# Homework Assignment 5

① $$H = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) - mg\ell\cos\theta$$

$$\begin{cases} \dot p_\varphi = 0 \quad \Rightarrow \quad p_\varphi = L = \text{constant} \\ \dot\varphi = \dfrac{\partial H}{\partial p_\varphi} = \dfrac{1}{m\ell^2}\dfrac{p_\varphi}{\sin^2\theta} \\ \dot\theta = \dfrac{\partial H}{\partial p_\theta} = \dfrac{1}{m\ell^2} p_\theta \quad \Rightarrow \quad \dot p_\theta = m\ell^2\ddot\theta \\ \dot p_\theta = m\ell^2\ddot\theta = -\dfrac{\partial H}{\partial\theta} = \dfrac{L^2}{2m\ell^2}\dfrac{\cos\theta}{\sin^3\theta} - mg\ell\sin\theta \end{cases}$$

For uniform circular motion: $\theta = \theta_0$, $\dot\theta = 0$, $\ddot\theta = 0$

$$\Rightarrow L^2 = (m\ell^2\sin^2\theta_0)^2 \frac{g}{\ell\cos\theta_0}$$

For deviations from circular motion: $\theta = \theta_0 + \Delta\theta$, $\dot\theta = \Delta\dot\theta$

$$\frac{1}{\sin^2\theta} = \left[\sin\theta_0\cos\Delta\theta + \cos\theta_0\sin\Delta\theta\right]^{-2}$$

$$\cong \left[\sin\theta_0\left(1 - \frac{\Delta\theta^2}{2}\right) + \cos\theta_0\,\Delta\theta\right]^{-2}$$

$$= \frac{1}{\sin^2\theta_0}\left[1 + \frac{\cos\theta_0}{\sin\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right]^{-2}$$

$$\cong \frac{1}{\sin^2\theta_0}\left[1 - 2\left(\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta - \frac{\Delta\theta^2}{2}\right) + 3\left(\frac{\cos\theta_0}{\sin\theta_0}\Delta\theta\right)^2\right]$$

$$= \frac{1}{\sin^2\theta_0}\left[1 - 2\Delta\theta\frac{\cos\theta_0}{\sin\theta_0} + \Delta\theta^2\left(1 + 3\frac{\cos^2\theta_0}{\sin^2\theta_0}\right)\right]$$

$$\cos\theta = \cos\theta_0 \cos\Delta\theta - \sin\theta_0 \sin\Delta\theta$$

$$\simeq \cos\theta_0 \left[ \left(1 - \frac{\Delta\theta^2}{2}\right) - \frac{\sin\theta_0}{\cos\theta_0} \Delta\theta \right]$$

$$= \cos\theta_0 \left[ 1 - \Delta\theta \frac{\sin\theta_0}{\cos\theta_0} - \frac{\Delta\theta^2}{2} \right]$$

$$\Rightarrow H \simeq \frac{1}{2ml^2} \left[ p_{\Delta\theta}^2 + \frac{L^2}{\sin^2\theta_0} \left( 1 - 2\Delta\theta \frac{\cos\theta_0}{\sin\theta_0} + \Delta\theta^2 \left(1 + 3\frac{\cos^2\theta_0}{\sin^2\theta_0}\right)\right)\right]$$

$$- mgl\cos\theta_0 \left(1 - \Delta\theta \frac{\sin\theta_0}{\cos\theta_0} - \frac{\Delta\theta^2}{2}\right)$$

$$= \frac{1}{2ml^2} p_{\Delta\theta}^2$$

$$+ \frac{1}{2ml^2} (ml^2)^2 \sin^2\theta_0 \frac{g}{l\cos\theta_0} \left[ 1 \cancel{- 2\Delta\theta \frac{\cos\theta_0}{\sin\theta_0}} + \Delta\theta^2 \left(1 + 3\frac{\cos^2\theta_0}{\sin^2\theta_0}\right)\right]$$

$$- mgl\cos\theta_0 \left(1 \cancel{- \Delta\theta \frac{\sin\theta_0}{\cos\theta_0}} - \frac{\Delta\theta^2}{2}\right)$$

$$= \frac{1}{2ml^2} p_{\Delta\theta}^2 + \frac{1}{2} ml^2 \frac{g}{l\cos\theta_0} \Delta\theta^2 \left[ \left(\sin^2\theta_0 + 3\cos^2\theta_0\right) + \cos^2\theta_0 \right]$$

$$+ \text{constant}$$

$$= \frac{1}{2ml^2} p_{\Delta\theta}^2 + \frac{1}{2} ml^2 \underbrace{\frac{g}{l\cos\theta_0} \left(1 + 3\cos^2\theta_0\right)}_{\omega^2} \Delta\theta^2 + \text{constant}$$

(2) $L = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - V(\bar{r})$

a) $\frac{\partial L}{\partial \bar{r}} = -\frac{\partial V}{\partial \bar{r}} = -\bar{\nabla} V(\bar{r})$

$$\bar{p} = \frac{\partial L}{\partial \bar{v}} = -mc^2 \frac{1}{2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}\left(-2\frac{\bar{v}}{c^2}\right) = m\frac{\bar{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

$$\Rightarrow \dot{\bar{p}} = -\bar{\nabla} V(\bar{r})$$

b) $\bar{p} = m\frac{\bar{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$

$$H = \bar{p}\cdot\bar{v} - L = m\frac{v^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} + V(\bar{r})$$

$$= \frac{mc^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + V(\bar{r})$$

$$(m^2c^4 + p^2c^2)^{1/2} = \left(m^2c^4 + \frac{m^2v^2c^2}{\left(1-\frac{v^2}{c^2}\right)}\right)^{1/2}$$

$$= \frac{m\ c^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

$$\Rightarrow H = (m^2c^4 + p^2c^2)^{1/2} + V(\bar{r}) = \text{constant of motion}$$

c) $\frac{d}{dt}(\bar{r}\times\bar{p}) = \dot{\bar{r}}\times\bar{p} + \bar{r}\times\dot{\bar{p}}$

$$= \bar{v}\times\frac{m\bar{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + \bar{r}\times\left(-\frac{\partial H}{\partial \bar{r}}\right)$$

$$= -\bar{r}\times\frac{\partial V}{\partial \bar{r}}$$

If $V(\bar{r}) = V(r) \rightarrow \frac{\partial V}{\partial \bar{r}} \sim \hat{r} \Rightarrow \frac{d}{dt}(\bar{r} \times \bar{p}) = 0$

$\rightarrow$ pick motion in plane defined by $\bar{r} \times \bar{p} \rightarrow (r, \varphi)$ only with $p^2 = p_r^2 + \frac{p_\varphi^2}{r^2}$

③ $[A,[B,C]] = [A, B_i C^i - B^i C_i] = [A, B_i C^i] - [A, B^i C_i]$

$= A_j B_i{}^j C^i + A_j B_i C^{ij} - A^j B_{ij} C^i - A^j B_i C^i{}_j$

$- A_j B^{ij} C_i - A_j B^i C_i{}^j + A^j B^i{}_j C_i + A^j B^i C_{ij}$

and similar for the other terms

$\rightarrow$ everything cancels out

④ $[L_1, L_2] = [x_2 p_3 - x_3 p_2, x_3 p_1 - x_1 p_3]$

$= [x_2 p_3, x_3 p_1] - [x_3 p_2, x_3 p_1] - [x_2 p_3, x_1 p_3] + [x_3 p_2, x_1 p_3]$

$= x_2 \underbrace{[p_3, x_3]}_{= -1} p_1 + x_1 \underbrace{[x_3, p_3]}_{= 1} p_2$

other terms only contain Poisson brackets with $[x_i, x_j]$, $[p_i, p_j]$ of $[x_i, p_j]$, $i \neq j$

$= -x_2 p_1 + x_1 p_2$

$= L_3$

$[L^2, L_1] = [L_1^2 + L_2^2 + L_3^2, L_1] = 2L_1 \underbrace{[L_1, L_1]}_{= 0} + 2L_2 \underbrace{[L_2, L_1]}_{= -L_3} + 2L_3 \underbrace{[L_3, L_1]}_{= L_2}$

$= -2L_2 L_3 + 2L_3 L_2 = 0$