3T1: Fourier Transform properties (1 of 2)

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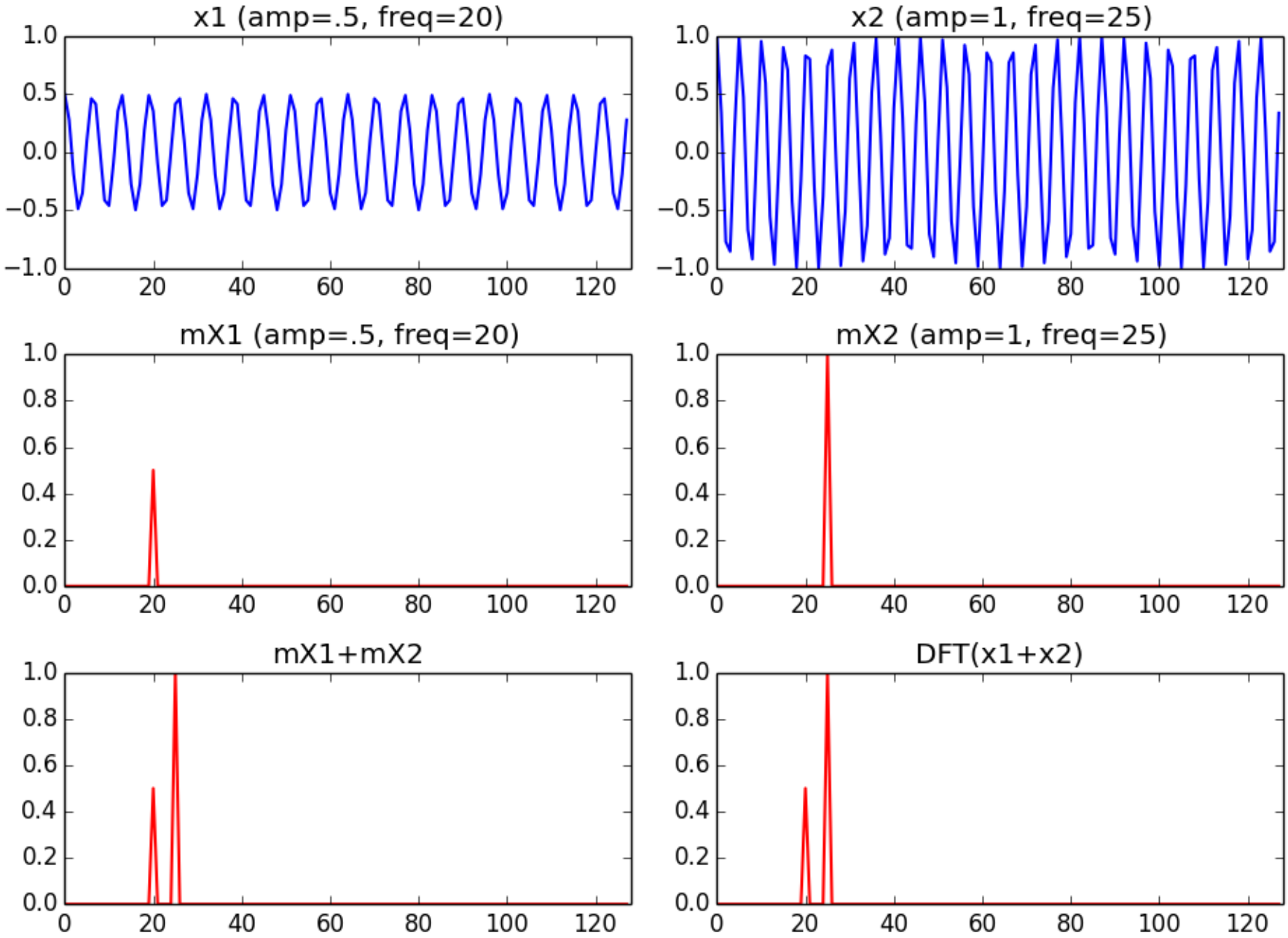
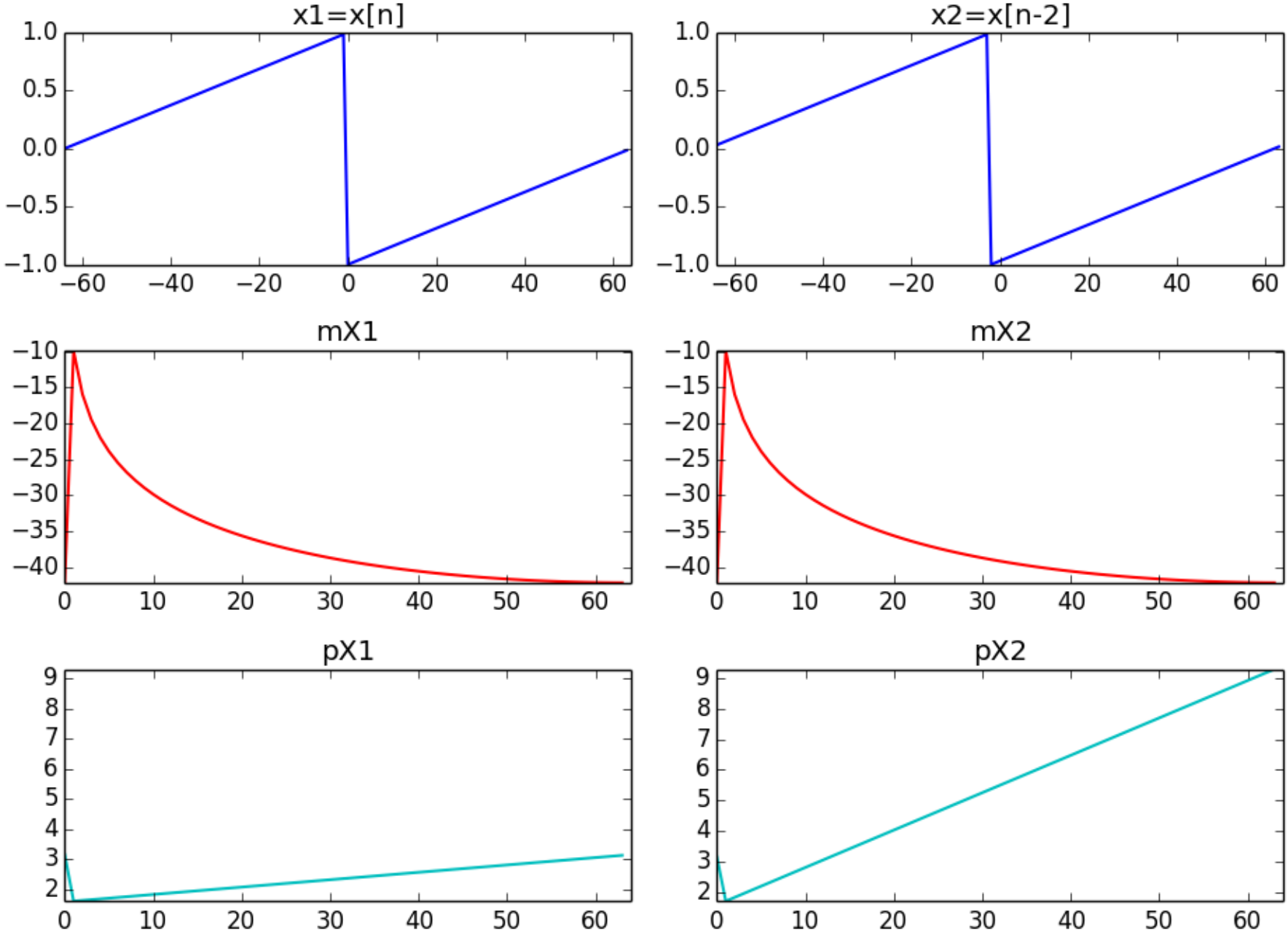
Linearity: $a\,x_1[n]+b\,x_2[n] \Leftrightarrow a\,X_1[k]+b\,X_2[k]$

$$
\begin{aligned}
&DFT\left(a\,x_1[n]+b\,x_2[n]\right) \\
&\quad=\sum_{n=0}^{N-1}\left(a\,x_1[n]+b\,x_2[n]\right)e^{-j\,2\pi\,kn/N} \\
&\quad=\sum_{n=0}^{N-1}a\,x_1[n]e^{-j\,2\pi\,kn/N}+\sum_{n=0}^{N-1}b\,x_2[n]e^{-j\,2\pi\,kn/N} \\
&\quad=a\sum_{n=0}^{N-1}x_1[n]e^{-j\,2\pi\,kn/N}+b\sum_{n=0}^{N-1}x_2[n]e^{-j\,2\pi\,kn/N} \\
&\quad=a\,X_1[k]+b\,X_2[k]
\end{aligned}
$$


x1 (amp=.5, freq=20)
x2 (amp=1, freq=25)
mX1 (amp=.5, freq=20)
mX2 (amp=1, freq=25)
mX1+mX2
DFT(x1+x2)

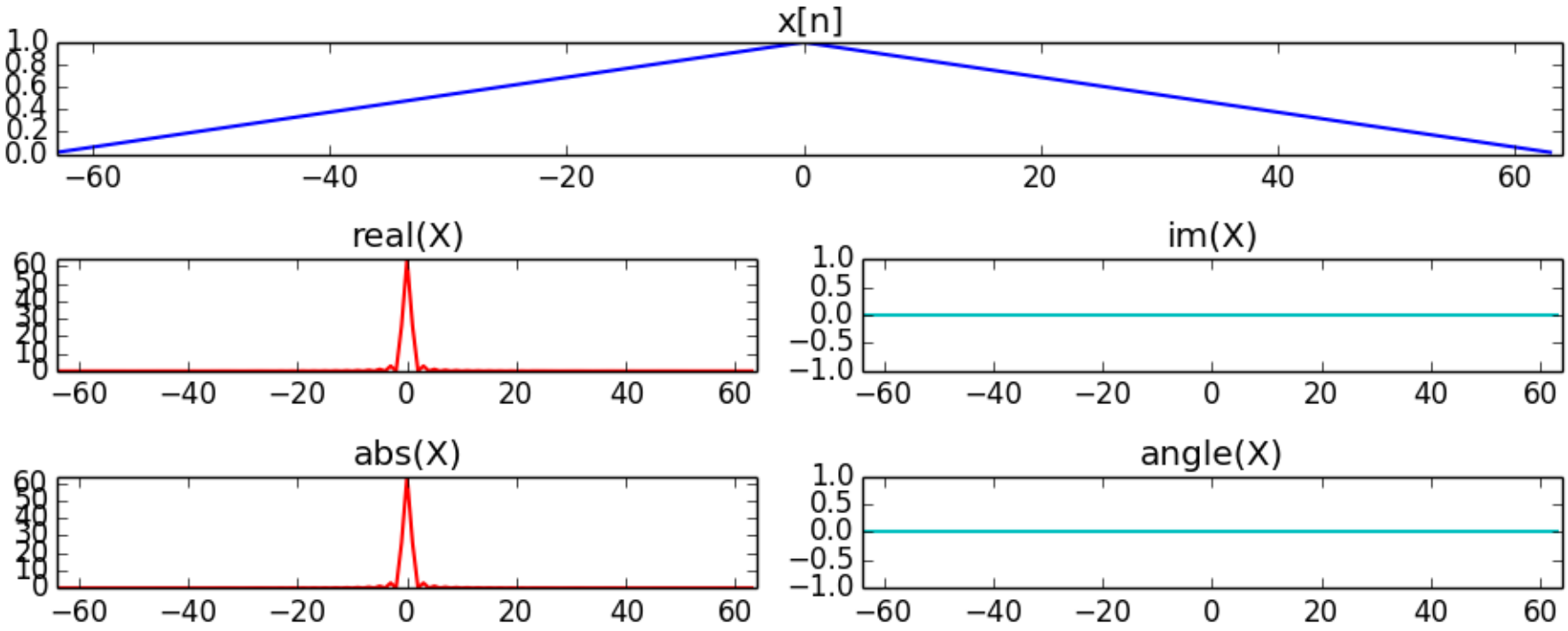
Shift: $x[n-n_0] \Leftrightarrow e^{-j2\pi k n_0/N} X[k]$

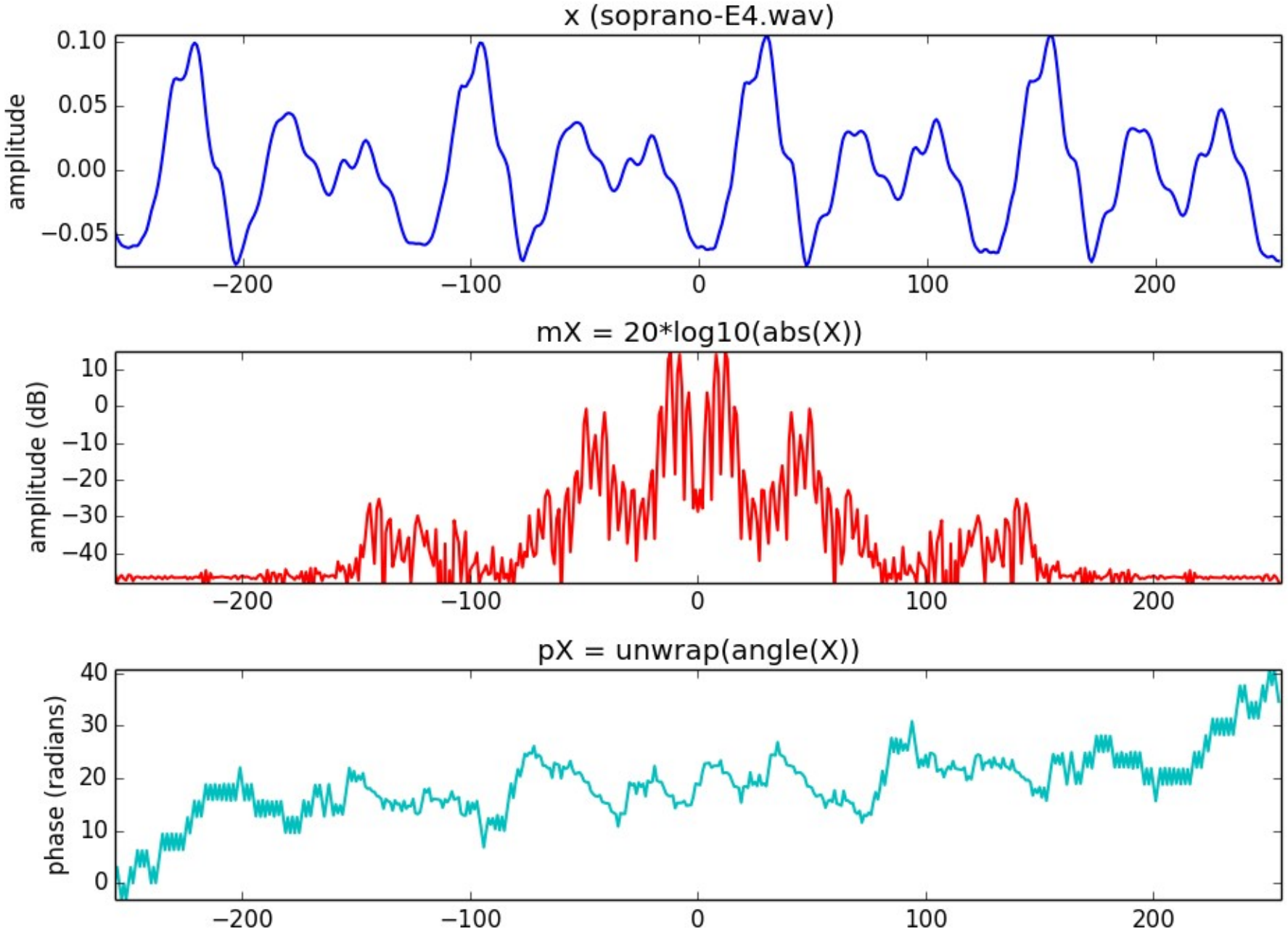
$$
\begin{aligned}
&DFT\left(x[n-n_0]\right) \\
&\quad = \sum_{n=0}^{N-1} x[n-n_0] e^{-j2\pi kn/N} \\
&\quad = \sum_{m=-n_0}^{N-1-n_0} x[m] e^{-j2\pi k(m+n_0)/N} \qquad (m = n-n_0) \\
&\quad = \sum_{m=0}^{N-1} x[m] e^{-j2\pi km/N} e^{-j2\pi kn_0/N} \\
&\quad = e^{-j2\pi kn_0/N} \sum_{m=0}^{N-1} x[m] e^{-j2\pi km/N} \\
&\quad = e^{-j2\pi kn_0/N} X[k]
\end{aligned}
$$

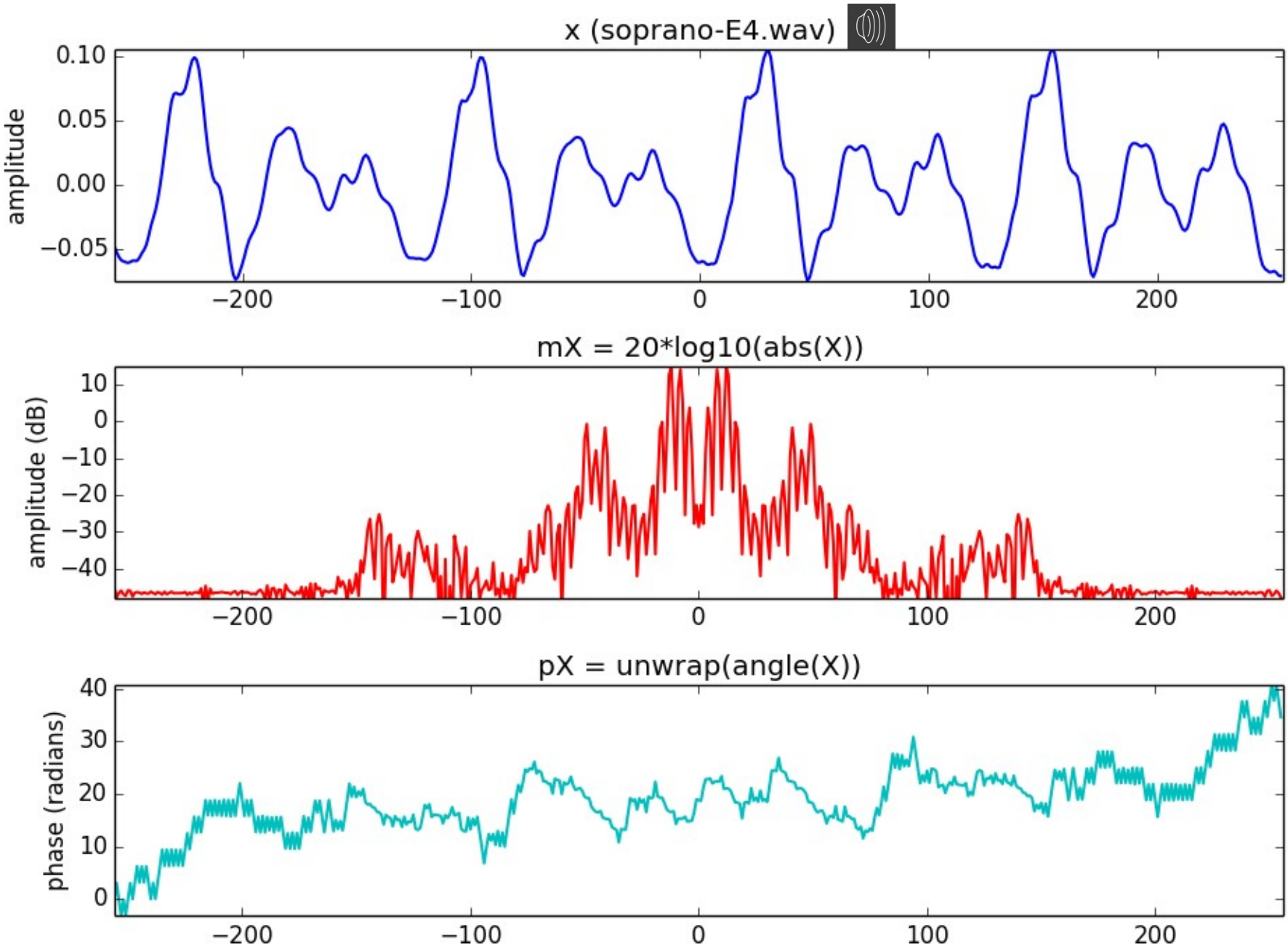

x1=x[n]
x2=x[n-2]
mX1
mX2
pX1
pX2

Symmetry:

$$x[n]\,real \;\Rightarrow\; \Re\{X[k]\}\,even \;\text{ and }\; \Im\{X[k]\}\,odd$$
$$\Rightarrow\; |X[k]|\,even \;\text{ and }\; \angle X[k]\,odd$$
$$x[n]\,real \text{ and } even \;\Rightarrow\; \Re\{X[k]\}\,even \;\text{ and }\; \Im\{X[k]\}=0$$
$$\Rightarrow\; |X[k]|\,even \;\text{ and }\; \angle X[k]=0$$




x (soprano-E4.wav)
amplitude
0.10
0.05
0.00
−0.05
−200
−100
0
100
200
mX = 20*log10(abs(X))
amplitude (dB)
10
0
−10
−20
−30
−40
−200
−100
0
100
200
pX = unwrap(angle(X))
phase (radians)
40
30
20
10
0
−200
−100
0
100
200


x (soprano-E4.wav)
amplitude
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mX = 20*log10(abs(X))
amplitude (dB)
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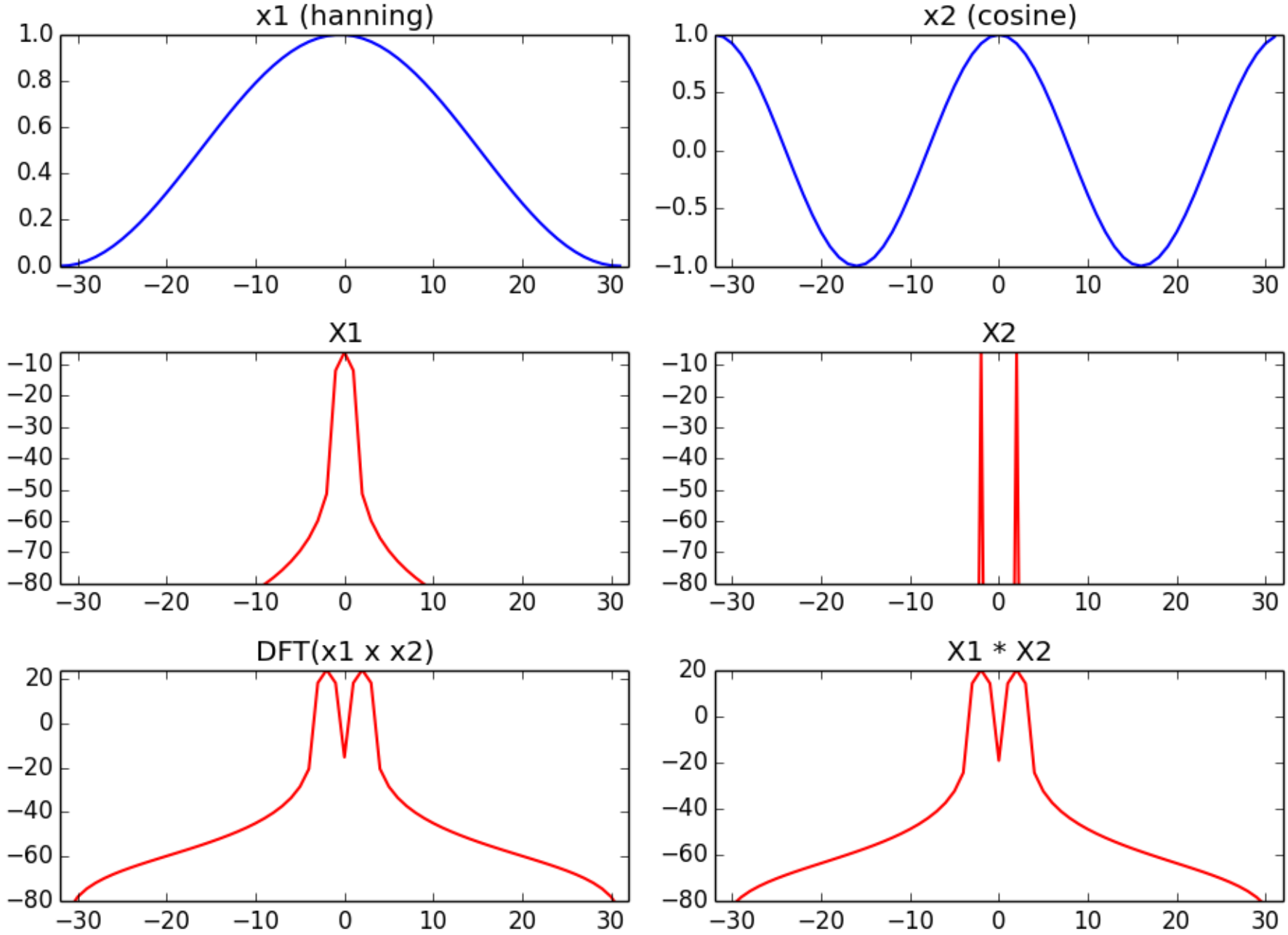
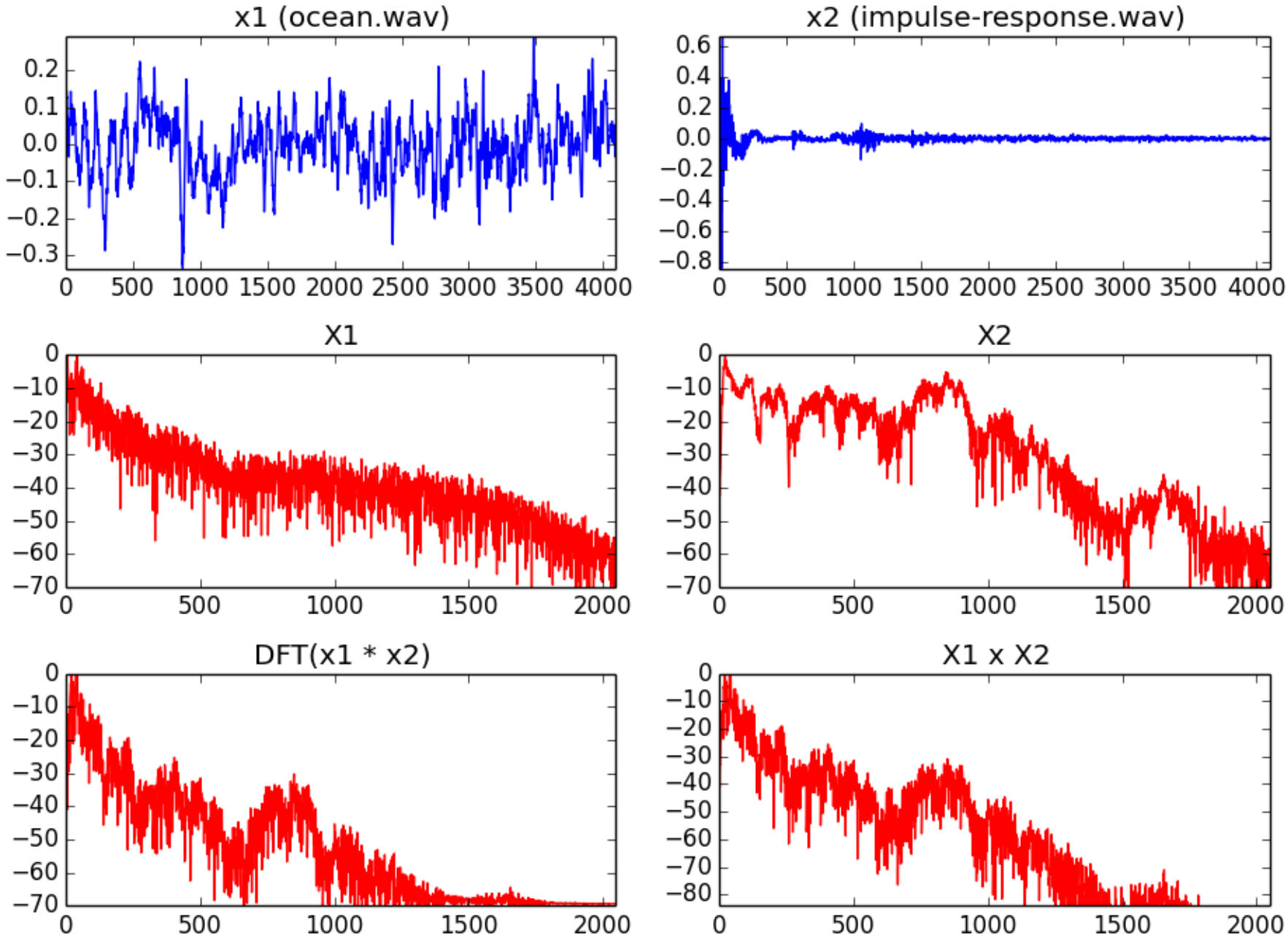
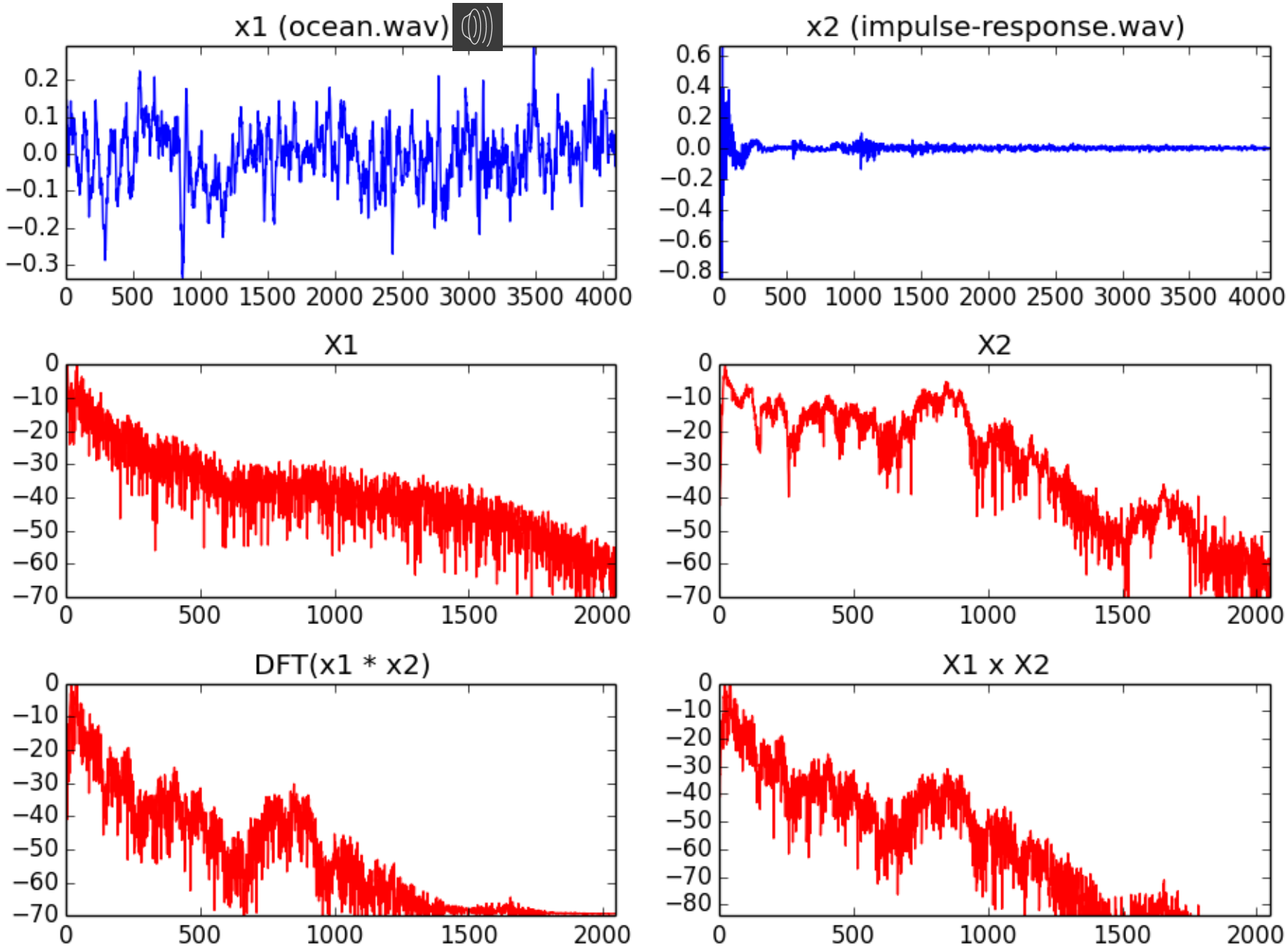
−200
−100
0
100
200
pX = unwrap(angle(X))
phase (radians)
40
30
20
10
0
−200
−100
0
100
200

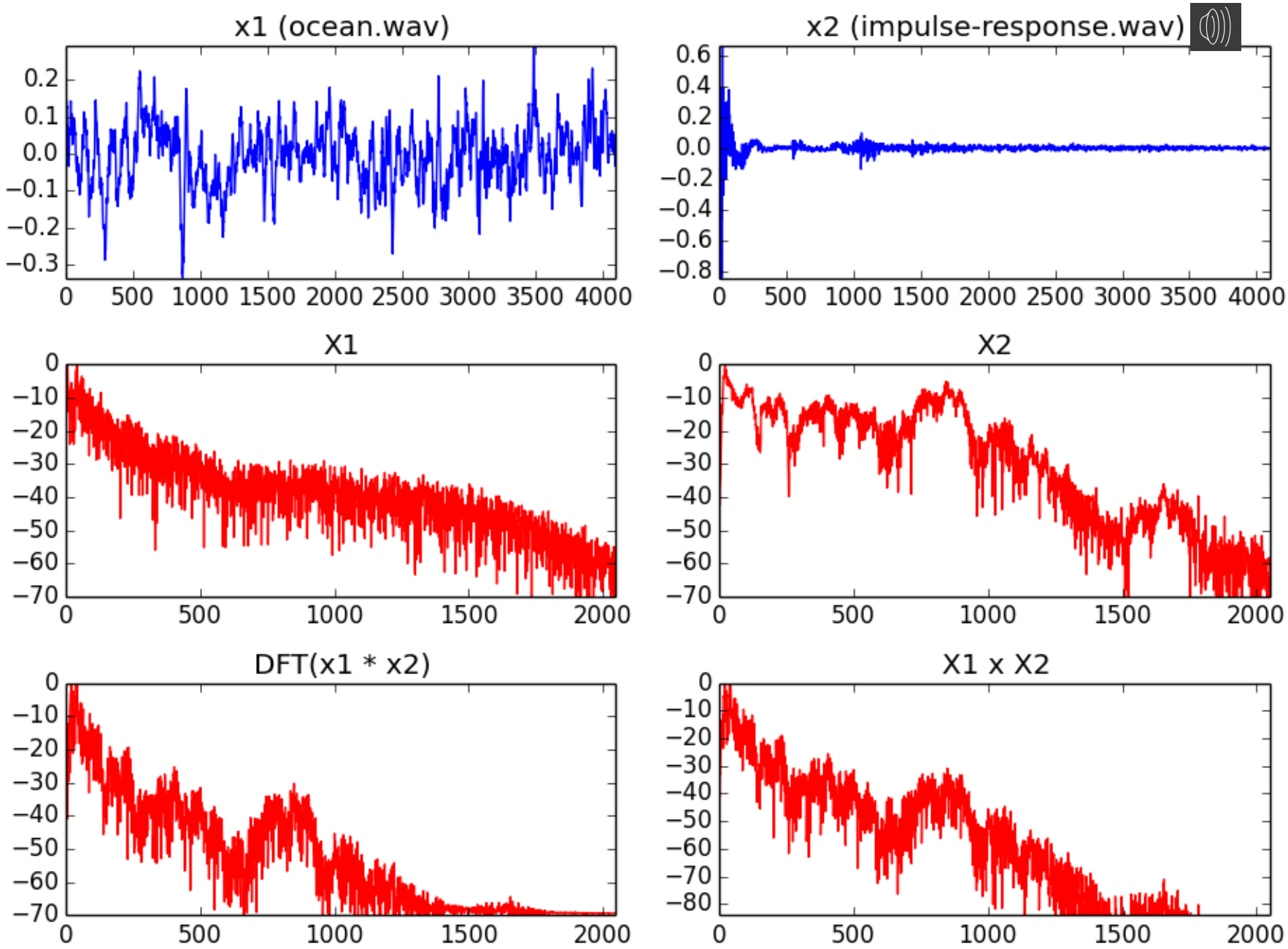
Convolution: $x_1[n] * x_2[n] \Leftrightarrow X_1[k] \times X_2[k]$

$$
\begin{aligned}
& DFT\left(x_1[n] * x_2[n]\right) \\
&= \sum_{n=0}^{N-1} \left(x_1[n] * x_2[n]\right) e^{-j 2\pi kn/N} \\
&= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] x_2[n-m] e^{-j 2\pi kn/N} \\
&= \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x_2[n-m] e^{-j 2\pi kn/N} \\
&= \left(\sum_{m=0}^{N-1} x_1[m] e^{-j 2\pi km/N} \right) X_2[k] \\
&= X_1[k] X_2[k]
\end{aligned}
$$


x1 (hanning)
x2 (cosine)
X1
X2
DFT(x1 x x2)
X1 * X2


x1 (ocean.wav)
x2 (impulse-response.wav)
X1
X2
DFT(x1 * x2)
X1 x X2


x1 (ocean.wav)
x2 (impulse-response.wav)
X1
X2
DFT(x1 * x2)
X1 x X2


x1 (ocean.wav)
x2 (impulse-response.wav)
X1
X2
DFT(x1 * x2)
X1 x X2

References and credits

- More information in:
  - https://en.wikipedia.org/wiki/Discrete_Fourier_transform
- Reference on the DFT by Julius O. Smith: https://ccrma.stanford.edu/~jos/mdft/
- Sounds from: http://www.freesound.org/people/xserra/packs/13038/
- Slides and code released using the CC Attribution-Non Commercial-Share Alike license or the Affero GPL license and available from https://github.com/MTG/sms-tools

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