Homework Assignment 6:

① $H(q,p,t)$ and $K(Q,P,t) = K\big(Q(q,p,t), P(q,p,t), t\big)$

$$= K(q,p,t) = H(q,p,t) + \frac{\partial F}{\partial t}(q,Q,t)$$

Hamilton's equations:

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} = \frac{\partial}{\partial P_j}\left(H + \frac{\partial F}{\partial t}\right) = \frac{\partial H}{\partial P_j} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial P_j} + \sum_i \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial P_j}$$

$$\parallel$$

$$\frac{dQ_j}{dt} = \sum_i \frac{\partial Q_j}{\partial q_i}\underset{\substack{\parallel \\ \frac{\partial H}{\partial p_i}}}{\dot{q}_i} + \sum_i \frac{\partial Q_j}{\partial p_i}\underset{\substack{\parallel \\ -\frac{\partial H}{\partial q_i}}}{\dot{p}_i} = -\sum_i \frac{\partial H}{\partial q_i}\frac{\partial Q_j}{\partial p_i} + \sum_i \frac{\partial H}{\partial p_i}\frac{\partial Q_j}{\partial q_i}$$

$$\Rightarrow \frac{\partial q_i}{\partial P_j} = -\frac{\partial Q_j}{\partial p_i} \quad \text{and} \quad \frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i}$$

$$\dot{P}_j = -\frac{\partial K}{\partial Q_j} = -\frac{\partial H}{\partial Q_j} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial Q_j} + \sum_i \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial Q_j}$$

$$\parallel$$

$$\frac{dP_j}{dt} = \sum_i \frac{\partial P_j}{\partial q_i}\underset{\substack{\parallel \\ \frac{\partial H}{\partial p_i}}}{\dot{q}_i} + \sum_i \frac{\partial P_j}{\partial p_i}\underset{\substack{\parallel \\ -\frac{\partial H}{\partial q_i}}}{\dot{p}_i} = -\sum_i \frac{\partial H}{\partial q_i}\frac{\partial P_j}{\partial p_i} + \sum_i \frac{\partial H}{\partial p_i}\frac{\partial P_j}{\partial q_i}$$

$$\Rightarrow \frac{\partial q_i}{\partial Q_j} = -\frac{\partial P_j}{\partial p_i} \quad \text{and} \quad \frac{\partial p_i}{\partial Q_j} = \frac{\partial P_j}{\partial q_i}$$

② Poisson brackets:

$$[P,P] = \frac{1}{-4\alpha^2}[p-i\alpha q, p-i\alpha q] = -\frac{1}{2\alpha^2}\left([p,-i\alpha q]+[-i\alpha q,p]\right)$$
$$= 0$$

$$[Q,Q] = [p+i\alpha q, p+i\alpha q] = [p,i\alpha q]+[i\alpha q,p] = 0$$

$$[P,Q] = \frac{1}{2i\alpha}[p-i\alpha q, p+i\alpha q] = \frac{1}{2i\alpha}[p,i\alpha q] - \frac{1}{2i\alpha}[i\alpha q,p]$$
$$= \frac{1}{i\alpha}[p,i\alpha q] = [p,q] = -1$$

$\Rightarrow$ canonical transformation

Invert: $\begin{cases} q = \frac{1}{2i\alpha}(Q-2i\alpha P) \\ p = \frac{1}{2}(Q+2i\alpha P) \end{cases}$ $\quad\begin{cases} Q = p+i\alpha q \\ P = \frac{1}{2i\alpha}(p-i\alpha q) \end{cases}$

$$p = Q - i\alpha q = \frac{\partial F}{\partial q} \Leftrightarrow F(q,Q,t) = qQ - \frac{1}{2}i\alpha q^2 + f(Q)$$

$$P = \frac{1}{2i\alpha}(Q-2i\alpha q) = -\frac{\partial F}{\partial Q} = -q - \frac{\partial f}{\partial Q} \Leftrightarrow f(Q) = -\frac{1}{4i\alpha}Q^2$$

$$\Rightarrow F(q,Q,t) = qQ - \frac{1}{2}i\alpha q^2 - \frac{1}{4i\alpha}Q^2$$

$$H = \frac{1}{2}(p^2+\alpha^2 q^2) \Rightarrow K = i\alpha PQ$$

$$\begin{cases} \dot{Q} = i\alpha Q \\ \dot{P} = -i\alpha P \end{cases} \Rightarrow \begin{cases} Q = A e^{i\alpha t} \\ P = B e^{-i\alpha t} \end{cases}$$

③ $H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)$

$$\begin{cases} \dot{q} = p q^4 \iff p^2 q^6 = \dot{q} q^2 \quad \text{and} \quad \ddot{q} \underset{\Downarrow}{=} \dot{p} q^4 + 4 p q^3 \\ \dot{p} = -\frac{1}{2}\left(-\frac{2}{q^3} + 4 p^2 q^3\right) \qquad\qquad \dot{p} \underset{\Downarrow}{=} \frac{1}{q^4}\left(\ddot{q} - 4\frac{\dot{q}}{q}\right) \end{cases}$$

$$\Rightarrow \dot{p} q^3 = 1 - 2 p^2 q^6 = 1 - 2\dot{q} q^2 = \dot{p} q^3 = \frac{\ddot{q}}{q} - 4\frac{\dot{q}}{q^2}$$

$$\begin{cases} Q = p q^2 \\ P = \frac{1}{q} \end{cases} \iff \begin{cases} q = \frac{1}{P} \\ p = Q P^2 \end{cases} \Rightarrow K = \frac{1}{2}\left(P^2 + Q^2\right)$$

Using problem 1 :

$$\frac{\partial P}{\partial q} = -\frac{1}{q^2} = -\frac{\partial p}{\partial Q} = -P^2$$

$$\frac{\partial P}{\partial p} = 0 = \frac{\partial q}{\partial Q} = 0$$

$$\frac{\partial Q}{\partial q} = 2pq = \frac{\partial p}{\partial P} = 2PQ$$

$$\frac{\partial Q}{\partial p} = q^2 = -\frac{\partial q}{\partial P} = \frac{1}{P^2}$$

$\Rightarrow$ canonical transformation

Solution :

$Q = A \sin(t+\beta)$, $P = A\cos(t+\beta)$

$$\Rightarrow q = \frac{1}{A\cos(t+\beta)} , \quad p = A^3 \sin(t+\beta)\cos^2(t+\beta)$$

$$\Rightarrow \dot{q} = \frac{\sin(t+\beta)}{A\cos^2(t+\beta)} , \quad \dot{p} = A^3\left(\cos^3(t+\beta) - 2\sin^2(t+\beta)\cos(t+\beta)\right)$$

$$p q^4 = \frac{A^3 \sin(t+\beta)\cos^2(t+\beta)}{A^4 \cos^4(t+\beta)} = \frac{\sin(t+\beta)}{A\cos^2(t+\beta)} = \dot{q} \quad \checkmark$$

$$\dot{p} = A^3\left(\cos^3(t+\beta) - 2\sin^2(t+\beta)\cos(t+\beta)\right)$$

$$= -\frac{1}{2}\left(-2A^3\cos^3(t+\beta) + 4\frac{A^6\sin^2(t+\beta)\cos^4(t+\beta)}{A^3\cos^3(t+\beta)}\right)$$

$$= A^3\cos^3(t+\beta) - 2A^3\sin^2(t+\beta)\cos(t+\beta)$$

$$= \frac{1}{q^3} - 2p^2q^3 \quad \checkmark$$

④ $H = \frac{1}{2m}(p_x^2 + p_z^2) + mgz = E = \alpha$

$$S = W_x(x,\alpha_x) + W_z(z,\alpha_z) - \alpha t$$

$$\underbrace{\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2}_{\alpha_x} + \underbrace{\frac{1}{2m}\left(\frac{\partial W_z}{\partial z}\right)^2 + mgz}_{\alpha_z} = \alpha$$

For x : $W_x = \pm\sqrt{2m\alpha_x}\,x$

$$\beta_x = \frac{\partial W_x}{\partial \alpha_x} - t = \pm\sqrt{\frac{m}{2\alpha_x}}\,x - t \iff x = \pm\sqrt{\frac{2\alpha_x}{m}}(t+\beta_x)$$

For z : $\frac{1}{2m}\left(\frac{\partial W_z}{\partial z}\right)^2 + mgz = \alpha_z$

$$\frac{\partial W_z}{\partial z} = \pm\sqrt{2m(\alpha_z - mgz)}$$

$$\iff W_z = \pm\frac{1}{3m^2g}\left[2m(\alpha_z - mgz)\right]^{3/2}$$

$$\beta_z = \frac{\partial W_z}{\partial \alpha_z} - t = \pm\sqrt{\frac{2}{g}}\sqrt{\alpha_z - mgz} - t$$

$$\Leftrightarrow \quad z = -\frac{1}{2}g(t+\beta_z)^2 + \frac{\alpha_z}{mg}$$

Initial conditions:

$x = 0$ at $t = 0 \rightarrow \beta_x = 0$

$\dot{x} = v_0 \cos\theta \rightarrow \frac{2\alpha_x}{m} = v_0$

$\hookrightarrow \quad x = v_0 \cos\theta . t$

$z = 0$ at $t = 0 \rightarrow \frac{\alpha_z}{mg} = \frac{1}{2}g\beta_z^2$

$\dot{z} = v_0 \sin\theta \longrightarrow v_0 \sin\theta = -g\beta_z$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \beta_z = -\frac{v_0}{g}\sin\theta, \quad \alpha_z = \frac{m}{2}v_0^2 \sin^2\theta$

$$\hookrightarrow \quad z = -\frac{1}{2}g\left(t - \frac{v_0}{g}\sin\theta\right) + \frac{1}{2g}v_0^2\sin^2\theta$$

$$z = -\frac{1}{2}gt^2 + v_0\sin\theta . t$$

(5) $-\frac{\partial V}{\partial x} = At \Rightarrow V(x) = -Atx$

$$H = \frac{p_x^2}{2m} - Axt$$

$$\rightarrow \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - Axt + \frac{\partial S}{\partial t} = 0$$

With $S = \frac{1}{2}At^2x + \alpha x - \varphi(t)$:

$$\frac{\partial S}{\partial x} = \frac{1}{2}At^2 + \alpha \qquad \frac{\partial S}{\partial t} = Atx - \varphi'(t)$$

$$\Leftrightarrow \frac{1}{2m}\left(\frac{1}{2}At^2+\alpha\right)^2 - \cancel{A\alpha t} + \cancel{At\alpha} - \varphi'(t) = 0$$

$$\Leftrightarrow \varphi'(t) = \frac{1}{2m}\left(\frac{1}{2}At^2+\alpha\right)^2$$

$$\Leftrightarrow \varphi(t) = \frac{1}{2m}\left(\frac{A^2t^5}{20} + \frac{A\alpha t^3}{3} + \alpha^2 t\right)$$

$$\beta = \frac{\partial S}{\partial \alpha} = x - \frac{\partial \varphi}{\partial \alpha} = x - \frac{1}{2m}\left(\frac{1}{3}At^3 + 2\alpha t\right)$$

$$\hookrightarrow x = \beta + \frac{\alpha}{m}t + \frac{A}{6m}t^3$$

$$p = \frac{\partial S}{\partial x} = \frac{1}{2}At^2 + \alpha$$