Coupling Towards The Past:
Local Sampling from Markov Chains

Poster presenter: Chunyang Wang[1]
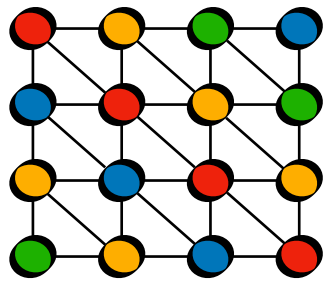
Based on joint works with Weiming Feng[2], Heng Guo[3], Hongyang Liu[1], Jingcheng Liu[1], Jiaheng Wang[3], Yitong Yin[1] and Yixiao Yu[1]

1 Nanjing University   2   the University of Hong Kong   3   the University of Edinburgh

Spin Systems and Gibbs sampling

graph $G = (V, E)$ $q \geq 2$ states

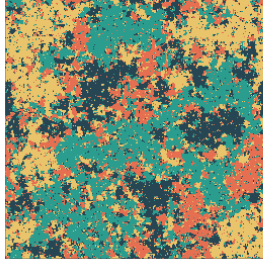configuration $\sigma \in [q]^V$

external fields $\lambda_v \in \mathbb{R}_{\geq 0}^q$ for each $v \in V$

interaction matrix $A_e \in \mathbb{R}_{\geq 0}^{q \times q}$ for each $e \in E$

weight: $w(\sigma) = \prod_{v \in V} \lambda_v(\sigma(v)) \prod_{(u,v) \in E} A_e(\sigma(u), \sigma(v))$

partition function: $Z = \sum_{\sigma \in [q]^V} w(\sigma)$

Gibbs distribution: $\mu(\sigma) = \dfrac{w(\sigma)}{Z}$

(Systematic Scan) Gibbs Sampling/Glauber Dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;
at each time $1 \leq t \leq T$:
  pick the vertex $v = v_{t \mod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$);
  resample $\sigma_v \sim \mu_v \left(\cdot \mid \sigma_{V \setminus \{v\}} \right)$;
return σ;   converges to μ as $T \to \infty$!

Coupling Towards The Past (CTTP)

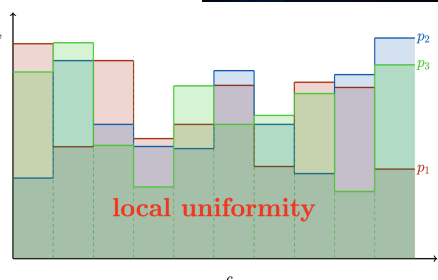Idea: Imagine the chain runs from the infinite past to time 0, can we somehow deduce its final state (distributed as μ) ?

If only we can resolve an update without knowing the current configuration…

Local Uniformity (Marginal Lower Bound)

$\forall c \in [q], \quad \mu_v^{\min}(c) \triangleq \min_{\mu(\sigma_{V \setminus \{v\}}) > 0} \mu_v \left(c \mid \sigma_{V \setminus \{v\}} \right)$

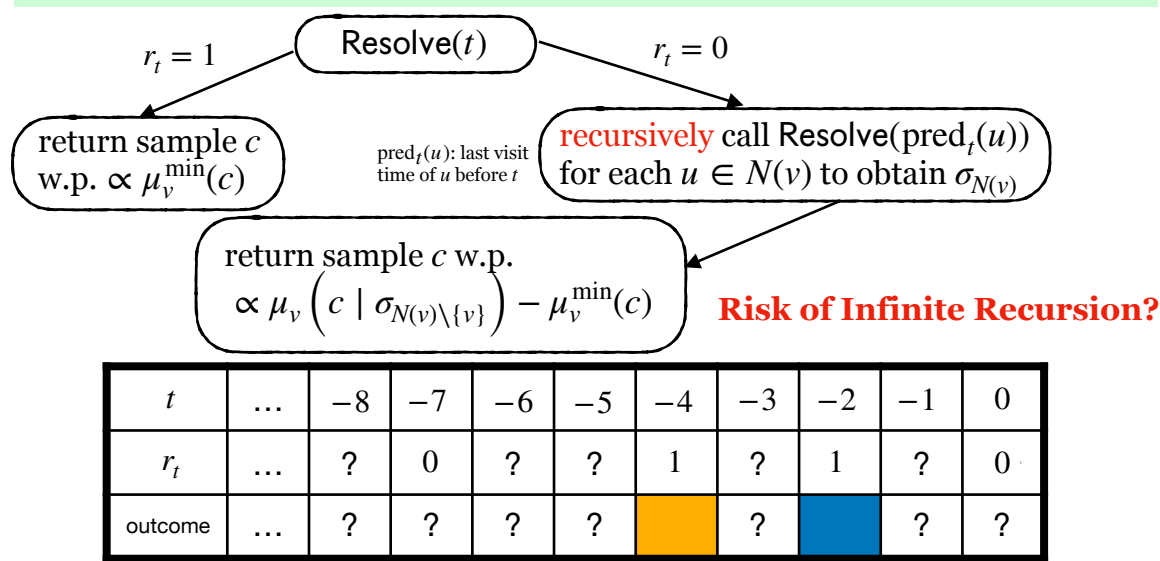$\theta-$(locally) uniform: $\sum_{c \in [q]} \mu_v^{\min}(c) \geq \theta$

local uniformity

$\begin{array}{c} \sigma_v \sim \mu_v \left(\cdot \mid \sigma_{V \setminus \{v\}} \right) \\ \parallel \\ \mu_v \left(\cdot \mid \sigma_{N(v) \setminus \{v\}} \right) \end{array}$ $\Longrightarrow$ draw $r_t \sim \text{Ber}(\theta)$ and σ_v such that

$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v \left(c \mid \sigma_{N(v) \setminus \{v\}} \right) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$

With probability θ, an update can be directly resolved!
Otherwise, we need to know its neighbors' states to determine…

$r_t = 1$ ← Resolve(t) → $r_t = 0$

return sample c
w.p. $\propto \mu_v^{\min}(c)$

$\text{pred}_t(u)$: last visit time of u before t

recursively call $\text{Resolve}(\text{pred}_t(u))$ for each $u \in N(v)$ to obtain $\sigma_{N(v)}$

return sample c w.p.
$\propto \mu_v \left(c \mid \sigma_{N(v) \setminus \{v\}} \right) - \mu_v^{\min}(c)$

Risk of Infinite Recursion?

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	…	?	0	?	?	1	?	1	?	0
outcome	…	?	?	?	?	🟧	?	🟦	?	?

sufficient condition for termination: $(1 - \theta)\Delta \leq 1$ (Δ: maximum degree of graph)

a perfect (no bias) local (produces local samples within local time) sampler

a direct-sum style decomposition of Markov chains:
resolving a single update takes $O(1/n)$ time of learning the entire configuration

naming: hidden (default) grand coupling + backward deduction of states

Application: Deterministic Counting

CTTP often gives exponential tail bound: $\Pr[t_{\text{run}} \geq T] \leq \exp(-O(T))$

truncate up to $K = O_\Delta \left(\log \dfrac{n}{\varepsilon} \right)$ random bits: ε-approximate marginals

brute force enumeration
→ efficient deterministic approximate counting matching MCMC bounds
also applies to problems with high-order constraints

Hypergraph Independent Sets (HIS)

Let $H = (V, \mathscr{E})$ be a hypergraph.
$S \subseteq V$ is a (weak) independent set if $S \cap e \neq e$ for all $e \in V$.

We obtain optimal (on the exponent) deterministic approximate counting algorithms for k-uniform ($|e| = k$ for all $e \in \mathscr{E}$) HIS:

Hypergraph independent sets	Reference	Bound	Running time
Randomised counting / sampling	[BDK08, BDK06]	$\Delta \leq k - 2$	$\tilde{O}(n^2) / O(n \log n)$
	[HSZ19, QWZ22]	$\Delta \lesssim 2^{k/2}$	$\tilde{O}(n^2) / O(n \log n)$
Deterministic counting	[BGG+19]	$\Delta \leq k$	$n^{O(\log(k\Delta))}$
	[JPV21b]	$\Delta \lesssim 2^{k/7}$	$n^{\text{poly}(k,\Delta)}$
	[HWY23]	$\Delta \lesssim 2^{k/5}$	$n^{\text{poly}(k,\Delta)}$
	Our result	$\mathbf{\Delta \lesssim 2^{k/2}}$	$n^{\text{poly}(k,\Delta)}$
Hardness	[BGG+19]	$\Delta \geq 5 \cdot 2^{k/2}$ assuming $\mathbf{P} \neq \mathbf{NP}$	

Application: Analytical Stability

CTTP provides a direct-sum style decomposition of the Gibbs measure.
lift to the complex plane
→ analytical stability of certain polynomials matching MCMC bounds

Hypergraph Independence Polynomial (HIP)

Let $H = (V, \mathscr{E})$ be a hypergraph and Ω collect its independent sets.
The (univariate) independence polynomial $Z_H : \mathbb{C} \to \mathbb{C}$ of H is given as:

$$Z_H(\lambda) = \sum_{S \subseteq \Omega} \lambda^{|S|}$$

Complex zeroes of $Z_H(\lambda)$ are often called Lee-Yang zeroes.

We obtain optimal (on the exponent) zero-free regions for k-uniform HIP:

[GMP+24]   0   Our result $\lambda_c \approx \dfrac{1}{\Delta^{2/k} - 1}$ ("sampling LLL condition") λ

[Zhang'23]

NP-Hard [BGG+19]

Rapid mixing of Markov chains
[HSZ19, HSW21, QWZ22, FGW+23]

Extending CTTP:
Local Sampling near Criticality

The requirement of local uniformity may be restrictive for certain models.
improved grand coupling and deduction rules
→ efficient local samplers near criticality for Ising model and $q-$colorings

Instance	Tractable regimes		Our result (local sampling)
	Global sampling	Local sampling	
Ising model	$\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2} \right)$	$\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)} \right)$	$\beta \in \left(\frac{\Delta - 0.5}{\Delta}, \frac{\Delta}{\Delta - 0.5} \right)$
q-colorings	$q \geq 1.809\Delta$	N/A	$q \geq 65\Delta$

- first local sampler for near-critical Ising model;
- first local sampler for q-coloring (also near-critical);
- perfect samplers;
- expected linear running time: $O(\Delta \cdot |\Lambda|)$ for Ising;
  $O(\Delta^2 q \cdot |\Lambda|)$ for q-colorings.

1. *Towards Derandomising Markov Chain Monte Carlo.* Weiming Feng, Heng Guo, Chunyang Wang, Jiaheng Wang, Yitong Yin. In **SICOMP** '25 (preliminary version in **FOCS** '23).
2. *Phase Transitions via Complex Extensions of Markov Chains.* Jingcheng Liu, Chunyang Wang, Yitong Yin, Yixiao Yu. In **STOC** '25.
3. *Local Gibbs Sampling beyond Local Uniformity.* Hongyang Liu, Chunyang Wang, Yitong Yin. To appear in **SODA** '26.