membrane

April 2, 2020

1 Membrane Deflection

Author: Stefan Wittwer
E-mail: info@wittwer-datatools.ch

```
In [1]: from wdtPy.WDFE import WDFE
        from wdpPy.WDataView import WDataView
        from wdpPy.WDomainFactory import WDomainFactory
        from wdpPy.WDMeshFactory import WDMeshFactory
```

1.0.1 Abstract

The class ***wdtPy.WDFE*** extends some functions of the packages ***fenics*** and ***mshr*** (www.fenics.org). It implements the properties and methods used to process partial differential equations (PDE) with the finite element method (FEM).

This article demonstrates how to solve the steady-state Poisson equation for calculating the deflection $z(x, y)$ of a circular membrane under a position-depending pressure $p(x, y)$. The deflection will be assumed to be small against the radius of the circular domain. Therefore the tension σ is set to be a constant.

```
In [2]: wd_fe = WDFE('FE20200324.pvd')
```

1.1 Problem Definition

The membrane is defined to be a two-dimensional circular domain Ω with radius R. As long as no external forces act $z = 0$ everywhere on the membrane. However, as soon as a pressure acts on the membrane there is a non-zero deflection on Ω. The deflection is determined by the Poisson equation and a boundary condition.

The corresponding Poisson equation reads

$$\Delta z(x, y) = -\frac{1}{\sigma} p(x, y) \tag{1}$$

and the Dirichlet boundary condition is

$$z_D(x, y) = 0 \qquad x, y \in \partial\Omega \tag{2}$$

The pressure $p(x, y)$ is the source function on the right side of the PDE and is given by a Gaussian shaped pressure load

$$p(x, y) = \frac{p_0}{2\pi\Delta p} \exp\left(-\frac{1}{2\Delta p^2}\left((x - x_0)^2 + (y - y_0)^2\right)\right) \tag{3}$$

The parameters are: * pressure amplitude p_0, * distribution width Δp of pressure distribution, * center (x_0, y_0) of pressure distribution (set to $(0, y_0)$ with $0 < y_0 < R$).

1.2 Dimensionless Equations

With the transformations

- $x \to x/R$
- $y \to y/R$
- $z \to z/z_c$
- $y_0 \to y_0/R$

the dimensionless PDE is obtained:

$$\Delta z = -a \exp\left(-b^2\left(x^2 + (y - y_0)^2\right)\right) \tag{4}$$

with

$$a = \frac{R^2 p_0}{2\pi\sigma z_c \Delta p} \tag{5}$$

$$b = \frac{R}{\sqrt{2}\Delta p} \tag{6}$$

In [3]:
```
# radius of domain
R = 0.8

# init domain factory
wdomain_factory = WDomainFactory()
wdomain_factory.SetRadius(R)

# create circle domain
wd_fe.Domain = wdomain_factory.CreateCircle()
```

1.3 Preprocessing Stage

A preprocessing stage is run before the finite element simulation can be started. The first thing to do here is to create a mesh on the domain Ω. Then a function space is created on the mesh domain. Finally the boundary conditions are defined.

In [4]:
```
wd_mesh = WDMeshFactory(wd_fe.Domain, n=48)  # change n for different resolutions
```

1.3.1 Create Mesh

The mesh is created with the ***MeshFactory*** class. Once the mesh is successfully created it is displayed and saved to a VTK file type which can be visualized post process with the paraview tool (www.paraview.org).

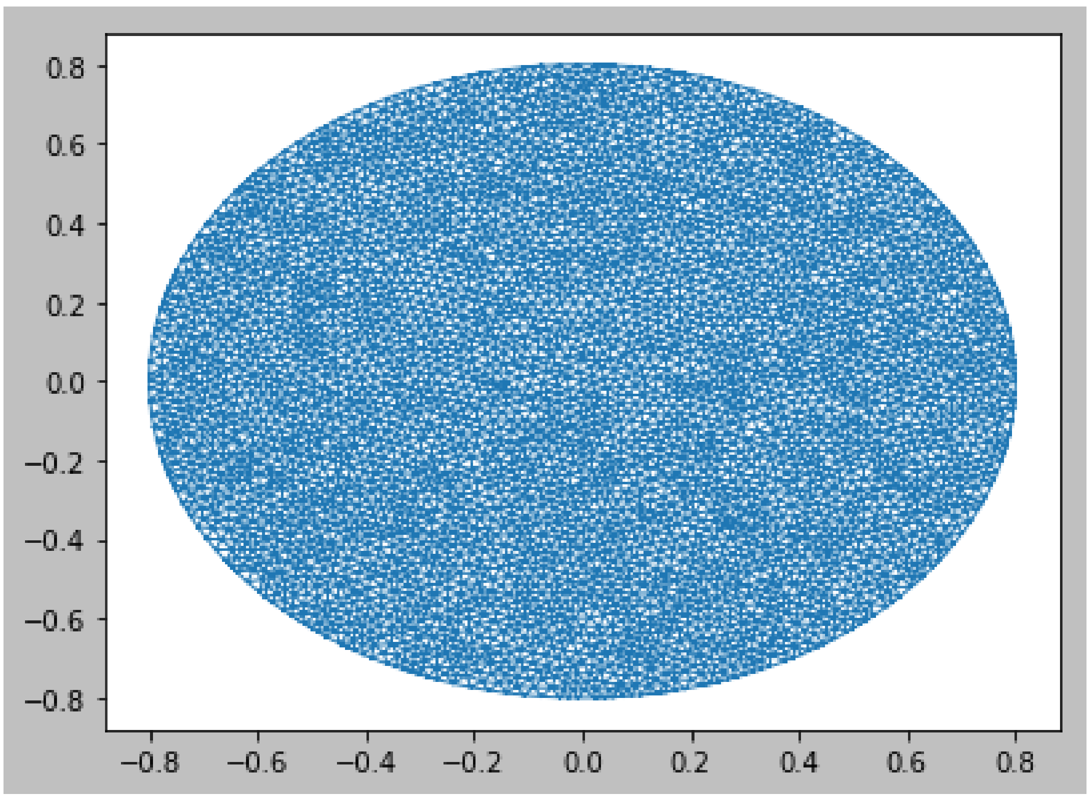
In [5]:
```
# create mesh
wd_fe.Mesh = wd_mesh.CreateMesh()
```

```python
# print number of vertices
vertex = wd_fe.Mesh.coordinates()
print('Number of coordinates: {0}'.format(vertex.shape))

# save mesh to VTK file
wd_fe.SaveToVTK(wd_fe.Mesh)  # view mesh with paraview

# show mesh
wdv = WDataView()
wdv.CreateTriPlot(vertex)
```

```
Number of coordinates: (4618, 2)
```



1.3.2 Create Function Space

In weak variational forms of PDE problems there are test and trial functions used to find the solution. The test function v and the trial function u are defined on the discrete function space of the mesh domain. The method ***DefineFunctionSpace()*** calls the ***FunctionSpace()*** method of Fenics (www.fenics.org) which takes the three following arguments: * the mesh, * the type of mesh element according www.femtable.org and * the degree of the mesh element.

```
In [6]: # define function space on mesh
        # default: cell_type='P', cell_degree=1 (see www.femtable.org)
        wd_fe.FSpace = wd_mesh.DefineFunctionSpace(wd_fe.Mesh)
```

1.3.3 Define Boundary Conditions

The Dirichlet boundary condition is determined by the following arguments: * a C-compatible string defining the formula of the boundary condition and * an accuracy degree (typically equal to the degree of the mesh element). Internally, the method defines a nested boolean function that decides whether a vertex is on a boundary or not.

In the present case we assume zero deflection on the entire boundary.

```
In [7]: z_D = '0.0'   # C-compatible string formula
        dbc = wd_fe.DefineDirichletBC(z_D, degree = 2)
```

1.4 Processing Stage: FEM Computation

Now that the FEM problem has been fully defined in the preprocessing stage the FEM computation of the solution can start. First the source function $f(\vec{x})$ is defined.

1.4.1 Define Pressure Load

The method ***DefineSourceFunction()*** is used to set the source term function on the right side of the PDE. Here this function is called $p(x,y)$ and has the physical meaning of a pressure load acting on the membrane surface.

```
In [8]: # define source function
        p = 'a*exp(-pow(b, 2.0)*(pow(x[0], 2.0) + pow(x[1] - c, 2.0)))'
        p = wd_fe.DefineSourceFunction(p, 2, {'a': 4.0, 'b': 8.0, 'c': 0.3})
```

1.4.2 Define Variational Form

Second, the variational problem is implemented by defining the test and trial functions. The method ***DefineVariationalForm*** returns the bilinear form as well as the linear operator in the PDE.

```
In [9]: # define variational problem
        a, L = wd_fe.DefineVariationalForm(-p)
```

1.4.3 Solve PDE

Finally the solution of the FEM problem is computed.

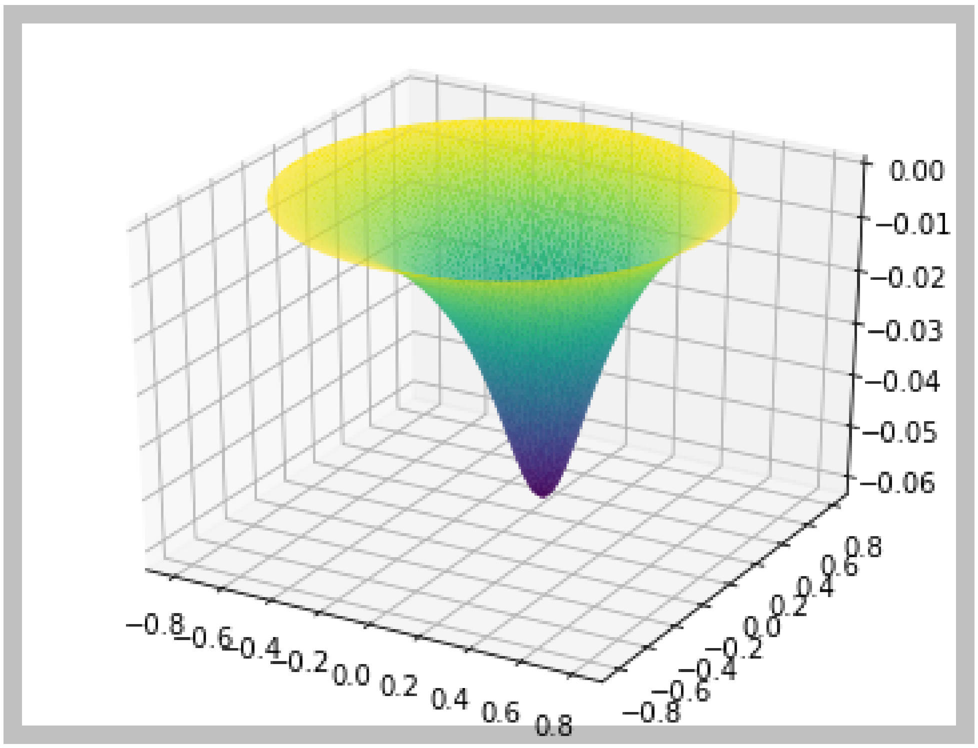
```
In [10]: # compute solution
         u = wd_fe.ComputeSteadyState(a, L)
```

1.5 Postprocessing Stage

1.5.1 Visualize Solution

The solution $z(x, y)$ is now ready to be investigated. Typically this is done by visualizing the numerical results of the FEM simulation. An interactive and versatile visualization can be done with the application ***paraview*** (www.paraview.org) and the created VTK file ***FE20200324.pvd***. The file contains the mesh in the first frame, and the scalar deflection field in the second frame.

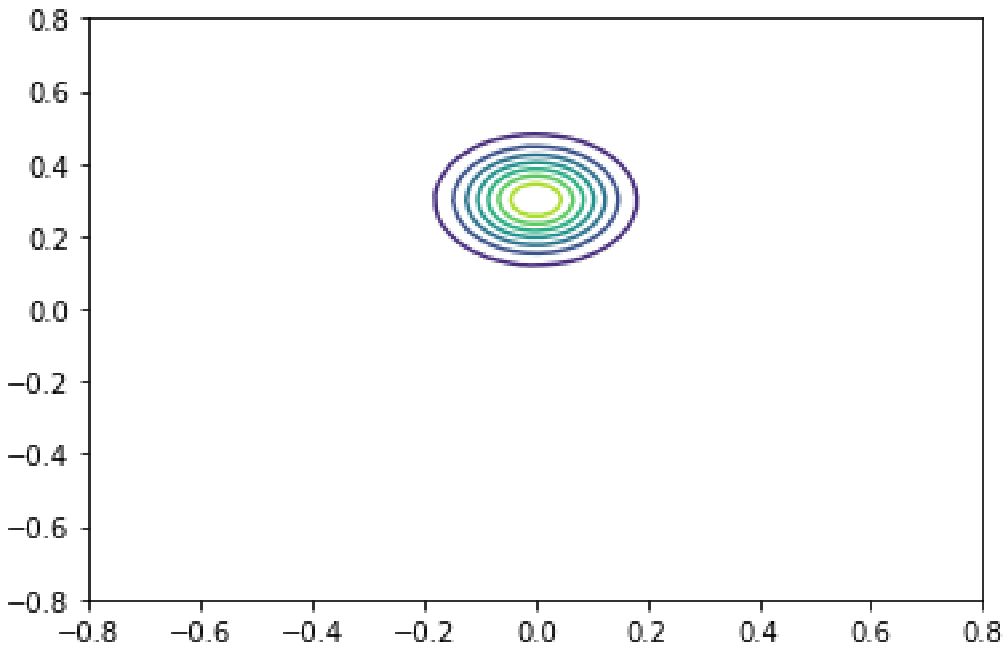
```
In [11]: z = wd_fe.ComputeVertexValues(u)
         wdv.CreateMeshSurface('z(x, y)', vertex[:, 0], vertex[:, 1], z)
```



1.5.2 Visualize Pressure Load

The pressure load represented by the function $p(x, y)$ must be interpolated over the function space of the mesh, before it can be visualized in the same way as the solution above.

```
In [12]: p = wd_fe.Interpolate(p)
         q = [p(x, y) for x, y in zip(vertex[:, 0], vertex[:, 1])]
         wdv.CreateMeshContour(vertex, q)
```



1.5.3 Computational Errors

Finally, let's compute the error in the L2 norm and the maximum error at the vertices. This helps to describe the quality of the simulation results.

```
In [13]: e_max = wd_fe.ComputeError(dbc, u)
         print('error max = {0}'.format(e_max))
```

```
error max = 0.062274627371311266
```

1.5.4 Save Results to VTK Files

```
In [14]: wd_fe.SaveToVTK(u)
         wd_fe.SaveToVTK(p)
```