Sampling Lovász local lemma for general constraint satisfaction solutions in near-linear time

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Joint work with Kun He(CAS) and Yitong Yin(Nanjing University)

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- Example(k-SAT): Boolean variables $V = \{x_1, x_2, x_3, x_4, x_5\}$

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_3 \vee x_4 \vee \neg x_5)$$

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- **Constructive(Algorithmic) LLL**[MT10]:

$$\mathrm{e}p\Delta \leq 1 \implies \text{solution can be found very efficiently}$$

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generate a uniform random $\boldsymbol{x} \in \mathcal{Q}$;
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- Satisfying solutions may be exponentially rare!

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- Application: inference in probabilistic graphical models

Gibbs distribution $\quad \mu(\boldsymbol{x}) \propto \Phi(\boldsymbol{x}) = \prod_{c \in \mathcal{C}} c(\boldsymbol{x}_{\mathrm{vbl}(c)})$

$$\text{where each } c : \bigotimes_{i \in \mathrm{vbl}(c)} Q_i \to \mathbb{R}_{\geq 0}$$

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[i] Monotone CNF: all variables appear positively, e.g. $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_3 \vee x_4 \vee x_5)$

[ii] $\lesssim$ hides lower order items, e.g., k, q.

[iii] s: two independent clauses share **at least** s variables

[iv] atomic means each constraint of the CSP has exactly one forbidden configuration

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- It can be seen there are mainly two lines of work, taking different methods:
  - The line of work [Moi19, GLLZ19, JPV21b] applies the coupling and linear programming method, initiated by Moitra. This method is deterministic and made to work for general CSP instances in [JPV21b]. However, this method suffers from a $n^{\mathrm{poly}(k,\Delta,\log q)}$ running time, which is exponential if $k, \Delta = \Omega(1)$.

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  - Another line of work [FGYZ21, FHY21, JPV21a, HSW21] focuses on fast sampling and uses the static mark/unmark paradigm(later refined and generalized to entropy compression/state tensorization) to overcome the connectivity barrier in solution spaces.

A missing piece

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- It is possible that the non-atomicity of general CSPs might have imposed greater challenges to the sampling LLL than to its constructive counterpart.
- A major open problem:
  Is there a fast algorithm for general CSP instances in the LLL regime?

Our results

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This work	general CSP	$p\Delta^{7} \lesssim 1$	$\mathrm{poly}(q,k,\Delta)\cdot n\log n$ expected	A new marginal sampler

Our results(sampling)

Theorem 1.1(informal)

There is an algorithm such that given as input any $\varepsilon \in (0,1)$ and any CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$ with n variables satisfying

$$q^2 \cdot k \cdot p \cdot \Delta^7 \le \frac{1}{150\mathrm{e}^3}, \tag{1}$$

the algorithm terminates within $\mathrm{poly}(q, k, \Delta) \cdot n \log\left(\frac{n}{\varepsilon}\right)$ time in expectation and outputs an almost uniform sample of satisfying assignments for Φ within ε total variation distance.

Our results(inference)

Theorem 1.5(informal)

There is an algorithm such that given as input any $\varepsilon \in (0, 1)$, any CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$ satisfying (1), and any $v \in V$, the algorithm returns a random value $x \in Q_v$ distributed approximately as μ_v within total variation distance ε, within $\mathrm{poly}\,(q, k, \Delta, \log(1/\varepsilon))$ time in expectation.

Theorem 1.6(informal)

There is an algorithm such that given as input any $\varepsilon, \delta \in (0, 1)$, any CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$ satisfying (1), and any $v \in V$, the algorithm returns for every $x \in Q_v$ an ε-approximation of the marginal probability $\mu_v(x)$ within $\mathrm{poly}\,(q, k, \Delta, 1/\varepsilon, \log(1/\delta))$ time with probability at least $1 - \delta$.

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- Our sampler relies on a local marginal sampler inspired from [AJ21], which achieves sublinear running time in inference problem.

The marginal distribution

- μ: uniform distribution over all CSP solutions Ω
- μ_v: distribution of X_v where $X \sim \mu$
- μ_v^σ: μ_v conditional on some partial assignment σ
- partial assignment: assignment only on a subset of variables

Local uniformity

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Local uniformity([HSS11])

Given a CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$, if $\mathrm{e}p\Delta < 1$, then for any variable $v \in V$ and any value $x \in Q_v$, it holds that

$$\frac{1}{q_v} - \eta \leq \mu_v(x) \leq \frac{1}{q_v} + \eta,$$

where $\eta = (1 - \mathrm{e}p)^{-\Delta} - 1$.

Local uniformity

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Figure: Uniform distribution over Q_v

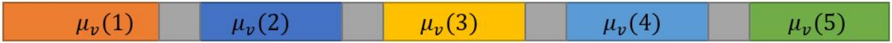

Figure: the marginal distribution $\mu_v(\cdot)$

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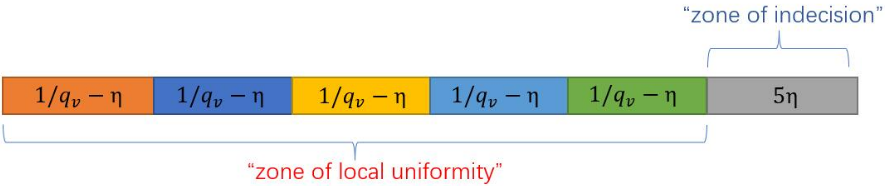

Figure: the marginal distribution $\mu_v(\cdot)$, after rearranging

Marginal sampler, first try

- Let $\theta_v = \frac{1}{q_v} - \eta$, then the "zone of local uniformity" has size $q_v\theta_v$, and the "zone of indecision" has size $1 - q_v\theta_v$.
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- Suppose we draw $r \in [0, 1)$ first, if $r \leq q_v\theta_v$, then we already know the result!
- What if $r > q_v\theta_v$?
- The distribution in the "zone of local indecision" is a linear transform $\mathcal{D}$ of the marginal distribution μ_v:

$$\forall x \in Q_v, \quad \mathcal{D}(x) = \frac{\mu_v(x) - \theta_v}{1 - q_v\theta_v}$$

Marginal sampler, first try

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- Sampling from $\mu_v(\cdot) \implies$ Sampling from $\mathcal{D}(\cdot)$ with probability $1 - q_v \theta_v$.
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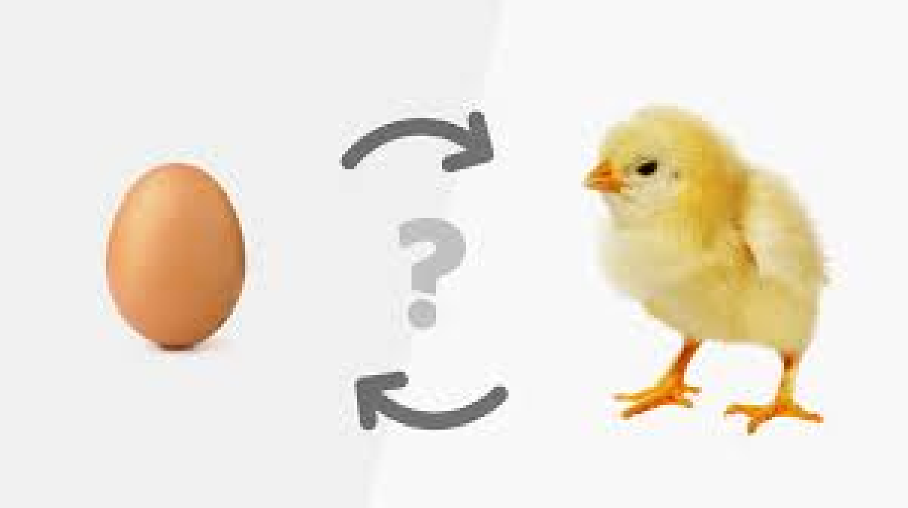
Chicken-egg dilemma?



Back to rejection sampling

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Sampling from $\mu_v(\cdot)$ using rejection sampling

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- This is efficient if the connected component containing v is logarithmically small!

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A factorizing process

- Repeatedly choose closest not assigned variable u in the same connected component as v, and use $\mathcal{O}$ to draw a value for u(conditional on current partial assignment) until no such variable exist.
- Ideally, if p is small enough, the connected component containing v is logarithmically small with high probability.

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- This is resolved by the idea of "freezing" constraints with a high violation probability, which dates back to [Bec91].
- Mark a constraint as frozen and all its unassigned variables as fixed if its conditional violation probability exceeds some threshold $p' \geq p$. Also we fix all assigned variables.

Factorizing(cont'd)

A factorizing process, adapted in LLL regime

- Repeatedly choose a closest not fixed variable u in the same component as v, and use $\mathcal{O}$ to draw a value for u(conditional on current partial assignment) until no such variable exist.
- It can be shown that at the end of this process, with properly chosen p', the connected component containing v is logarithmically small with high probability.

Putting things together

- Sampling from $\mu_v(\cdot) \Longrightarrow$ Sampling from $\mathcal{D}(\cdot)$ with probability $1 - q_v\theta_v$.
- Sampling from $\mathcal{D}(\cdot) \Longrightarrow$ Sampling from $\mu_v(\cdot)$.

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- Recurse!
- Rejection Sampling serves as the basis of the recursion.
- The recursion converges if $1 - q_v\theta_v$ is small relative to the number of recursive calls needed in the factorizing process.

The marginal sampler

Algorithm for sampling from $\mu_v^\sigma(\cdot)$

1. Choose $r \in [0, 1)$ uniformly at random
2. If $r < q_v\theta_v$, return the $\lceil r/\theta_v \rceil$-th value in Q_v
3. Otherwise, return a sample from $\mathcal{D}_v^\sigma = \frac{\mu_v^\sigma - \theta_v}{1 - q_v\theta_v}$

The marginal sampler(cont'd)

Algorithm for sampling from $\mathcal{D}_v^\sigma(\cdot)$

1. If v is already factorized conditioning on σ, return a sample from $\mathcal{D}_v^\sigma(\cdot)$ using Bernoulli factory algorithm with rejection sampling procedure as input coins.
2. Otherwise,
   2.1 Properly choose some variable u.
   2.2 Choose $r \in [0, 1)$ uniformly at random.
   2.3 If $r < q_v\theta_v$, set $\sigma(u)$ as the $\lceil r/\theta_v \rceil$-th value in Q_u.
   2.4 Otherwise, set $\sigma(u)$ as a sample from $\mathcal{D}_u^\sigma$ by recursively calling this algorithm.
   2.5 return a sample from $\mathcal{D}_v^\sigma$ by recursively calling this algorithm.

From marginal sampler to a full sampler

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From marginal sampler to a full sampler

- Sequential sampling while using the same idea of "freezing" with the same parameter p' to guarantee the properties in the local lemma regime.
- It can be shown that after a first sequential sampling, the whole formula scatters into connected components with logarithmic sizes with high probability.
- Then use rejection sampling to complete the assignment.

The sampling algorithm

Algorithm for sampling from μ

1. Set X as the empty assignment
2. For each $v \in V$
   2.1 If v is not fixed conditioning on X, sample $X(v)$ from μ_v^X using the marginal sampler introduced before.
3. Complete X using rejection sampling.

Analysis of efficiency

- Construct an abstract data structure recursive cost tree that for each (σ, v), if one calls the algorithm for sampling from μ_v^σ, calculates for each (X, u) the probability that the algorithm for sampling from $\mathcal{D}_u^X$ is called

Analysis of efficiency

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- Still many technicalities are omitted.

Thank you!

arXiv:2204.01520

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