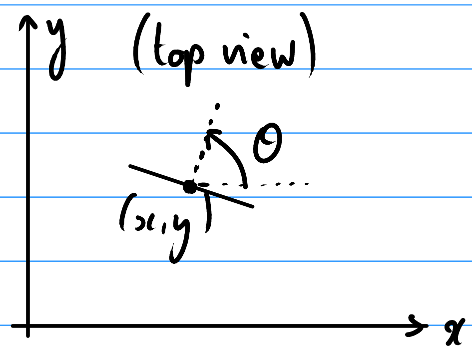
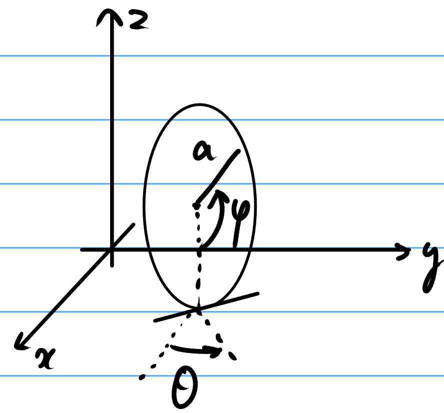
Homework Assignment 1 (Phys 601)

1 



Speed of disk without slipping:

$a\dot{\varphi} = v$

$$\sum A_i \dot{q}_i + B = 0$$

$$\begin{cases} \dot{x} = v\sin\theta \\ \dot{y} = -v\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = a\dot{\varphi}\sin\theta \\ \dot{y} = -a\dot{\varphi}\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} - a\sin\theta\,\dot{\varphi} = 0 & (1) \\ \dot{y} + a\cos\theta\,\dot{\varphi} = 0 & (2) \end{cases}$$

(1) $A_x = 1$, $A_\varphi = -a\sin\theta$, $A_y = 0$, $A_\theta = 0$

$\hookrightarrow \dfrac{\partial A_\varphi}{\partial \theta} = -a\cos\theta$, $\dfrac{\partial A_\theta}{\partial \varphi} = 0$

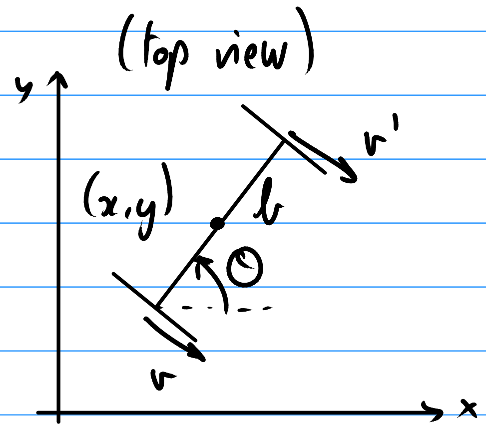
$\Rightarrow$ nonholonomic

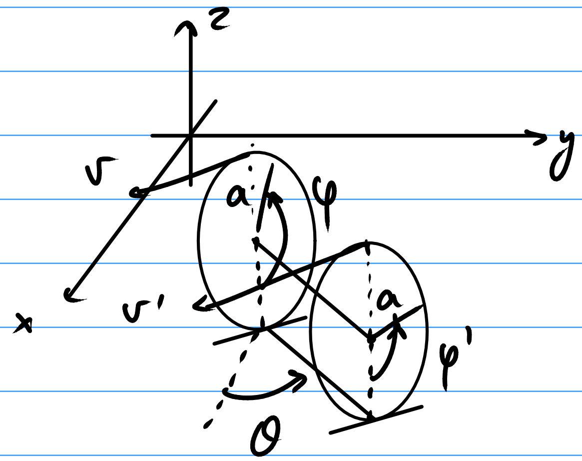
(2) $A_y = 1$, $A_\varphi = a\cos\theta$, $A_x = 0$, $A_\theta = 0$

$\hookrightarrow \dfrac{\partial A_\varphi}{\partial \theta} = -a\sin\theta$, $\dfrac{\partial A_\theta}{\partial \varphi} = 0$

$\Rightarrow$ nonholonomic

②





$$\begin{cases} v = a\dot{\varphi} \\ v' = a\dot{\varphi}' \end{cases} \Rightarrow \text{ speed of center of mass } = \frac{v+v'}{2}$$

Constraint relating v, v' and $\dot{\theta}$ (notice correct signs)

$$v' - v = a\dot{\varphi} - a\dot{\varphi}' = -l\dot{\theta} \Rightarrow \theta = \text{constant} - \frac{a}{l}(\varphi - \varphi')$$

Constraints on components of v (similar to problem 1)

$$\begin{cases} \dot{x} = \frac{v+v'}{2}\sin\theta = \frac{a}{2}(\dot{\varphi}+\dot{\varphi}')\sin\theta \\ \dot{y} = -\frac{v+v'}{2}\cos\theta = -\frac{a}{2}(\dot{\varphi}+\dot{\varphi}')\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} \sin\theta\,\dot{x} - \cos\theta\,\dot{y} = \frac{a}{2}(\dot{\varphi}+\dot{\varphi}') \Rightarrow \sin\theta\,dx - \cos\theta\,dy = \frac{a}{2}(d\varphi + d\varphi') \\ \cos\theta\,\dot{x} + \sin\theta\,\dot{y} = 0 \Rightarrow \cos\theta\,dx + \sin\theta\,dy = 0 \end{cases}$$

Holonomic?

1^{st} constraint only involves coordinates

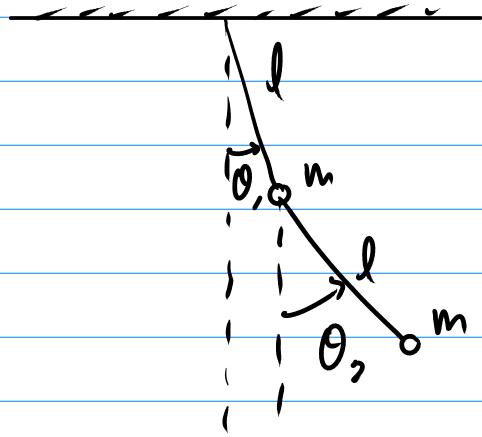
$\rightarrow$ so is already in form $f(\{q\},t) = c$

Other constraints (both are analogous)

$$A_x = \sin\theta,\ A_y = -\cos\theta,\ A_\varphi = -\frac{a}{2},\ A_{\varphi'} = -\frac{a}{2},\ A_\theta = 0$$

$\hookrightarrow \frac{\partial A_x}{\partial \theta} \neq \frac{\partial A_\theta}{\partial x} \Rightarrow$ non holonomic

③



$$\begin{cases} x_1 = l\sin\theta_1 \\ y_1 = -l\cos\theta_1 \end{cases}$$

$$\begin{cases} x_2 = l\sin\theta_1 + l\sin\theta_2 \\ y_2 = -l\cos\theta_1 - l\cos\theta_2 \end{cases}$$

$$V = mg\,y_1 + mg\,y_2 = -mg\left(2l\cos\theta_1 + l\cos\theta_2\right)$$

$$\Rightarrow \begin{cases} \dot{x}_1 = l\dot{\theta}_1\cos\theta_1 \\ \dot{y}_1 = l\dot{\theta}_1\sin\theta_1 \end{cases}, \quad \begin{cases} \dot{x}_2 = l\dot{\theta}_1\cos\theta_1 + l\dot{\theta}_2\cos\theta_2 \\ \dot{y}_2 = l\dot{\theta}_1\sin\theta_1 + l\dot{\theta}_2\sin\theta_2 \end{cases}$$

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\left(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\right)\right)$$

$$= \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

$$L = T - V = \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

$$+ mgl\left(2\cos\theta_1 + \cos\theta_2\right)$$

$$\theta_1 : \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - ml^2\dot{\theta}_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - 2mgl\sin\theta_1$$

$$\Rightarrow ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - ml^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 2mgl\sin\theta_1 = 0$$

$$\theta_2 : \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m\ell^2\ddot\theta_2 + m\ell^2\ddot\theta_1\cos(\theta_1-\theta_2) - m\ell^2\dot\theta_1(\dot\theta_1+\dot\theta_2)\sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = -m\ell^2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2) - mg\ell\sin\theta_2$$

$$\Rightarrow m\ell^2\ddot\theta_2 + m\ell^2\ddot\theta_1\cos(\theta_1-\theta_2) - m\ell^2\dot\theta_1^2\sin(\theta_1-\theta_2) + mg\ell\sin\theta_2 = 0$$

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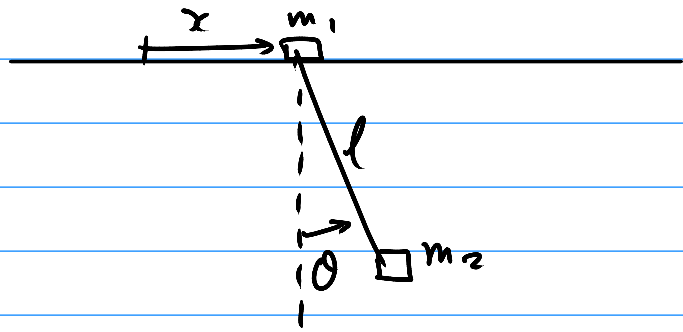

$$T = \frac{1}{2}m_1\dot x^2 + \frac{1}{2}m_2(\dot x_2^2 + \dot y_2^2)$$

$$V = m_2 g y_2$$

$$\begin{cases} x_2 = x + \ell\sin\theta \\ y_2 = -\ell\cos\theta \end{cases} \Rightarrow \begin{cases} \dot x_2 = \dot x + \ell\dot\theta\cos\theta \\ \dot y_2 = \ell\dot\theta\sin\theta \end{cases}$$

$$\Rightarrow L = T - V = \frac{1}{2}m_1\dot x^2 + \frac{1}{2}m_2\left(\dot x^2 + \ell^2\dot\theta^2 + 2\ell\dot x\dot\theta\cos\theta\right) + m_2 g\ell\cos\theta$$

$$x : \frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) = (m_1+m_2)\ddot x + m_2\ell\ddot\theta\cos\theta - m_2\ell\dot\theta^2\sin\theta$$

$$\frac{\partial L}{\partial x} = 0$$

$$\Rightarrow (m_1+m_2)\ddot x + m_2\ell\ddot\theta\cos\theta - m_2\ell\dot\theta^2\sin\theta = 0$$

$$\theta : \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = m_2\ell^2\ddot\theta + m_2\ell\ddot x\cos\theta - m_2\ell\dot x\dot\theta\sin\theta$$

$$\frac{\partial L}{\partial \theta} = -m_2\ell\dot x\dot\theta\sin\theta - m_2 g\ell\sin\theta$$

$$\Rightarrow \quad m_2 l^2 \ddot{\theta} + m_2 l \ddot{x} \cos\theta + m_2 g l \sin\theta = 0$$

For small oscillations : $\theta \ll 1 \rightarrow \sin\theta \simeq \theta, \ \cos\theta \simeq 1$

$$\Rightarrow \begin{cases} (m_1+m_2)\ddot{x} + m_2 l \ddot{\theta} - m_2 l \dot{\theta}^2 \theta = 0 \\ m_2 l^2 \ddot{\theta} + m_2 l \ddot{x} + m_2 g l \theta = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \ddot{x} = - l \ddot{\theta} - g\theta \\ (m_1+m_2)(-l\ddot{\theta} - g\theta) + m_2 l \ddot{\theta} - m_2 l \dot{\theta}^2 \theta = 0 \end{cases}$$

If θ is small and harmonic (assumption), then

$$\theta \simeq A \sin \omega t$$

$$\hookrightarrow \dot{\theta} \simeq A\omega \cos \omega t \longrightarrow \dot{\theta}^2 \theta \simeq A^3 \omega^2 \sin(\omega' t + \delta)$$

$$\hookrightarrow \ddot{\theta} \simeq A\omega^2 \sin \omega t$$

$\hookrightarrow A^3 \omega^2 \ll A\omega^2$ when A is small

$$\Rightarrow - m_1 l \ddot{\theta} - (m_1+m_2) g \, \theta = 0$$

$$\ddot{\theta} + \frac{m_1+m_2}{m_1} \frac{g}{l} \theta = 0$$

$$\omega^2 = \frac{m_1+m_2}{m_1} \frac{g}{l}$$