2T2: The Discrete Fourier Transform (2 of 2)

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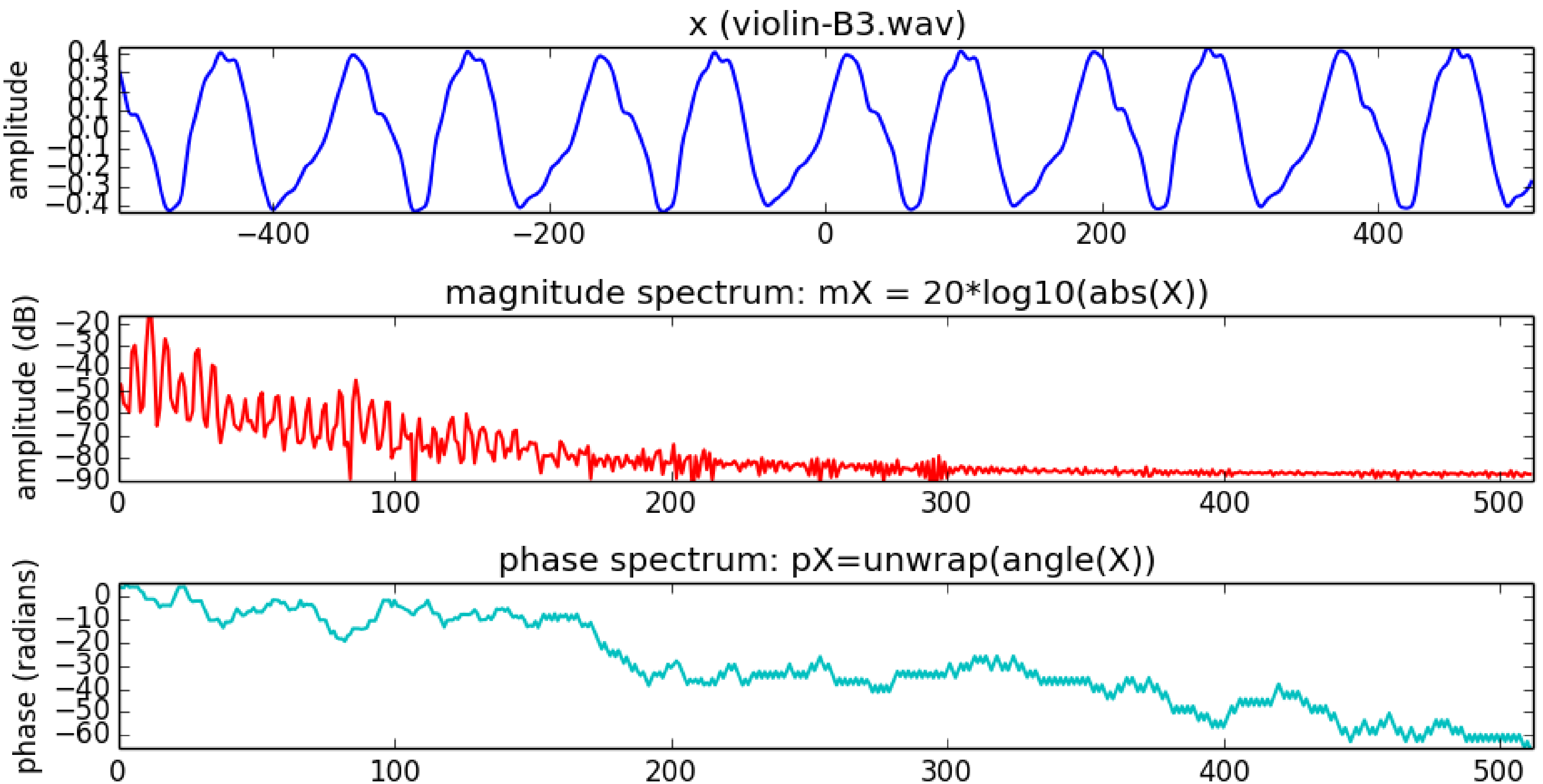
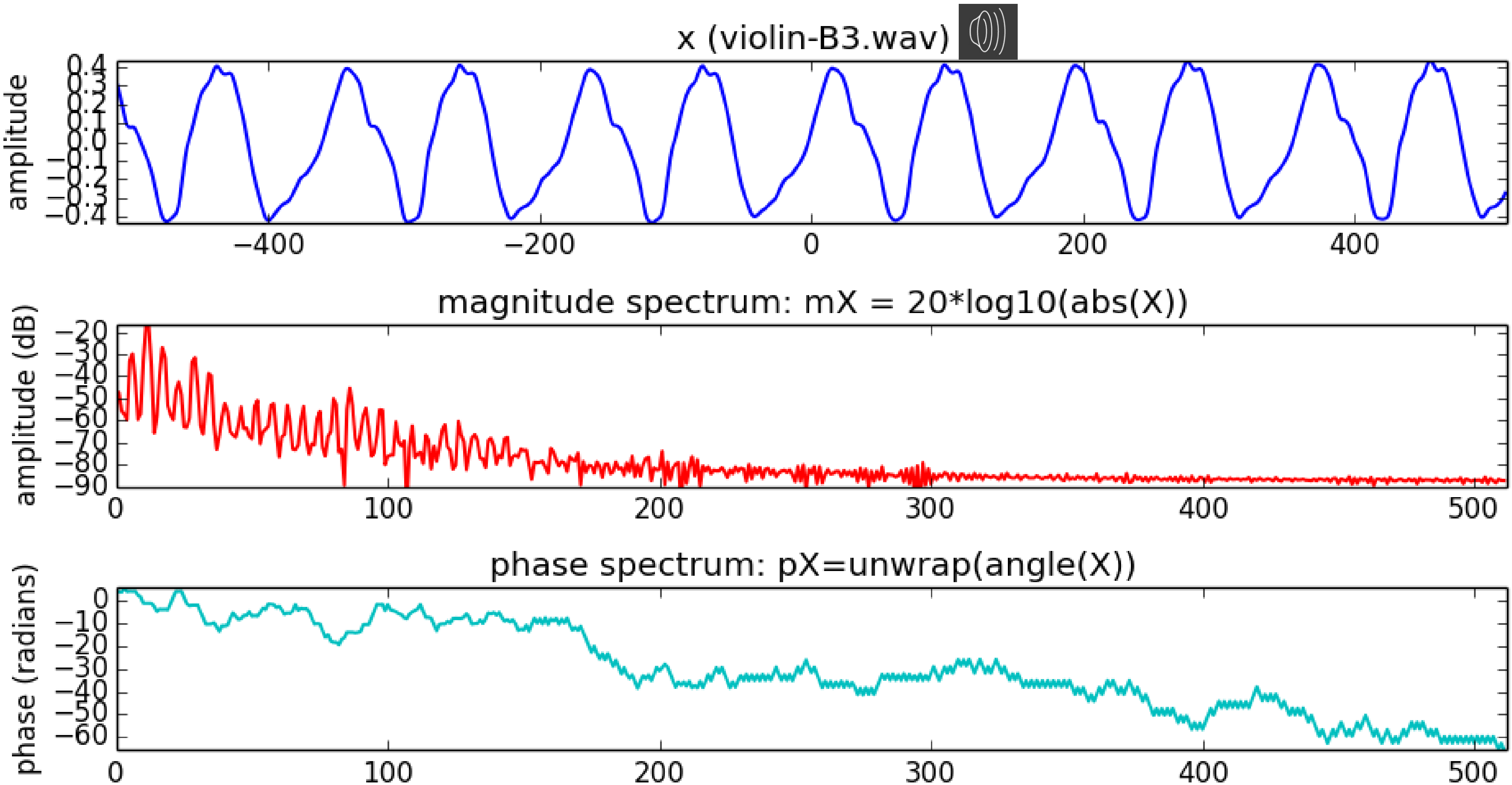
Discrete Fourier Transform

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad k=0,\ldots,N-1$$



Discrete Fourier Transform

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad k=0,\ldots,N-1$$



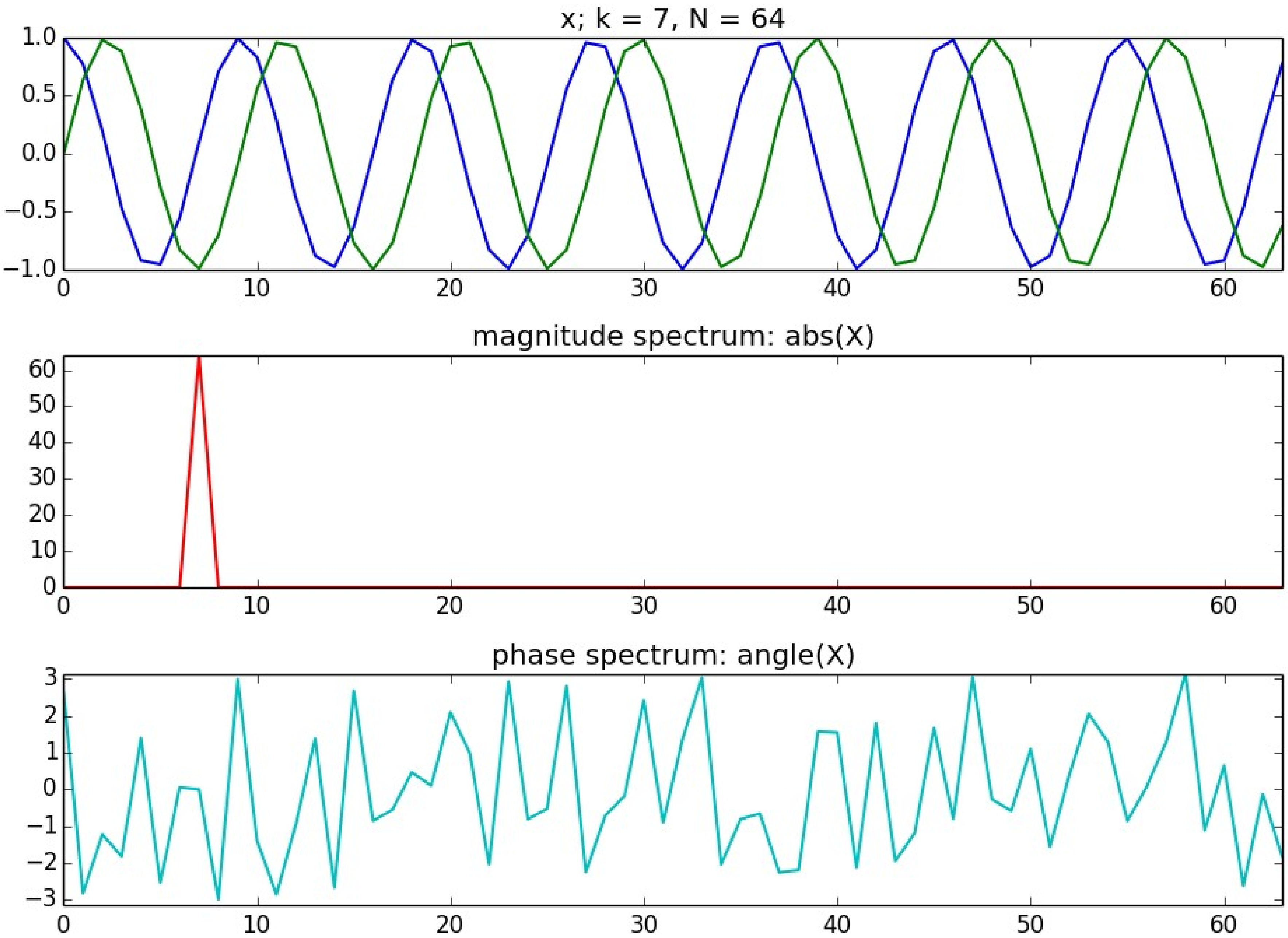
DFT of complex sinusoid

$$x_1[n] = e^{j2\pi k_0 n/N} \quad \text{for } n = 0, \ldots, N-1$$

$$\begin{aligned}
X_1[k] &= \sum_{n=0}^{N-1} x_1[n] e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} e^{j2\pi k_0 n/N} e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} e^{-j2\pi(k-k_0)n/N} \\
&= \frac{1-e^{-j2\pi(k-k_0)}}{1-e^{-j2\pi(k-k_0)/N}} \quad \text{(sum of a geometric series)}
\end{aligned}$$

if $k \neq k_0$, denominator $\neq 0$ and numerator $= 0$

thus $X_1[k] = N$ for $k = k_0$ and $X_1[k] = 0$ for $k \neq k_0$



x; k = 7, N = 64
magnitude spectrum: abs(X)
phase spectrum: angle(X)

DFT of any complex sinusoid

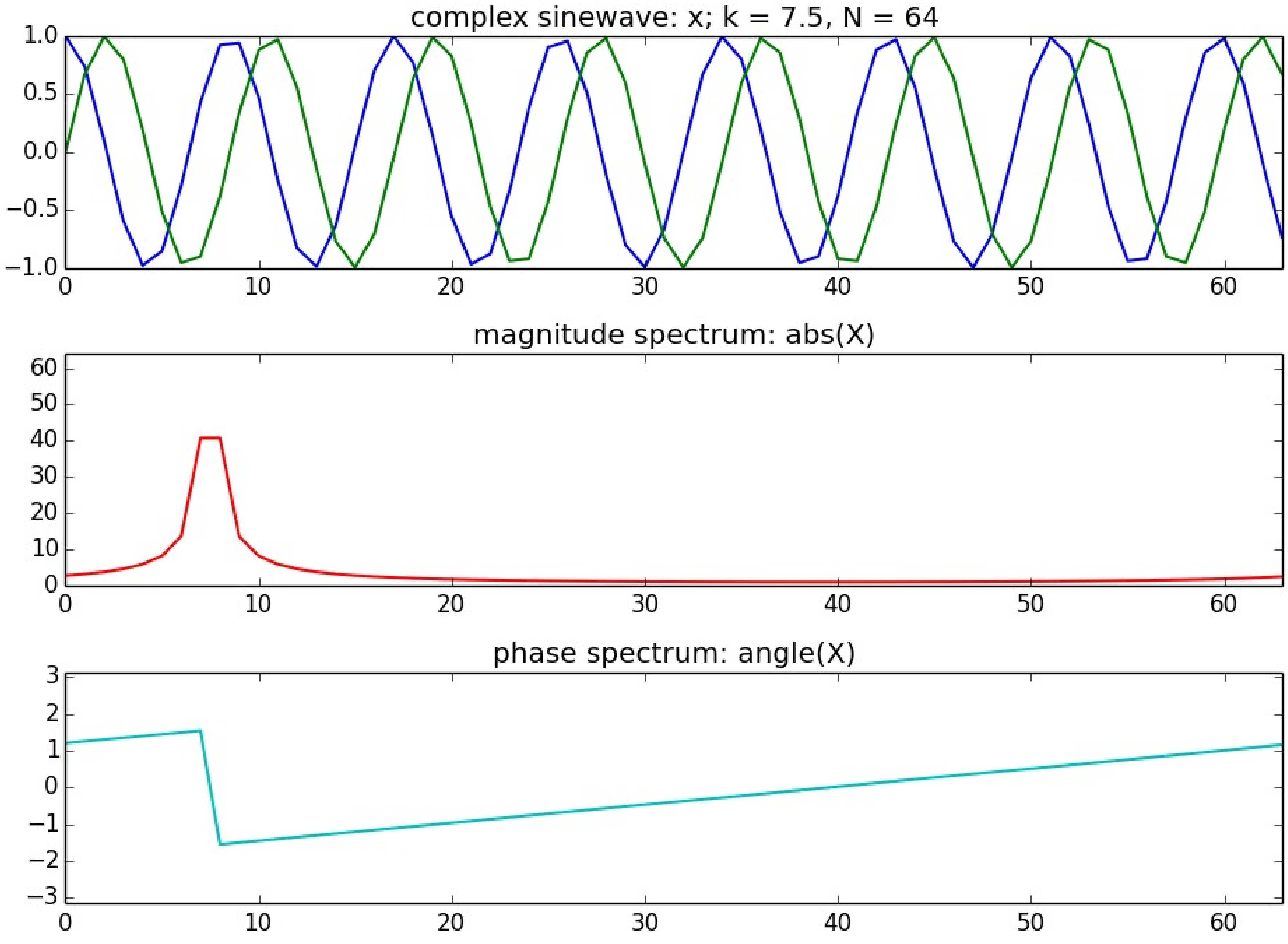
$$x_2[n] = e^{j2\pi f_0 n + \varphi} \quad \text{for } n = 0, \ldots, N-1$$

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n] e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi f_0 n + \varphi} e^{-j2\pi kn/N}$$

$$= e^{j\varphi} \sum_{n=0}^{N-1} e^{-j2\pi(k/N - f_0)n}$$

$$= e^{j\varphi} \frac{1 - e^{-j2\pi(k/N - f_0)N}}{1 - e^{-j2\pi(k/N - f_0)}}$$



complex sinewave: x; k = 7.5, N = 64
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
60
magnitude spectrum: abs(X)
60
50
40
30
20
10
0
0
10
20
30
40
50
60
phase spectrum: angle(X)
3
2
1
0
−1
−2
−3
0
10
20
30
40
50
60

DFT of real sinusoids

$$x_3[n] = A_0 \cos(2\pi k_0 n/N) = \frac{A_0}{2} e^{j2\pi k_0 n/N} + \frac{A_0}{2} e^{-j2\pi k_0 n/N}$$

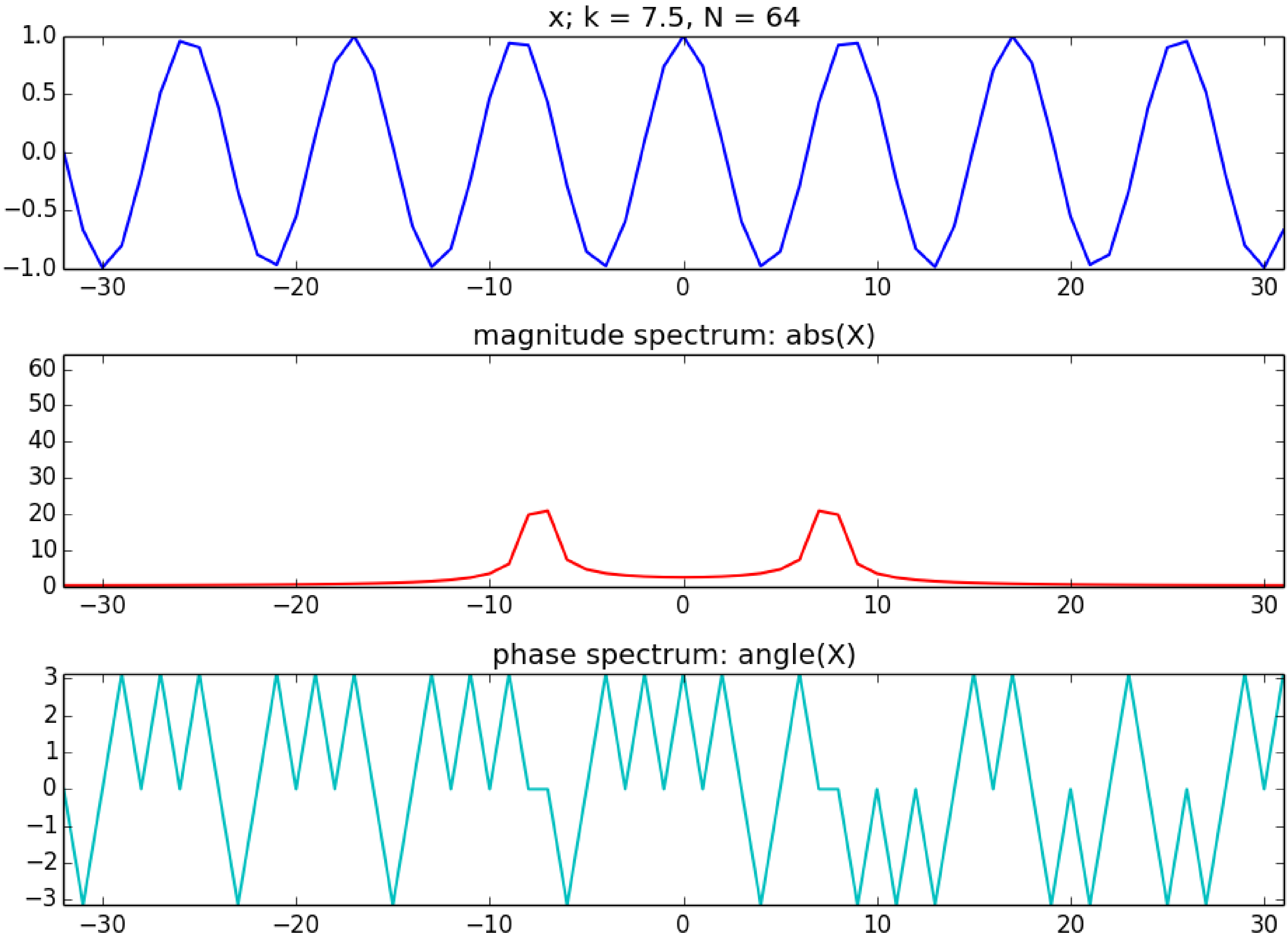
$$X_3[k] = \sum_{n=-N/2}^{N/2-1} x_2[n] e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \left(\frac{A_0}{2} e^{j2\pi k_0 n/N} + \frac{A_0}{2} e^{-j2\pi k_0 n/N}\right) e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{j2\pi k_0 n/N} e^{-j2\pi kn/N} + \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi k_0 n/N} e^{-j2\pi kn/N}$$

$$= \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi(k-k_0)n/N} + \sum_{n=-N/2}^{N/2-1} \frac{A_0}{2} e^{-j2\pi(k+k_0)n/N}$$

$$= \frac{A_0}{2} \text{ for } k = k_{0,} -k_0;\ 0 \text{ for rest of } k$$



x; k = 7.5, N = 64
1.0
0.5
0.0
−0.5
−1.0
−30
−20
−10
0
10
20
30
magnitude spectrum: abs(X)
60
50
40
30
20
10
0
phase spectrum: angle(X)
3
2
1
0
−1
−2
−3

Inverse DFT

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]s_k[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \quad n=0,1,\ldots,N-1$$

Example:

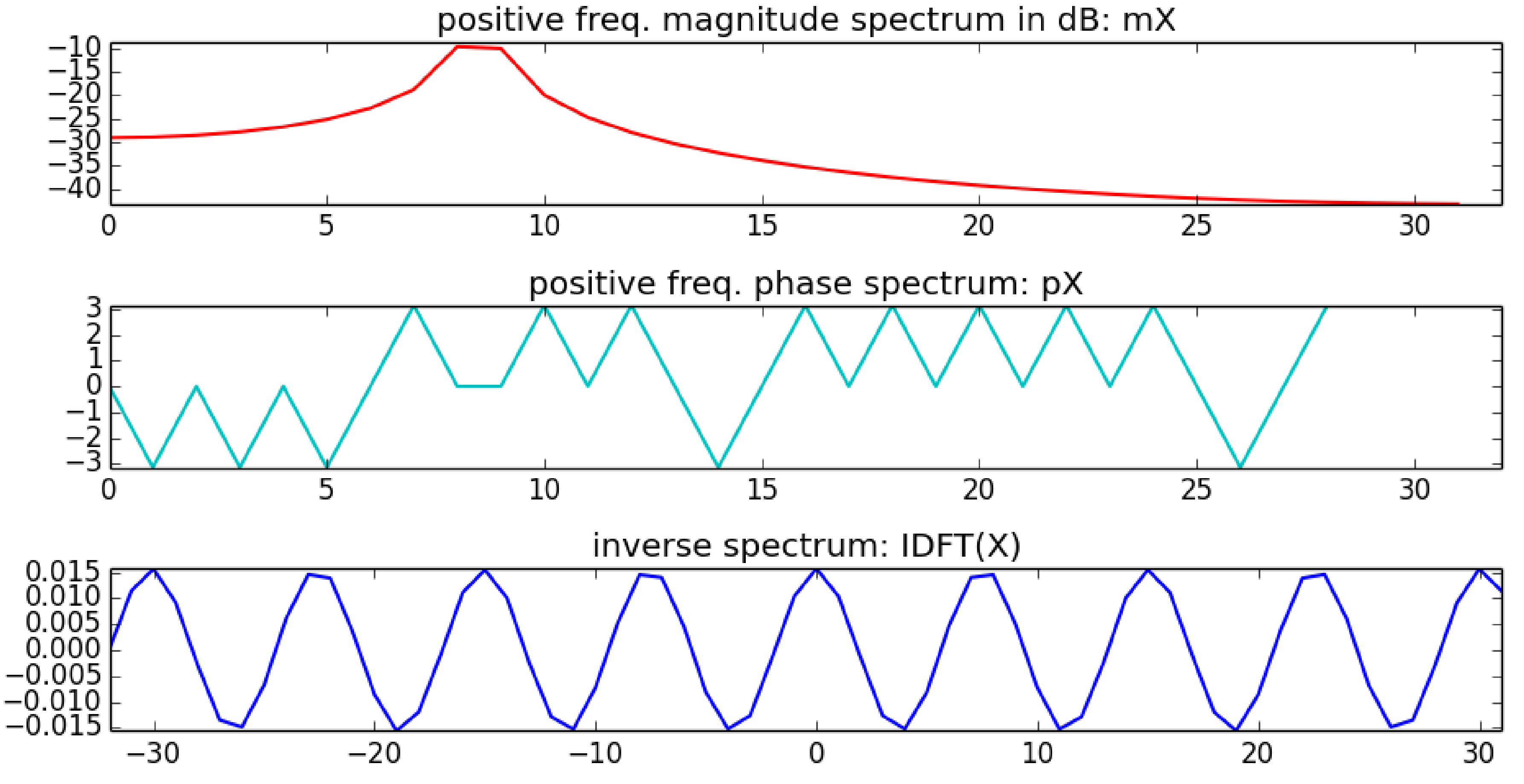
$$X[k]=[0,4,0,0];N=4$$

$$x[0]=X*s[n=0]=0*1+4*1+0*1+0*1=4$$
$$x[1]=X*s[n=1]=0*1+4*j+0*(-1)+0*(-j)=4j$$
$$x[2]=X*s[n=2]=0*1+4*(-1)+0*1+0*(-1)=-4$$
$$x[3]=X*s[n=3]=0*1+4*(-j)+0*(-1)+0*j=-4j$$

Inverse DFT for real signals

$X[k] = |X[k]| e^{j \sphericalangle X[k]}$ and $X[-k] = |X[k]| e^{-j \sphericalangle X[k]}$

for $k = 0, 1, \ldots, N/2 - 1$



References and credits

- More information in:
  - https://en.wikipedia.org/wiki/Discrete_Fourier_transform
- Reference on the DFT by Julius O. Smith: https://ccrma.stanford.edu/~jos/mdft/
- Sounds from: http://www.freesound.org/people/xserra/packs/13038/
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