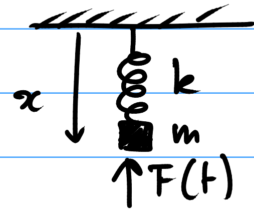
Classical Mechanics (Phys 601) - November 17, 2011

* **Final exam:**
  Wednesday December 7, 2011, between 9 am and 12 pm
  in Small Hall 122

* **Non-linear systems** $\longrightarrow$ non-linear equations of motion

Mass on a spring:



$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2$$

$$\hookrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F(t) \Leftrightarrow m\ddot{q} + m\omega^2 q = F(t)$$

Linear system: only terms linear in derivatives of $q(t)$ in the differential equations of motion.

No driving force:

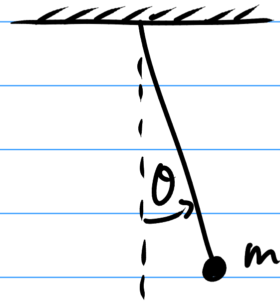
$$m\ddot{q} + m\omega^2 q = 0 \rightarrow \text{homogeneous, linear}$$

If $q_1(t)$ and $q_2(t)$ are solutions (with different initial conditions), then $a q_1(t) + b q_2(t)$ is also a solution.

Driving force:

$$m\ddot{q} + m\omega^2 q = F(t) \rightarrow \text{non-homogeneous, still linear}$$

Pendulum :



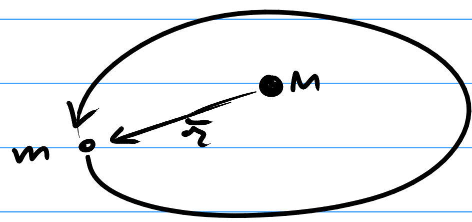
$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta$$

↳ non-linear system

Linearize for θ small : $\sin\theta \simeq \theta \Rightarrow \ddot{\theta} = -\frac{g}{l}\theta$

Orbital motion :



$$L = \frac{1}{2} m \bar{v}^2 - GMm\frac{1}{r}$$

$$\Rightarrow m\ddot{\bar{r}} = -GM\frac{\hat{r}}{r^2}$$

→ non-linear system

Linear equations of motion → typically solvable
Non-linear equations of motion → typically <u>not</u> solvable
↳ some exceptions : orbital motion for two-body system

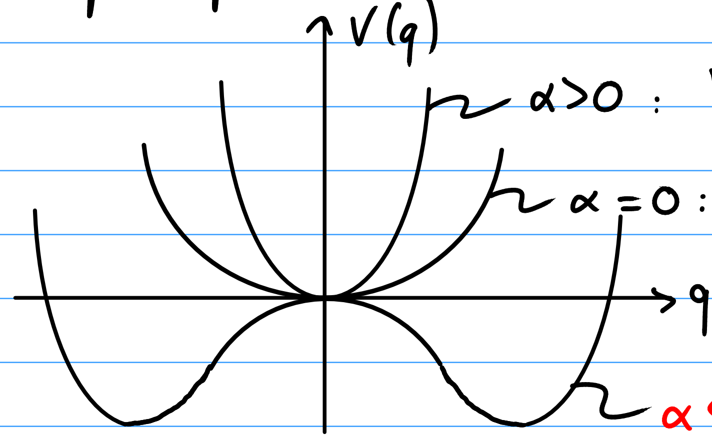
* Duffing oscillator: general form

Consider the following modification to the harmonic oscillator:

$$V(q) = \frac{1}{2} m\alpha q^2 + \frac{1}{4} m\beta q^4$$ ← quartic term

At $|q| \to \infty$ physical reasons require $V(q)$ to be positive.
$\Rightarrow \beta$ positive

In principle α can be either positive or negative.



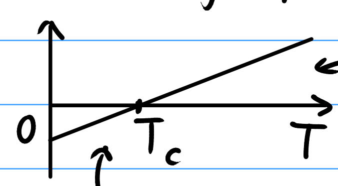
$\alpha > 0$: $V(q) = \frac{1}{2} m \overset{>0}{\alpha} q^2 + \frac{1}{4} \overset{>0}{\beta} q^4$

$\alpha = 0$: $V(q) = \frac{1}{4} m\beta q^4$

$\alpha < 0$: $V(q) = \frac{1}{2} m \overset{<0}{\alpha} q^2 + \frac{1}{4} \overset{>0}{\beta} q^4$

Applications:

- superconductivity phase transitions: $\alpha(T)$



← regular

superconducting

- Higgs mechanism for electroweak symmetry breaking
  ↳ minimum energy state is <u>not</u> at $q=0$
  ↳ "hat potential" $V(\varphi)$ where φ is complex

$V(q)$ is stationary when $\frac{\partial V}{\partial q} = 0 \Leftrightarrow m\alpha q + m\beta q^3 = 0$

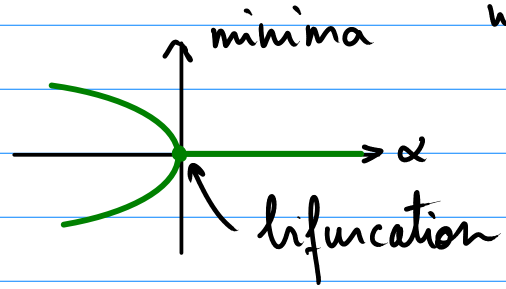
$$\Leftrightarrow q = 0 \quad \text{or} \quad q^2 = -\frac{\alpha}{\beta} \begin{cases} 0, & \text{for } \alpha = 0 \\ <0, & \text{for } \alpha > 0 \\ >0, & \text{for } \alpha < 0 \end{cases}$$

$\alpha > 0$: only one real root as absolute minimum

$\alpha = 0$: $q = 0$ is a triple root as absolute minimum

$\alpha < 0$: $\underbrace{q = 0}_{\text{local maximum}}$ and $\underbrace{q = \pm\sqrt{-\frac{\alpha}{\beta}}}_{\text{absolute minima}} = \pm\sqrt{\frac{|\alpha|}{\beta}}$ are roots

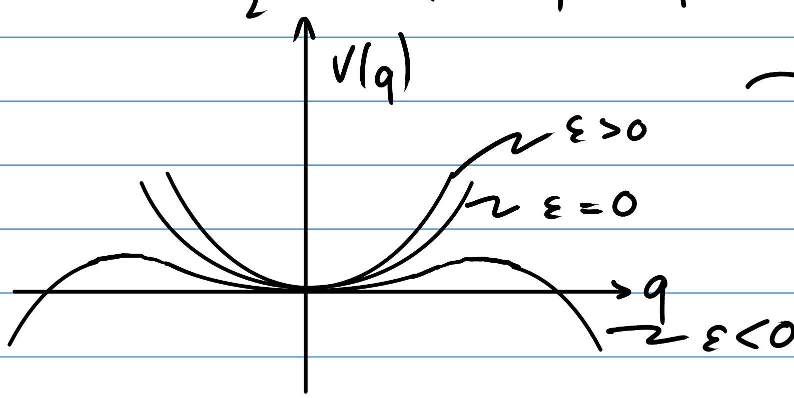
$\Rightarrow$ continuous evolution of α leads to bifurcation of minimum with energy barrier between them :



$$V(q^{\pm}) = \frac{1}{2} m\alpha\left(-\frac{\alpha}{\beta}\right) + \frac{1}{4} m\beta\left(-\frac{\alpha}{\beta}\right)^2$$
$$= -\frac{1}{4} m \frac{\alpha^2}{\beta}$$

* Duffing oscillator as perturbation of harmonic oscillator :

$$V(q) = \frac{1}{2} m\omega_0^2 q^2 + \frac{1}{4} \varepsilon m q^4$$ with ε small (positive or negative)



$\rightarrow$ we will focus on small region around $q = 0$
$\rightarrow$ no problems with $q \rightarrow \infty$

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega_0^2 q^2 - \frac{1}{4} \varepsilon m q^4$$

$$\Rightarrow \ddot{q} + \omega_0^2 q + \varepsilon q^3 = 0 \longrightarrow \text{non-linear!}$$

$$\Rightarrow p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 + \frac{1}{4} \varepsilon m q^4$$

Perturbation theory: assume solution of form

$$q(t) = q_0(t) + \varepsilon q_1(t) + \ldots$$

$$\begin{cases} \text{1st order: } \ddot{q}_0 + \omega_0^2 q_0 = 0 \\ \text{2nd order: } \ddot{q}_1 + \omega_0^2 q_1 + q_0^3 = 0 \\ \vdots \end{cases} \quad \text{with} \quad \begin{cases} q_0(0) = a \\ \dot{q}_0(0) = 0 \\ \vdots \\ q_i(0) = 0 \\ \dot{q}_i(0) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} q_0(t) = a \cos \omega_0 t \\ \ddot{q}_1 + \omega_0^2 q_1 = -a^3 \cos^3 \omega_0 t \\ \vdots \end{cases}$$

$$(e^{i\varphi})^3 = \cos 3\varphi + i \sin 3\varphi = \cos^3\varphi + 3i\cos^2\varphi \sin\varphi - 3\cos\varphi \sin^2\varphi - i\sin^3\varphi$$

$$\overset{\text{Re}}{\Leftrightarrow} \cos 3\varphi = \cos^3\varphi - 3\cos\varphi(1 - \cos^2\varphi)$$

$$= -3\cos\varphi + 4\cos^3\varphi$$

$$\Leftrightarrow \cos^3\varphi = \frac{1}{4}(\cos 3\varphi + 3\cos\varphi)$$

$$\Leftrightarrow \begin{cases} q_0(t) = a \cos \omega_0 t \\ \ddot{q}_1 + \omega_0^2 q_1 = \underbrace{-\frac{3}{4} a^3 \cos \omega_0 t}_{\text{driving term}} - \underbrace{\frac{1}{4} a^3 \cos 3\omega_0 t}_{\text{driving term}} \\ \vdots \end{cases}$$

Solution of second order differential equation:

$$a_2 y'' + a_1 y' + a_0 y = e^{ikx}$$

↳ characteristic equation → poles k_1 and k_2
$a_2 k^2 + a_1 k + a_0 = 0$

1) if $k \neq k_1, k_2$ → $y = A e^{ik_1 x} + B e^{ik_2 x}$

2) if $k = k_1 \neq k_2$ → $y = At e^{ik_1 x} + B e^{ik_2 x}$

3) if $k = k_1 = k_2$ → $y = (At^2 + Bt) e^{ik_1 x}$

resonance

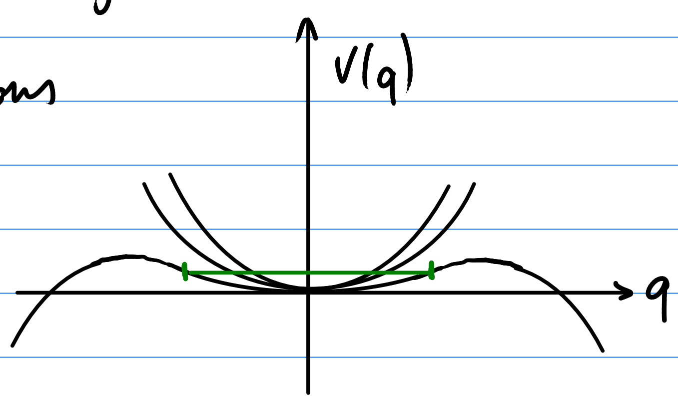
↳ substitute to determine A and B of particular solution

$$\Rightarrow \quad q_1(t) = -\frac{a^3}{8\omega_0^2}\left(3\omega_0 t \sin\omega_0 t + \frac{1}{4}(\cos\omega_0 t - \cos 3\omega_0 t)\right)$$

$$\Rightarrow q(t) = a\cos\omega_0 t - \frac{\varepsilon a^3}{8\omega_0^2}\left(3\omega_0 t \sin\omega_0 t + \frac{1}{4}(\cos\omega_0 t - \cos 3\omega_0 t)\right) + O(\varepsilon^2)$$

This solution is impossible because it indicates that $q(t)$ will increase linearly.

However, energy considerations limit the size of $q(t)$
(if $E <$ barrier height)



Need to be more careful: $\omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \cdots$

$\qquad\qquad\qquad\qquad\qquad\qquad a(\varepsilon) = a + \varepsilon a_1 + \cdots$

$$\Rightarrow \quad q(t) = a(\varepsilon)\cos\big(\omega(\varepsilon)t\big)$$

$$= a(\varepsilon)\cos(\omega_0 t + \varepsilon\omega_1 t + \cdots) \quad \text{coefficient } \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0}$$

$$= a(\varepsilon)\left(\cos\omega_0 t - \varepsilon\omega_1 t\sin\omega_0 t + \cdots\right)$$

With this $\omega = \omega_0 + \varepsilon\omega_1 + \cdots$ we can write:

$$\ddot{q} + \omega_0^2 q + \varepsilon q^3 = 0 \qquad \tau = \omega t$$

$$\Leftrightarrow \omega^2\frac{d^2q}{d\tau^2} + \omega_0^2 q + \varepsilon q^3 = 0$$

$$\Leftrightarrow (\omega_0 + \varepsilon\omega_1 + \cdots)^2\frac{d^2q}{d\tau^2} + \omega_0^2 q + \varepsilon q^3 = 0$$

$$\Leftrightarrow \left(\omega_0^2\frac{d^2q}{d\tau^2} + \omega_0^2 q\right) + \left(2\varepsilon\omega_0\omega_1\frac{d^2q}{d\tau^2} + \varepsilon q^3\right) + \cdots = 0$$

$$\Leftrightarrow \left(\omega_0^2\frac{d^2q_0}{d\tau^2} + \omega_0^2 q_0\right) + \left(\omega_0^2\varepsilon\frac{d^2q_1}{d\tau^2} + \omega_0^2\varepsilon q_1\right)$$

$$+\left(2\varepsilon\omega_0\omega_1\frac{d^2q_0}{d\tau^2} + \varepsilon q_0^3\right) + \cdots = 0$$

$$\Leftrightarrow \begin{cases} \omega_0^2\left(\dfrac{d^2q_0}{d\tau^2} + q_0\right) = 0 \\ \omega_0^2\left(\dfrac{d^2q_1}{d\tau^2} + q_1\right) = -2\omega_0\omega_1\dfrac{d^2q_0}{d\tau^2} - q_0^3 \\ \vdots \end{cases}$$

first order equation → harmonic oscillator

$$\Rightarrow \quad q_0(\tau) = a\cos\tau$$

second order equation after substituting $q_0(\tau)$

$$\omega_0^2\left(\frac{d^2 q_1}{d\tau^2} + q_1\right) = \Big(\underbrace{2\omega_0\omega_1}_{\text{additional term}} - \frac{3}{4}a^2\Big)a\cos\tau - \frac{1}{4}a^3\cos 3\tau$$

resonant driving term

Remove resonant driving term by choosing $\omega_1 = \dfrac{3}{8}\dfrac{a^2}{\omega_0}$

$$\hookrightarrow \omega_0^2\left(\frac{d^2 q_1}{d\tau^2} + q_1\right) = -\frac{1}{4}a^3\cos 3\tau$$

$$\Rightarrow \quad q_1(\tau) = -\frac{a^3}{32\omega_0^2}\left(\cos\tau - \cos 3\tau\right) \quad \tau \simeq \omega_0 t$$

$$\text{or} \quad q_1(t) \simeq -\frac{a^3}{32\omega_0^2}\left(\cos\omega_0 t - \cos 3\omega_0 t\right) + O(\varepsilon)$$

Full solution:

$$q(t) = a\cos\left(\left(\omega_0 + \varepsilon\frac{3}{8}\frac{a^2}{\omega_0}\right)t\right) - \frac{\varepsilon a^3}{32\omega_0^2}\left(\cos\omega_0 t - \cos 3\omega_0 t\right)$$

→ no term blows up anymore, but expand this in ε:

$$q(t) = a\cos\omega_0 t - \varepsilon\frac{3a^3}{8\omega_0}t\sin(\omega_0 t) - \frac{\varepsilon a^3}{32\omega_0^2}\left(\cos\omega_0 t - \cos 3\omega_0 t\right) + O(\varepsilon^2)$$

$\Rightarrow$ by changing $\omega_0 \to \omega_0 + \varepsilon\omega$, for a specific ω, we could avoid infinities in our perturbation theory calculation

↳ concept behind parts of quantum field theory
and renormalization !
and running couplings

+ Perturbations of periodic Hamiltonian systems

If we have a Hamiltonian of the form

$$H(p,q,t) = H_0(p,q,t) + \varepsilon V(p,q,t)$$

Can we use the methods we developed for linear systems on non-linear perturbations?

Review:

Canonical transformation:

$$\begin{cases} H_0 \\ p,q \end{cases} \xrightarrow{S_0} \begin{cases} \tilde{H}_0 \\ P,Q \end{cases} \qquad \text{with } S_0(q,P,t)$$

$$\begin{cases} \tilde{H}_0(P,Q,t) = H_0(p,q,t) + \dfrac{\partial S_0}{\partial t}(q,P,t) \\ p = \dfrac{\partial S_0}{\partial q}(q,P,t) \\ Q = \dfrac{\partial S_0}{\partial P}(q,P,t) \end{cases}$$

Canonical transformation preserves Hamilton's equations:

$$\begin{cases} \dot{Q} = \dfrac{\partial \tilde{H}_0}{\partial P}(P,Q,t) \\ \dot{P} = -\dfrac{\partial \tilde{H}_0}{\partial Q}(P,Q,t) \end{cases}$$

Hamilton-Jacobi equation:

Now transform to P, Q such that $\tilde{H}_0(P,Q,t) \equiv 0$
$\Rightarrow$ P and Q are constants of motion

Condition on S_0: $H_0(p,q,t) + \dfrac{\partial S_0}{\partial t}(q,P,t) = 0$

$$\text{with } p = \frac{\partial S_0}{\partial q}$$

$$\Rightarrow \quad H_0\left(\frac{\partial S_0}{\partial q}, q, t\right) + \frac{\partial S_0}{\partial t}(q,P,t) = 0$$

If $\dfrac{\partial H}{\partial t} = 0 \rightarrow S_0(q,P,t) = W_0(q,P) - Et$

(note: we called $P_i = \alpha_i$ earlier, now we will just use P)

For one-dimensional problems: $S_0(q,E,t) = W(q,E) - Et$

$$\Rightarrow \quad H_0\left(\frac{\partial S_0}{\partial q}, q, t\right) = E$$

Action-angle variables for periodic systems:

Define action variable (now with 2π)

$$J = \frac{1}{2\pi}\int_{\text{cycle}} p\, dq = \frac{1}{2\pi}\int_{\text{cycle}} \frac{\partial W_0}{\partial q}(q,E)\, dq = J(E)$$

$$\uparrow$$

$$p = \frac{\partial S_0}{\partial q} = \frac{\partial W_0}{\partial q}$$

If we assume that $J(E)$ can be inverted, then

$$\bar{S}_0(q, J, t) = S_0(q, E(J), t)$$

describes a canonical transformation to p and

$$\bar{Q} = \frac{\partial \bar{S}_0}{\partial J}(q, J, t) = \underbrace{\frac{\partial \bar{W}_0}{\partial J}(q, J)}_{\text{angle variable } \varphi} - \underbrace{\frac{\partial E_0}{\partial J}(J)}_{\text{frequency } \omega_0} t \equiv \bar{\beta}$$

$$\Rightarrow \quad \varphi = \omega_0 t + \bar{\beta}$$

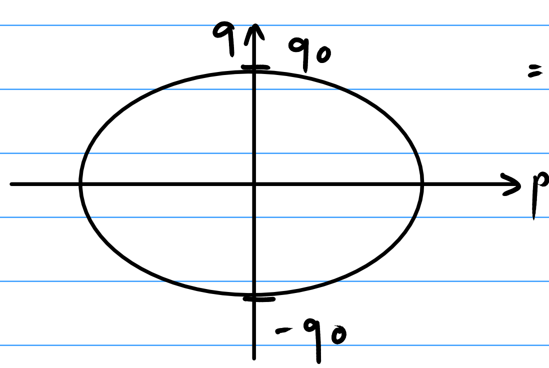
$\Rightarrow$ action variable is constant
angle variable increases linearly with time

For perturbative system this will not be true anymore!

Harmonic oscillator :

$$H_0 = E_0 = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 \quad \Rightarrow \quad p = \pm \sqrt{2m} \sqrt{E_0 - \frac{1}{2} m \omega_0^2 q^2}$$

$$\Rightarrow J = \frac{1}{2\pi} \int_{\text{cycle}} p \, dq = \frac{4\sqrt{2m}}{2\pi} \int_{\text{quadrant}} \sqrt{\frac{1}{2} m \omega_0^2 (q_0^2 - q^2)}$$



$$= \frac{2 q_0 \sqrt{2mE}}{\pi} \int_0^1 dx \sqrt{1 - x^2}$$

$$= \frac{E_0}{\omega_0} \quad \Rightarrow \quad E_0(J) = \omega_0 J$$

$$\Rightarrow \quad \omega = \frac{dE_0}{\partial J} = \omega_0$$

$$H_0\left(\frac{\partial W}{\partial q}, q, t\right) = E_0$$

$$\Leftrightarrow W(q,E) = \sqrt{2m}\int dq\sqrt{E_0 - \frac{1}{2}m\omega_0^2 q^2} + W_0(E_0)$$

$$\Rightarrow \bar{W}(q, E_0(J)) = \sqrt{2m}\int dq\sqrt{\omega_0 J - \frac{1}{2}m\omega_0^2 q^2} + \bar{W}_0(J)$$

$$\Rightarrow \varphi = \frac{\partial \bar{W}}{\partial J}(q,J) = \frac{\omega_0}{2}\sqrt{2m}\int dq \frac{1}{\sqrt{\omega_0 J - \frac{1}{2}m\omega_0^2 q^2}} + \varphi_0(J)$$

$$= \sin^{-1}\frac{q}{q_0}$$

$$\Rightarrow \quad q = \sqrt{\frac{2J}{m\omega_0}}\sin\varphi \quad \text{and} \quad p = \sqrt{2m\omega_0 J}\cos\varphi \quad , \varphi = \omega_0 t$$

Perturbation to H_0

$$H(J,\varphi) = H_0(J,\varphi) + \varepsilon H_1(J,\varphi)$$

$$= E_0(J,\varphi) + \varepsilon H_1(J,\varphi)$$

$$= \omega_0 J + \varepsilon \frac{1}{4} m \left(\frac{2J}{m\omega_0}\right)^2 \sin^4\varphi$$

$\Rightarrow$ Hamilton's equations :

$$\begin{cases} \dot{J} = -\dfrac{\partial H}{\partial \varphi} = -\varepsilon m\left(\dfrac{2J}{m\omega_0}\right)^2 \sin^3\varphi\cos\varphi \\ \dot{\varphi} = \dfrac{\partial H}{\partial J} = \omega_0 + \varepsilon\dfrac{2J}{m\omega_0^2}\sin^4\varphi \end{cases}$$