Homework Assignment 11 :

(1) $$H = \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} = T + V_{eff}$$

$$V'_{eff}(\beta) = \frac{2(p_\alpha - p_\gamma \cos\beta)\, p_\gamma \sin\beta}{2I_1 \sin^2\beta} - 2\,\frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^3\beta}\cos\beta$$

$$V'_{eff}(\beta) = 0 \iff p_\gamma \sin\beta = (p_\alpha - p_\gamma \cos\beta)\frac{\cos\beta}{\sin\beta}$$

$$\iff p_\gamma(\sin^2\beta + \cos^2\beta) = p_\alpha \cos\beta$$

$$\iff \cos\beta = \frac{p_\gamma}{p_\alpha}$$

$$\iff \beta_0 = \pm \operatorname{acos}\left(\frac{p_\gamma}{p_\alpha}\right)$$

$$\Rightarrow V_{eff}(\beta_0) = \frac{\left(p_\alpha - \frac{p_\gamma^2}{p_\alpha}\right)^2}{2I_1\left(1 - \frac{p_\gamma^2}{p_\alpha^2}\right)} + \frac{p_\gamma^2}{2I_3} = \frac{p_\alpha^2}{2I_1} + \frac{p_\gamma^2}{2I_3}$$

(2) a) Rotation around $\hat{e}_1$ in $\dot{\theta}$

Rotation around $\hat{e}_3^0$ in $-\dot{\varphi}$ and $\hat{e}_3^0 = -\cos\theta\, \hat{e}_2 + \sin\theta\, \hat{e}_3$

Rotation around $\hat{e}_3$ in $\dot{\gamma}$ and $a\, d\gamma = b\, d\varphi \Leftrightarrow \dot{\gamma} = \frac{b}{a}\dot{\varphi}$

$$\Rightarrow \quad \bar{\omega} = \dot{\theta}\hat{e}_1 - \dot{\varphi}\hat{e}_3^0 + \dot{\gamma}\,\hat{e}_3$$

$$\Leftrightarrow \quad \bar{\omega} = \dot{\theta}\hat{e}_1 + \dot{\varphi}\cos\theta\, \hat{e}_2 - \dot{\varphi}\sin\theta\, \hat{e}_3 + \frac{b}{a}\dot{\varphi}\, \hat{e}_3$$

$$= \dot{\theta}\,\hat{e}_1 + \dot{\varphi}\cos\theta\, \hat{e}_2 + \left(\frac{b}{a} - \sin\theta\right)\dot{\varphi}\,\hat{e}_3$$

b) $$T = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}M\left(V_1^2 + V_3^2\right) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\varphi}^2\cos^2\theta + \frac{1}{2}I_3\left(\frac{b}{a} - \sin\theta\right)^2\dot{\varphi}^2$$

$$= \frac{1}{2}M\left(b - a\sin\theta\right)^2\dot{\varphi}^2 + \frac{1}{2}Ma^2\dot{\theta}^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\varphi}^2\cos^2\theta + \frac{1}{2}I_3\left(\frac{b}{a} - \sin\theta\right)^2\dot{\varphi}^2$$

c) $$V = -M\bar{g}\cdot\bar{R}$$

with $\bar{g} = g\left(\sin\alpha\, \hat{e}_2^0 - \cos\alpha\, \hat{e}_3^0\right)$

and $\bar{R} = \left(b - a\sin\theta\right)\left(\sin\varphi\, \hat{e}_1^0 + \cos\varphi\, \hat{e}_2^0\right) + a\cos\theta\, \hat{e}_3^0$

$$\Rightarrow V = -M\bar{g}\cdot\bar{R} = -Mg\left[\left(b - a\sin\theta\right)\sin\alpha\cos\varphi - a\cos\alpha\cos\theta\right]$$

d) $L = \frac{1}{2} M (b - a\sin\theta)^2 \dot{\varphi}^2 + \frac{1}{2} M a^2 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\varphi}^2 \cos^2\theta$

$+ \frac{1}{2} I_3 \left(\frac{b}{a} - \sin\theta\right)^2 \dot{\varphi}^2 + Mg\left[(b - a\sin\theta)\sin\alpha\cos\varphi - a\cos\alpha\cos\theta\right]$

$+ \lambda_\theta \left(\theta - \arcsin\frac{a}{b}\right)$

e) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$ for constant $\theta = \arcsin\frac{a}{b}$

$\Leftrightarrow \frac{d}{dt}\left(M(b - a\sin\theta)^2 \dot{\varphi} + I_2 \dot{\varphi}\cos^2\theta + I_3\left(\frac{b}{a} - \sin\theta\right)^2 \dot{\varphi}\right)$

$= - Mg(b - a\sin\theta)\sin\alpha\sin\varphi$

$b - a\sin\theta = \frac{a}{\sin\theta} - a\sin\theta = a\frac{\cos^2\theta}{\sin\theta} = a\cos\theta\cot\theta$

$\Leftrightarrow \left[\left(M + \frac{I_3}{a^2}\right)(a\cos\theta\cot\theta)^2 + I_2\cos^2\theta\right]\ddot{\varphi}$

$= - Mg(a\cos\theta\cot\theta)\sin\alpha\,\underline{\sin\varphi}$

Pendulum with $\Omega = \frac{Mg\sin\alpha(a\cos\theta\cot\theta)}{\left(M + \frac{I_3}{a^2}\right)(a\cos\theta\cot\theta)^2 + I_2\cos^2\theta}$

$I_3 = \frac{1}{2} M a^2 \quad , \quad I_2 = \frac{1}{4} M a^2$

$\Rightarrow \Omega = \frac{g}{a} \frac{4\sin\alpha(\cos\theta\cot\theta)}{6(\cos\theta\cot\theta)^2 + \cos^2\theta}$

$= \frac{g}{a} \frac{4\sin\alpha}{\cos\theta\cot\theta} \frac{1}{6 + \frac{\cos^2\theta}{(\cos\theta\cot\theta)^2}}$

$= \frac{g}{a} \frac{4\sin\alpha}{\cos\theta\cot\theta} \frac{1}{6 + \tan^2\theta}$

f) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

For small $\varphi \to \varphi \simeq 0,\ \dot{\varphi} \simeq 0$ and constant $\theta \to \dot{\theta} = 0$

$L \simeq Mg\left[(b - a\sin\theta)\sin\alpha - a\cos\alpha\cos\theta\right] + \lambda_\theta\left(\theta - \arcsin\frac{a}{b}\right)$

$\Leftrightarrow \frac{\partial L}{\partial \theta} = 0 \Leftrightarrow \lambda_\theta = -Mg\,(a\cos\theta\sin\alpha - a\cos\alpha\sin\theta)$
$= -Mga\sin(\theta - \alpha)$

λ_θ is the torque of constraint due the normal force at the contact point.

The tack doesn't tip over as long as $\lambda_\theta > 0$

$\Leftrightarrow \cos\theta\sin\alpha < \cos\alpha\sin\theta$

$\Leftrightarrow \tan\alpha < \tan\theta$

③ $$\begin{cases} I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2E \\ I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} I_1\omega_1^2 + I_3\omega_3^2 = 2E - I_2\omega_2^2 \\ I_1^2\omega_1^2 + I_3^2\omega_3^2 = L^2 - I_2^2\omega_2^2 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} I_1 & I_3 \\ I_1^2 & I_3^2 \end{pmatrix} \begin{pmatrix} \omega_1^2 \\ \omega_3^2 \end{pmatrix} = \begin{pmatrix} 2E - I_2\omega_2^2 \\ L^2 - I_2^2\omega_2^2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \omega_1^2 \\ \omega^2 \end{pmatrix} = \frac{1}{I_1 I_3(I_3 - I_1)} \begin{pmatrix} I_3^2 & -I_3 \\ -I_1^2 & I_1 \end{pmatrix} \begin{pmatrix} 2E - I_2\omega_2^2 \\ L^2 - I_2^2\omega_2^2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \omega_1^2 = \dfrac{2E I_3^2 - I_3^2 I_2 \omega_2^2 - L^2 I_3 + I_2^2 I_3 \omega_2^2}{I_1 I_3 (I_3 - I_1)} \\ \omega_3^2 = \dfrac{-2E I_1^2 + I_1^2 I_2 \omega_2^2 + I_1 L^2 - I_1 I_2^2 \omega_2^2}{I_1 I_3 (I_3 - I_1)} \end{cases}$$

$$\Leftrightarrow \begin{cases} \omega_1^2 = \dfrac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)} \left(\dfrac{2E}{I_2} - \omega_2^2 \right) \\ \omega_3^2 = \dfrac{I_2(I_2 - I_1)}{I_3(I_3 - I_1)} \left(\dfrac{2E}{I_2} - \omega_2^2 \right) \end{cases}$$

$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$

$$\tau^{-1} = \omega_\infty \left(\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3} \right)^{1/2}$$

$$= \frac{I_2}{\tau \omega_\infty} \left(\frac{2E}{I_2} - \omega_2^2 \right)$$

$$\Leftrightarrow \quad \dot{\omega}_2 = \frac{1}{\tau \omega_\infty}\left(\frac{2E}{I_2} - \omega_2^2\right)$$

$$\omega_\infty^2 = \frac{(2E)^2}{L^2} = \frac{2E}{I_2}$$

$$\Leftrightarrow \quad \dot{\omega}_2 = \frac{d}{dt}\omega_2 = \frac{1}{\tau \omega_\infty}\left(\omega_\infty^2 - \omega_2^2\right)$$

$$u = t/\tau \,, \quad v = \frac{\omega_2}{\omega_\infty}$$

$$\Leftrightarrow \quad \frac{dv}{du} = 1 - v^2$$

$$\Leftrightarrow \quad \int \frac{dv}{1 - v^2} = \int du$$

$$\Leftrightarrow \quad u = \tanh^{-1} v$$

$$\Leftrightarrow \quad v = \tanh u$$

$$\Leftrightarrow \quad \omega_2 = \omega_\infty \tanh \frac{t}{\tau}$$

and $1 - \tanh^2 \frac{t}{\tau} = \operatorname{sech}^2 \frac{t}{\tau}$