Coupling Towards The Past: Local Sampling from Markov Chains

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Based on joint works with Weiming Feng,[2] Heng Guo,[3] Hongyang Liu,[1] Jingcheng Liu,[1] Jiaheng Wang,[3] Yitong Yin[1] and Yixiao Yu[1]

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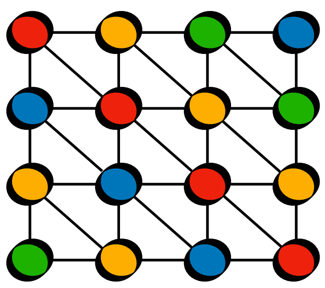
Spin Systems and Gibbs sampling



graph $G = (V, E)$ $q \geq 2$ states

configuration $\sigma \in [q]^V$

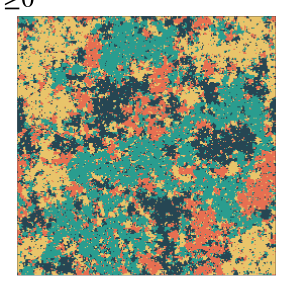
external fields $\lambda_v \in \mathbb{R}^q_{\geq 0}$ for each $v \in V$

interaction matrix $A_e \in \mathbb{R}^{q \times q}_{\geq 0}$ for each $e \in E$

weight: $w(\sigma) = \prod_{v \in V} \lambda_v(\sigma(v)) \prod_{(u,v) \in E} A_e(\sigma(u), \sigma(v))$

partition function: $Z = \sum_{\sigma \in [q]^V} w(\sigma)$

Gibbs distribution: $\mu(\sigma) = \dfrac{w(\sigma)}{Z}$



(Systematic Scan) Gibbs Sampling/Glauber Dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;
at each time $1 \leq t \leq T$:
  pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \ldots, v_{n-1}\}$);
  resample $\sigma_v \sim \mu_v\left(\cdot \mid \sigma_{V \setminus \{v\}}\right)$;
return σ;

converges to μ as $T \to \infty$!

Coupling Towards The Past (CTTP)

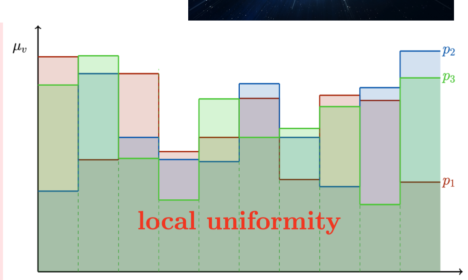
Idea: Imagine the chain runs from the infinite past to time 0, can we somehow deduce its final state (distributed as μ) ?

If we can resolve an update without needing to know the current configuration…

Local Uniformity (Marginal Lower Bound)

$\forall c \in [q], \quad \mu_v^{\min}(c) \triangleq \min_{\mu(\sigma_{V \setminus \{v\}}) > 0} \mu_v\left(c \mid \sigma_{V \setminus \{v\}}\right)$

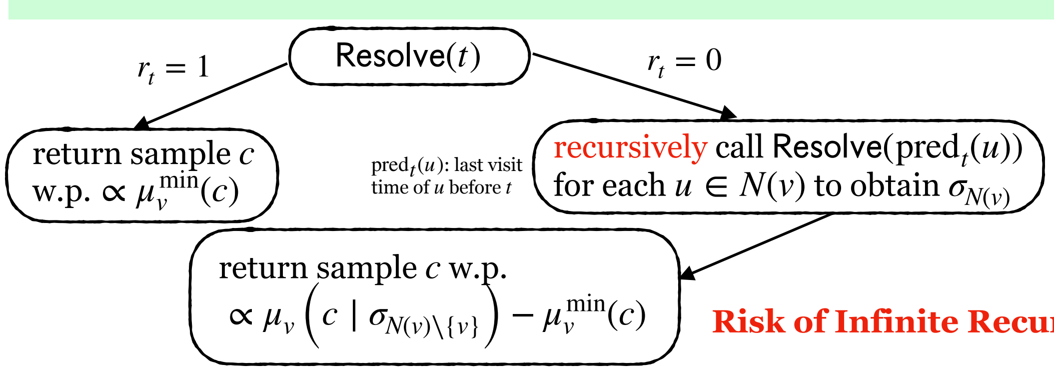
$\theta-$(locally) uniform: $\sum_{c \in [q]} \mu_v^{\min}(c) \geq \theta$



$$\sigma_v \sim \mu_v\left(\cdot \mid \sigma_{V \setminus \{v\}}\right) = \mu_v\left(\cdot \mid \sigma_{N(v) \setminus \{v\}}\right) \implies \text{draw } r_t \sim \mathrm{Ber}(\theta) \text{ and } \sigma_v \text{ such that } \Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v\left(c \mid \sigma_{N(v) \setminus \{v\}}\right) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

With probability θ, an update can be directly resolved!

Otherwise, we need to know its neighbors' states to determine…



Risk of Infinite Recursion?

t	…	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	…	?	0	?	?	1	?	1	?	0
outcome	…	?	?	?	?		?		?	?

sufficient condition for termination: $(1 - \theta)\Delta \leq 1$ (Δ: maximum degree of graph)

a perfect (no bias) local (produces local samples within local time) sampler

a direct-sum style decomposition of Markov chains:
resolving a single update takes $O(1/n)$ time of learning the entire configuration

naming: hidden (default) grand coupling + backward deduction of states

Application: Deterministic Counting

CTTP often gives exponential tail bound: $\Pr[t_{\text{run}} \geq T] \leq \exp(-O(T))$

truncate up to $K = O_\Delta\left(\log \dfrac{n}{\varepsilon}\right)$ random bits: ε-approximate marginals

brute force enumeration

→ efficient deterministic approximate counting matching MCMC bounds

also applies to problems with high-order constraints

Hypergraph Independent Sets (HIS)

Let $H = (V, \mathscr{E})$ be a hypergraph.
$S \subseteq V$ is a (weak) independent set if $S \cap e \neq e$ for all $e \in V$.

We obtain optimal (on the exponent) deterministic approximate counting algorithms for k-uniform ($|e| = k$ for all $e \in \mathscr{E}$) HIS:

Hypergraph independent sets	Reference	Bound	Running time
Randomised counting / sampling	[BDK08, BDK06]	$\Delta \leq k - 2$	$\tilde{O}(n^2)$ / $O(n \log n)$
	[HSZ19, QWZ22]	$\boldsymbol{\Delta \lesssim 2^{k/2}}$	$\tilde{O}(n^2)$ / $O(n \log n)$
Deterministic counting	[BGG+19]	$\Delta \leq k$	$n^{O(\log(k\Delta))}$
	[JPV21b]	$\Delta \lesssim 2^{k/7}$	$n^{\mathrm{poly}(k,\Delta)}$
	[HWY23]	$\Delta \lesssim 2^{k/5}$	$n^{\mathrm{poly}(k,\Delta)}$
	Our result	$\boldsymbol{\Delta \lesssim 2^{k/2}}$	$n^{\mathrm{poly}(k,\Delta)}$
Hardness	[BGG+19]	$\Delta \geq 5 \cdot 2^{k/2}$ assuming $\mathbf{P} \neq \mathbf{NP}$	

Application: Analytical Stability

CTTP provides a direct-sum style decomposition of the Gibbs measure.

lift to the complex plane

→ analytical stability of certain polynomials matching MCMC bounds

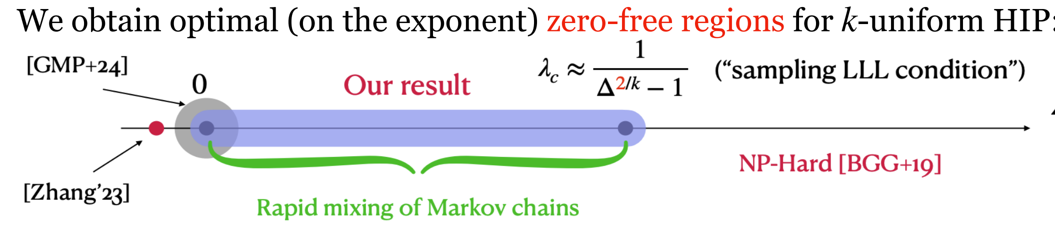
Hypergraph Independence Polynomial (HIP)

Let $H = (V, \mathscr{E})$ be a hypergraph and Ω collect its independent sets.
The (univariate) independence polynomial $Z_H : \mathbb{C} \to \mathbb{C}$ of H is given as:

$$Z_H(\lambda) = \sum_{S \subseteq \Omega} \lambda^{|S|}$$

Complex zeroes of $Z_H(\lambda)$ are often called Lee-Yang zeroes.

We obtain optimal (on the exponent) zero-free regions for k-uniform HIP:



Extending CTTP: Local Sampling near Criticality

The requirement of local uniformity may be restrictive for certain models.

improved grand coupling and deduction rules

→ efficient local samplers near criticality for Ising model and $q-$colorings

Instance	Tractable regimes		Our result (local sampling)
	Global sampling	Local sampling	
Ising model	$\beta \in \left(\frac{\Delta-2}{\Delta}, \frac{\Delta}{\Delta-2}\right)$	$\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)}\right)$	$\beta \in \left(\frac{\Delta-0.5}{\Delta}, \frac{\Delta}{\Delta-0.5}\right)$
q-colorings	$q \geq 1.809\Delta$	N/A	$q \geq 65\Delta$

- first local sampler for near-critical Ising model;
- first local sampler for q-coloring (also near-critical);
- perfect samplers;
- expected linear running time: $O(\Delta \cdot |\Lambda|)$ for Ising;
  $O(\Delta^2 q \cdot |\Lambda|)$ for q-colorings.

1. *Towards derandomising Markov Chain Monte Carlo.* Weiming Feng, Heng Guo, Chunyang Wang, Jiaheng Wang, Yitong Yin. In **SICOMP** '25 (preliminary version in **FOCS** '23).
2. *Phase transitions via Complex Extensions of Markov Chains.* Jingcheng Liu, Chunyang Wang, Yitong Yin, Yixiao Yu. In **STOC** '25.
3. *Local Gibbs Sampling beyond Local Uniformity.* Hongyang Liu, Chunyang Wang, Yitong Yin. To appear in **SODA** '26.