Classical Mechanics (Phys 601) - September 29, 2011

* Canonical transformations leave Hamilton's equations unchanged

$\{q_i, p_i\} \longrightarrow \{Q_j, P_j\}$

with $p_i = \frac{\partial F}{\partial q_i}$, $P_j = -\frac{\partial F}{\partial Q_j}$

and $K(Q_j, P_j, t) = H(q_i, p_i, t) + \frac{\partial F}{\partial t}(q_i, Q_j, t)$

$F(q_i, Q_j, t)$ is the generating function of the canonical transformation

Note: under canonical transformation:

$[F, G]_{pq} = [F, G]_{PQ}$ (by direct calculation)

$\Rightarrow$ transformation is canonical if:

- there exist a generating function $F(q_i, Q_j, t)$

OR

- Poisson brackets are preserved

Motivation: example of harmonic oscillator

$$H = \frac{1}{2m} p^2 + \frac{1}{2} k q^2 \curvearrowright K = \frac{1}{2m} P^2$$

$(P^2 = p^2 + q^2)$

cyclic in Q

$\dot{P} = 0$

Can we make all coordinates and momenta cyclic?

Remember that if H is cyclic for all q_i → $\frac{\partial H}{\partial q_i} = 0$
then $\dot{p}_i = 0 \Rightarrow p_i = \alpha_i$ = constant
$\Rightarrow H = H(\alpha_i, q_i, t)$

If H is cyclic for all p_i and q_i → $\frac{\partial L}{\partial p_i} = \frac{\partial L}{\partial q_i} = 0$
then $\dot{p}_i = 0$ and $\dot{q}_i = 0$
and: $p_i = \alpha_i$ and $q_i = \beta_i$
$\Rightarrow H = H(\alpha_i, \beta_i, t)$ with α_i, β_i constants (of integration) set by initial conditions

Note: Also other forms of $F(q_i, Q_j, t)$ exist
↳ Legendre transformation:

$$F = F_1(q_i, Q_j, t)$$

↳ $F = -\sum_j P_j Q_j + F_2(q_i, P_j, t)$

with $Q_j = \frac{\partial F_2}{\partial P_j}$, $p_i = \frac{\partial F_2}{\partial q_i}$

↳ $F = \sum_i p_i q_i + F_3(Q_j, p_i, t)$

with $q_i = -\frac{\partial F_3}{\partial p_i}$, $P_j = -\frac{\partial F_3}{\partial Q_j}$

↳ $F = \sum_i p_i q_i - \sum_j P_j Q_j + F_4(p_i, q_i, t)$

with $q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_j = \frac{\partial F_4}{\partial P_j}$

* Hamilton - Jacobi theory :

$\Rightarrow$ Can we simultaneously make all coordinates cyclic ?

$$\hookrightarrow \frac{\partial H}{\partial q_i} = 0 \text{ for all } i \Rightarrow H = H(p_i)$$

$$\hookrightarrow \begin{cases} p_i = \alpha_i = \text{constant} \Rightarrow H = H(\alpha_i) \\ q_i = \frac{\partial H}{\partial \alpha_i}(\alpha_i) \end{cases}$$

Introduce generating function S of second kind : $S(q_i, P_j, t)$

$$\rightarrow F(q_i, Q_j, t) = -\sum_j P_j Q_j + S(q_i, Q_j, t)$$

$$\hookrightarrow P_j = -\frac{\partial F}{\partial Q}$$

$\mathcal{L}$: replace Q_j by P_j (Legendre transformation)

$$dF = -\sum_j P_j dQ_j - \sum_j Q_j dP_j + \sum_i \frac{\partial S}{\partial q_i} dq_i + \sum_j \frac{\partial S}{\partial P_j} dP_j + \frac{\partial S}{\partial t} dt$$

$$dF = \sum_j \frac{\partial F}{\partial Q_j} dQ_j + \sum_i \frac{\partial F}{\partial q_i} dq_i + \frac{\partial F}{\partial t} dt$$

$$P_j = -\frac{\partial F}{\partial Q_j}$$

$$\Rightarrow \cancel{-\sum_j P_j dQ_j} - \sum_j Q_j dP_j + \sum_i \frac{\partial S}{\partial q_i} dq_i + \sum_j \frac{\partial S}{\partial P_j} dP_j + \frac{\partial S}{\partial t} dt$$

$$= \cancel{\sum_j \frac{\partial F}{\partial Q_j} dQ_j} + \sum_i \frac{\partial F}{\partial q_i} dq_i + \frac{\partial F}{\partial t} dt \longrightarrow p_i = -\frac{\partial F}{\partial q_i}$$

This is satisfied when, for $S(q_i, P_j, t)$:

$$Q_j = \frac{\partial S}{\partial P_j} \quad , \quad p_i = \frac{\partial S}{\partial q_i}$$

$$\text{and} \quad \frac{\partial S}{\partial t} = \frac{\partial F}{\partial t}$$

$$\Rightarrow K = H + \frac{\partial S}{\partial t}$$

$\Rightarrow$ S is also a generator of a canonical transformation

Choose S such that
all coordinates and momenta are cyclic, for $K \equiv 0$!

$$\Rightarrow P_i = \alpha_i, \; Q_i = \beta_i \text{ are constant}$$

$$K(P, Q, t) = H(p, q, t) + \frac{\partial S}{\partial t}(P, q, t) = 0$$

With $p_i = \frac{\partial S}{\partial q_i}$ $\Rightarrow$ $H\left(\frac{\partial S}{\partial q_i}, q_i, t\right) + \frac{\partial S}{\partial t}(P_j, q_i, t) = 0$

Hamilton-Jacobi equation

S is Hamilton's principal function

First order PDE in $S(q_i, P_j, t)$ $\rightarrow$ one integration constant

$$S = \underbrace{S(\alpha_i}_{n}, q_i, t) \;\overset{}{+}\; \underbrace{S_0}_{1}$$

$n + 1$ integration constants
but one is overall offset $\rightarrow$ irrelevant

From Hamilton's principal function $\rightarrow$ transformation.

$$S(P_j, q_i, t) \rightarrow \quad p_i = \frac{\partial S}{\partial q_i} \quad \text{and} \quad Q_j = \frac{\partial S}{\partial P_j}$$

This indeed leads to $K \equiv 0$.

$$\hookrightarrow \quad \dot{P}_j = 0 \quad \text{and} \quad \dot{Q}_j = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{P_j = \alpha_j}_{n} \quad + \quad \underbrace{Q_j = \beta_j}_{n}$$

independent constants of motion

* Physical meaning of Hamilton's principal function :

$$S = S(q_i, \alpha_i, t)$$

$\hookrightarrow \frac{dS}{dt} = \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t}$ along a physical trajectory

but $\frac{\partial S}{\partial q_i} = p_i$

and $\frac{\partial S}{\partial t} = K - H = -H$

$$\Rightarrow \frac{dS}{dt} = \sum_i p_i \dot{q}_i - H = L$$

$$\Rightarrow S(t) = \int_1^2 L(t)\,dt + S(t_1)$$

$\Rightarrow$ S(t) is the action along the trajectory

* Time-independent (conserved) Hamiltonian :

$$H\left(\frac{\partial S}{\partial q_i}, q_i, \cancel{t}\right) + \frac{\partial S}{\partial t} = 0$$

$$\Rightarrow \underbrace{S(q_i, \alpha_i, t)}_{\text{principal function}} = \underbrace{W(q_i, \alpha_i)}_{\text{characteristic function}} - \alpha_1 t$$

Momentum is $p_i = \frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i}$

Example: General one-dimensional potential $V(q)$

$$H = \frac{p^2}{2m} + V(q) = E$$

1) Hamilton-Jacobi equation for $S(q,\alpha,t)$

$$\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V(q)\right] + \frac{\partial S}{\partial t} = 0$$

Because Hamiltonian is time-independent:

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t$$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2 + V(q) = \alpha = \text{const.}$$

$$\Leftrightarrow \left(\frac{dW}{dq}\right)^2 = 2m\left[\alpha - V(q)\right]$$

$$\Leftrightarrow W(q,\alpha) = \pm \int dq \sqrt{2m\left[\alpha - V(q)\right]}$$

2) $$\beta = \frac{\partial S}{\partial \alpha} = \pm\sqrt{\frac{m}{2}} \int dq \frac{1}{\alpha - V(q)} - t = \text{const.}$$

↳ this can be used to relate q and t with α, β determined from the initial conditions.

* Example: simple harmonic oscillator

$$H = \frac{1}{2m} p^2 + \underbrace{\frac{1}{2} k q^2}_{V(q)} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \overset{\text{conserved}}{=} E$$

Hamilton-Jacobi equation for $S(q,P,t)$:

$$H + \frac{\partial S}{\partial t} = \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

With $S(q,\alpha,t) = W(q,\alpha) - \alpha t$:

$$H = \frac{1}{2m}\left[\left(\frac{dW}{dq}\right)^2 + m^2\omega^2 q^2\right] = \alpha = \text{total energy } E$$

$$\frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}$$

$$W = \sqrt{2m\alpha} \int dq \sqrt{1 - \frac{m\omega^2}{2\alpha} q^2}$$

$$S = \sqrt{2m\alpha} \int dq \sqrt{1 - \frac{m\omega^2}{2\alpha} q^2} - \alpha t$$

Integral can be solved, but we are interested in $Q_j = \frac{\partial S}{\partial P_j}$

1) $$\beta' = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}} \int dq \frac{1}{\sqrt{1 - \frac{m\omega^2}{2\alpha} q^2}} - t$$

$$\Leftrightarrow \quad t + \beta' = \frac{1}{\omega} \sin^{-1}\left(q\sqrt{\frac{m\omega^2}{2\alpha}}\right) \qquad \beta'\omega = \beta$$

$$\Leftrightarrow \quad q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t + \beta)$$

And in $p_i = \frac{\partial S}{\partial q_i}$:

2) $p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2}$

$\Leftrightarrow \quad p = \sqrt{2m\alpha}\ \cos(\omega t + \beta)$

Finally, α and β determined by initial conditions:

$$\begin{cases} 2m\alpha = p_0^2 + m^2\omega^2 q_0^2 \\ \tan\beta = m\omega \frac{q_0}{p_0} \end{cases}$$

$\Rightarrow$ we transformed the simple harmonic oscillator to a new canonical coordinate "phase angle" and a new canonical momentum "total energy".

Example: uncoupled double harmonic oscillator

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 + m^2\omega_y^2 y^2\right)$$

$\hookrightarrow S(x, y, \alpha_x, \alpha_y, t) = W_x(x, \alpha_x) + W_y(y, \alpha_y) - \underset{\alpha_x + \alpha_y}{\underset{\parallel}{\alpha}} t$

$\Rightarrow$ Hamilton-Jacobi equation:

$$\frac{1}{2m}\Bigg[\underbrace{\left(\frac{\partial W_x}{\partial x}\right)^2 + m^2\omega_x^2 x^2}_{\text{only } x\text{-dependent}} + \underbrace{\left(\frac{\partial W_y}{\partial y}\right)^2 + m\,\omega_y^2 y^2}_{\text{only } y\text{-dependence}}\Bigg] = \underset{\text{constant}}{\alpha}$$

$$\text{“}\ \alpha_x \quad + \quad \text{“}\ \alpha_y = \alpha$$

$$\Rightarrow \begin{cases} x = \sqrt{\frac{2\alpha_x}{m\omega_x^2}} \sin(\omega_x t + \beta_x) & \leftarrow \beta_x = \frac{\partial S}{\partial \alpha_x} \\ p_x = \sqrt{2m\alpha_x} \cos(\omega_x t + \beta_x) & \leftarrow p_x = \frac{\partial S}{\partial x} \\ y = \sqrt{\frac{2\alpha_y}{m\omega_y^2}} \sin(\omega_y t + \beta_y) \\ p_y = \sqrt{2m\alpha_y} \cos(\omega_y t + \beta_y) \end{cases}$$

with $\alpha_x + \alpha_y = \alpha = E$

Example: isotopic uncoupled double harmonic oscillator

$\hookrightarrow \omega_x = \omega_y$

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + m^2\omega^2(x^2+y^2)\right)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + m^2\omega^2 r^2\right)$$

$p_x = m\dot{x}$
$= m\dot{r}\cos\theta$
$\vdots \quad - mr\dot{\theta}\sin\theta$
$\rightarrow p_x^2 + p_y^2 = p_r^2 + \frac{p_\theta^2}{r^2}$

$$S = W_r(r, \alpha_r) + W_\theta(\theta, \alpha_\theta) - \alpha t$$

θ is cyclic $\rightarrow p_\theta = \frac{\partial S}{\partial \theta} = \alpha_\theta \Rightarrow \underbrace{W_\theta = \theta\alpha_\theta}_{\text{general form for cyclic coordinates}}$

$$S = W_r(r, \alpha_r) + \theta\alpha_\theta - \alpha t$$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 = \alpha$$

Could solve this equation for $W(r, \alpha_r)$, but we use solutions for x, y, p_x, p_y obtained earlier to get:

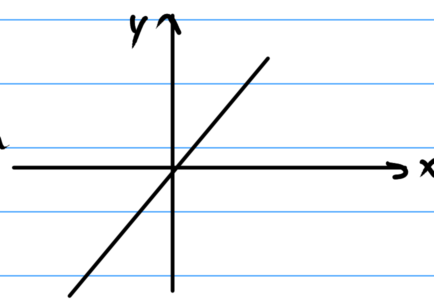
$(\beta = \beta_x - \beta_y)$

$$\Rightarrow \begin{cases} r = \sqrt{\dfrac{2\alpha}{m\omega^2}} \sqrt{\sin^2(\omega t) + \sin^2(\omega t + \beta)} \\ p_r = m\dot{r} \\ \theta = \tan^{-1}\left(\dfrac{\sin \omega t}{\sin(\omega t + \beta)}\right) \\ p_\theta = m r^2 \dot{\theta} \end{cases}$$

$$\beta = 0 \rightarrow \quad r = \frac{4\alpha}{m\omega^2} \sin \omega t \quad , \theta = \frac{\pi}{4}$$

$$p_r = \sqrt{2m\alpha} \cos \omega t \quad , p_\theta = 0$$

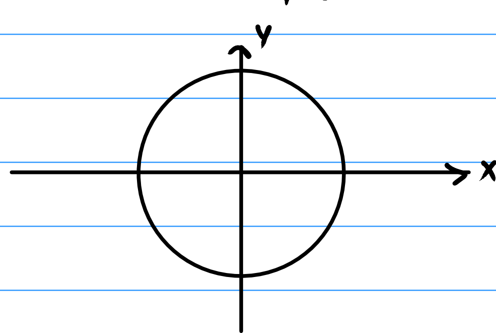
linear motion



$$\beta = \frac{\pi}{2} \rightarrow \quad r = \sqrt{\frac{2\alpha}{m\omega^2}} \quad , \quad \theta = \tan^{-1}(\tan \omega t) = \omega t$$

$$p_r = 0 \quad , \quad p_\theta = m r^2 \omega$$

circular motion



* Connection with Quantum Mechanics:

$$H(p_i, q_i, t)\,\psi = i\hbar \frac{\partial \psi}{\partial t} \quad \text{with } p_i = \frac{\hbar}{i}\frac{\partial}{\partial q_i}$$

If we let $\psi(q_i, t) \sim A \exp\left[\frac{i}{\hbar} S(q_i, t)\right]$ (plane wave assumption)

$\downarrow$

$$H\left(\frac{\hbar}{i}\frac{\partial}{\partial q_i}\left(\frac{i}{\hbar}S(q_i,t)\right), q_i, t\right)\psi = i\hbar\frac{\partial}{\partial t}\left(\frac{i}{\hbar}S(q_i,t)\right)\psi$$

$$\Leftrightarrow \left[H\left(\frac{\partial S}{\partial q_i}, q_i, t\right) + \frac{\partial S}{\partial t}\right]\psi = 0$$

$$\Leftrightarrow H\left(\frac{\partial S}{\partial q_i}, q_i, t\right) + \frac{\partial S}{\partial t} = 0 \quad \text{(Hamilton - Jacobi eqn)}$$

$\Rightarrow$ $S(q_i, t)$ is phase of plane wave function
$\quad\ \ \|$ $\quad$ is action in classical context
Hamilton's principal function