# Sampling Lovász local lemma for general constraint satisfaction solutions in near-linear time

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# Constraint Satisfaction Problem

- $\Phi = (V, \mathcal{Q}, \mathcal{C})$
- **Variables**: $V = \{x_1, x_2, \ldots, x_n\}$ with finite domains $Q_v$ for each $v \in V$
- (local) **Constraints**: $\mathcal{C} = C_1, C_2, \ldots, C_m$
  - each $c \in \mathcal{C}$ is defined on a subset $\mathsf{vbl}(c)$ of variables

$$c : \bigotimes_{v \in \mathsf{vbl}(c)} Q_v \to \{\texttt{True}, \texttt{False}\}$$

- **CSP formula**: $\forall \boldsymbol{x} \in \mathcal{Q} = \bigotimes_{v \in V} Q_v$

$$\Phi(\boldsymbol{x}) = \bigwedge_{c \in \mathcal{C}} c(\boldsymbol{x}_{\mathsf{vbl}(c)})$$

- Example($k$-SAT): Boolean variables $V = \{x_1, x_2, x_3, x_4, x_5\}$

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_3 \vee x_4 \vee \neg x_5)$$

# Lovász Local Lemma

- Variables take independent random values $X_1, X_2, \ldots, X_n$
- **Violation probability**: each $c \in \mathcal{C}$ is violated with probability $\leq p$
- **Constraint Degree**: each $c \in \mathcal{C}$ shares variable with $\leq \Delta$ constraints $c' \in \mathcal{C}$(including $c$ itself), i.e., $\mathrm{vbl}(c) \cap \mathrm{vbl}(c') \neq \emptyset$
- **LLL**[EL75]:

$$\mathrm{e}p\Delta \leq 1 \implies \text{ solution exists}$$

- **Constructive(Algorithmic) LLL**[MT10]:

$$\mathrm{e}p\Delta \leq 1 \implies \text{ solution can be found very efficiently}$$

# Sampling & Counting LLL

## Sampling & Counting LLL

Input: a CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$
Output:

- (sampling): uniform random satisfying solution
- (counting): number of satisfying solutions

- $\mu$: uniform distribution over all satisfying solutions of $\Phi$

## Rejection Sampling

generate a uniform random $\boldsymbol{x} \in \mathcal{Q}$;
if $\Phi(\boldsymbol{x}) = \texttt{True}$ then accept else reject;
$\mu$ is the distribution of $(\boldsymbol{x} \mid \text{accept})$

- Satisfying solutions may be exponentially rare!

# Sampling & Counting LLL

- Exact counting is #P-hard
- 

- Application: inference in probabilistic graphical models

$$\text{Gibbs distribution} \quad \mu(\boldsymbol{x}) \propto \Phi(\boldsymbol{x}) = \prod_{c \in \mathcal{C}} c(\boldsymbol{x}_{\mathsf{vbl}(c)})$$

$$\text{where each } c : \bigotimes_{i \in \mathsf{vbl}(c)} Q_i \to \mathbb{R}_{\geq 0}$$

- Inference: $\Pr_{\boldsymbol{X} \sim \mu} [X_i = \cdot \mid \boldsymbol{X}_S = \boldsymbol{x}_S]$
- Sampling almost uniform constraint satisfcation solutions under LLL-like condition?

# Sampling & Counting LLL

| Paper | Instance | Condition | Complexity | Technique |
|---|---|---|---|---|
| [HSZ19] | monotone CNF[i] | $p\Delta^2 \lesssim 1$[ii] | $\mathrm{poly}(k,\Delta)\cdot n\log n$ | MCMC |
| [GJL19] | general CSP | $s \geq \min(\log \Delta k, k/2)$[iii] $p\Delta^2 \lesssim 1$ | $\mathrm{poly}(k,\Delta)\cdot n$ | PRS |
| [BGG+19] | monotone CNF | $p\Delta^2 \gtrsim 1$ | **NP**-hard | lower bound |
| [Moi19] | CNF | $p\Delta^{60} \lesssim 1$ | $n^{\mathrm{poly}(k,\Delta)}$ | Mark/unmark+Coupling+LP |
| [GLLZ19] | Hypergraph coloring | $p\Delta^{16} \lesssim 1$ | $n^{\mathrm{poly}(k,\Delta,\log q)}$ | Adaptive mark/unmark+Coupling+LP |
| [FGYZ21] | CNF | $p\Delta^{20} \lesssim 1$ | $\mathrm{poly}(k,\Delta)\cdot \tilde{O}(n^{1.001})$ | Mark/unmark+MCMC |
| [FHY21] | atomic CSP[iv] | $p\Delta^{350} \lesssim 1$ | $\mathrm{poly}(q,k,\Delta)\cdot \tilde{O}(n^{1.001})$ | Entropy Compression+MCMC |
| [JPV21b] | general CSP | $p\Delta^{7} \lesssim 1$ | $n^{\mathrm{poly}(k,\Delta,\log q)}$ | Adaptive mark/unmark+Coupling+LP |
| [JPV21a] | atomic CSP | $p\Delta^{7.043} \lesssim 1$ | $\mathrm{poly}(q,k,\Delta)\cdot \tilde{O}(n^{1.001})$ | Entropy Compression+MCMC |
| [HSW21] | atomic CSP | $p\Delta^{5.714} \lesssim 1$ | $\mathrm{poly}(q,k,\Delta)\cdot n\log n$ expected | State tensorization+MCMC |

[i] Monotone CNF: all variables appear positively, e.g. $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_3 \vee x_4 \vee x_5)$

[ii] $\lesssim$ hides lower order items, e.g., $k$, $q$.

[iii] $s$: two independent clauses share **at least** $s$ variables

[iv] atomic means each constraint of the CSP has exactly one forbidden configuration

# Sampling & Counting LLL

- It can be seen there are mainly two lines of work, taking different methods:
  - The line of work [Moi19, GLLZ19, JPV21b] applies the coupling and linear programming method, initiated by Moitra. This method is deterministic and made to work for general CSP instances in [JPV21b]. However, this method suffers from a $n^{\mathrm{poly}(k,\Delta,\log q)}$ running time, which is exponential if $k, \Delta = \Omega(1)$.
  - Another line of work [FGYZ21, FHY21, JPV21a, HSW21] focuses on fast sampling and uses the static mark/unmark paradigm(later refined and generalized to entropy compression/state tensorization) to overcome the connectivity barrier in solution spaces.

# A missing piece

- All existing fast algorithms for sampling LLL relied on some projection of the solution space to a much smaller space where the barrier of disconnectivity could be circumvented because the images of the projection might collide and were well connected.
- Such projection may be hard to find (or not exist) for general CSP instances
- It is possible that the non-atomicity of general CSPs might have imposed greater challenges to the sampling LLL than to its constructive counterpart.
- A major open problem:
  Is there a fast algorithm for general CSP instances in the LLL regime?

# Our results

| Paper | Instance | Condition | Complexity | Technique |
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| This work | general CSP | $p\Delta^{7} \lesssim 1$ | $\mathrm{poly}(q,k,\Delta)\cdot n\log n$ expected | A new marginal sampler |

# Our results(sampling)

**Theorem 1.1(informal)**

There is an algorithm such that given as input any $\varepsilon \in (0,1)$ and any CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$ with $n$ variables satisfying

$$q^2 \cdot k \cdot p \cdot \Delta^7 \leq \frac{1}{150\mathrm{e}^3}, \tag{1}$$

the algorithm terminates within $\mathrm{poly}(q, k, \Delta) \cdot n \log\left(\frac{n}{\varepsilon}\right)$ time in expectation and outputs an almost uniform sample of satisfying assignments for $\Phi$ within $\varepsilon$ total variation distance.

# Our results(inference)

**Theorem 1.5(informal)**

There is an algorithm such that given as input any $\varepsilon \in (0,1)$, any CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$ satisfying (1), and any $v \in V$, the algorithm returns a random value $x \in Q_v$ distributed approximately as $\mu_v$ within total variation distance $\varepsilon$, within $\mathrm{poly}\,(q, k, \Delta, \log(1/\varepsilon))$ time in expectation.

**Theorem 1.6(informal)**

There is an algorithm such that given as input any $\varepsilon, \delta \in (0,1)$, any CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$ satisfying (1), and any $v \in V$, the algorithm returns for every $x \in Q_v$ an $\varepsilon$-approximation of the marginal probability $\mu_v(x)$ within $\mathrm{poly}\,(q, k, \Delta, 1/\varepsilon, \log(1/\delta))$ time with probability at least $1 - \delta$.

# Some remarks

- As in the case of algorithmic LLL [MT10, HV15], we assume an abstraction of constraint evaluations, because arbitrary constraint functions defined on a super-constant number of variables can be highly nontrivial to express and evaluate.
  - We assume we can efficiently check if some constraint $c$ is already satisfied under some assignment $\mathcal{Q}_\Lambda = \bigotimes_{v \in \Lambda} Q_v$ specified on a subset $\Lambda \subseteq \mathrm{vbl}(c)$ of variables
- Our sampler is perfect if we can further determinsitically estimate the probability that a constraint $c$ is violated given a partially specified assignment $\sigma$, possibly with gaps.
- Our sampler relies on a local marginal sampler inspired from [AJ21], which achieves sublinear running time in inference problem.

# The marginal distribution

- $\mu$: uniform distribution over all CSP solutions $\Omega$
- $\mu_v$: distribution of $X_v$ where $X \sim \mu$
- $\mu_v^\sigma$: $\mu_v$ conditional on some partial assignment $\sigma$
- partial assignment: assignment only on a subset of variables

# Local uniformity

- A crucial property in the local lemma regime is the marginal distribution on any variable is close to uniform.

**Local uniformity([HSS11])**

Given a CSP formula $\Phi = (V, \mathcal{Q}, \mathcal{C})$, if $\mathrm{e}p\Delta < 1$, then for any variable $v \in V$ and any value $x \in Q_v$, it holds that

$$\frac{1}{q_v} - \eta \le \mu_v(x) \le \frac{1}{q_v} + \eta,$$

where $\eta = (1 - \mathrm{e}p)^{-\Delta} - 1$.

# Local uniformity

- Let's see an example first. Assume $Q_v = \{1, 2, 3, 4, 5\}$.


Figure: Uniform distribution over $Q_v$

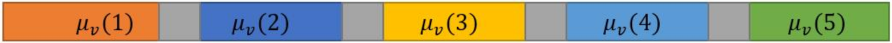



Figure: the marginal distribution $\mu_v(\cdot)$

# Local uniformity

- Sampling from $\mu_v(\cdot)$ can be viewed as choosing $r \in [0, 1)$ randomly and taking the interval $r$ falls into as the desired sample

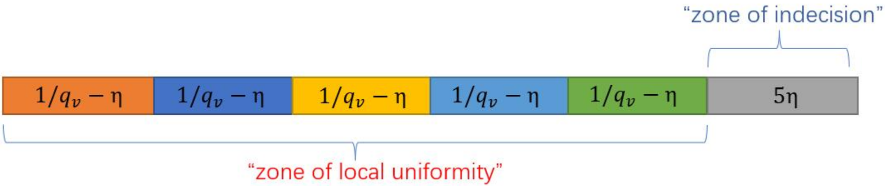



Figure: the marginal distribution $\mu_v(\cdot)$, after rearranging

# Marginal sampler, first try

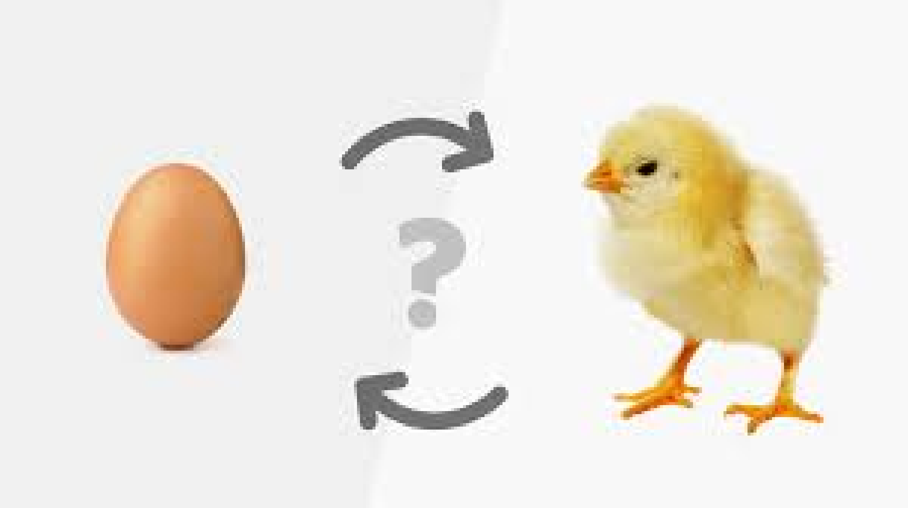
- Let $\theta_v = \frac{1}{q_v} - \eta$, then the "zone of local uniformity" has size $q_v\theta_v$, and the "zone of indecision" has size $1 - q_v\theta_v$.
- Suppose we draw $r \in [0, 1)$ first, if $r \le q_v\theta_v$, then we already know the result!
- What if $r > q_v\theta_v$?
- The distribution in the "zone of local indecision" is a linear transform $\mathcal{D}$ of the marginal distribution $\mu_v$:

$$\forall x \in Q_v, \quad \mathcal{D}(x) = \frac{\mu_v(x) - \theta_v}{1 - q_v\theta_v}$$

# Marginal sampler, first try

- How to sample from $\mathcal{D}(\cdot) = \frac{\mu_v(\cdot) - \theta_v}{1 - q_v\theta_v}$?
- Suppose we have access to an oracle to can sample from $\mu_v(\cdot)$, we can use known constructions of Bernoulli Factory algorithms[NP05, Hub16, DHKN17] to sample from $\mathcal{D}(\cdot)$.
- Sampling from $\mu_v(\cdot) \Longrightarrow$ Sampling from $\mathcal{D}(\cdot)$ with probability $1 - q_v\theta_v$.
- Sampling from $\mathcal{D}(\cdot) \Longrightarrow$ Sampling from $\mu_v(\cdot)$.

# Chicken-egg dilemma?



# Back to rejection sampling

- Note that we have a marginal sampler equipped using the idea of rejection sampling.

**Sampling from $\mu_v(\cdot)$ using rejection sampling**

Repeat the following procedure:

- Use rejection sampling to draw a sample $X \sim \mu$
- return $X_v$ as the sample

- Constraints satisfied by the partial assignment deconstructs $\Phi$ into connected components.
- This is efficient if the connected component containing $v$ is logarithmically small!

# Factorizing(with a marginal oracle)

- If we have an access of an oracle $\mathcal{O}$ that samples the (possibly conditional on partial assignments) marginal distribution on some variables other than $v$, can we use this oracle to sample from $\mu_v(\cdot)$?

## A factorizing process

- Repeatedly choose closest not assigned variable $u$ in the same connected component as $v$, and use $\mathcal{O}$ to draw a value for $u$(conditional on current partial assignment) until no such variable exist.
- Ideally, if $p$ is small enough, the connected component containing $v$ is logarithmically small with high probability.

# Frozen and fixed

- In our setting of local lemma regime, there's some issue when realizing such factorizing process: we may fall out of local lemma regime when conditioning on partial assignments.
- This is resolved by the idea of "freezing" constraints with a high violation probability, which dates back to [Bec91].
- Mark a constraint as frozen and all its unassigned variables as fixed if its conditional violation probability exceeds some threshold $p' \geq p$. Also we fix all assigned variables.

# Factorizing(cont'd)

**A factorizing process, adapted in LLL regime**

- Repeatedly choose a closest not fixed variable $u$ in the same component as $v$, and use $\mathcal{O}$ to draw a value for $u$(conditional on current partial assignment) until no such variable exist.
- It can be shown that at the end of this process, with properly chosen $p'$, the connected component containing $v$ is logarithmically small with high probability.

# Putting things together

- Sampling from $\mu_v(\cdot) \implies$ Sampling from $\mathcal{D}(\cdot)$ with probability $1 - q_v\theta_v$.
- Sampling from $\mathcal{D}(\cdot) \implies$ Sampling from $\mu_v(\cdot)$.
- Sampling from $\mu_v(\cdot) \implies$ First sampling from other variables to factorize the formula, then use rejection sampling to sample from $\mu_v(\cdot)$.
- Recurse!
- Rejection Sampling serves as the basis of the recursion.
- The recursion converges if $1 - q_v\theta_v$ is small relative to the number of recursive calls needed in the factorizing process.

# The marginal sampler

**Algorithm for sampling from $\mu_v^\sigma(\cdot)$**

1. Choose $r \in [0, 1)$ uniformly at random
2. If $r < q_v \theta_v$, return the $\lceil r/\theta_v \rceil$-th value in $Q_v$
3. Otherwise, return a sample from $\mathcal{D}_v^\sigma = \frac{\mu_v^\sigma - \theta_v}{1 - q_v \theta_v}$

# The marginal sampler(cont'd)

**Algorithm for sampling from $\mathcal{D}_v^\sigma(\cdot)$**

1. If $v$ is already factorized conditioning on $\sigma$, return a sample from $\mathcal{D}_v^\sigma(\cdot)$ using Bernoulli factory algorithm with rejection sampling procedure as input coins.
2. Otherwise,
   2.1 Properly choose some variable $u$.
   2.2 Choose $r \in [0, 1)$ uniformly at random.
   2.3 If $r < q_v\theta_v$, set $\sigma(u)$ as the $\lceil r/\theta_v \rceil$-th value in $Q_u$.
   2.4 Otherwise, set $\sigma(u)$ as a sample from $\mathcal{D}_u^\sigma$ by recursively calling this algorithm.
   2.5 return a sample from $\mathcal{D}_v^\sigma$ by recursively calling this algorithm.

# From marginal sampler to a full sampler

- Sequential sampling while using the same idea of "freezing" with the same parameter $p'$ to guarantee the properties in the local lemma regime.
- It can be shown that after a first sequential sampling, the whole formula scatters into connected components with logarithmic sizes with high probability.
- Then use rejection sampling to complete the assignment.

# The sampling algorithm

**Algorithm for sampling from $\mu$**

1. Set $X$ as the empty assignment
2. For each $v \in V$
   2.1 If $v$ is not fixed conditioning on $X$, sample $X(v)$ from $\mu_v^X$ using the marginal sampler introduced before.
3. Complete $X$ using rejection sampling.

# Analysis of efficiency

- Construct an abstract data structure recursive cost tree that for each $(\sigma, v)$, if one calls the algorithm for sampling from $\mu_v^\sigma$, calculates for each $(X, u)$ the probability that the algorithm for sampling from $\mathcal{D}_u^X$ is called
- Use linearity of expectation to analyze the possible length of path generated by another process $\mathrm{Path}(\sigma)$.
- A classical technique dates back to [Alo91]: If the length of path is too long, then there exists a large $\{2,3\}$-tree of several bad events, which happens with exponentially low probability.
- Still many technicalities are omitted.

# Thank you!

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