Homework Assignment 12 :

① $$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$
$$= \frac{p_\gamma^2}{2I_1} \frac{(1-\cos\beta)^2}{\sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$
$$= \frac{p_\gamma^2}{2I_1} \frac{(1-\cos\beta)}{(1+\cos\beta)} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$
$$\approx \frac{p_\gamma^2}{2I_1}\left(\frac{\beta^2}{4} + \frac{\beta^4}{24}\right) + \frac{p_\gamma^2}{2I_3} + Mgl\left(1 - \frac{\beta^2}{2} + \frac{\beta^4}{24}\right) + O(\beta^5)$$
$$= \left(\frac{p_\gamma^2}{2I_3} + Mgl\right) + \left(\frac{p_\gamma^2}{4I_1} - Mgl\right)\frac{\beta^2}{2} + \left(\frac{p_\gamma^2}{2I_1} + Mgl\right)\frac{\beta^4}{24}$$

This describes a parabola with minimum at $\beta=0$ for small β when $p_\gamma^2 > 4I_1 Mgl$. Explicitly:

$$\frac{\partial V_{eff}}{\partial\beta} = \frac{p_\gamma^2}{2I_1}\left(\frac{\beta}{2} + \frac{\beta^3}{6}\right) - Mgl\left(\beta - \frac{\beta^3}{6}\right) = 0 \Rightarrow \beta = 0 \text{ is solution}$$
$$\left.\frac{\partial^2 V_{eff}}{\partial\beta^2}\right|_{\beta=0} = \frac{p_\gamma^2}{2I_1}\cdot\frac{1}{2} - Mgl > 0 \Leftrightarrow p_\gamma^2 > 4I_1 Mgl$$

$$\hookrightarrow V_{eff}(\beta) \simeq \frac{1}{2}\underbrace{\left(\frac{p_\gamma^2}{4I_1} - Mgl\right)}\beta^2 + O(\beta^3)$$

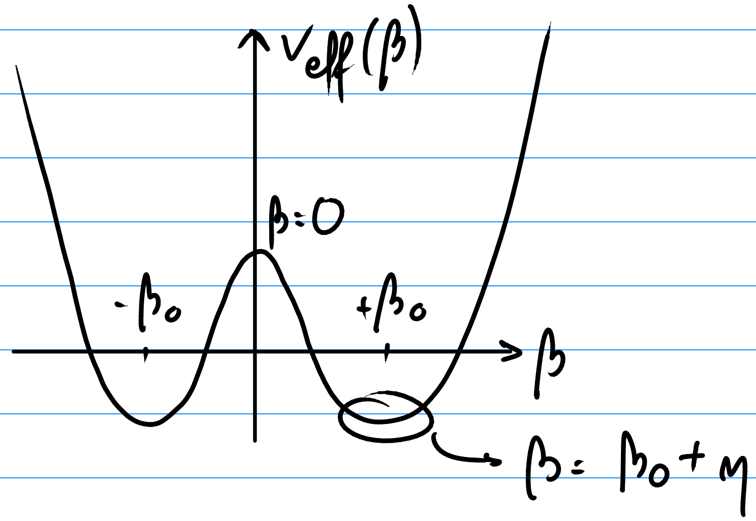
$$\Omega^2 = \frac{1}{I_1}\left(\frac{p_\gamma^2}{4I_1} - Mgl\right) = \frac{1}{4I_1^2}\left(p_\gamma^2 - 4I_1 Mgl\right)$$

The minima $\pm\beta_0$ for $p_\gamma^2 < 4I_1 Mgl$ are determined by

$$\left(\frac{p_\gamma^2}{4I_1} - Mgl\right) + \left(\frac{p_\gamma^2}{2I_1} + Mgl\right)\frac{\beta_0^2}{6} = 0$$

$$\Rightarrow \quad \beta_0^2 = -3\,\frac{p_\gamma^2 - 4I_1 Mgl}{\underbrace{p_\gamma^2 + 2I_1 Mgl}_{p_\gamma^2 \simeq 4I_1 Mgl}} \cong -3\,\frac{p_\gamma^2 - 4I_1 Mgl}{6I_1 Mgl}$$

$$= 2 - \frac{p_\gamma^2}{2I_1 Mgl}$$



$$V_{eff}(\eta) = \left(\frac{p_\gamma^2}{2I_3} + Mgl\right) + \left(\frac{p_\gamma^2}{4I_1} - Mgl\right)\frac{\beta^2}{2} + \left(\frac{p_\gamma^2}{2I_1} + Mgl\right)\frac{\beta^4}{24}$$

$$= \text{constant} + \left(\frac{p_\gamma^2}{4I_1} - Mgl\right)\beta_0\eta + \left(\frac{p_\gamma^2}{2I_1} + Mgl\right)\frac{4\beta_0^3\eta}{24}$$

$$+ \left(\frac{p_\gamma^2}{4I_1} - Mgl\right)\frac{\eta^2}{2} + \left(\frac{p_\gamma^2}{2I_1} + Mgl\right)\frac{\beta_0^2\eta^2}{4} + O(\eta^3)$$

Constant term is irrelevant.
First order term is zero by definition of β_0.
Second order term :

$$\Omega^2 = -\frac{2}{I_1}\left[\left(\frac{p_\gamma^2}{4I_1} - Mgl\right)\frac{1}{2} + \frac{1}{4}\left(\frac{p_\gamma^2}{2I_1} + Mgl\right)\left(2 - \frac{p_\gamma^2}{2I_1 Mgl}\right)\right]$$

$$= -\frac{1}{4I_1^2}\left[p_\gamma^2 - 4I_1 Mgl + (p_\gamma^2 + 2I_1 Mgl)\left(2 - \frac{p_\gamma^2}{2I_1 Mgl}\right)\right]$$

$p_\gamma^2 \approx 4I_1 Mgl$

$$= -\frac{1}{4I_1^2}\left[3p_\gamma^2 - p_\gamma^2 - \frac{(p_\gamma^2)^2}{2I_1 Mgl}\right] = \frac{1}{2I_1^2}\left[4I_1 Mgl - p_\gamma^2\right]$$

(2) $$L = \frac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \frac{1}{2} I_3 \left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgl\cos\beta$$

$$H = \frac{p_\beta^2}{2I_1} + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$

$$= \frac{p_\beta^2}{2I_1} + V_{eff}(\beta)$$

α, γ cyclic $\rightarrow p_\alpha =$ constant, $p_\gamma =$ constant

$$S = \alpha p_\alpha + W_\beta + \gamma p_\gamma - Et$$

$$\hookrightarrow H\left(\frac{\partial W_\beta}{\partial \beta}, \beta\right) = \frac{1}{2I_1}\left(\frac{\partial W_\beta}{\partial \beta}\right)^2 + V_{eff}(\beta) = E$$

$$\rightarrow W_\beta(\beta, E) = \pm \int \sqrt{2I_1(E - V_{eff}(\beta))}\, d\beta$$

$$\Rightarrow S(\alpha, \beta, \gamma, p_\alpha, E, p_\gamma, t) =$$

$$\int \sqrt{2I_1(E - V_{eff}(\beta))}\, d\beta - Et$$

$$\beta = \frac{\partial S}{\partial E} = \sqrt{\frac{I_1}{2}} \int \frac{1}{\sqrt{E - V_{eff}(\beta)}}\, d\beta - t$$

③ $H = E = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega_0^2 q^2 + \frac{1}{4} m \varepsilon q^4$

$$\hookrightarrow \frac{dq}{dt} = \dot{q}(E,q) = \sqrt{\frac{2}{m}\left(E - \frac{1}{2} m\omega_0^2 q^2 - \frac{1}{4} m \varepsilon q^4\right)}$$

$$\tau = \int_{cycle} dt = \int_{cycle} \sqrt{\frac{m}{2}} \frac{dq}{\left(E - \frac{1}{2} m\omega_0^2 q^2 - \frac{1}{4} m\varepsilon q^4\right)^{1/2}}$$

$$\tau = 2\int_{-a}^{a} \sqrt{\frac{m}{2}} \frac{dq}{\left[\frac{1}{2} m\omega_0^2 (a^2 - q^2) + \frac{1}{4} m\varepsilon (a^4 - q^4)\right]^{1/2}}$$

$q = a \sin\varphi \rightarrow dq = a\cos\varphi \, d\varphi$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

$$\tau = 2\int_{-\pi/2}^{\pi/2} \sqrt{\frac{m}{2}} \frac{a\cos\varphi \, d\varphi}{\left[\frac{1}{2} m\omega_0^2 a^2 \cos^2\varphi + \frac{1}{4} m\varepsilon a^4 (1 - \sin^4\varphi)\right]^{1/2}}$$

$$\simeq \sqrt{2m} \int_{-\pi/2}^{\pi/2} d\varphi \cos\varphi \left(\frac{1}{2} m\omega_0^2 \cos^2\varphi\right)^{-1/2} \left(1 - \frac{1}{4}\frac{\varepsilon}{\omega_0^2} a^2 \frac{(1 - \sin^4\varphi)}{\cos^2\varphi}\right)$$

$$= \frac{2}{\omega_0} \int_{-\pi/2}^{\pi/2} d\varphi \left(1 - \frac{1}{4}\frac{\varepsilon a^2}{\omega_0^2} \frac{(1 - \sin^4\varphi)}{\cos^2\varphi}\right)$$

$$= \frac{2\pi}{\omega_0} - \frac{2\pi}{\omega_0} \cdot \frac{3\varepsilon a^2}{8\omega_0}$$

$$\Rightarrow \omega = \frac{2\pi}{\tau} = \omega_0\left(1 + \frac{3\varepsilon a^2}{8\omega_0}\right)$$

Perturbations in multiple dimensions :

$$H(p,q,t) = H_0(p,q,t) + \varepsilon V(p,q,t)$$

$$\hookrightarrow H(J,\varphi) = \sum_{i=1}^{m} E_i(J) + \varepsilon V(J,\varphi)$$

$$\begin{cases} \dot{\varphi}_i = \dfrac{\partial H}{\partial J_i} \\ \dot{J}_i = -\dfrac{\partial H}{\partial \varphi_i} \end{cases}$$

Example : $V(p,q,t) = m^2 \omega_1^2 \omega_2^2 q_1^2 q_2^2$

$$V(J,\varphi) = m^2 \omega_1^2 \omega_2^2 \frac{2J_1}{m\omega_1} \frac{2J_2}{m\omega_2} \sin^2\varphi_1 \sin^2\varphi_2$$

$$= 4\omega_1 \omega_2 J_1 J_2 \sin^2\varphi_1 \sin^2\varphi_2$$

$$E_0(J) = \omega_1 J_1 + \omega_2 J_2$$

$$\Rightarrow \dot{\varphi}_1 = \omega_1 + 4\varepsilon \omega_1 \omega_2 J_2 \sin^2\varphi_1 \sin^2\varphi_2$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi_1 \frac{1}{2\pi}\int_0^{2\pi} d\varphi_2 \sin^2\varphi_1 \sin^2\varphi_2 = \frac{1}{4}$$

$$\Rightarrow \begin{cases} \langle \dot{\varphi}_1 \rangle = \omega_1 + \varepsilon \omega_1 \omega_2 J_2 \\ \langle \dot{\varphi}_2 \rangle = \omega_2 + \varepsilon \omega_1 \omega_2 J_1 \end{cases}$$