High-Order Spectral Element Transport Scheme on the Cubed Sphere using SAT

Implementation and Validation

Wang, Chi-Wei

December 14, 2025

Department of Applied Mathematics
National Chung Hsing University (NCHU)

Outline

Introduction

Motivation & Concept

The Problem:

- **Pole Singularity:** Traditional Latitude-Longitude grids restrict time steps severely near the poles (CFL condition).
- **Scalability:** Global spectral methods require expensive global communications.

The Solution: The Cubed Sphere

- Decomposes the sphere into 6 identical patches.
- Results in a quasi-uniform grid with no singular points.

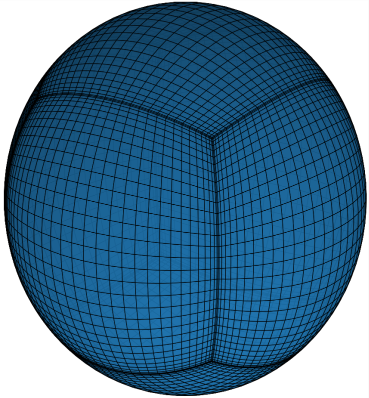


Figure: Projecting a cube onto a sphere

Objective & Methodology

The primary goal is to implement a conservative transport solver for the advection equation on the cubed sphere:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{u}\phi) = 0 \tag{1}$$

Key Implementation Techniques:

1. **Geometry:** Equiangular Gnomonic Projection to optimize grid uniformity.
2. **Discretization:** Spectral Element Method (SEM) on Legendre-Gauss-Lobatto (LGL) nodes for high-order accuracy.
3. **Boundary Treatment:** Simultaneous Approximation Term (SAT) with **upwind numerical flux** for stable interface coupling.
4. **Time Integration:** Low-Storage Runge-Kutta (LSRK) scheme for memory-efficient explicit time stepping.

Methodology

Geometry: Equiangular Projection

Mapping: Points on the cube faces (x, y) are mapped to the sphere using central angles (α, β):

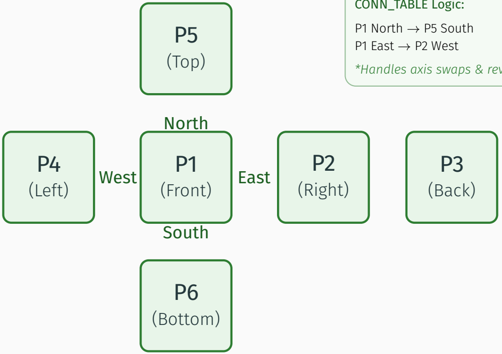
$$x = a \tan \alpha, \quad y = a \tan \beta$$

Jacobian ($\sqrt{g}$): Analytically computed to ensure conservation:

$$\sqrt{g} = \frac{R^2}{\rho^3 \cos^2 \alpha \cos^2 \beta}$$

Connectivity: The Unfolded Cube

Handling boundary data transfer requires an unfolded map (Connectivity Table) to handle coordinate rotations.



CONN_TABLE Logic:

P1 North → P5 South
P1 East → P2 West

**Handles axis swaps & reversals*

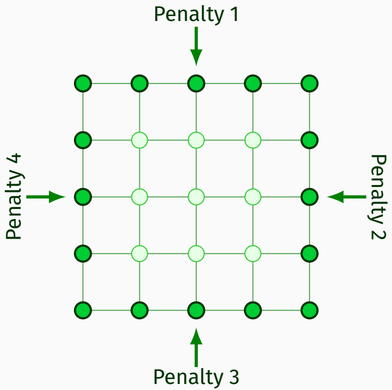
The Penalty Term (SAT)

Concept: Instead of imposing hard boundaries, we add a **Penalty Term** to the RHS equation.

Formula: Applied *only* on inflow boundaries ($V_n < 0$):

$$\text{Pen} = \frac{1}{w}\frac{V_n - |V_n|}{2}(q_{out} - q_{in})$$

- **Interior:** Penalty = 0.
- **Boundary:** Forces $q_{in} \to q_{out}$.



Matrix View of a Face: Penalty is only non-zero at boundaries.

Time Integration: LSRK Scheme

To advance the semi-discrete system $\frac{d\mathbf{U}}{dt} = \mathbf{R}(\mathbf{U})$, we employ a **Low-Storage Runge-Kutta (LSRK)** scheme.

Scheme Details (Carpenter & Kennedy, 1994):

- **Order:** 4th-order accurate (matches high-order spatial accuracy).
- **Stages:** 5 stages per time step (LSRK54).

Time Step Selection:

$$\Delta t = \frac{\mathrm{CFL}}{2\nu_{\max}} \frac{2}{N^2}$$

Note that the time step scales with $O(N^{-2})$ due to the clustering of LGL nodes near the element boundaries.

Numerical Results

Test Case: Solid Body Rotation

Setup:

- Advection of a Gaussian Bell.
- Velocity $u_0 = 2\pi$ (One full revolution in $T = 1.0$).
- Flow angle $\alpha = \pi/4$ (Flows over corners/edges).

Visual Result:

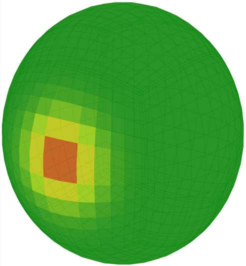


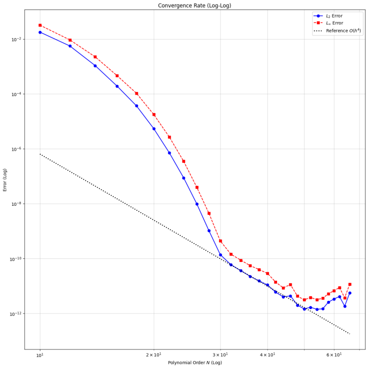
Figure 1: 3D Scalar field ϕ advecting over the Cubed Sphere.

Convergence Analysis

We evaluated L_2 and L_∞ errors for polynomial orders $N = 10$ to 66.

Regime Analysis:

- **Spectral Regime ($10 \leq N \leq 32$):**
- Dominant spectral convergence.
- Error decays exponentially from 10^{-2} to 10^{-10}.
- **Temporal Saturation ($32 < N \leq 42$):**
- Spatial error becomes negligible.
- Total error is dominated by the 4th-order accuracy of the LSRK time-stepping scheme.



Log-Log Convergence Plot

Conclusion

Summary & Conclusion

1. **Framework:** Successfully implemented a High-Order Spectral Element Solver on the Cubed Sphere.
2. **Geometry:** Equiangular projection avoids pole singularities.
3. **Stability:** The **SAT Penalty Term** effectively handles the discontinuous grid interfaces (as visualized in the matrix diagram).
4. **Accuracy:** Numerical experiments confirm **spectral convergence**, validating the implementation against theoretical expectations.

R. D. Nair, S. J. Thomas, and R. D. Loft.
A Discontinuous Galerkin Transport Scheme on the Cubed Sphere.
Monthly Weather Review, 133:814–828, 2005.

Penalty Term Code Implementation Tutorial.
Internal Course Material, 2025.