Classical Mechanics (Phys 601) - September 13, 2011

* (Non-)holonomic constraints $\Rightarrow$ Forces of constraint

$$\delta S = \delta \int_{t_1}^{t_2} L(\{q_j\},\{\dot{q}_j\},t)\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} dt \sum_j^n \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j = 0$$

- Holonomic $f_\ell(\{q_j\},t) = c_\ell \,,\ \ell = 1,\dots,k$

$$\Rightarrow \delta f_\ell = \sum_j^n \frac{\partial f_\ell}{\partial q_j}\delta q_j = 0 \,,\ \ell = 1,\dots,k$$

- Non-holonomic : $\sum_j^n a_{\ell_j}\dot{q}_j + b_\ell = 0$ virtual displacement

$$\Rightarrow \cancel{\delta f_\ell} = \sum_j^n a_{\ell_j}\delta q_j = 0$$

Multiply by $\lambda_\ell(\{q_j\},t)$ and sum

$$\Rightarrow \int_{t_1}^{t_2} dt \sum_j^n \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_\ell^k \lambda_\ell a_{\ell_j} \right) \delta q_j$$

(with $a_{\ell_j} = \frac{\partial f_\ell}{\partial q_j}$ if holonomic)

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} - \dfrac{\partial L}{\partial q_j} = \sum_\ell^k \lambda_\ell a_{\ell_j} & (n) \\ \sum_j^n a_{\ell_j}\dot{q}_j + b_\ell = 0 & (k) \end{cases} \Bigg\}\ n+k \text{ unknowns}$$

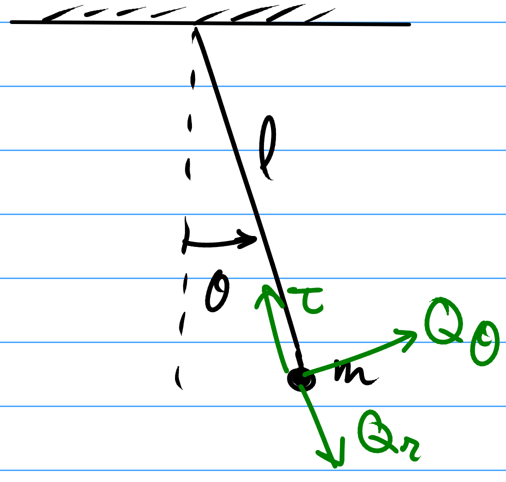
Using $L = T - V$, and $V(\{q_j\}, t)$ we find:

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + \sum_{\ell}^{k} \lambda_\ell a_{\ell j}$$

Original form of Lagrange's equation:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j = \text{total force}$$

$$\Rightarrow \quad Q_j = \underbrace{-\frac{\partial V}{\partial q_j}}_{\text{applied force}} + \underbrace{\sum_{\ell}^{k} \lambda_\ell a_{\ell j}}_{\text{force of constraint}} = Q_j^{\text{applied}} + Q_j^{\text{constraint}}$$



* Pendulum:

Constraint: $f(r, \theta) = r = \ell$

Lagrangian:

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} - \dfrac{\partial L}{\partial r} = Q_r = \lambda \dfrac{\partial f}{\partial r} = -\tau \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} - \dfrac{\partial L}{\partial \theta} = Q_\theta = 0 \\ r = \ell \end{cases}$$

$(f(r,\theta) = r)$

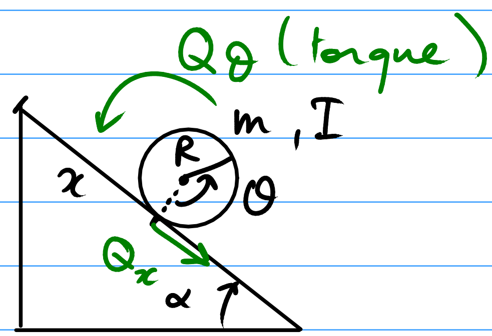
$$\frac{\partial f}{\partial r} = 1 \qquad \frac{\partial f}{\partial \theta} = 0$$

$$\Rightarrow \begin{cases} m(\ddot{r} - r\dot{\theta}^2) - mg\cos\theta = \lambda = -\tau \\ \frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0 \\ r = l \Rightarrow \dot{r} = 0,\ \ddot{r} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \tau = ml\dot{\theta}^2 + mg\cos\theta \\ \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \end{cases}$$ solve for $\theta(t)$

τ = centrifugal force + gravitational force

* Disk rolling on slope :



Constraints: $R\theta = x$

$$\Rightarrow f(x,\theta) = x - R\theta = 0$$

$$a_x = 1\ ,\ a_\theta = -R$$

Lagrangian : $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + mg\,x\sin\alpha$

$$\Rightarrow \begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda\frac{\partial f}{\partial x} = \lambda = Q_x \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f}{\partial \theta} = -R\lambda = Q_\theta \\ x = R\theta \end{cases}$$

$$\Rightarrow \begin{cases} m\ddot{x} - mg\sin\alpha = \lambda \\ I\ddot{\theta} = -R\lambda \Rightarrow \frac{1}{2}mR^2\frac{\ddot{x}}{R} = \frac{1}{2}Rm\ddot{x} = -R\lambda \\ x = R\theta \end{cases}$$

$$\Rightarrow \begin{cases} \frac{3}{2}m\ddot{x} - mg\sin\alpha = 0 \\ \frac{1}{2}m\ddot{x} = -\lambda \\ x = R\theta \end{cases}$$

$$\Rightarrow \quad \ddot{x} = \frac{2g\sin\alpha}{3} \Rightarrow x = \frac{1}{3}g\sin\alpha \cdot t^2 + v_0 t + x_0$$

If $v_0 = 0$ and $x_0 = 0 \Rightarrow x = \frac{1}{3}g\sin\alpha\, t^2$

$$\Rightarrow \begin{cases} Q_x = \lambda = -\frac{1}{2}m\ddot{x} = -\frac{1}{3}g\sin\alpha & \text{(friction)} \\ Q_\theta = -R\lambda = \frac{1}{3}Rg\sin\alpha & \text{(torque)} \end{cases}$$

Notice that $\sum_j Q_j \delta q_j = Q_x \delta x + Q_\theta \delta\theta$

$$= -\frac{1}{3}g\sin\alpha\,(\underbrace{\delta x - R\delta\theta}_{x = R\theta \,\to\, \delta x = R\delta\theta})$$

$$= 0$$

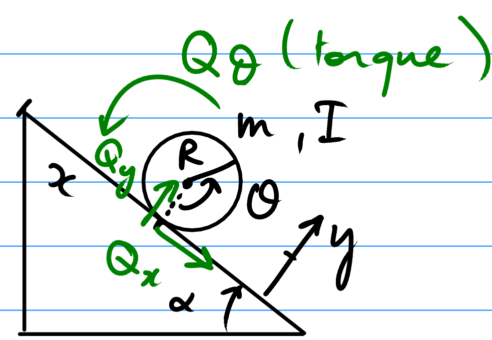
$\Rightarrow$ no work is done by forces of constraint!

* Disk rolling on slope (with normal force):

Constraints :

$f_1(x,y,\theta) = x - R\theta = 0$

$f_2(x,y,\theta) = y = R$



Lagrangian : $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\dot{\theta}^2 + mg(x\sin\alpha - y\cos\alpha)$

$$\Rightarrow \begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda\frac{\partial f_1}{\partial x} + \mu\frac{\partial f_2}{\partial x} = \lambda = Q_x \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda\frac{\partial f_1}{\partial y} + \mu\frac{\partial f_2}{\partial y} = \mu = Q_y \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f_1}{\partial \theta} + \mu\frac{\partial f_2}{\partial \theta} = -R\lambda = Q_\theta \\ x = R\theta \\ y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} m\ddot{x} - mg\sin\alpha = \lambda = Q_x \\ m\ddot{y} + mg\cos\alpha = \mu = Q_y \\ I\ddot{\theta} = -R\lambda = Q_\theta \\ x = R\theta \\ y = 0 \Rightarrow \dot{y} = 0 \Rightarrow \ddot{y} = 0 \end{cases}$$

$\Rightarrow$ same results for Q_x and Q_θ

$Q_y = \mu = mg\cos\alpha$ (normal force)

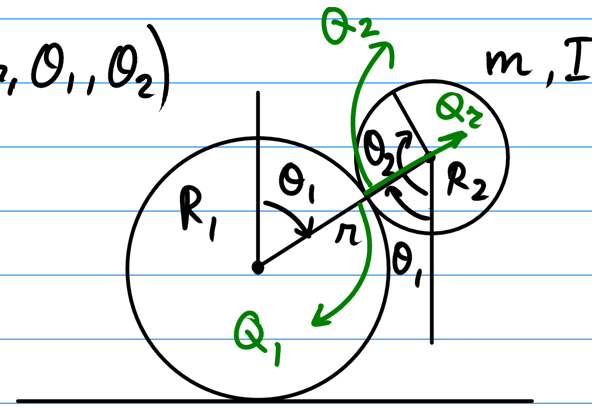
* One cylinder rolling on another:

Goal: find point where contact is lost.

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}_1^2\right) \qquad (r, \theta_1, \theta_2)$$

$$+ \frac{1}{2}\left(\frac{1}{2}mR_2^2\right)\dot{\theta}_2^2$$

$$- mgr\cos\theta_1$$



Constraints: (holonomic)

$r = R_1 + R_2 \longrightarrow \lambda_1 \rightarrow$ normal force

$R_1\theta_1 = R_2(\theta_2 - \theta_1) \longrightarrow \lambda_2 \rightarrow$ frictional torque

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} - \dfrac{\partial L}{\partial r} = Q_r \Rightarrow m(\ddot{r} - r\dot{\theta}_1^2) + mg\cos\theta_1 = \lambda_1 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}_1} - \dfrac{\partial L}{\partial \theta_1} = Q_{\theta_1} \Rightarrow \dfrac{d}{dt}\left(mr^2\dot{\theta}_1\right) - mgr\sin\theta_1 = \lambda_2(R_1 + R_2) \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}_2} - \dfrac{\partial L}{\partial \theta_2} = Q_{\theta_2} \Rightarrow \dfrac{1}{2}mR_2^2\ddot{\theta}_2 = -\lambda_2 R_2 \\ r = R_1 + R_2 \Rightarrow \dot{r} = 0 \Rightarrow \ddot{r} = 0 \quad (1) \\ (R_1 + R_2)\theta_1 - R_2\theta_2 = 0 \Rightarrow (R_1 + R_2)\ddot{\theta}_1 = R_2\ddot{\theta}_2 \quad (2) \end{cases}$$

$$\Rightarrow \begin{cases} -m(R_1 + R_2)\dot{\theta}_1^2 + mg\cos\theta_1 = \lambda_1 \quad (1) \\ \lambda_2 = -\dfrac{1}{2}m(R_1 + R_2)\ddot{\theta}_1 \quad (2) \\ m(R_1 + R_2)^2\ddot{\theta}_1 - mg(R_1 + R_2)\sin\theta_1 = -\dfrac{1}{2}m(R_1 + R_2)^2\ddot{\theta}_1 \end{cases}$$

$$\Rightarrow \quad \frac{3}{2}(R_1+R_2)\,\ddot{\theta}_1 = g \sin\theta_1 \quad \Big| \cdot \dot{\theta}_1$$

$$\Leftrightarrow \quad \frac{3}{2}(R_1+R_2)\int_0^t \ddot{\theta}_1\,\dot{\theta}_1\,dt = g\int_0^t \dot{\theta}_1 \sin\theta_1\,dt$$

$$\Leftrightarrow \quad \frac{3}{2}(R_1+R_2)\left(\frac{1}{2}\dot{\theta}_1^2 - \frac{1}{2}\dot{\theta}_0^2\right) = -g\cos\theta_1 + g\cos\theta_0$$

$$\Leftrightarrow \quad (R_1+R_2)\,\dot{\theta}_1^2 = \frac{4}{3}g(1-\cos\theta_1)$$

$$\theta_0 = 0$$
$$\dot{\theta}_0 = 0$$

$$\Rightarrow \quad -\frac{4}{3}mg(1-\cos\theta_1) + mg\cos\theta_1 = \lambda_1$$

$$\Leftrightarrow \quad \lambda_1 = \frac{1}{3}mg\left(7\cos\theta_1 - 4\right)$$

λ_1 is the normal force : $Q_r = \lambda_1 \frac{\partial f_1}{\partial r} = \lambda_1$

$\Rightarrow$ if $\lambda_1 = 0$, then the cylinders lose contact :

$$7\cos\theta_1 = 4 \Leftrightarrow \theta_1 = \cos^{-1}\frac{4}{7}$$

* Dissipative forces:

General form of Lagrange's equation

$$\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}}_{\text{dynamics including conservative forces and velocity dependence}} = Q_j + \sum_{\ell}^{k} \lambda_\ell a_{\ell j}$$

constraints

forces not derivable from potential

$$Q_j^{pot} = -\frac{\partial V}{\partial q_j}$$

friction, e.g. $\bar{F} = -k\bar{v}$

↳ introduce $\mathcal{F} = \frac{1}{2}\sum_j k_j \dot{q}_j^2 = \frac{1}{2}\sum_i \left(k_x \dot{x}^2 + k_y \dot{y}^2 + k_z \dot{z}^2\right)$

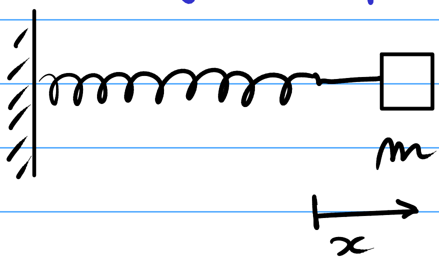
$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \qquad F_x = -k_x \dot{x}, \ldots$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j + \sum_{\ell}^{k} \lambda_\ell a_{\ell j} - \frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

Can also be used to introduce driving forces.

* Spring with friction and driving force :



$$V = \frac{1}{2} k x^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\mathcal{F} = \frac{1}{2} \beta \dot{x}^2 - F(t)\, \dot{x}$$

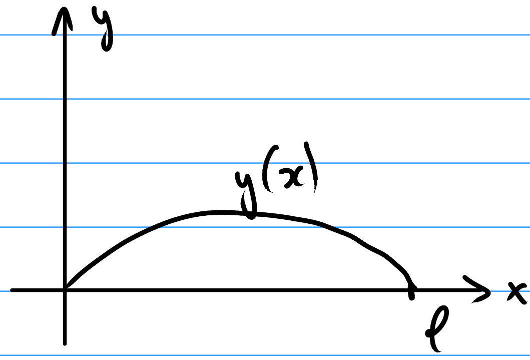
$$\Rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = - \frac{\partial \mathcal{F}}{\partial \dot{x}}$$

$$\Leftrightarrow \quad m \ddot{x} + k x = - \beta \dot{x} + F(t)$$

$$\Leftrightarrow \quad m \ddot{x} + \beta \dot{x} + k x = F(t)$$

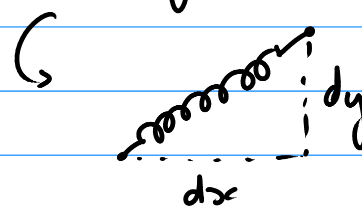
↳ damped harmonic oscillator with driving function $F(t)$

* Transition to continuous systems : string



$$L = T - V$$

$$\begin{cases} T = \frac{1}{2}\int_0^\ell dx\, \mu(x)\, \dot{y}^2 & \text{mass per unit length} \\ V = \frac{1}{2}\int_0^\ell dx\, \sigma(x)\, y'^2 & \text{tension per unit length} \end{cases}$$



$$L(t) = \frac{1}{2}\int_0^\ell dx \left(\mu(x)\dot{y}^2 - \sigma(x) y'^2\right)$$

$$S = \frac{1}{2}\int_{t_1}^{t_2} dt \int_0^\ell dx \left(\mu(x)\dot{y}^2 - \sigma(x) y'^2\right) \qquad dF = -\sigma(x)\, dy$$

$$\delta S = 0 = \frac{1}{2}\int_{t_1}^{t_2} dt \int_0^\ell dx \left(\mu(x)\,\dot{y}\,\delta\dot{y} - \sigma(x)\, y'\delta y'\right)$$

$$= \frac{1}{2}\int_{t_1}^{t_2} dt \int_0^\ell dx \left(\mu(x)\,\dot{y}\,\frac{\partial}{\partial t}\delta y - \sigma(x)\, y'\frac{\partial}{\partial x}\delta y\right)$$

$$\downarrow \quad \frac{\partial}{\partial x}\left(\sigma(x) y'\delta y\right) = \frac{\partial}{\partial x}\left(\sigma(x) y'\right)\delta y + \sigma(x) y' \frac{\partial}{\partial x}\delta y$$

$$= -\frac{1}{2}\int_{t_1}^{t_2} dt \left[\sigma(x)\, y'\,\delta y\right]_0^\ell + \frac{1}{2}\int_0^\ell dx \left[\mu(x)\dot{y}\,\delta y\right]_{t_1}^{t_2}$$

$$-\frac{1}{2}\int_{t_1}^{t_2} dt \int_0^\ell dx \left(\frac{\partial}{\partial t}\left(\mu(x)\dot{y}\right) - \frac{\partial}{\partial x}\left(\sigma(x) y'\right)\right)\delta y$$

Boundary conditions : * $y(0) = y(\ell) = \text{constant}$

$\hookrightarrow \delta y(0) = \delta y(\ell) = 0$

* $y'(0) = y'(\ell) = 0$

End points fixed in time: $\delta y(t_1) = \delta y(t_2) = 0$

$$\Rightarrow \quad \mu(x) \frac{\partial \dot{y}}{\partial t} = \frac{\partial}{\partial x}\left(\sigma(x) y'\right)$$

if $\mu(x) = \mu$, $\sigma(x) = \sigma$ $\Rightarrow$ $\mu \frac{\partial^2 y}{\partial t^2} = \sigma \frac{\partial^2 y}{\partial x^2}$

* Transition to continuous systems: Lagrangian density $\mathcal{L}$

Field $\varphi(\bar{x}, t)$ in 3 dimensions

$$\Rightarrow \begin{cases} T = \frac{1}{2}\int d^3x \; \dot{\varphi}^2(x,t) = \int d^3x \; \mathcal{T}(\bar{x},t) \\ V = \frac{1}{2}\int d^3x \; (\nabla \varphi)^2 = \int d^3x \; \mathcal{V}(\bar{x},t) \end{cases}$$

$$\Rightarrow L(t) = \frac{1}{2}\int d^3x \left[\dot{\varphi}^2 - (\nabla\varphi)^2\right] = \int d^3x \; \mathcal{L}(\bar{x},t)$$

$$S = \frac{1}{2}\int dt \int d^3x \left[\dot{\varphi}^2 - (\nabla\varphi)^2\right]$$

$$\delta S = -\frac{1}{2}\int dt \int d^3x \left[\ddot{\varphi} - \nabla^2\varphi\right]\delta\varphi$$

$$\Downarrow$$

$$\ddot{\varphi} - \nabla^2 \varphi = 0 \qquad \text{(Laplace equation)}$$

In general: (D = dimensions)

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{\mu}^{D} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0$$

$$\hookrightarrow \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial t}\right)}, \; \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x}\right)}$$