Classical Mechanics (Phys 601) - September 1, 2011

Remember: 3n cartesian coordinates for n particles $\{x_i\}$

constraints

$\downarrow$

3n - k generalized coordinates $\{q_i\}$

$$x_j = x_j(\{q_i\}, t) \quad , \quad \begin{array}{l} j = 1, \ldots, 3n \\ i = 1, \ldots, 3n-k \end{array}$$

→ coordinates $\{q_i\}$ are a minimal, complete set that is consistent with all constraints

→ independent set that completely specifies the system

Full differential:

$$dx_j = \sum_i \frac{\partial x_j}{\partial q_i} dq_i + \frac{\partial x_j}{\partial t} dt$$ → infinitesimal displacement

e.g. given dt → changes in dx_j

given dq_i → changes in dx_j even if dt zero, but consistent with constraints

Virtual displacement:

$$\delta x_j = \sum_i \frac{\partial x_j}{\partial q_i} \delta q_i$$

→ infinitesimal displacement

→ at a given instant in time!

→ consistent with the constraints

(system frozen in time)

(Example of double pendulum → appendix)

Question by Anne: "what motivates the search for holonomic / nonholonomic constraints?"

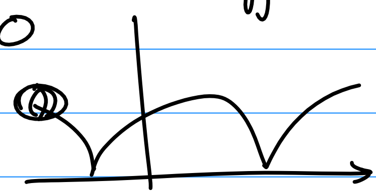
* 3N cartesian coordinates $\longrightarrow$ generalized coordinates

holonomic constraint: $x_i = x_i(\{q_j\}, t) = c_i$

$\hookrightarrow$ one variable becomes irrelevant for each holonomic constraint

nonholonomic constraint $\rightarrow$ more difficult case

e.g. $y \geq 0$



* From $\sum_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - Q_j \right] \delta q_j = 0$

follow the Lagrange equations only when the set of δq_j are independent

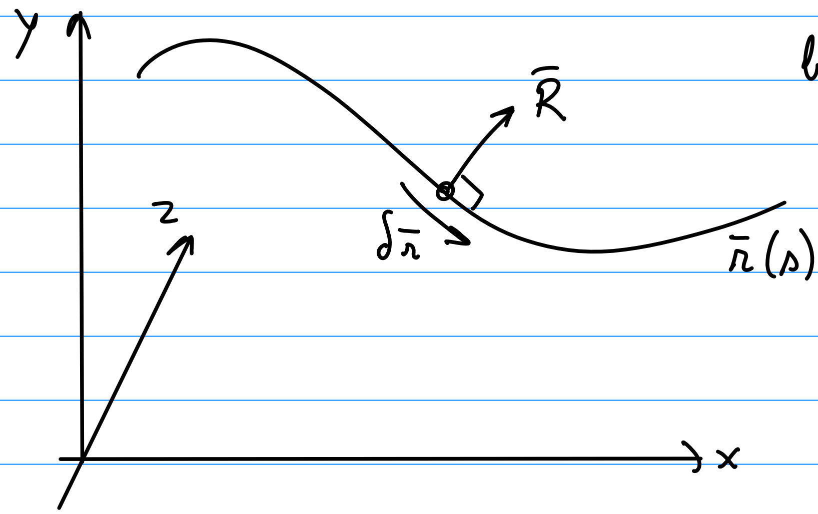
* d'Alembert's Principle (absent frictional forces)

Forces of constraint do no work under virtual displacement.

$$\delta W = \sum_i \bar{R}_i \cdot \delta \bar{r}_i = 0$$

Justification:

- Particle constrained to a line



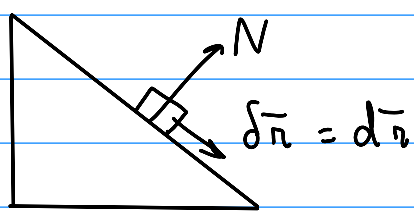
bead on frictionless wire

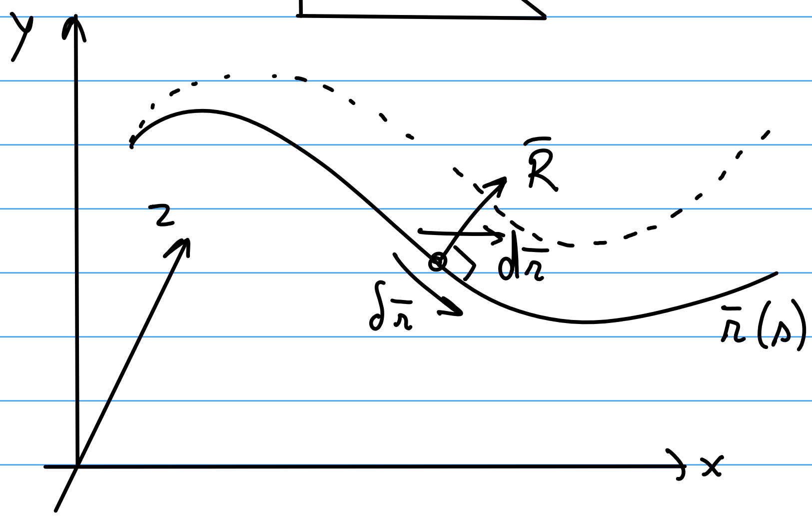
↳ $\bar{R} \perp \delta\bar{r}$

$\Downarrow$

$\delta W = \bar{R} \cdot \delta\bar{r} = 0$

e.g.



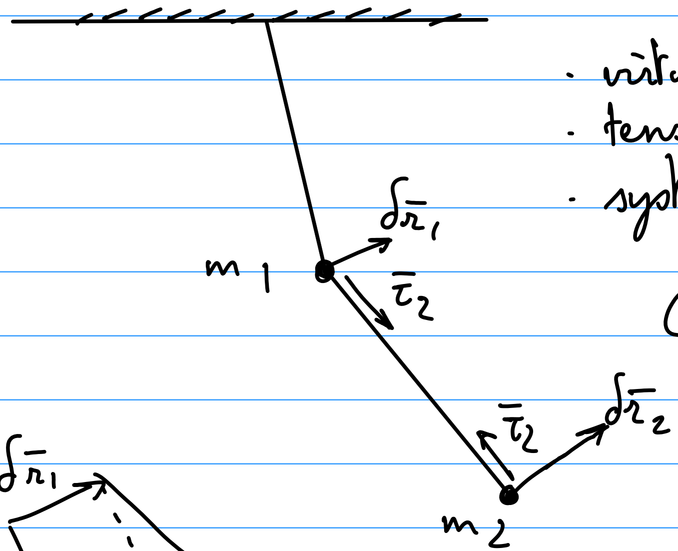


bead on moving frictionless wire

$\bar{R} \cdot d\bar{r}$ may be $\neq 0$
but $\bar{R} \cdot \delta\bar{r} = 0$

- Similar arguments for particle constrained to a surface.

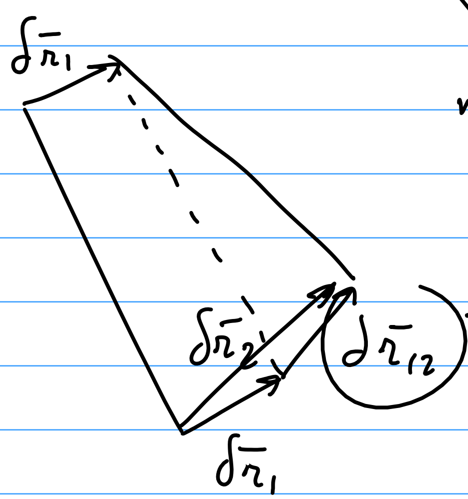
- Case of double planar pendulum



. virtual displacements $\delta \bar{r}_1$, $\delta \bar{r}_2$
. tension in string is τ
. system frozen in time

$\hookrightarrow \delta \bar{r}_2 = \delta \bar{r}_1 + \delta \bar{r}_{12}$

but $\delta \bar{r}_{12}$ is rotation around point m_1



$\delta \bar{r}_{12} \perp \bar{\tau}_2$

$$\delta W = \underbrace{\bar{\tau}_2 \cdot \delta \bar{r}_1}_{} + \underbrace{\bar{\tau}_2 \cdot \delta \bar{r}_2}_{} = \bar{\tau}_2 \cdot \delta \bar{r}_{12} = 0$$

work on m_1 + work on m_2 = 0

- Newton's law : $\dot{\bar{p}}_i = \bar{F}_i = \underbrace{\bar{F}_i^{(a)}}_{\text{applied}} + \underbrace{\bar{R}_i}_{\text{reaction}}$, $i = 1, \dots, n$

(or component-wise : $\dot{p}_i = F_i^{(a)} + R_i$, $i = 1, \dots, 3n$)

Multiply with $\delta \bar{r}_i$ and sum over all i :

$$\sum_i^n \left(\dot{\bar{p}}_i - \bar{F}_i^{(a)} \right) \cdot \delta \bar{r}_i = \sum_i^{3n} \left(\dot{p}_i - F_i^{(a)} \right) \delta x_i = 0$$

$\hookrightarrow$ forces of constraint have disappeared

* Lagrange's Equations :

$$\underbrace{\sum_i^{3n} \dot{p}_i}\ \underbrace{- F_i) \delta x_i}_{\text{generalized forces } Q_j} \quad \text{with } \delta x_i = \sum_j \frac{\partial x_i}{\partial q_j} \delta q_j$$

$$\delta W = \sum_i^{3n} F_i \delta x_i = \sum_j^{3n-k} Q_j \delta q_j$$

↳ virtual work by only applied forces

$$\sum_i^{3n} m_i \ddot{x}_i \sum_j^{3n-k} \frac{\partial x_i}{\partial q_j} \delta q_j = \sum_j \delta q_j \underbrace{\sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j}}$$

$$\sum_i m_i \frac{d\dot{x}_i}{dt} \frac{\partial x_i}{\partial q_j} = \sum_i m_i \Bigg[\underbrace{\frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_j} \right)}_{(2)} - \dot{x}_i \underbrace{\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j} \right)}_{(1)} \Bigg]$$

$\Rightarrow$ (1) $$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{dx_i}{dt}\right) \Rightarrow \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j} = \frac{\partial}{\partial q_j}(T)$$

(2) $\sum_i m_i \dot{x}_i \dfrac{\partial x_i}{\partial q_j}$: $\quad x_i = x_i(\{q_j\}, t)$

$$\dot{x}_i = \sum_j \underbrace{\frac{\partial x_i}{\partial q_j}} \dot{q}_j + \underbrace{\frac{\partial x_i}{\partial t}}$$

functions of $(\{q_j\}, t)$, not of $\{\dot{q}_j\}$

$\Downarrow$

$$\sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \Leftarrow \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$$ picks out only one term

$\|$

$$\Rightarrow \sum_i^{3n} \left(\dot{p}_i - F_i\right) \delta x_i = \sum_i^{3n} \left(\dot{p}_i - F_i\right) \sum_j^{3n-k} \frac{\partial x_i}{\partial q_j} \delta q_j$$

$$= \sum_j \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i \dot{x}_i^2 \right) - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i \dot{x}_i^2 \right) - Q_j \right] \delta q_j = 0$$

Since all δq_j are independent, this means that

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \qquad \text{(Lagrange's equation)}$$

where $T = \sum_i \frac{1}{2} m \dot{x}_i^2 = T(\{q_i\}, \{\dot{q}_i\}, t)$

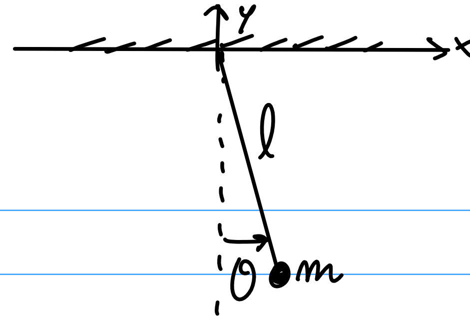
↳ specified in generalized coordinates!

* Recipe:

- identify the external forces
- identify the generalized coordinates
- express kinetic energy T as function of generalized coordinates
- compute generalized forces as

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}$$

- solve the equations!



* Example: simple pendulum:

  - external force: $\bar{F} = -m\bar{g}$ (tension is a force of constraint!)
  - generalized coordinates $\{q_i\} = \{\theta\}$
  - $T = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\theta})^2$

$y = -l\cos\theta$

first: - $Q_\theta = \sum_i F_i \frac{\partial x_i}{\partial \theta} = -mg\frac{\partial y}{\partial \theta} = -mgl\sin\theta$

  - $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$

$$\Leftrightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\right) - 0 = -mgl\sin\theta$$

$$\Leftrightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\Leftrightarrow l\ddot{\theta} + g\sin\theta = 0 \Rightarrow l\ddot{\theta} + g\theta = 0 \text{ for small } \theta$$

later: - $V = -mgl\cos\theta + \text{constant}$

$$\Rightarrow L = T - V = \frac{1}{2} m (l\dot{\theta})^2 + mgl\cos\theta - \text{constant}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Leftrightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl\sin\theta = 0$$

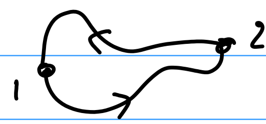
$$\Leftrightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\Leftrightarrow l\ddot{\theta} + g\sin\theta = 0$$

* Conservative forces: $\bar{F}_i = -\bar{\nabla}_i V$

$$V(\{x_i\}) = V(\{q_j\}, t)$$

in cartesian coordinates: V does not depend on time (otherwise



would depend on how long you are underway)

but $q_j(\{x_i\}, t)$ introduces time-dependence!

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = \sum_i -\frac{\partial}{\partial x_i}\Big(V(\{x_i\})\Big) \frac{\partial x_i}{\partial q_j}$$

$$\Rightarrow \quad Q_j = -\frac{\partial V}{\partial q_j}(\{q_j\}, t)$$

$\Rightarrow$ Lagrange's equation: $\displaystyle \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$

* For velocity-independent potentials: $\displaystyle \frac{\partial V}{\partial \dot{q}_j} = 0$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$L = T - V =$ Lagrangian

* For velocity-dependent "generalized potentials" $U(q_j, \dot{q}_j)$
(NOT conservative !)

Define $Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$

then $L = T - U$ is still valid
and gives equations of motion.

e.g. electromagnetic fields on moving charges

$$\bar{F} = e\left(\bar{E} + \bar{v} \times \bar{B}\right)$$

If $\bar{B} = \bar{0} \rightarrow \bar{F} = -\bar{\nabla}\varphi$ with φ the conservative electromagnetic potential

$\bar{E}$ and $\bar{B}$ can be derived from
- the scalar potential φ
- the vector potential $\bar{A}$

$$\bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

$$\Rightarrow U = e\varphi - e\bar{A} \cdot \bar{v}$$

and $L = \frac{1}{2} m v^2 - e\varphi + e\bar{A} \cdot \bar{v}$

* Uniqueness of the Lagrangian:

multiple Lagrangians can lead to the same equations of motion

e.g. $L = \frac{1}{2} m \ell^2 \dot{\theta}^2 + m g \ell \cos\theta$ − constant

↳ free to choose

$$L'(\{q_j\},\{\dot{q}_j\},t) = L(\{q_j\},\{\dot{q}_j\},t) + \frac{dF}{dt}$$

with $F(\{q_j\},t)$

$$\downarrow$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt}\right)$$

$$- \frac{\partial L}{\partial q_j} - \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_k \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t}\right)$$

$$= \frac{d}{dt}\frac{\partial F}{\partial q_j} = \sum_k \frac{\partial}{\partial q_k}\frac{\partial F}{\partial q_j}\dot{q}_k + \frac{\partial}{\partial t}\frac{\partial F}{\partial q_j}$$

$$\frac{\partial}{\partial q_j}\frac{dF}{dt} = \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t}\right)$$

$$= \sum_k \frac{\partial}{\partial q_j}\frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial}{\partial q_j}\frac{\partial F}{\partial t}$$

↳ equal and opposite

$\Longrightarrow$ F does not show up in the equations of motion

* Double planar pendulum :

constraints : $\begin{cases} x_1^2 + y_1^2 = \ell_1^2 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = \ell_2^2 \end{cases}$ $\begin{cases} f_1(\{x_i\}, t) = c_1 \\ f_2(\{x_i\}, t) = c_2 \end{cases}$

$\hookrightarrow \{q_i\} = \{\theta_1, \theta_2\}$, $\begin{cases} x_1 = \ell_1 \sin\theta_1 \\ y_1 = -\ell_1 \cos\theta_1 \\ x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \\ y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 \end{cases}$

$$f_1(\{x_i\}, t) = x_1^2 + y_1^2 = \ell_1^2 = c_1$$

$$\hookrightarrow A_{x_1} = \frac{\partial f}{\partial x_1} = 2x_1 \qquad A_{x_2} = 0$$

$$A_{y_1} = 2y_1 \qquad A_{y_2} = 0$$

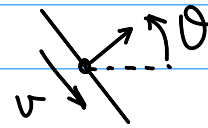
$$\hookrightarrow \frac{\partial A_{x_1}}{\partial y_1} = 0 = \frac{\partial A_{y_1}}{\partial x_1}, \text{ etc} \dots \Rightarrow \text{holonomic}$$

e.g.: $dx_2 = \sum_i \frac{\partial x_2}{\partial q_i} dq_i + \frac{\partial x_2}{\partial t} dt$

$= \ell_1 \cos\theta_1 \, d\theta_1 + \ell_2 \cos\theta_2 \, d\theta_2$ (analog for δx_2)

* Comments on homework problems

① $R\dot{\varphi} = v = \sqrt{\dot{x}^2 + \dot{y}^2}$



$$\begin{cases} \dot{x} = v \sin\theta = R\dot{\varphi}\sin\theta \\ \dot{y} = -v\cos\theta = -R\dot{\varphi}\cos\theta \end{cases}$$

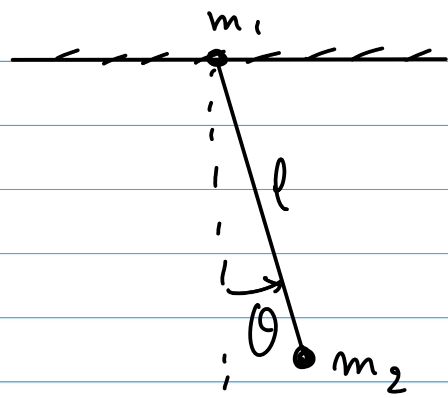
(e.g. $\theta = 0 \to \dot{y} = -v$)

$\to \sum_i A_i \dot{q}_i + B = 0$

with $A_x = 1$, $A_y = 0$, $A_\varphi = -R\sin\theta$, $A_\theta = 0$ for $\dot{x}$

$\hookrightarrow \dfrac{\partial A_\varphi}{\partial\theta} = -R\cos\theta \quad , \quad \dfrac{\partial A_\theta}{\partial\varphi} = 0 \to$ non-holonomic

(3.3)



$\to \{q_j\} = \{x, \theta\}$

(constraints: $y_1 = 0$, $x_2^2 + y_2^2 = \ell^2$)

$\bar{r}_2 = (x + \ell\sin\theta, -\ell\cos\theta)$

$\dot{\bar{r}}_2 = (\dot{x} + \ell\dot{\theta}\cos\theta, \ell\dot{\theta}\sin\theta)$

$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + \ell^2\dot{\theta}^2\cos^2\theta + \ell^2\dot{\theta}^2\sin^2\theta + 2\dot{x}\ell\dot{\theta}\cos\theta\right) + m_2 g\ell\cos\theta$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2\ell^2\dot{\theta}^2 + m_2\ell\dot{x}\dot{\theta}\cos\theta + m_2 g\ell\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = (m_1 + m_2)\ddot{x} + m_2\ell\ddot{\theta}\cos\theta - m_2\ell\dot{\theta}^2\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = m_2\ell^2\ddot{\theta} + m_2\ell\ddot{x}\cos\theta - \cancel{m_2\ell\dot{x}\dot{\theta}\sin\theta} + \cancel{m_2\ell\dot{x}\dot{\theta}\sin\theta} - m_2 g\ell\sin\theta = 0$$

$$\begin{cases} (m_1 + m_2)\ddot{x} + m_2 l \ddot{\theta}\cos\theta - m_2 l \dot{\theta}^2 \sin\theta = 0 \\ m_2 l^2 \ddot{\theta} + m_2 l \ddot{x}\cos\theta - m_2 g l \sin\theta = 0 \end{cases}$$

$$\left(\frac{m_1 + m_2}{m_2}\right)\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta = 0$$

$$l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = 0$$

Small θ : $\cos\theta \simeq 1$, $\sin\theta \simeq \theta$

$$\begin{cases} (m_1 + m_2)\ddot{x} + m_2 l \ddot{\theta} - m_2 l \dot{\theta}^2 \theta = 0 \\ m_2 l^2 \ddot{\theta} + m_2 l \ddot{x} - m_2 g l \theta = 0 \end{cases}$$

$$\begin{cases} \left(\frac{m_1 + m_2}{m_2}\right)\ddot{x} + l\ddot{\theta} - l\dot{\theta}^2\theta = 0 \\ l\ddot{\theta} + \ddot{x} - g\theta = 0 \end{cases}$$

$$\left(\frac{m_1 + m_2}{m_2}\right)\left(g\theta - l\ddot{\theta}\right) + l\ddot{\theta} - l\dot{\theta}^2\theta = 0$$

Consider that:

$\theta(t) = A\sin\omega t$, $\dot{\theta}(t) = A\omega\cos\omega t$, $\ddot{\theta}(t) = -A\omega^2\sin\omega t$

$\ddot{\theta} \sim A\omega^2$, $\dot{\theta}^2 \sim A^2\omega^2$, $\underbrace{\dot{\theta}^2\theta \sim A^3\omega^2}_{\text{negligible}}$

$$(m_1 + m_2) g\theta + m_1 l\ddot{\theta} = 0$$

$$\hookrightarrow \omega^2 \sim \left(\frac{m_1 + m_2}{m_1}\right)\left(\frac{g}{l}\right)$$