathrm{T}}[\mathrm{k}]\phi[\mathrm{k}], \tag{2}$$

where ϕ and θ mean the parameter vectors, the former is available to sense and the latter is to be observed. The purpose of a least squares method is to minimize a cost function

$$J(\theta) = \sum (y[\mathrm{k}] - \hat{y}[\mathrm{k}])^2 . \tag{3}$$

If a dataset was provided, you can find $\hat{\theta}$ by

$$\hat{\theta} = \left(\sum \phi[\mathrm{k}]\phi^{\mathrm{T}}[\mathrm{k}]\right)^{-1} \sum \phi[\mathrm{k}]y[\mathrm{k}], \tag{4}$$

1.1 ARX model

For example, a prediction system using an ARX model is constructed as

$$x[\mathrm{k}+1] = a_0 x[\mathrm{k}] + a_1 x[\mathrm{k}-1] + \cdots + a_{\mathrm{n}} x[\mathrm{k}-\mathrm{n}] + b_0 u[\mathrm{k}] + \cdots + b_{\mathrm{m}} u[\mathrm{k}-\mathrm{m}], \tag{5}$$

and the parameters we want are a_{k} ($\mathrm{k} \in [0, \mathrm{n}]$) and b_{k} ($\mathrm{k} \in [0, \mathrm{n}]$). The parameter can be obtained by solving (3) setting

$$y[k+1] = x[\mathrm{k}+1] \tag{6}$$

$$\theta = \begin{bmatrix} a_0 & \cdots & a_{\mathrm{n}} & b_0 & \cdots & b_{\mathrm{m}} \end{bmatrix}^{\mathrm{T}} \tag{7}$$

$$\phi = \begin{bmatrix} x[k] & \cdots & x[\mathrm{k}-\mathrm{n}] & u[k] & \cdots & u[\mathrm{k}-\mathrm{m}] \end{bmatrix}^{\mathrm{T}} . \tag{8}$$

1.2 Whitebox model

If a whitebox model was provided as

$$y[k+1] = a_0 x_0[k] + a_1 x_1[k] + \cdots + a_{\mathrm{n}} x_1[\mathrm{n}], \tag{9}$$

where a_{k} ($\mathrm{k} \in [0, \mathrm{n}]$) are the parameter and x_{k} ($\mathrm{k} \in [0, \mathrm{n}]$) are available, the parameters can be obtained by solving (3) setting

$$\theta = \begin{bmatrix} a_0 & \cdots & a_{\mathrm{n}} \end{bmatrix}^{\mathrm{T}} \tag{10}$$

$$\phi = \begin{bmatrix} x_0[k] & \cdots & x_{\mathrm{n}}[k] \end{bmatrix}^{\mathrm{T}} . \tag{11}$$

1.3 Polynomial model

If a polynomial model was provided as

$$y[k] = a_{\mathrm{n}} x^{\mathrm{n}} + a_{\mathrm{n}-1} x^{\mathrm{n}} + \cdots + a_1 x + a_0 \tag{12}$$

where a_{k} ($\mathrm{k} \in [0, \mathrm{n}]$) are the parameter, the parameters can be obtained by solving (3) setting

$$\theta = \begin{bmatrix} a_{\mathrm{n}} & \cdots & a_1 & a_0 \end{bmatrix}^{\mathrm{T}} \tag{13}$$

$$\phi = \begin{bmatrix} x^{\mathrm{n}} & \cdots & x & 1 \end{bmatrix}^{\mathrm{T}} . \tag{14}$$

2 Kalman Filter

Let us consider a state-space representation

$$\theta[\mathrm{k}+1] = \theta[\mathrm{k}] + v \tag{15}$$

$$y[\mathrm{k}] = \phi^{\mathrm{T}}[\mathrm{k}]\theta[\mathrm{k}] + w, \tag{16}$$

where ϕ is the observation matrix and θ, y, v, and w denote the parameter to be estimate, available output, process noise, and observation noise. This system stands for the parameter fluctuate like random walk. A Kalman filter estimates a parameter following these steps:

- Prediction step

$$\hat{\theta}_{\mathrm{p}}[\mathrm{k}+1] = \hat{\theta}[\mathrm{k}] \tag{17}$$

$$P_{\mathrm{p}}[\mathrm{k}+1] = P_{\mathrm{p}}[\mathrm{k}] + Q \tag{18}$$

- Filtering step

$$e[k] = y[\mathrm{k}] - \phi^{\mathrm{T}}[\mathrm{k}]\theta[\mathrm{k}] \tag{19}$$

$$S[\mathrm{k}] = R + \phi^{\mathrm{T}}[\mathrm{k}]P_{\mathrm{p}}[k]\phi[\mathrm{k}] \tag{20}$$

$$\hat{\theta}[\mathrm{k}+1] = \hat{\theta}_{\mathrm{p}}[\mathrm{k}] + P_{\mathrm{p}}[\mathrm{k}]\phi[\mathrm{k}]S[\mathrm{k}]^{-1}e[\mathrm{k}] \tag{21}$$

$$P[\mathrm{k}+1] = P_{\mathrm{p}}[\mathrm{k}] - P_{\mathrm{p}}[\mathrm{k}]\phi[\mathrm{k}]S[\mathrm{k}]^{-1}\phi^{\mathrm{T}}[\mathrm{k}]P_{\mathrm{p}}[\mathrm{k}] \tag{22}$$

Here, P, Q, R, and S denotes the auto-covariance, covariance of process noise and observation noise, and covariance of observation, and the subscript $_{\mathrm{p}}$ stands for the predicted value.

2.1 ARX model

The parameter can be estimated by setting

$$y[k+1] = x[\mathrm{k}+1] \tag{23}$$

$$\theta = \begin{bmatrix} a_0 & \cdots & a_{\mathrm{n}} & b_0 & \cdots & b_{\mathrm{m}} \end{bmatrix}^{\mathrm{T}} \tag{24}$$

$$\phi = \begin{bmatrix} x[k] & \cdots & x[\mathrm{k}-\mathrm{n}] & u[k] & \cdots & u[\mathrm{k}-\mathrm{m}] \end{bmatrix}^{\mathrm{T}}. \tag{25}$$

2.2 Whitebox model

The parameter can be estimated by setting

$$\theta = \begin{bmatrix} a_0 & \cdots & a_{\mathrm{n}} \end{bmatrix}^{\mathrm{T}} \tag{26}$$

$$\phi = \begin{bmatrix} x_0[k] & \cdots & x_{\mathrm{n}}[k] \end{bmatrix}^{\mathrm{T}}. \tag{27}$$

2.3 Polynomial model

The parameter can be estimated by setting

$$\theta = \begin{bmatrix} a_{\mathrm{n}} & \cdots & a_1 & a_0 \end{bmatrix}^{\mathrm{T}} \tag{28}$$

$$\phi = \begin{bmatrix} x^{\mathrm{n}} & \cdots & x & 1 \end{bmatrix}^{\mathrm{T}}. \tag{29}$$