Local Gibbs Sampling Beyond Local Uniformity

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Joint work with:

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(Nanjing University)

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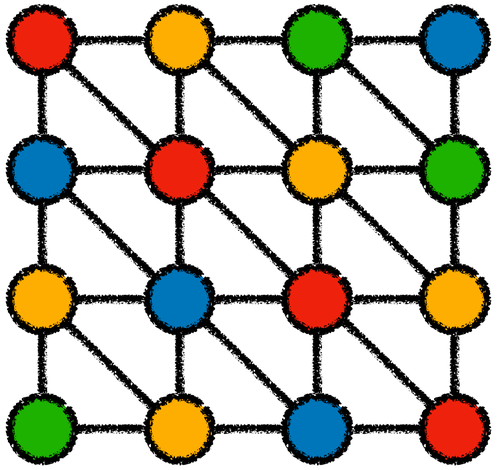
(Multi-) Spin Systems



graph $G = (V, E)$ $\quad q \geq 2$ states

configuration $\sigma \in [q]^V$

external fields $\lambda_v \in \mathbb{R}^q_{\geq 0}$ for each $v \in V$

interaction matrix $A_e \in \mathbb{R}^{q \times q}_{\geq 0}$ for each $e \in E$

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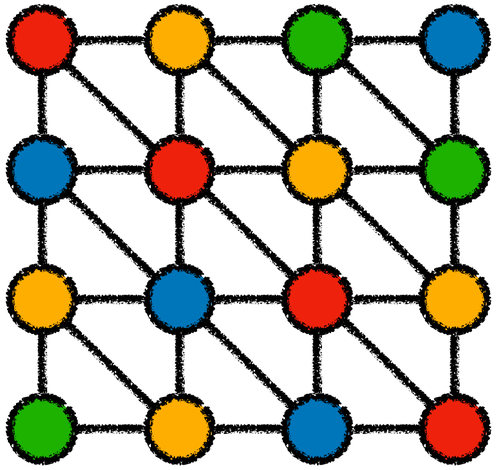

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Gibbs distribution: $\mu(\sigma) = \dfrac{w(\sigma)}{Z}$

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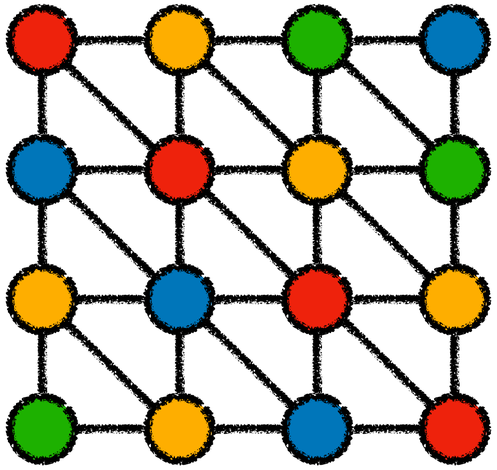

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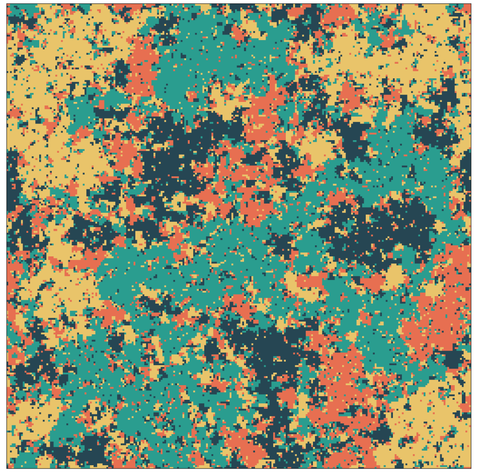

(Multi-) Spin Systems

Ising model

two states

external fields λ 1

interaction β 1

models weighted graph cuts

hardcore model

two states

external fields λ 1

interaction 1 0

models weighted independent sets

Potts model

q states

external fields 1

interaction β

others 1

$\beta > 1$ (ferromagnetic): models clusterings

$\beta < 1$ (antiferromagnetic): models (soft) $q-$colorings

Sampling from Spin Systems

Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$

Output: a random configuration $X \sim \mu$

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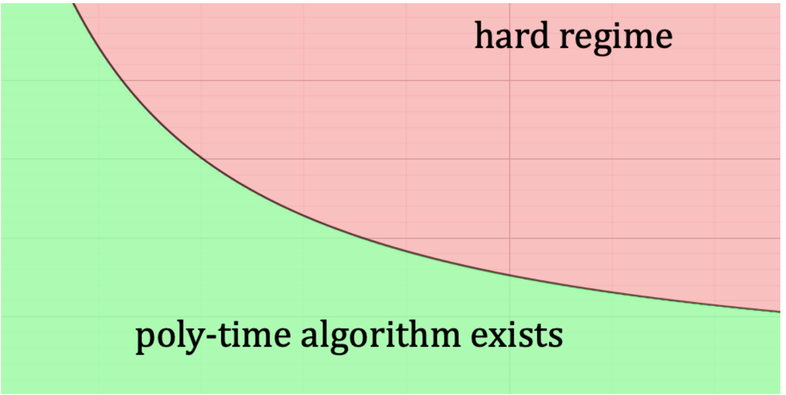

Computational Phase Transition

computational complexity of the sampling problem

changes sharply around some parameters of $\mathcal{S}$

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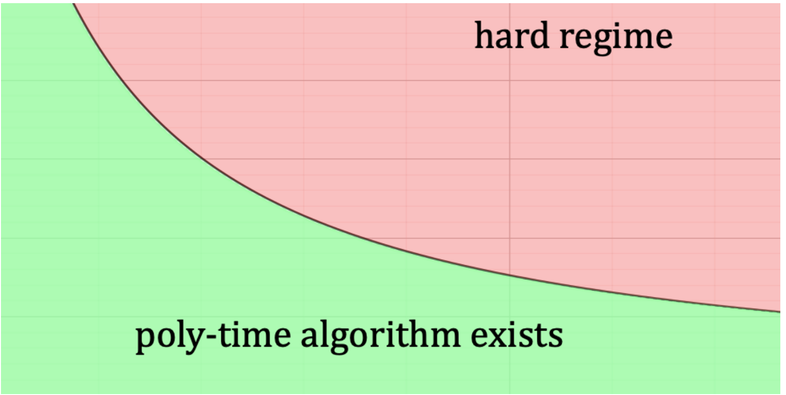


Computational Phase Transition

For some typical models (Ising, hardcore):

outside the "uniqueness regime": sampling is **NP**-hard! [SS '14, GŠV '16]

within the "uniqueness regime": poly-time algorithms exist! [Weitz '06, CLV '20]

Sampling from Spin Systems

Glauber dynamics

Start with arbitrary configuration σ with $w(\sigma) > 0$;

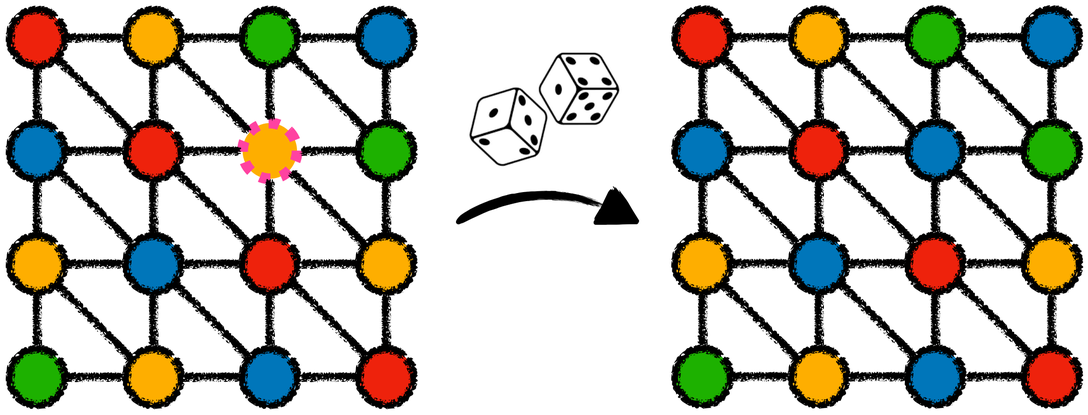
At each time $1 \leq t \leq T$:

Pick a vertex $v \in V$ uniformly at random;

Resample $\sigma_v \sim \mu_v\left(\cdot \mid \sigma_{V \setminus \{v\}}\right)$;

Return σ;

converges to μ as $T \to \infty$!



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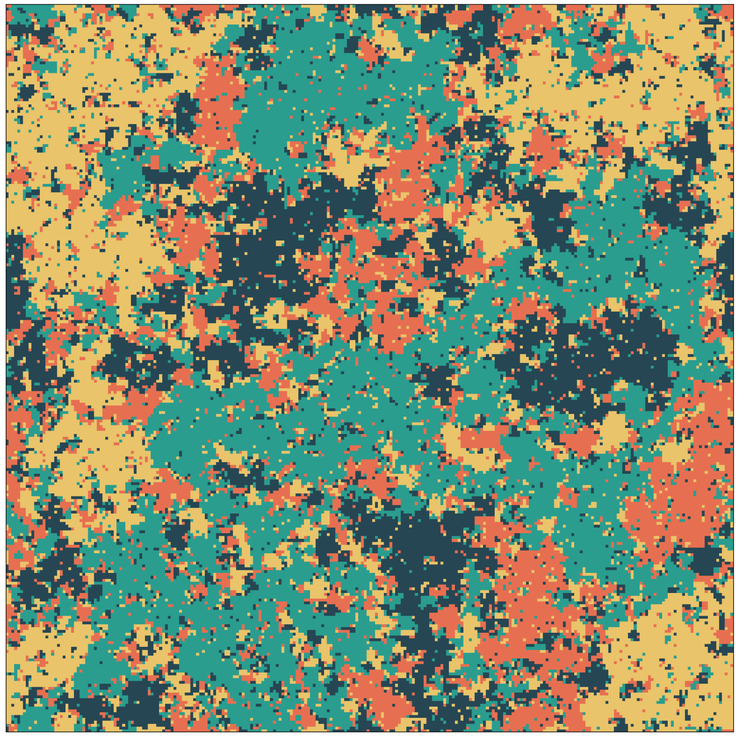
within the "uniqueness regime": **Glauber dynamics** mixes in near-linear time! [CLV '20]

(Bayesian) probabilistic inference

Given a partial configuration σ_Λ over $\Lambda \subseteq V$ and a vertex $v \in V \setminus \Lambda$,

estimate the conditional marginal probability distribution $\mu_v^{\sigma_\Lambda}(\,\cdot\,)$



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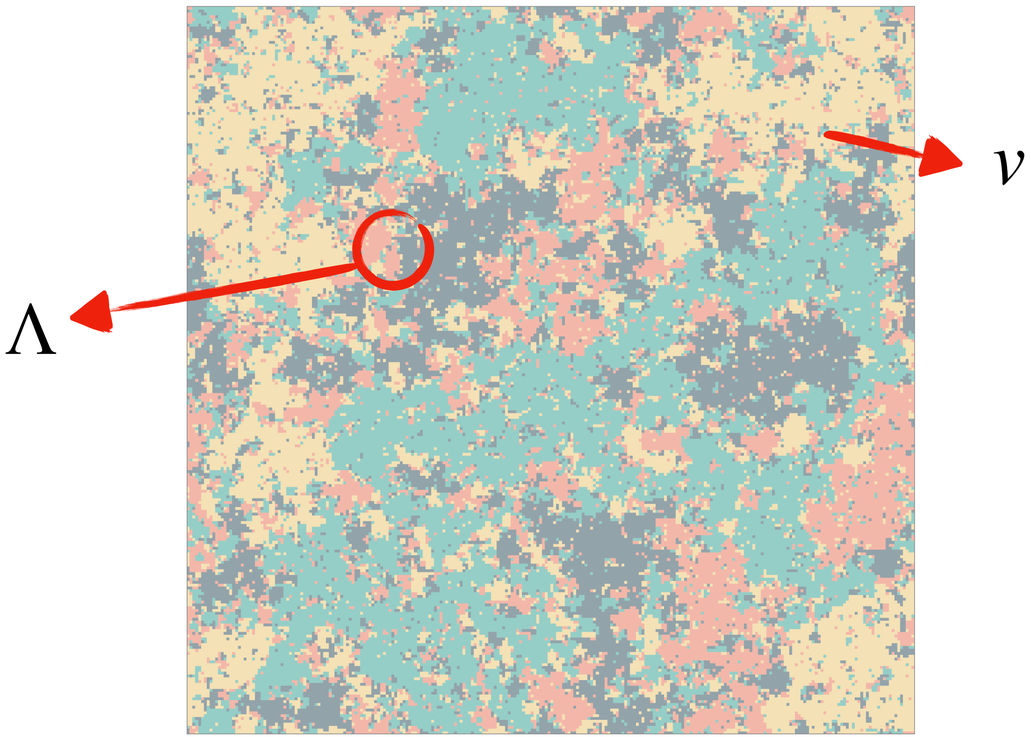

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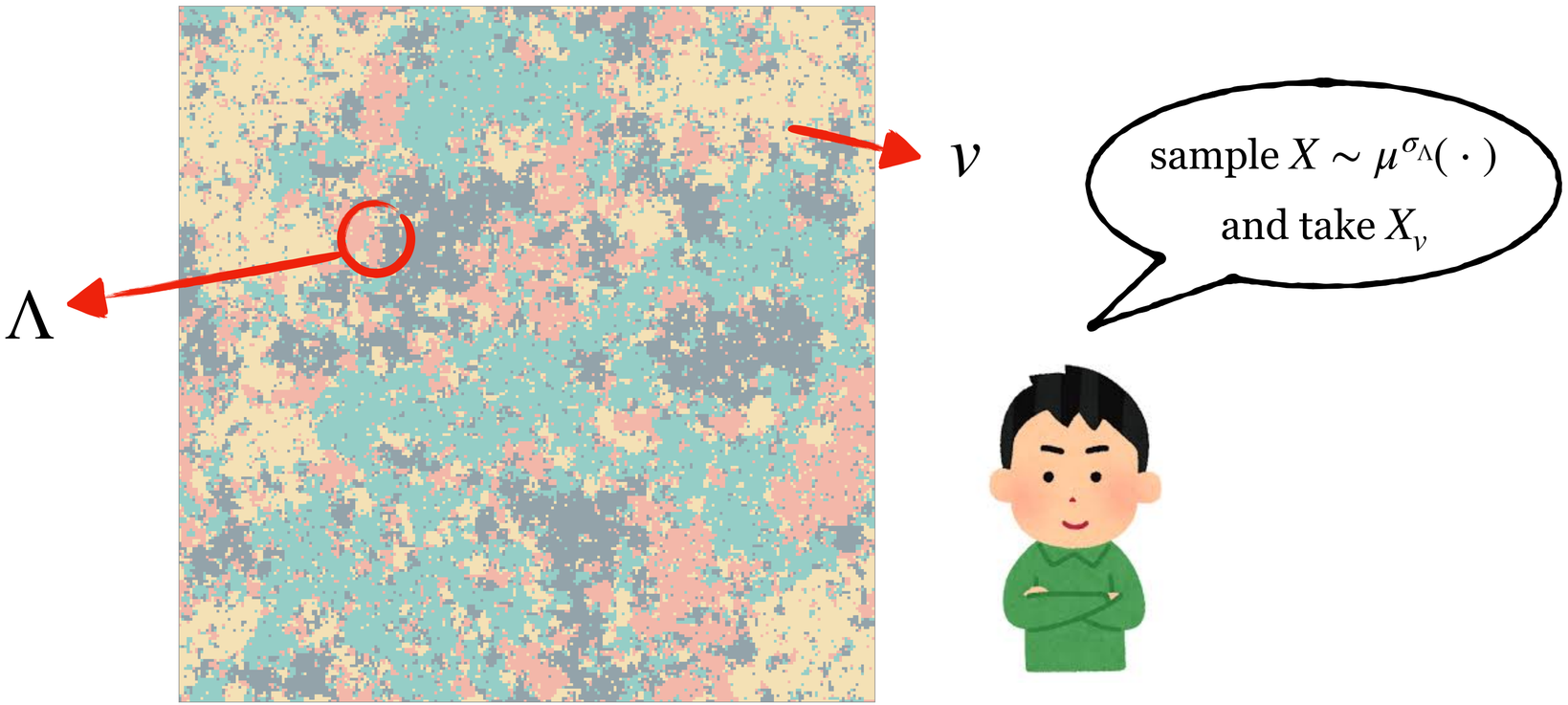

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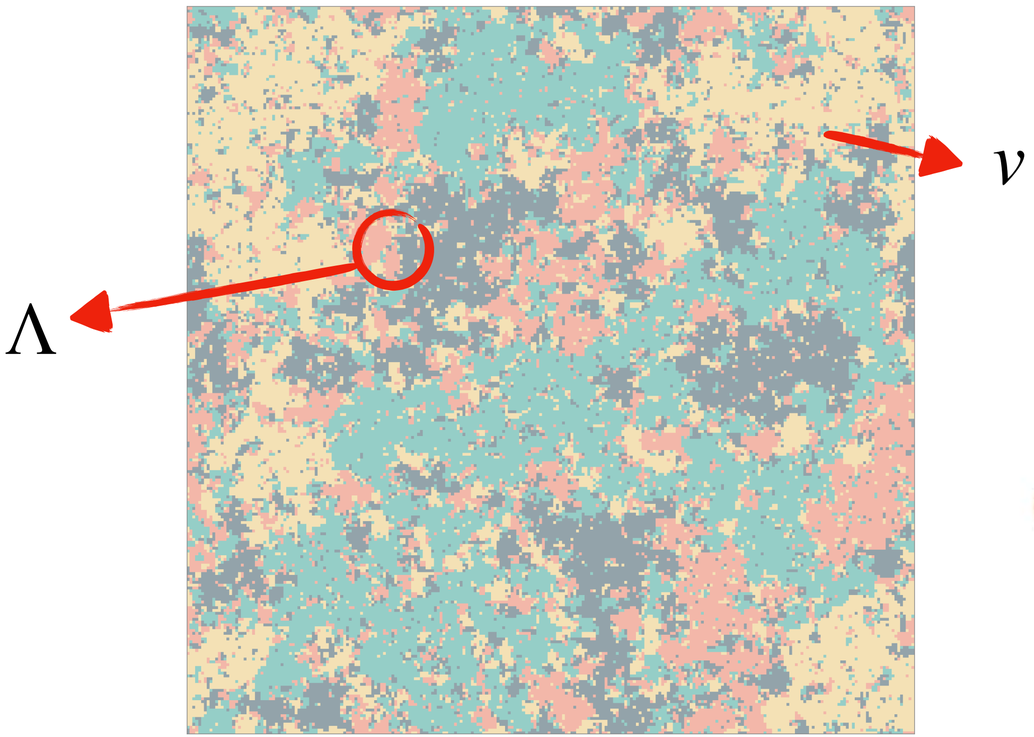

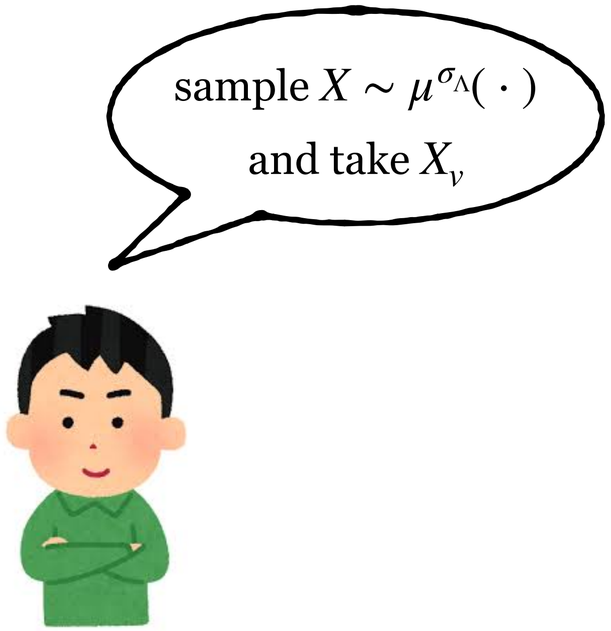

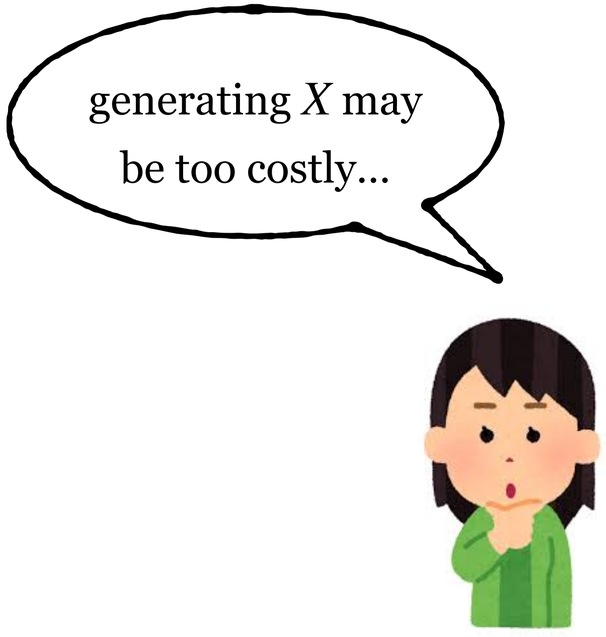

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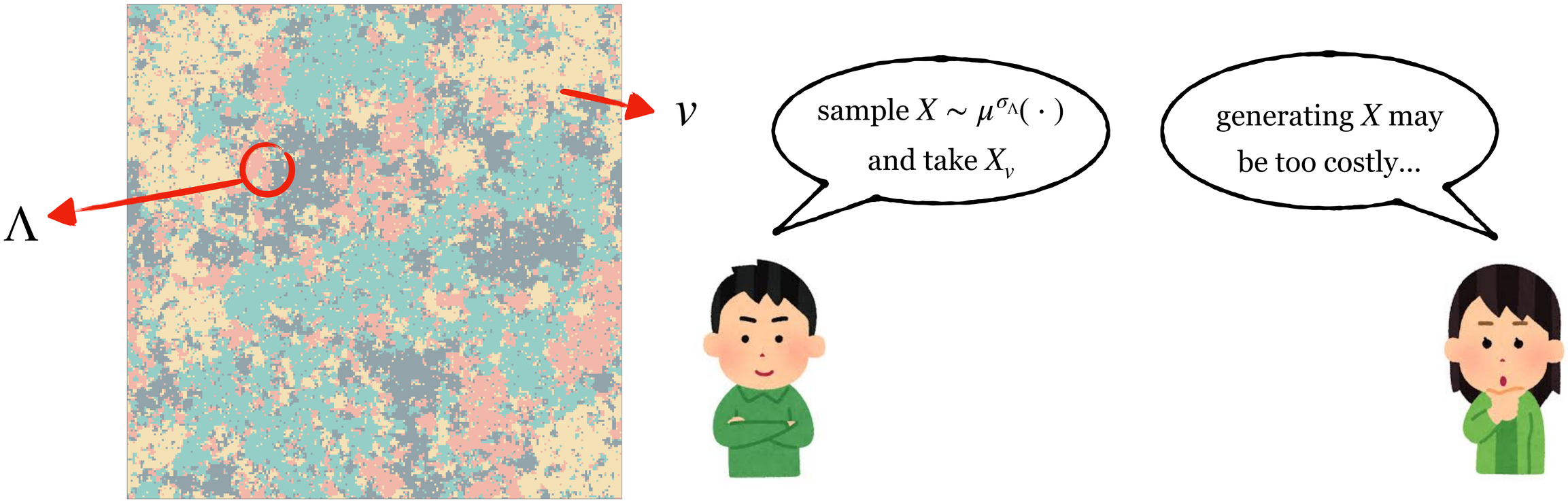

Can these locally defined sampling problems be solved at at a local cost?

Local Sampling

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Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$, a vertex subset $\Lambda \subseteq V$

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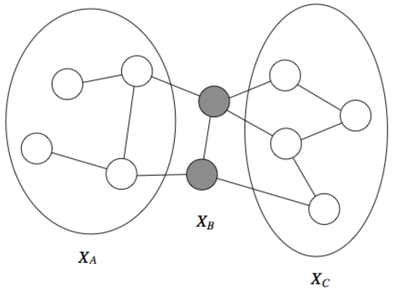
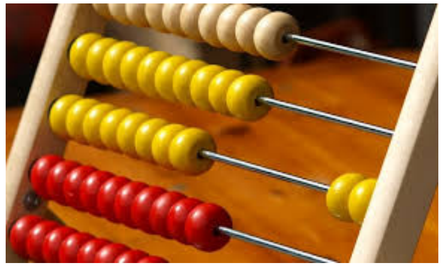
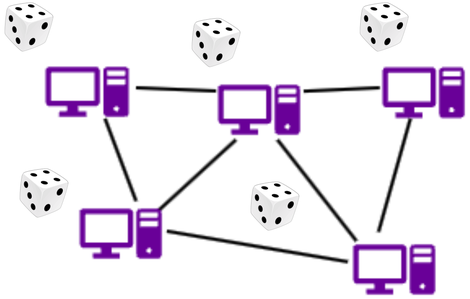
Applications:



probabilistic inference



deterministic (approx.) counting



distributed sampling

Related definition: local access to random objects [BRY '20]
(sampling variant of Local Computation Algorithms)

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Previous local sampling algorithms [AJ '22, FGWWY '23] require
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Instance	**Tractable regimes** (Δ: maximum degree)	
	Global sampling	**Local sampling**
Ising model	[1] $\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2}\right)$	[2] $\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)}\right)$
q-colorings	[3] $q \geq 1.809\Delta$	N/A

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Is local sampling significantly harder than global sampling?

Our results

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- first local sampler for near-critical Ising model;
- first local sampler for q-coloring (also near-critical);
- perfect samplers (no bias);
- expected linear running time: $O(\Delta \cdot |\Lambda|)$ for Ising;

  $O(\Delta^2 q \cdot |\Lambda|)$ for q-colorings.

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- extension of the "Coupling Towards The Past" framework in [FGWWY '23]

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$)

resample $\sigma_v \sim \mu_v\left(\cdot \mid \sigma_{V \setminus \{v\}}\right)$;

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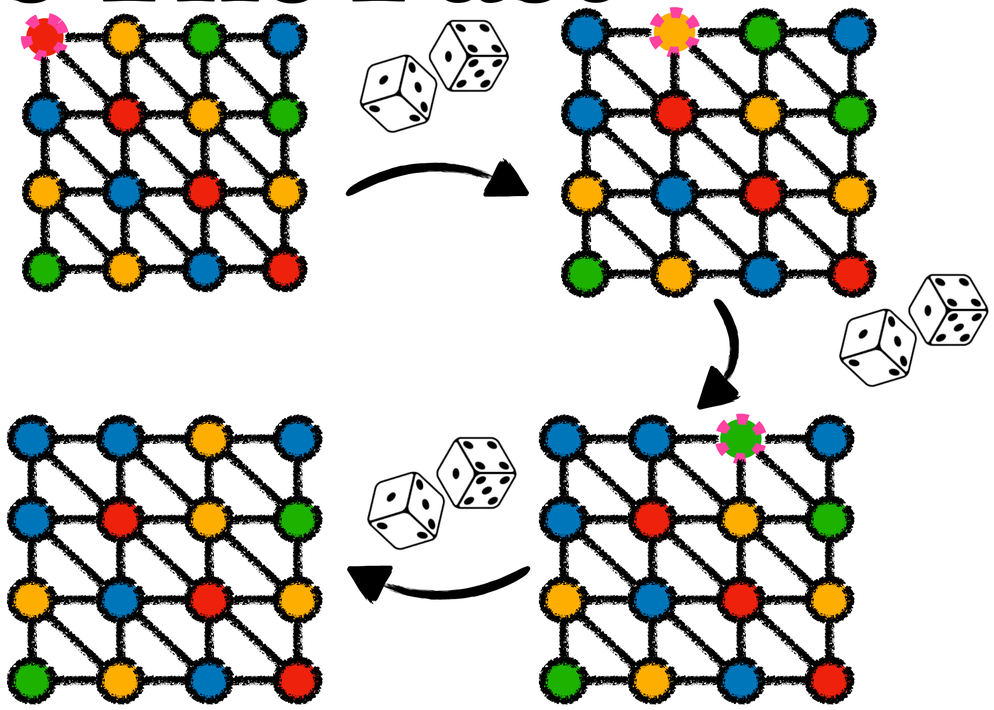

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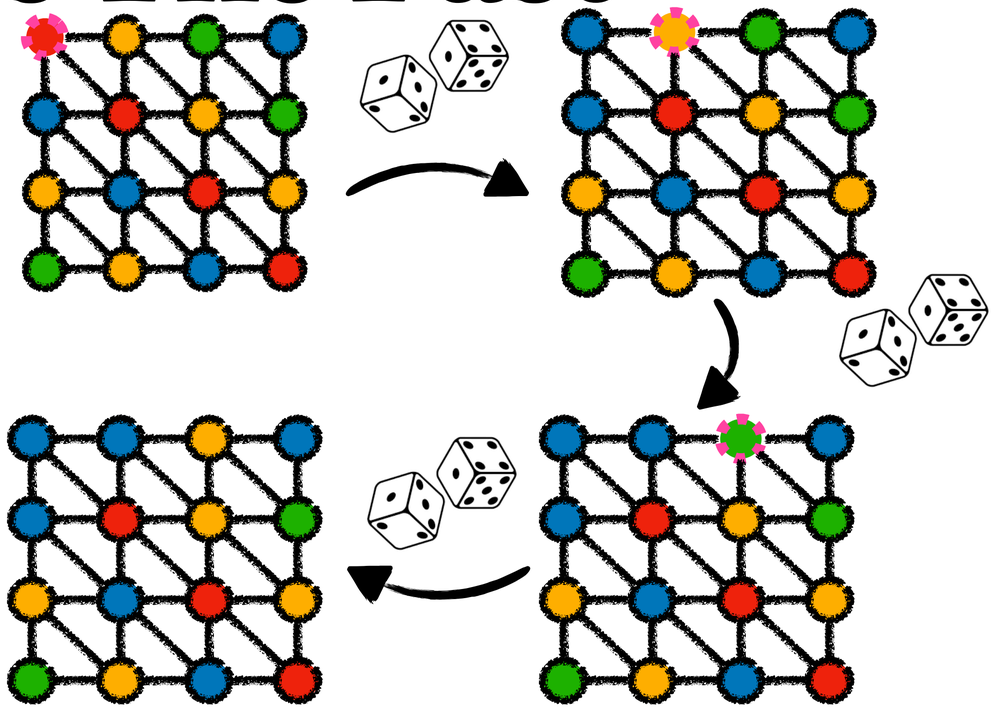
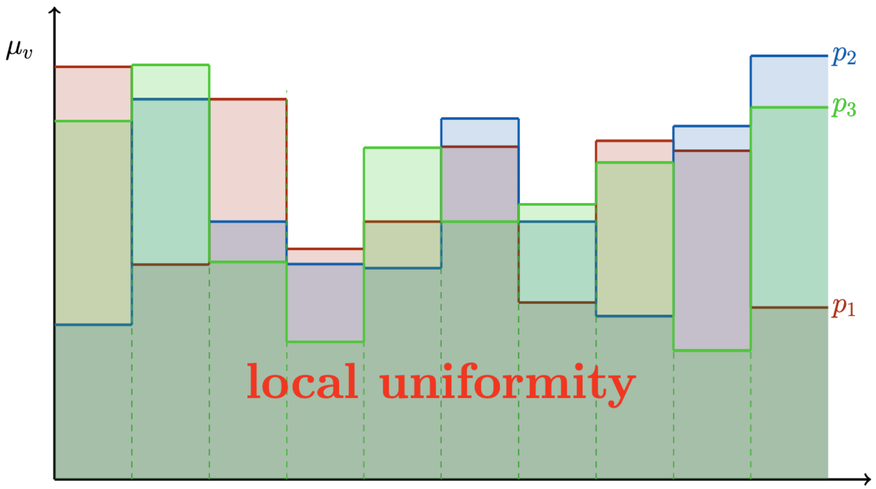


Local uniformity (marginal lower bound)

$$\forall c \in [q], \quad \mu_v^{\min}(c) \triangleq \min_{\mu(\sigma_{V\setminus\{v\}})>0} \mu_v\left(c \mid \sigma_{V\setminus\{v\}}\right)$$

$$\theta\text{−(locally)uniform: } \sum_{c\in[q]} \mu_v^{\min}(c) \ge \theta$$



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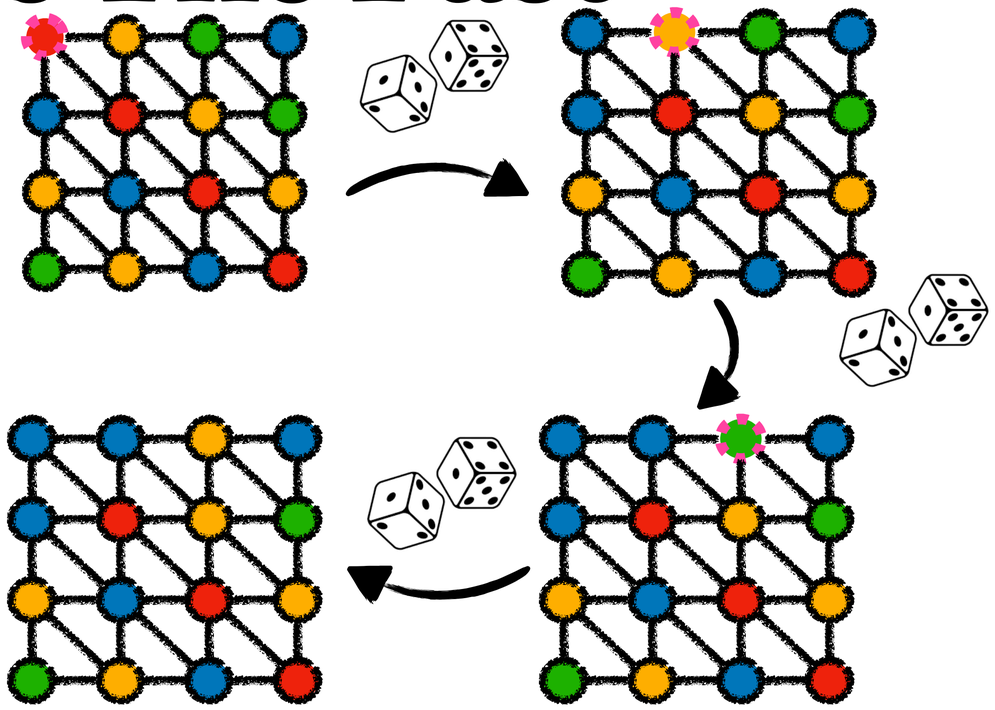
$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v\left(c \mid \sigma_{V\setminus\{v\}}\right) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ;



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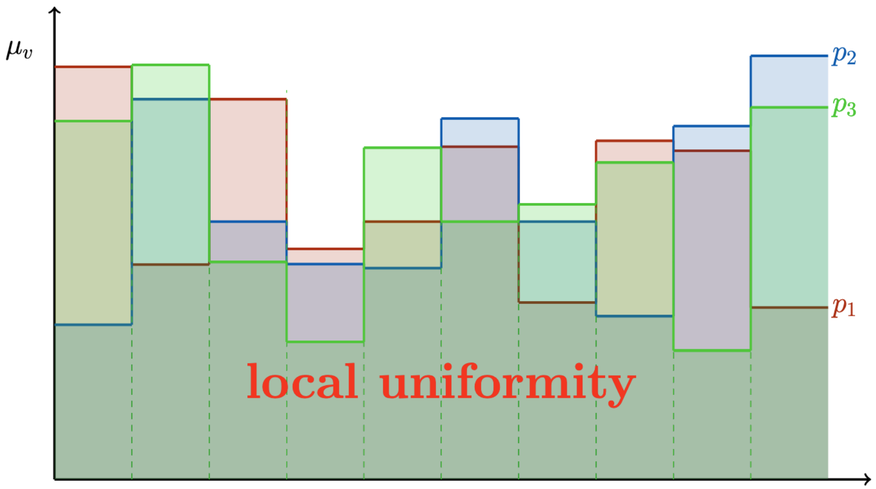

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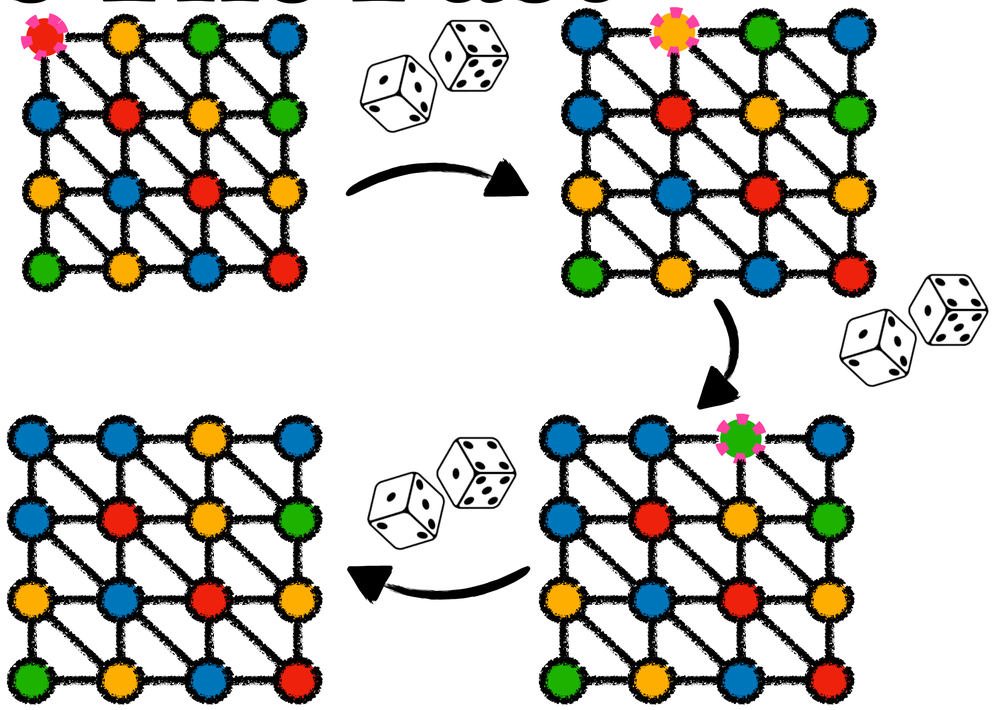
return σ;

With prob. θ, update can be directly resolved!



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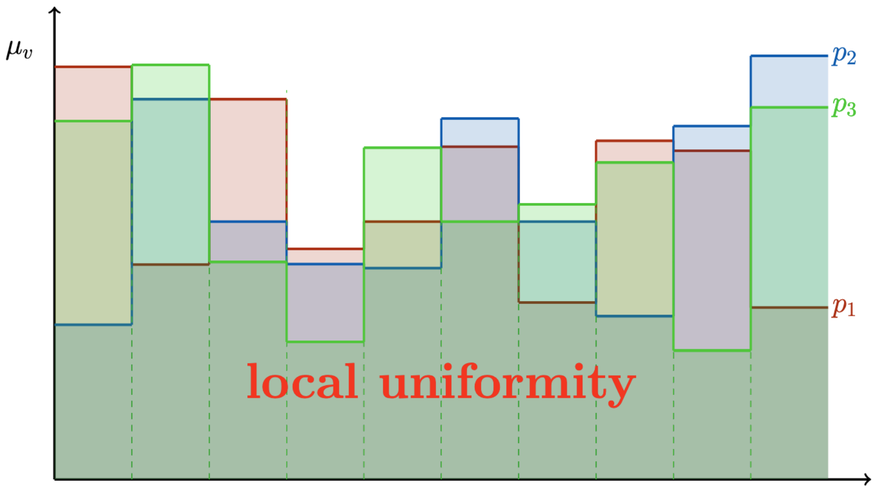

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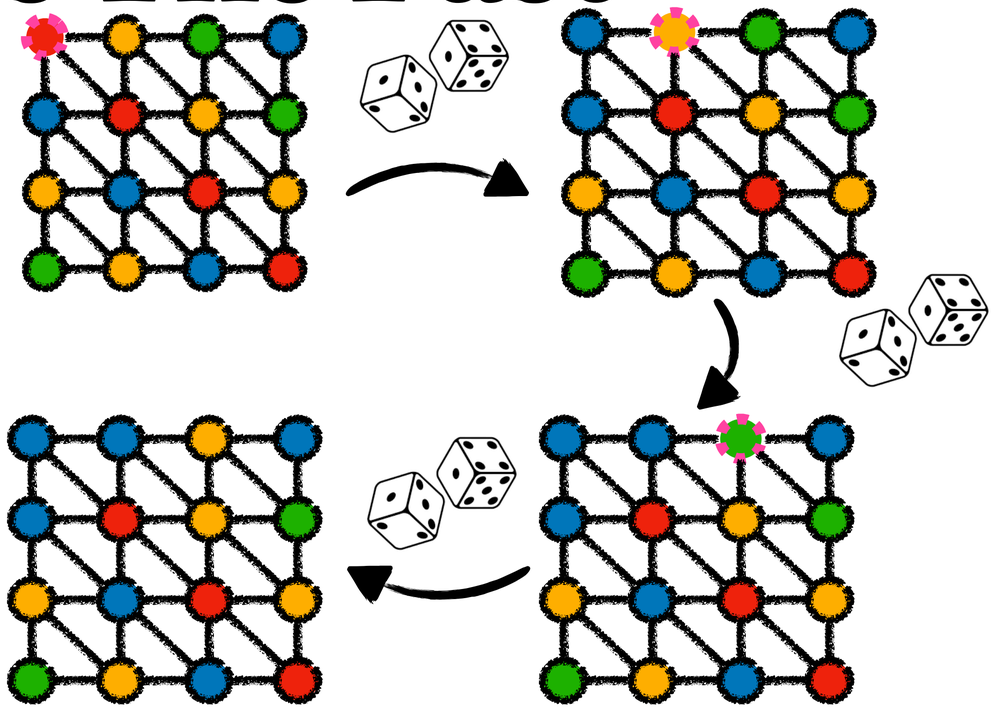

t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	?	?	?	?	?	?	?	?
outcome	…	?	?	?	?	?	?	?	?	?

final state → (approx) local sampler

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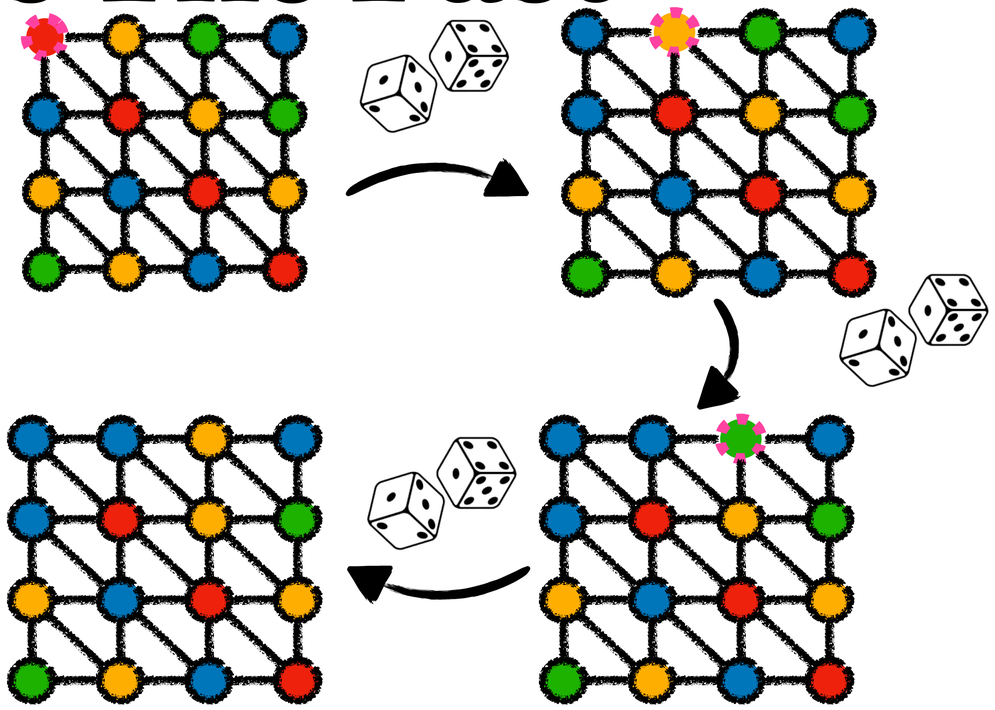
pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$)

sample $r_t \sim \mathrm{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v\left(c \mid \sigma_{V\setminus\{v\}}\right) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ;



t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	?	?	?	?	?	?	?	?
outcome	…	?	?	?	?	?	?	?	?	?

final state → (approx) local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \le t \le T$:

pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$)

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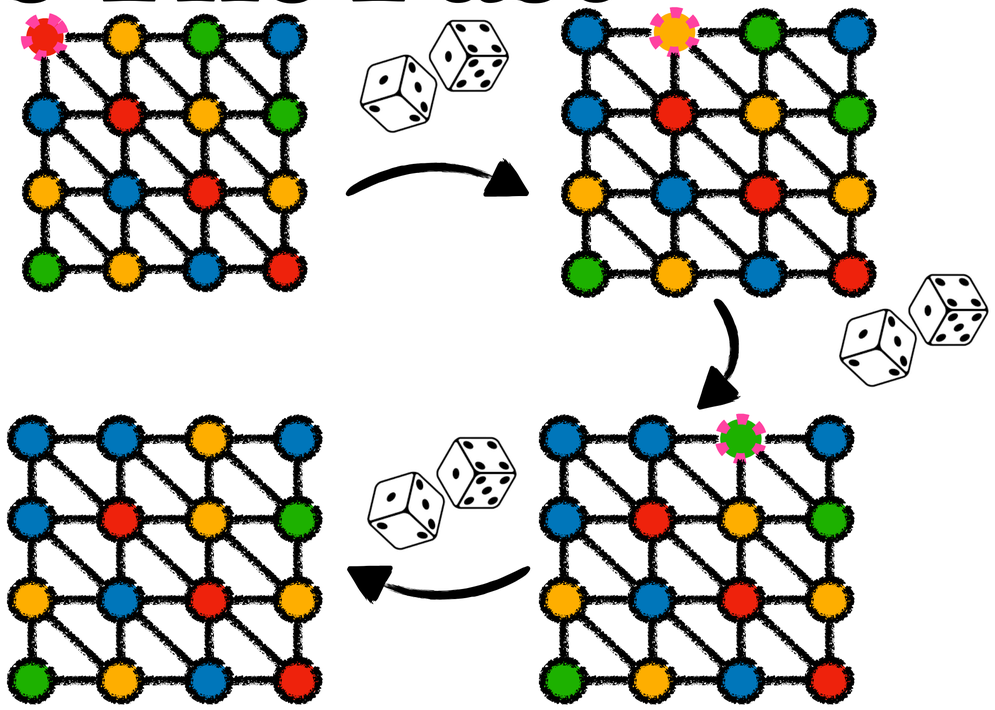
return σ;



t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	?	?	?	?	?	?	?	?
outcome	…	?	?	?	?	?	?	?	?	?

final state → (approx) local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

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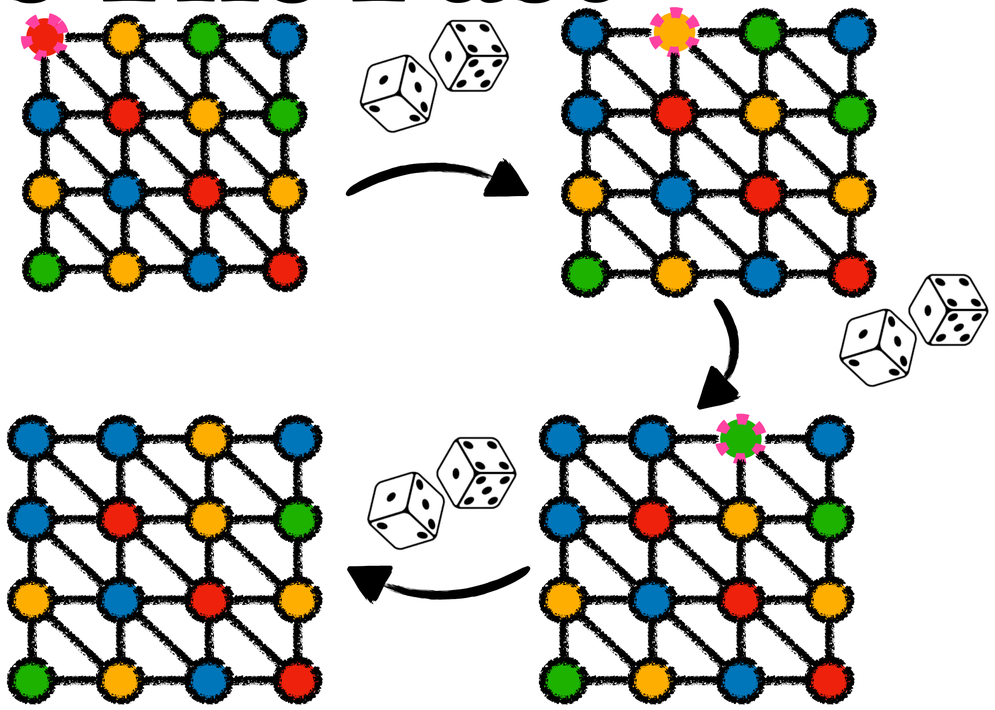

t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	?	?	?	?	?	?	?	1
outcome	…	?	?	?	?	?	?	?	?	

final state → (approx) local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \le t \le T$:

pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$)

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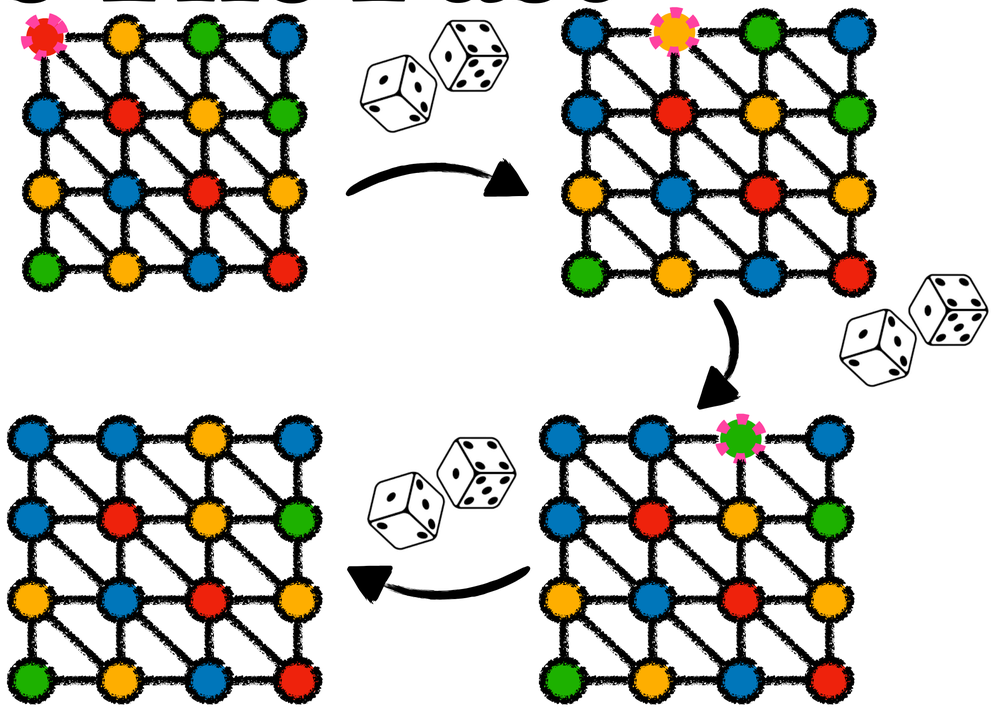

t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	?	?	?	?	?	?	?	0
outcome	…	?	?	?	?	?	?	?	?	?

final state → (approx) local sampler

backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \le t \le T$:

pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$)

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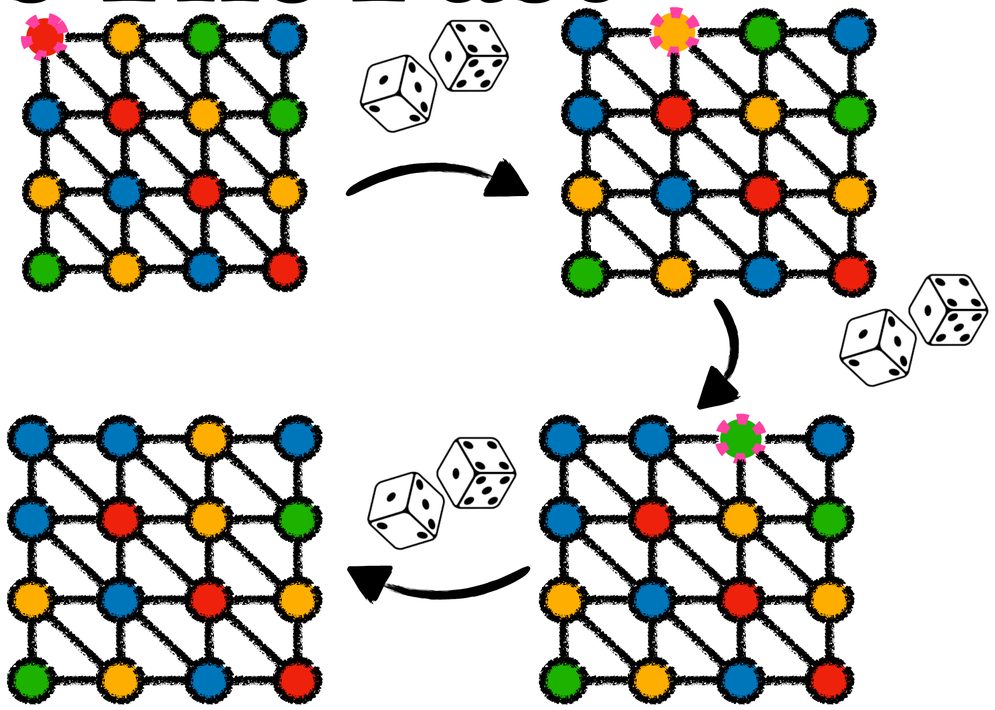

t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	1	?	?	1	?	1	?	0
outcome	…	?		?	?		?		?	?

final state → (approx) local sampler

backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

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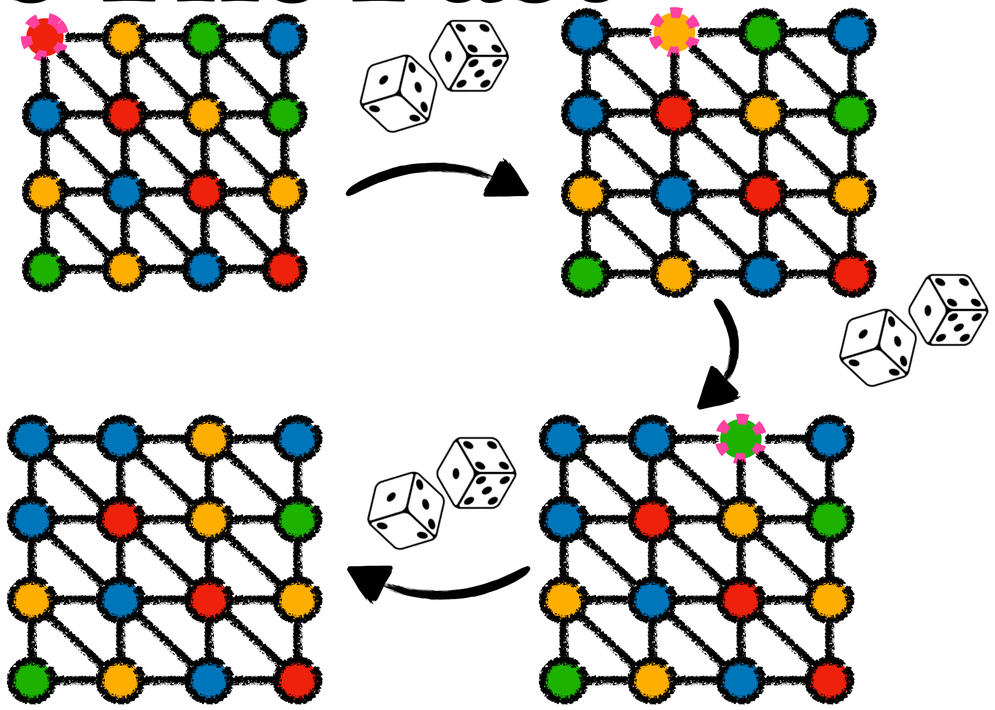

t	…	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	…	?	1	?	?	1	?	1	?	0
outcome	…	?		?	?		?		?	

final state → (approx) local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

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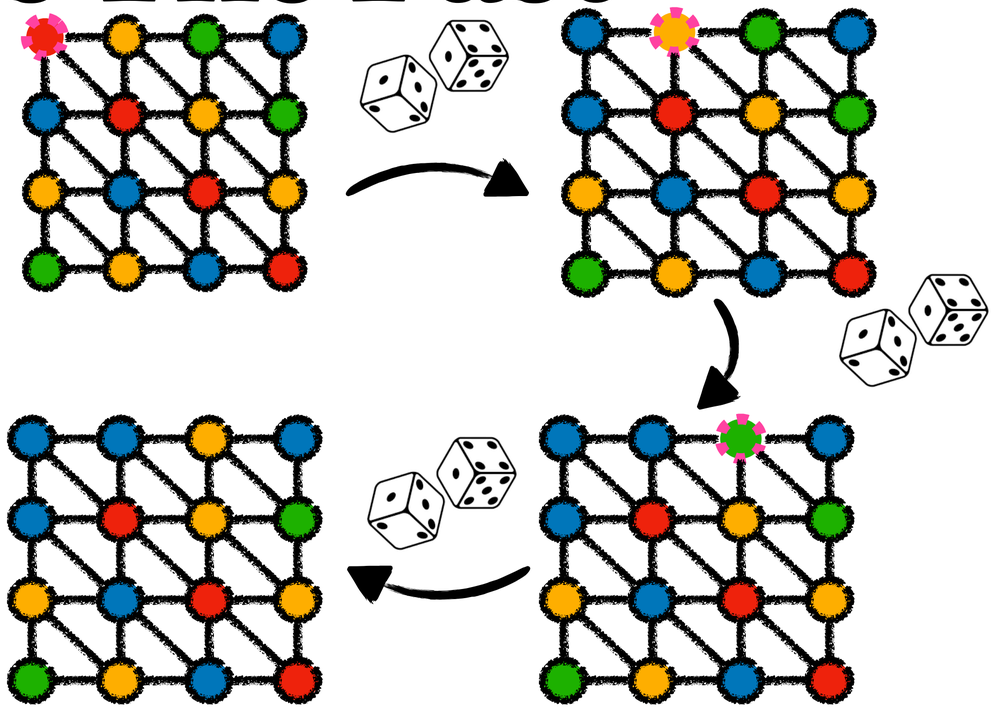

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	…	?	1	?	?	1	?	1	?	0
outcome	…	?		?	?		?		?	

final state → perfect local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \le t \le T$:

pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$)

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$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v\left(c \mid \sigma_{V\setminus\{v\}}\right) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

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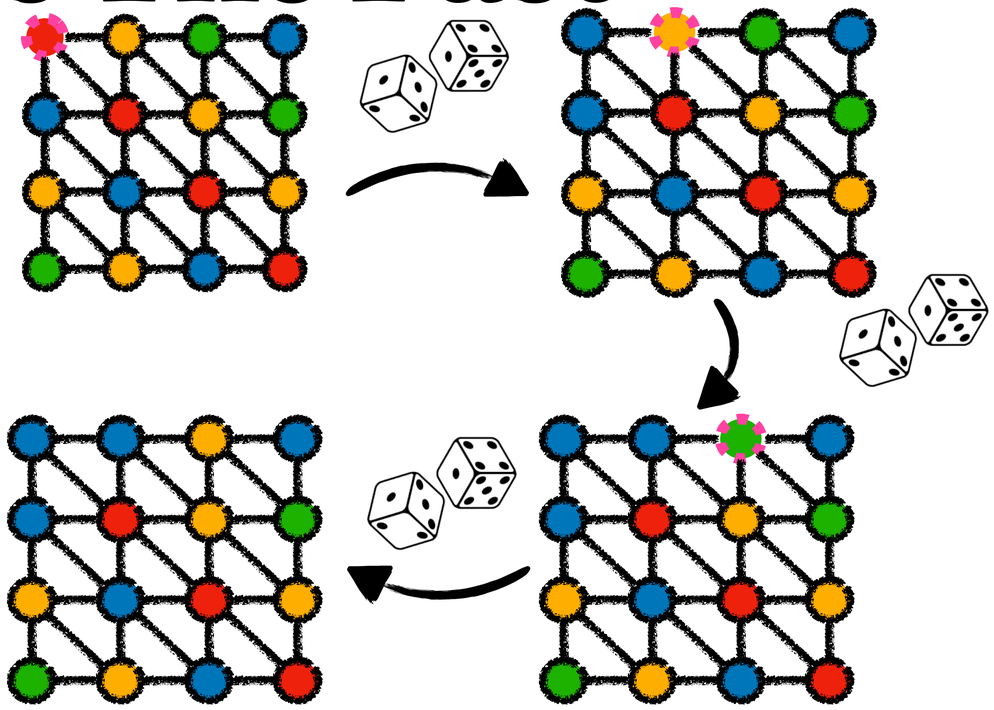

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	…	?	1	?	?	1	?	1	?	0
outcome	…	?		?	?		?		?	

final state → perfect local sampler

need $(1-\theta)\Delta \le 1$ to terminate

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \le t \le T$:

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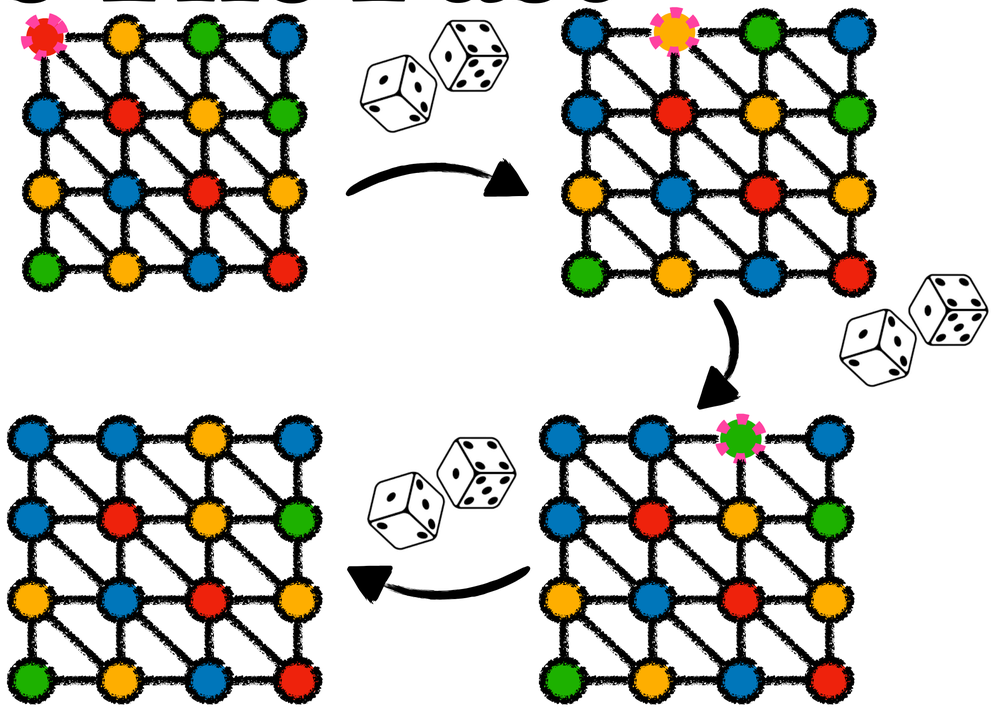
return σ;



t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	…	?	1	?	?	1	?	1	?	0
outcome	…	?		?	?		?		?	

final state → perfect local sampler

need $(1-\theta)\Delta \le 1$ to terminate

hidden (default) grand coupling!

Warmup: Local Ising Sampler

Ising model

two states

external fields λ 1

interaction β 1

marginal

$$\Pr[\bullet] = \frac{\lambda\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

$$\Pr[\bullet] = \frac{\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

Warmup: Local Ising Sampler

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A rejection sampler for $\beta \leq 1$

for $i = 1,2,\cdots$:

propose c_i: $\bullet$ w.p. $\frac{\lambda}{1+\lambda}$; $\bullet$ w.p. $\frac{1}{1+\lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0,1]$;

accept iff $r_{i,u} < \beta$ for each u with same color;

$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}$!

Warmup: Local Ising Sampler

Ising model

two states

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$$\Pr[\bullet] = \frac{\lambda\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

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$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}$!

Resolve an update

for $i = 1,2,\cdots$

for each $u \in N(v)$:

if $r_{i,u} \ge \beta$:

resolve u, go to next outer iter. if same color

return c_i

need $\mathbb{E}[\text{neighbors checked}] \le 1$

Warmup: Local Ising Sampler

Ising model

two states

external fields λ 1

interaction β 1

marginal

$$\Pr[\bullet] = \frac{\lambda\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

$$\Pr[\bullet] = \frac{\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

A rejection sampler for $\beta \geq 1$

for $i = 1,2,\cdots$:

propose c_i: $\bullet$ w.p. $\frac{\lambda}{1+\lambda}$; $\bullet$ w.p. $\frac{1}{1+\lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0,1]$;

accept iff $r_{i,u} < 1/\beta$ for each u with opposite color;

$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}$!

Resolve an update

for $i = 1,2,\cdots$:

for each $u \in N(v)$:

if $r_{i,u} \geq 1/\beta$:

resolve u, go to next outer iter. if opposite color;

return c_i;

need $\mathbb{E}[\text{neighbors checked}] \leq 1$

Warmup: Local Ising Sampler

Ising model

two states

external fields λ 1

interaction β 1

marginal

$$\Pr[\bullet] = \frac{\lambda\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

$$\Pr[\bullet] = \frac{\beta^{\#N(\bullet)}}{\lambda\beta^{\#N(\bullet)} + \beta^{\#N(\bullet)}}$$

A rejection sampler for $\beta \geq 1$

for $i = 1,2,\cdots$:

propose c_i: $\bullet$ w.p. $\frac{\lambda}{1+\lambda}$; $\bullet$ w.p. $\frac{1}{1+\lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0,1]$;

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Resolve an update

for $i = 1,2,\cdots$:

for each $u \in N(v)$:

if $r_{i,u} \geq 1/\beta$:

resolve u, go to next outer iter. if opposite color;

return c_i;

need $\mathbb{E}[\text{neighbors checked}] \leq 1$

smoother deduction rules for soft constraints!

Local q-coloring Sampler

Colorings

marginal $\Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \bullet \notin S_t \end{cases}$

$S_t =$ set of neighboring colors

Local q-coloring Sampler

Colorings

marginal $\Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \bullet \notin S_t \end{cases}$

$S_t =$ set of neighboring colors

Resolve an update

for $i = 1, 2, \cdots$:

propose a **uniform** color $\bullet$ from $[q]$;

accept $\bullet$ if $\bullet \notin S_t$;

Local q-coloring Sampler

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$$\text{marginal } \Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q]\setminus S_t|} & \bullet \notin S_t \end{cases}$$

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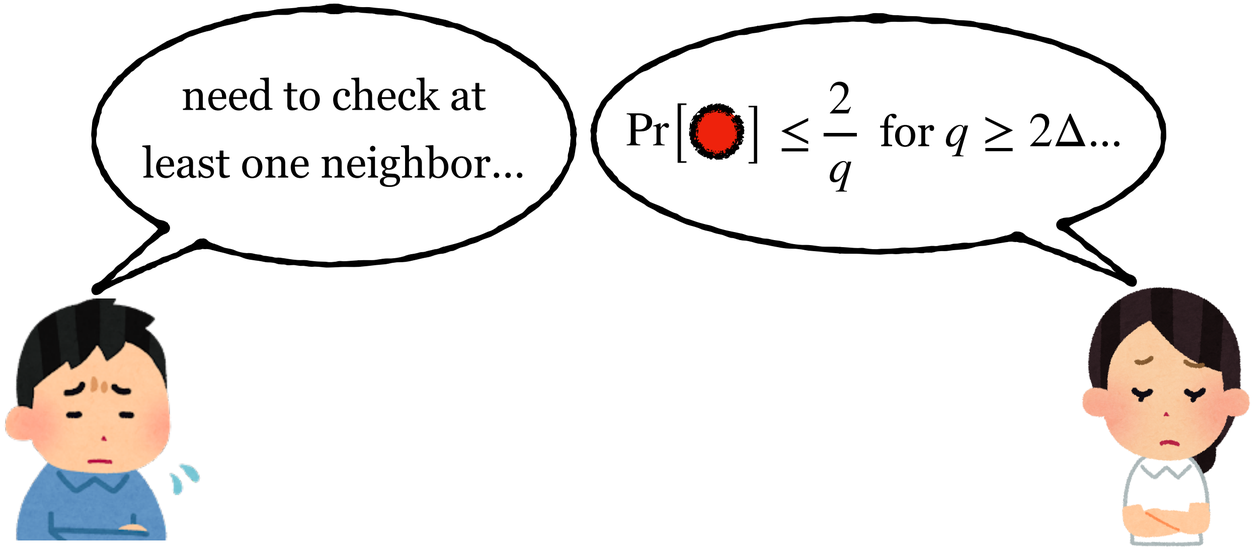

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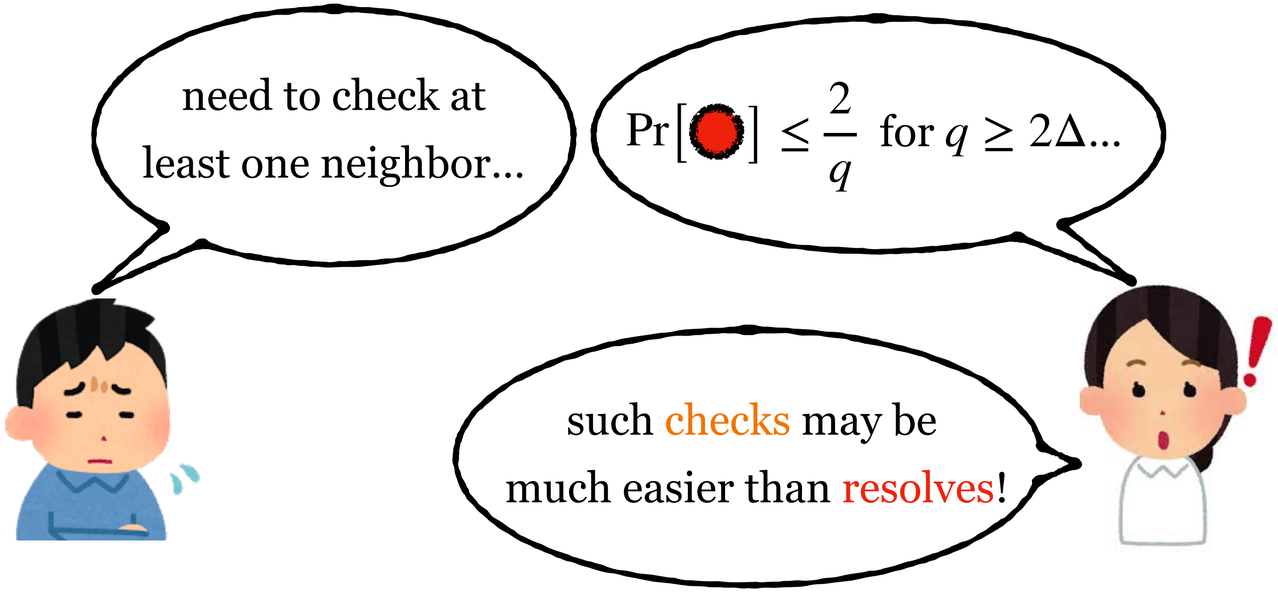

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S_t = set of neighboring colors

Resolve an update

for $i = 1,2,\cdots$:

propose a **uniform** color $\bullet$ from $[q]$;

accept $\bullet$ if $\bullet \notin S_t$;

Check if an update is $\bullet$

w.p. $1 - \frac{2}{q}$, directly certify that update is not $\bullet$;

otherwise:

check if $\bullet \in S_t$:

yes: certify that update is not $\bullet$;

no: certify that update: is $\bullet$ w.p. $\frac{q/2}{|[q]\setminus S_t|}$

is not $\bullet$ w.p. $\frac{|[q]\setminus S_t| - q/2}{|[q]\setminus S_t|}$

Local q-coloring Sampler

Colorings

$$\text{marginal } \Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q]\setminus S_t|} & \bullet \notin S_t \end{cases}$$

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is not $\bullet$ w.p. $\frac{|[q]\setminus S_t| - q/2}{|[q]\setminus S_t|}$

How to certify?

Local q-coloring Sampler

Colorings

$$\text{marginal } \Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q]\setminus S_t|} & \bullet \notin S_t \end{cases}$$

S_t = set of neighboring colors

Resolve an update

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Check if an update is $\bullet$

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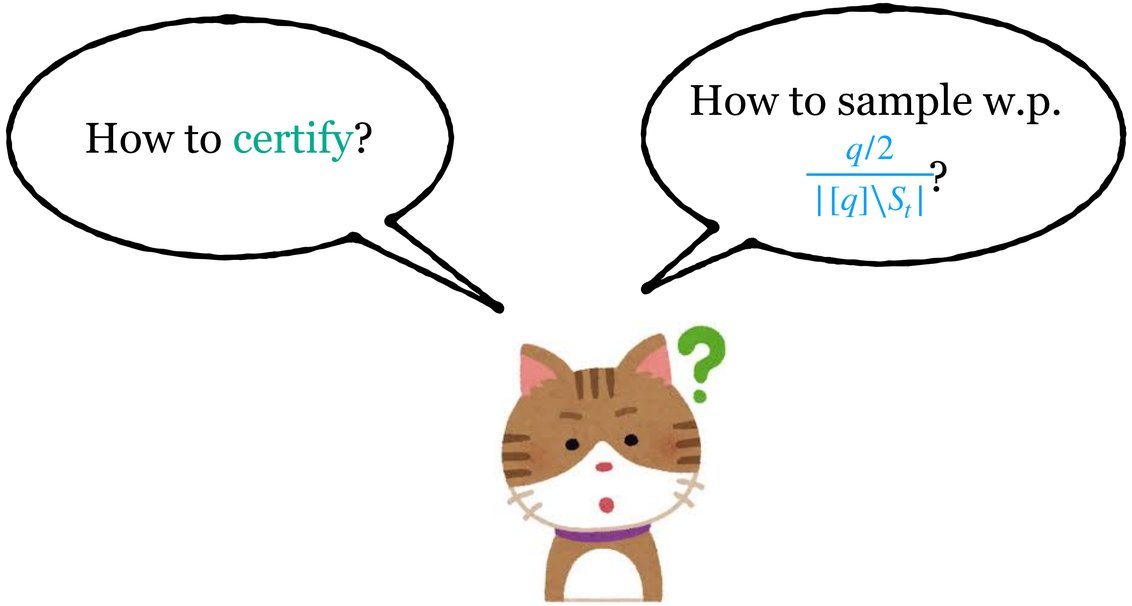

Local q-coloring Sampler

Resolve an update

for $i = 1,2,\cdots$:

propose a **uniform** color from $[q]$;

accept if $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{q}$, directly certify that update is not ;

otherwise:

check if $\in S_t$:

yes: certify that update is not ;

no: certify that update: is w.p. $\frac{q/2}{|[q]\setminus S_t|}$

is not w.p. $\frac{|[q]\setminus S_t| - q/2}{|[q]\setminus S_t|}$

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
outcome	…	?	?	?	?	?	?	?	?	?

Local q-coloring Sampler

Resolve an update

for $i = 1,2,\cdots$:

propose a **uniform** color from L_t;

accept if $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly certify that update is not ;

otherwise:

check if $\in S_t$:

yes: certify that update is not ;

no: certify that update: is w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$

is not w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	…									

Local q-coloring Sampler

Resolve an update

for $i = 1,2,\cdots$:

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accept if $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly certify that update is not ;

otherwise:

check if $\in S_t$:

yes: certify that update is not ;

no: certify that update: is w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$

is not w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	…									

certify update is : $L_t \leftarrow \{\}$

certify update is not : $L_t \leftarrow L_t \setminus \{\}$

***will ensure that $|L_t| \geq 2\Delta$ for $|L_t| \neq 1$**

Local q-coloring Sampler

Resolve an update

for $i = 1,2,\cdots$:

propose a **uniform** color from L_t;

accept if $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly certify that update is not ;

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check if $\in S_t$:

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is not w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	…									

certify update is : $L_t \leftarrow \{\}$

certify update is not : $L_t \leftarrow L_t \setminus \{\}$

***will ensure that $|L_t| \geq 2\Delta$ for $|L_t| \neq 1$**

Local q-coloring Sampler

Resolve an update

for $i = 1,2,\cdots$:

propose a **uniform** color from L_t;

accept if $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly certify that update is not ;

otherwise:

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feels strange my state depends on future proposals...

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non-Markovian grand coupling!

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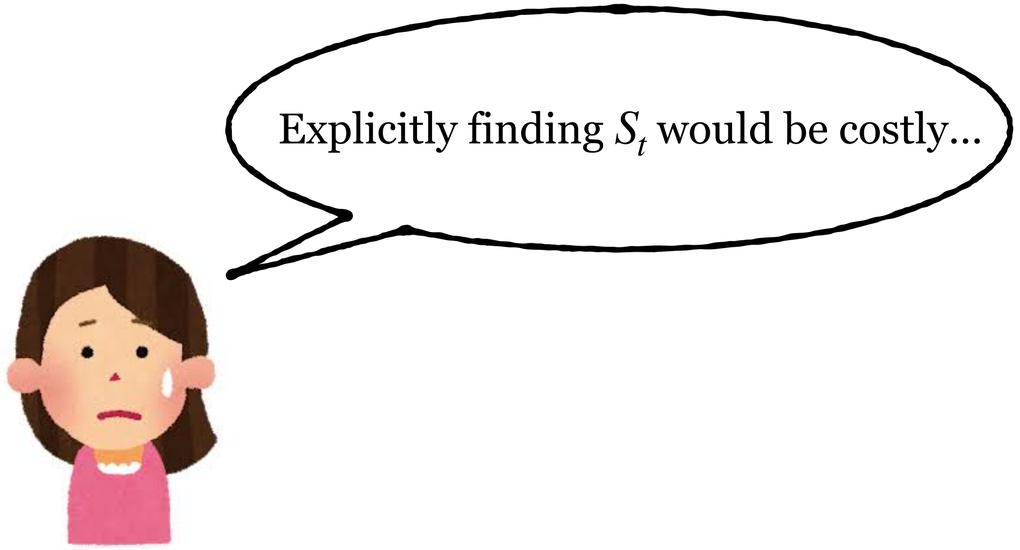

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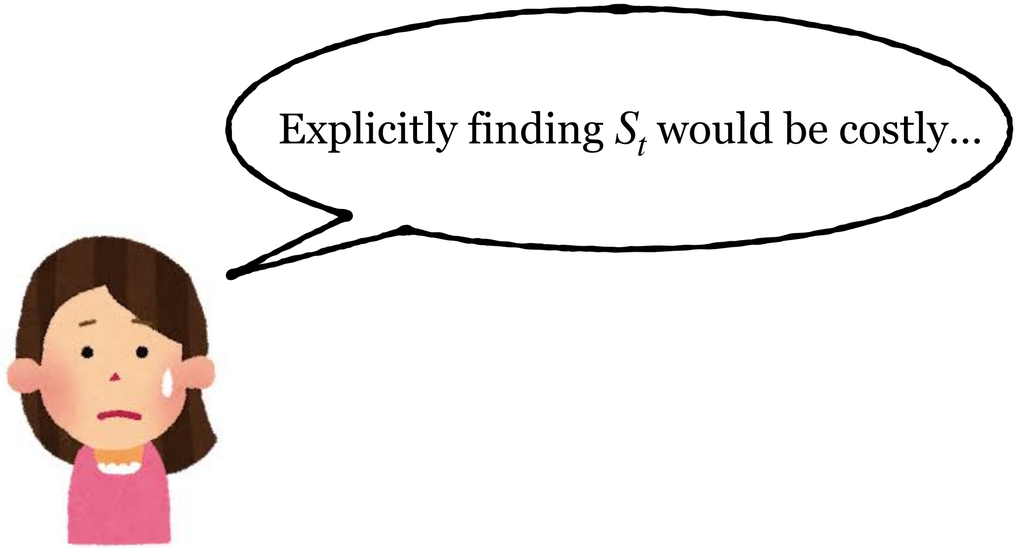

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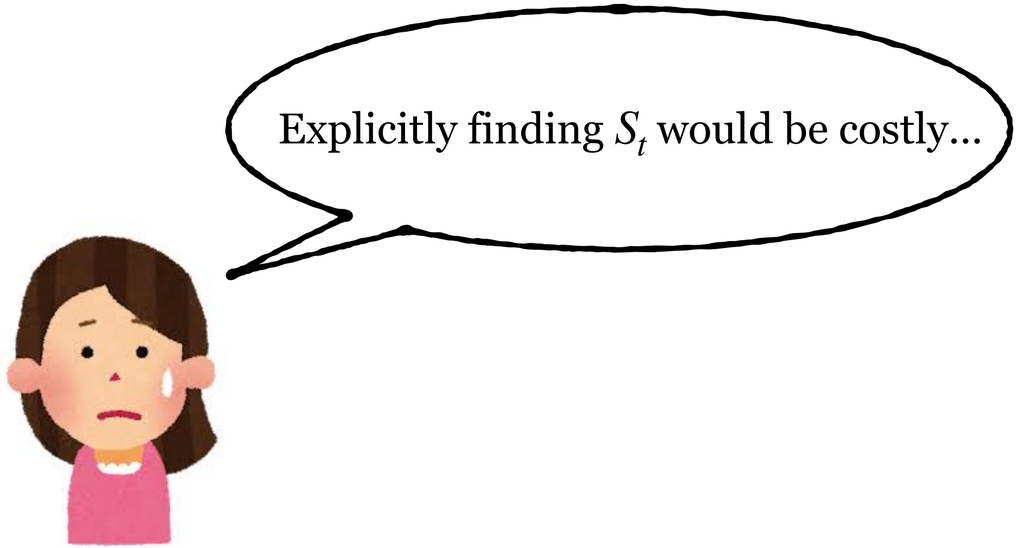
$\frac{|L_t \backslash S_t|}{|L_t|}$-coin $\longrightarrow$ $\frac{|L_t|/2}{|L_t \backslash S_t|}$-coin

Bernoulli factory for division

[NP '05, Hub '16, Mor '21]



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Bernoulli factory for division

[NP '05, Hub '16, Mor '21]

Efficiency Analysis

We carefully design a potential function Φ that

- $\Phi = \Phi(\Pi) \geq 0$ depends on the state of the algorithm;
- $\Phi = 0$ upon termination;
- Φ evolves as a supermartingale with bounded differences.

Summary

We propose

- the first local sampler for near-critical Ising model;
- the first local sampler for q-coloring (with $q = O(\Delta)$);

Our local samplers are perfect, runs in expected linear time, and is based on an extension of the Coupling Towards The Past (CTTP) framework.

Open problems

- Can local samplers achieve the exact critical thresholds/lower bounds ?
- Can local samplers achieve the thresholds for perfect sampling? (i.e., bounds achievable by Coupling From The Past method for q-colorings)
- More applications of local samplers?

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Thanks! Any questions?

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