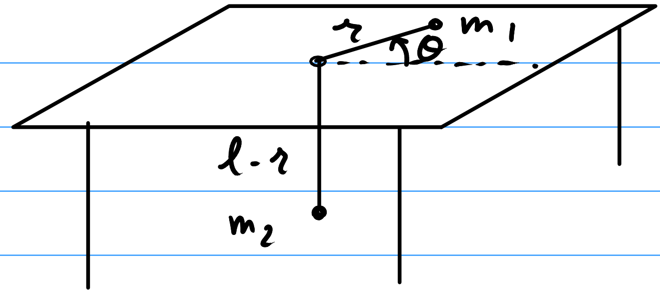
1



$$T = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m_2 \dot{r}^2$$

$$V = m_2 g r$$

$$L = T - V = \frac{1}{2} m_1 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m_2 \dot{r}^2 - m_2 g r$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}} \right) = (m_1 + m_2) \ddot{r} \,, \quad \frac{\partial L}{\partial r} = m_1 r \dot{\theta}^2 - m_2 g$$

$$\Rightarrow (m_1 + m_2) \ddot{r} - m_1 r \dot{\theta}^2 + m_2 g = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta} = \text{constant angular momentum } l$$

$$\Rightarrow \dot{\theta} = \frac{l}{m_1 r^2} \qquad (2)$$

Eliminate $\dot{\theta}$ from (1) and (2)

$$(m_1 + m_2) \ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0$$

Hanging mass is stationary if $\dot{r} = 0$ and $\dot{\theta} = \omega_0$

$$(1) \Rightarrow -m_1 r_0 \omega_0^2 + m_2 g = 0 \Rightarrow r_0 = \frac{m_2 g}{m_1 \omega_0^2} \Leftrightarrow \omega_0^2 = \frac{m_2 g}{m_1 r_0}$$

$$(2) \Rightarrow \omega_0 = \frac{l}{m_1 r_0^2} \Rightarrow l = m_1 r_0^2 \omega_0 \Rightarrow l^2 = m_1^2 r_0^4 \omega_0^2$$

$$= \cancel{m_1^2} r_0^4 \frac{m_2^2 g^2}{\cancel{m_1^2} r_0^2}$$

$$= (m_2 g r_0)^2$$

Small deviations:

$r = r_0 + \Delta r$ and $\theta = \omega_0 t + \Delta\theta$

$\dot{r} = \Delta\dot{r}$ $\qquad \dot{\theta} = \omega_0 + \Delta\dot{\theta}$

$\ddot{r} = \Delta\ddot{r}$ $\qquad \ddot{\theta} = \Delta\ddot{\theta}$

$$(m_1+m_2)\Delta\ddot{r} - \frac{\ell^2}{m_1} r^{-3} + m_2 g = 0$$

$$r^{-3} = (r_0+\Delta r)^{-3} = r_0^{-3}\left(1 - 3\frac{\Delta r}{r_0} + O\left(\frac{\Delta r^2}{r_0^2}\right)\right)$$

$$\Rightarrow (m_1+m_2)\Delta\ddot{r} + 3\frac{\ell^2}{m_1 r_0^4}\Delta r \cancel{- \frac{\ell^2}{m_1} r_0^{-3} + m_2 g} = 0$$

$$\Rightarrow \text{frequency } \omega = \sqrt{3\frac{\ell^2}{m_1(m_1+m_2)r_0^4}} = \sqrt{3\frac{m_1}{m_1+m_2}}\,\omega_0$$

Lagrange multiplier

z, r, θ : $L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{z}^2 - m_2 g z$

$$z = l - r \rightarrow \dot{z} = -\dot{r}$$

$$\begin{cases} (m_1+m_2)\ddot{r} - m_1 r\dot{\theta}^2 + \lambda = 0 \\ m_1 r^2\dot{\theta} = \ell \qquad \longrightarrow \frac{d}{dt}(m_1 r^2\dot{\theta}) = 0 \\ m_2\ddot{z} + m_2 g + \lambda = 0 \end{cases}$$

Stationary: $\dot{r} = 0$, $\dot{\theta} = \omega_0$

$$\Rightarrow -m_1 r_0\omega_0^2 + \lambda = 0$$

$$\Rightarrow \lambda = m_1 r_0\omega_0^2 = m_2 g \rightarrow \text{tension in string}$$

② $T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$ $\qquad \dot{z} = a\dot{\theta}$

$\dot{r} = 0$

$T = \frac{1}{2} m \left(a^2 + \ell^2 \right) \dot{\theta}^2$

$V = mgz = mga\theta$

$\Rightarrow L = \frac{1}{2} m \left(a^2 + \ell^2 \right) \dot{\theta}^2 - mga\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = m(a^2+\ell^2)\ddot{\theta} \ , \ \frac{\partial L}{\partial \theta} = -mga$

$\Rightarrow m(a^2+\ell^2)\ddot{\theta} + mga = 0 \quad \Rightarrow \quad \theta = -\frac{1}{2} g \frac{a}{a^2+\ell^2} t^2$

Constraints : $\lambda_1 : z - a\theta = 0 \rightarrow \frac{\partial f_1}{\partial z} = 1 \ , \ \frac{\partial f_1}{\partial \theta} = -a$

$\lambda_2 : r = b \rightarrow \frac{\partial f_2}{\partial r} = b$

$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) - mgz$

$$\begin{cases} \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m\ddot{r} \ , \ \frac{\partial L}{\partial r} = mr\dot{\theta}^2 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} \ , \ \frac{\partial L}{\partial \theta} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}} \right) = m\ddot{z} \ , \ \frac{\partial L}{\partial z} = -mg \end{cases}$$

$$\Rightarrow \begin{cases} m\ddot{r} - mr\dot{\theta}^2 + b\lambda_2 = 0 \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - a\lambda_1 = 0 \\ m\ddot{z} + mg + \lambda_1 = 0 \\ z = a\theta \quad \Rightarrow \ddot{z} = a\ddot{\theta} \\ r = b \quad \Rightarrow \dot{r} = 0 \end{cases}$$

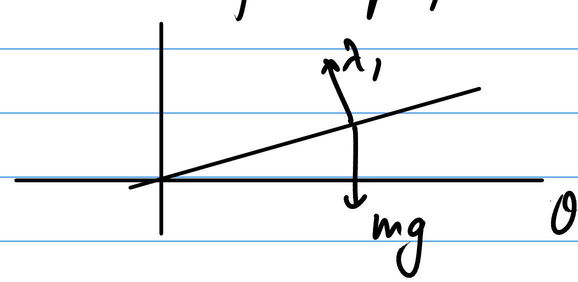
$$\Rightarrow \begin{cases} \lambda_1 = -mg - ma\ddot{\theta} \\ mr^2\ddot{\theta} + amg + ma^2\ddot{\theta} = 0 \Rightarrow (r^2+a^2)\ddot{\theta} = -ag \\ -mb\dot{\theta}^2 + b\lambda_2 = 0 \end{cases}$$

$$\Downarrow$$

$$\ddot{\theta} = -g\frac{a}{a^2+b^2}$$

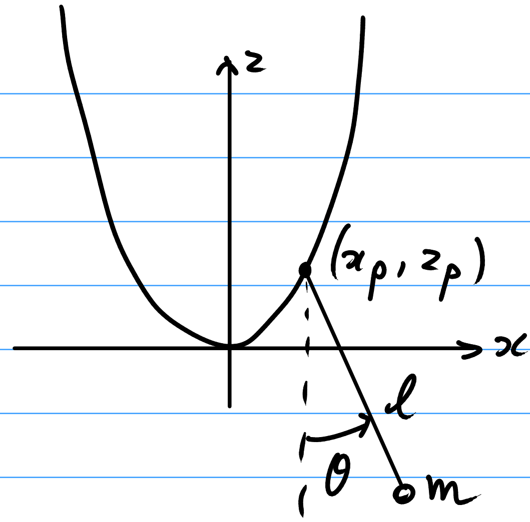
$$\Rightarrow \begin{cases} \lambda_1 = -mg\left(1 - \frac{a^2}{a^2+b^2}\right) = -mg\frac{b^2}{a^2+b^2} \\ \lambda_2 = m\dot{\theta}^2 = -mg^2\frac{a^2t^2}{(a^2+b^2)^2} \end{cases}$$

λ_1 is the fraction of the gravitational force perpendicular to the helix :



λ_2 is the centripetal force

(3)



$$\begin{cases} x_m = x + l\sin\theta \\ z_m = z - l\cos\theta \\ \quad\; = a x^2 - l\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_m = \dot{x} + l\dot{\theta}\cos\theta \\ \dot{z}_m = 2a x\dot{x} + l\dot{\theta}\sin\theta \end{cases}$$

$$T = \frac{1}{2}m\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m\left(2ax\dot{x} + l\dot{\theta}\sin\theta\right)^2$$

$$V = mg z_m = mg\left(ax^2 - l\cos\theta\right)$$

$$\Rightarrow L = T - V = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{\theta}l\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2\cos^2\theta$$

$$+ 2ma^2x^2\dot{x}^2 + 2max\dot{x}\dot{\theta}l\sin\theta + \frac{1}{2}ml^2\dot{\theta}\sin^2\theta$$

$$- mg\left(ax^2 - l\cos\theta\right)$$

$$= \frac{1}{2}m\left(1 + 4a^2x^2\right)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\left(\cos\theta + 2ax\sin\theta\right)\dot{x}\dot{\theta}$$

$$- mg\left(ax^2 - l\cos\theta\right)$$

$$T = \frac{1}{2}\dot{q}^T \underbrace{\begin{pmatrix} m(1+4a^2x^2) & ml(\cos\theta + 2ax\sin\theta) \\ ml(\cos\theta + 2ax\sin\theta & ml^2 \end{pmatrix}}_{M} \dot{q}$$

$$= \frac{1}{2}\dot{q}^T M \dot{q}$$

$$\left.\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} \end{aligned}\right\} \Rightarrow p = \begin{pmatrix} m(1+4a^2x^2) & m\ell(\cos\theta + 2ax\sin\theta) \\ m\ell(\cos\theta + 2ax\sin\theta) & m\ell^2 \end{pmatrix} \dot{q}$$

$$p = M\dot{q} \quad \Rightarrow \quad \dot{q} = M^{-1} p$$

$$\Rightarrow \quad T = \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2} p^T M^{-1} M M^{-1} p$$

$$= \frac{1}{2} p^T M^{-1} p$$

$$\Rightarrow \quad H = \frac{1}{2} p^T M^{-1} p + mg(ax^2 - \ell\cos\theta)$$

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} m\ell^2 & -m\ell(\cos\theta + 2ax\sin\theta) \\ -m\ell(\cos\theta + 2ax\sin\theta) & m(1+4a^2x^2) \end{pmatrix}$$

$$\begin{aligned} \det M &= (1+4a^2x^2)m^2\ell^2 - m^2\ell^2(\cos\theta + 2ax\sin\theta)^2 \\ &= (1-\cos^2\theta)m^2\ell^2 + 4a^2x^2(1-\sin^2\theta)m^2\ell^2 \\ &\quad -4m^2\ell^2 ax\cos\theta\sin\theta \\ &= \sin^2\theta\, m^2\ell^2 + 4a^2x^2\cos^2\theta\, m^2\ell^2 \\ &\quad - 4m^2\ell^2 ax\cos\theta\sin\theta \\ &= m^2\ell^2(\sin\theta - 2ax\cos\theta)^2 \end{aligned}$$

1) $$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{1}{2} p^T \frac{\partial M^{-1}}{\partial x} p - 2mgax$$

2) $$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{1}{2} p^T \frac{\partial M^{-1}}{\partial \theta} p - mg\ell\sin\theta$$

$$\frac{\partial M^{-1}}{\partial x} : \frac{\partial}{\partial x}(M M^{-1}) = \frac{\partial M}{\partial x} M^{-1} + M \frac{\partial M^{-1}}{\partial x} = 0$$

$$\Rightarrow \frac{\partial M^{-1}}{\partial x} = -M^{-1} \frac{\partial M}{\partial x} M^{-1}$$

$$\Rightarrow \begin{cases} \dfrac{\partial M}{\partial x} = \begin{pmatrix} 8ma^2x & 2mla\sin\theta \\ 2mla\sin\theta & 0 \end{pmatrix} \\ \dfrac{\partial M}{\partial \theta} = \begin{pmatrix} 0 & ml(2ax\cos\theta - \sin\theta) \\ ml(2ax\cos\theta - \sin\theta) & 0 \end{pmatrix} \end{cases}$$

$$\left.\begin{aligned} &3)\ \dot{x} = \frac{\partial H}{\partial p_x} = (M^{-1}p)_x \\ &4)\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = (M^{-1}p)_\theta \end{aligned}\right\} \quad \dot{q} = M^{-1}p \text{, as already found}$$

Algebra: $-\frac{1}{2} p^T \frac{\partial M^{-1}}{\partial x_i} p = \frac{1}{2} p^T M^{-1} \frac{\partial M}{\partial x_i} M^{-1} p$

$$= \frac{1}{2(\det M)^2} (p_x \ p_\theta) \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} u & v \\ v & 0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_\theta \end{pmatrix}$$

$$= \frac{1}{2(\det M)^2} (p_x \ p_\theta) \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} u m_{11} + v m_{12} & u m_{12} + v m_{22} \\ v m_{11} & v m_{12} \end{pmatrix} \begin{pmatrix} p_x \\ p_\theta \end{pmatrix}$$

$$= \frac{1}{2(\det M)^2} (p_x \ p_\theta) \begin{pmatrix} u m_{11}^2 + 2v m_{11} m_{12}, & u m_{11} m_{12} + v m_{11} m_{22} + v m_{12}^2 \\ u m_{11} m_{12} + v m_{12}^2 + v m_{11} m_{22}, & u m_{12}^2 + 2v m_{12} m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_\theta \end{pmatrix}$$

$$= \frac{1}{2(\det M)^2} \Big[(u m_{11}^2 + 2v m_{11} m_{12}) p_x^2 + (u m_{12}^2 + 2v m_{12} m_{22}) p_\theta^2 + 2(u m_{11} m_{12} + v m_{12}^2 + v m_{11} m_{22}) p_x p_\theta \Big]$$

$$m_{11} = m\ell^2, \quad m_{12} = -m\ell(\cos\theta + 2ax\sin\theta), \quad m_{22} = m(1+4a^2x^2)$$

$$-\frac{1}{2}p^T \frac{\partial M^{-1}}{\partial x} p : \quad u = 8ma^2x, \quad v = 2m\ell a \sin\theta$$

$$p_x^2 : \quad u m_{11}^2 + 2v m_{11} m_{12} = 8m^2\ell^4 m a^2 x - 4m\ell a \sin\theta\, m^2\ell^2(\cos\theta + 2ax\sin\theta)$$

$$= 4m^2\ell^3 a\cos\theta \sqrt{\det M}$$

$$p_\theta^2 : \quad u m_{12}^2 + 2v m_{12} m_{22} = m\ell(\cos x + 2ax\sin\theta)\left[8m^2a^2\ell x(\cos\theta + \cancel{2ax\sin\theta}) - 4m\ell a\sin\theta(1+\cancel{4a^2x^2})\right]$$

$$= 4m^2\ell a(\cos x + 2ax\sin\theta)\sqrt{\det M}$$

$$p_x p_\theta : \quad 2(u m_{11} m_{12} + v m_{12}^2 + v m_{11} m_{22}) = 2\left[-8m^3a^2x\ell^3(\cos\theta + 2ax\sin\theta) + 2m^3\ell^2 a\sin\theta(\cos\theta + 2ax\sin\theta)^2 + 2m^3\ell^3 a\sin\theta\,(1+4a^2x^2)\right]$$

$$= 2m^3\ell^2 a\left[-8ax\cos\theta - 16a^2x^2\sin\theta + 2\sin\theta\cos^2\theta + 8ax\cos\theta\sin^2\theta + 8a^2x^2\sin^3\theta + 2\sin\theta + 8a^2x^2\sin\theta\right]$$

$$= -4m\ell^2 a\sqrt{\det M}\left[2 - \sin^2\theta + 2ax\sin\theta\cos\theta\right]$$

$$-\frac{1}{2}p^T \frac{\partial M^{-1}}{\partial \theta} p : \quad u = 0, \quad v = m\ell(2ax\cos\theta - \sin\theta) = \sqrt{\det M}$$

$$p_x^2 : \quad 2v m_{11} m_{12} = -2\sqrt{\det M}\, m\ell(\cos\theta + 2ax\sin\theta)\, m\ell^2$$

$$p_\theta^2 : \quad 2v m_{22} m_{12} = -2\sqrt{\det M}\, m\ell(\cos\theta + 2ax\sin\theta)\, m(1+4a^2x^2)$$

$$p_x p_\theta : \quad 2v(m_{12}^2 + m_{11}m_{22}) = 2\sqrt{\det M}\left[m^2\ell^2(\cos\theta + 2ax\sin\theta)^2 + m^2\ell^2(1+4a^2x^2)\right]$$

$$\Rightarrow \dot{p}_x = \frac{2a}{ml^2(\sin\theta - 2ax\cos\theta)^3}\Big[\cos\theta \;\; l^2 p_x^2 + (\cos\theta + 2ax\sin\theta)\, p_\theta^2$$
$$- (2 - \sin^2\theta + 2ax\sin\theta\cos\theta)\; l\, p_x p_\theta\Big] - 2mgax$$

$$\Rightarrow \dot{p}_\theta = \frac{1}{ml^2(\sin\theta - 2ax\cos\theta)^3}\Big[(\cos\theta + 2ax\sin\theta)\big(l^2 p_x^2 + (1+4a^2x^2)\, p_\theta^2\big)$$
$$+ \big((\cos\theta + 2ax\sin\theta)^2 + (1+4a^2x^2)\big)\; l\, p_x p_\theta\Big] - mgl\sin\theta$$

(6) $T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)(q_1^2 + q_2^2)$

$V = \frac{1}{q_1^2 + q_2^2}$

$\Rightarrow L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)(q_1^2 + q_2^2) - \frac{1}{q_1^2 + q_2^2}$

$p_i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i (q_1^2 + q_2^2) \Rightarrow \dot{q}_i = \frac{p_i}{q_1^2 + q_2^2}$

$\Rightarrow H = \frac{1}{2} \frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} + \frac{1}{q_1^2 + q_2^2}$

$\frac{1}{2} \frac{1}{q_1^2 + q_2^2} \left[\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2 + 2 \right] = \alpha$

$\Leftrightarrow \left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2 + 2 = 2\alpha (q_1^2 + q_2^2)$

$W(q_1, q_2, \alpha_1, \alpha_2) = W_1(q_1, \alpha_1) + W_2(q_2, \alpha_2)$

$\Leftrightarrow \underbrace{\left[\left(\frac{\partial W_1}{\partial q_1}\right)^2 - 2\alpha q_1^2 \right]}_{\alpha_1} + \underbrace{\left[\left(\frac{\partial W}{\partial q_2}\right)^2 - 2\alpha q_2^2 \right]}_{\alpha_2} + 2 = 0$

$\alpha_1 + \alpha_2 = -2$

$$\Leftrightarrow \begin{cases} W_1(q_1) = \pm \int \sqrt{\alpha_1 + 2\alpha q_1^2}\, dq_1 \\ W_2(q_2) = \pm \int \sqrt{\alpha_2 + 2\alpha q_2^2}\, dq_2 = \pm \int \sqrt{-2 - \alpha_1 + 2\alpha q_2^2}\, dq_2 \end{cases}$$

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial W_1}{\partial \alpha} + \frac{\partial W_2}{\partial \alpha} - t = \pm \int \frac{q_1^2 \, dq_1}{\sqrt{\alpha_1 + 2\alpha q_1^2}} \pm \int \frac{q_2^2 \, dq_2}{\sqrt{-2 - \alpha_1 + 2\alpha q_2^2}} - t$$

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial W_1}{\partial \alpha_1} + \frac{\partial W_2}{\partial \alpha_1} = \pm \int \frac{dq_1}{2\sqrt{\alpha_1 + 2\alpha q_1^2}} \pm \int \frac{dq_2}{2\sqrt{-2 - \alpha_1 + 2\alpha q_2^2}}$$