Classical Mechanics (Phys 601) - November 10, 2011

* Euler equations :

$$\begin{cases} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2-I_3) = \Gamma_1 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3-I_1) = \Gamma_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1-I_2) = \Gamma_3 \end{cases}$$

↳ in tumbling rigid body frame

⇒ need to be able to uniquely describe orientation of rigid body frame in external inertial frame.

Kinetic energy $T_{rot} = \frac{1}{2}\omega^T I \omega$ = scalar independent of coordinate frame

⇓

$$T_{rot} = \frac{1}{2}\omega^T I \omega\Big|_{inertial} = \frac{1}{2}\omega^T I \omega\Big|_{body}$$

where $\omega|_{inertial}$ is related to $\omega|_{body}$ by an orthogonal rotation matrix that is time-dependent:

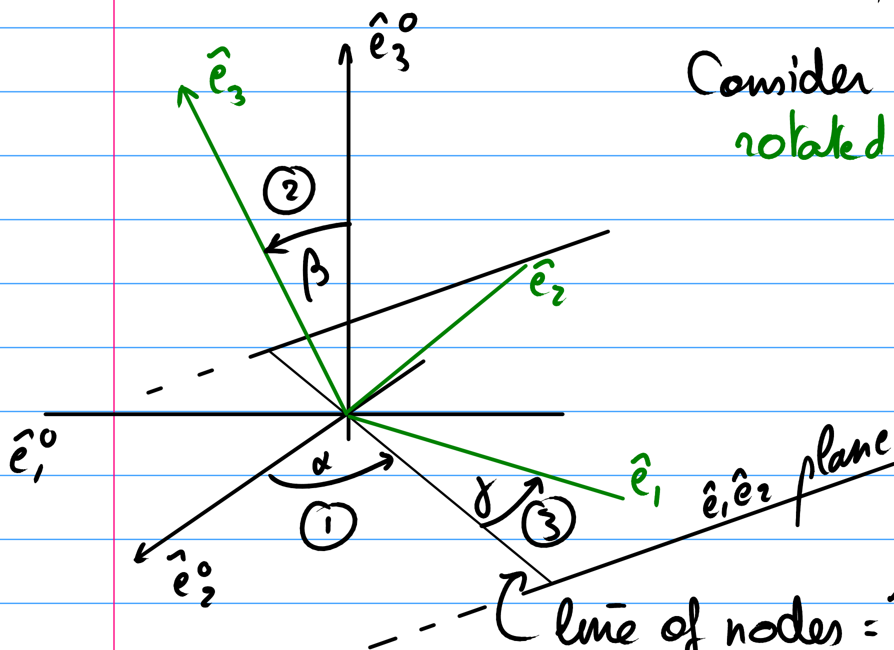
$$\omega|_{inertial} = U(t)\cdot\omega|_{body}$$

Orthogonal matrix U has 9 elements, but $U^TU = \mathbb{1}$ adds 3 normality constraints on diagonal and 3 independent orthogonality constraints off diagonal

⇒ $9-3-3 = 3$ degrees of freedom in $U(t)$

* Euler angles (α, β, γ)

= most convenient 3 degrees of freedom to describe rotation



Consider inertial frame $\{\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0\}$
rotated body frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

↓

split transformation in three steps, each associated with an Euler angle

① Rotation around $\hat{e}_\alpha = \hat{e}_3^0$ over angle α :

line of nodes : the intersection of plane $\hat{e}_1^0 \hat{e}_2^0$ and $\hat{e}_1 \hat{e}_2$
↳ name originates in celestial mechanics :
{ ascending / descending } nodes are points where orbit crosses ecliptic plane

This will rotate $\hat{e}_2^0$ to the line of nodes, $\hat{e}_2'$

$$\Rightarrow \quad U_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: other convention defines first Euler angle φ to rotate $\hat{e}_1^0$ to the line of nodes, with $\varphi = \alpha + \frac{\pi}{2}$

② Rotate around line of nodes $\hat{e}_\beta = \hat{e}_2'$ over angle β :

This will rotate $\hat{e}_3^0$ to $\hat{e}_3$:

$$\Rightarrow \quad U_\beta = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

Note : other convention now has rotation over θ around $\hat{e}_1'$:

$$U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad \text{with } \theta = \beta$$

③ Rotate around $\hat{e}_\gamma = \hat{e}_3$ over angle γ :

This will rotate $\hat{e}_2'$ from the line of nodes to the final $\hat{e}_2$:

$$U_\gamma = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: other convention now has rotation over ψ around $\hat{e}_3$ that rotates $\hat{e}_1'$ from the line of nodes to the final $\hat{e}_1$:

$$\psi = \gamma - \frac{\pi}{2}$$

Total rotation is product of individual rotations :

$$x^{(\alpha)} = U_\alpha x \quad , \quad x^{(\beta)} = U_\beta x^{(\alpha)} \quad , \quad x^{(\gamma)} = x' = U_\gamma x^{(\beta)}$$

$$\Rightarrow \quad x' = U_\gamma U_\beta U_\alpha x = U(\alpha, \beta, \gamma) \, x$$

$$U = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\beta\cos\gamma \\ \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \ldots\ \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix} = U(\alpha,\beta,\gamma)$$

(Note: similar expression for $\varphi = \frac{\pi}{2} + \alpha$, $\theta = \beta$, $\psi = \gamma - \frac{\pi}{2}$)

* Angular velocities and Euler angles

When rigid body is rotating $\rightarrow$ angular velocities

$\underbrace{\dot\alpha, \dot\beta, \dot\gamma}_{\dot q_i}$ (as time-derivatives of degrees of freedom $\underbrace{\alpha, \beta, \gamma}_{q_i}$)

Kinetic energy $T = \frac{1}{2} \omega^T I \omega = \frac{1}{2} \sum_i I_i \omega_i^2$,

with ω_i the components of $\vec\omega$ in the principal axes frame

↳ need to use $\dot\alpha, \dot\beta, \dot\gamma$ to express $\vec\omega$ in principal axes

Remember that infinitesimal rotations commute:

$$\bar{r}'' = \bar{r}' + d\theta_2 \hat{e}_2 \times \bar{r}' \quad \text{and} \quad \bar{r}' = \bar{r} + d\theta_1 \hat{e}_1 \times \bar{r}$$

$$\Rightarrow \quad \bar{r}'' = \bar{r} + d\theta_1 \hat{e}_1 \times \bar{r} \quad + d\theta_2 \hat{e}_2 \times \left(\bar{r} + d\theta_1 \hat{e}_1 \times \bar{r} \right)$$

$$= \bar{r} + \left(d\theta_1 \hat{e}_1 + d\theta_2 \hat{e}_2 \right) \times \bar{r} \quad + O(d\theta^2)$$

$\Rightarrow$ for infinitesimal time interval dt:

$$\frac{d\theta_1}{dt} \hat{e}_1 + \frac{d\theta_2}{dt} \hat{e}_2 = \bar{\omega}_1 + \bar{\omega}_2$$

Angular velocities in Euler angles:

$$\bar{\omega} = \bar{\omega}_\alpha + \bar{\omega}_\beta + \bar{\omega}_\gamma = \dot{\alpha} \hat{e}_\alpha + \dot{\beta} \hat{e}_\beta + \dot{\gamma} \hat{e}_\gamma$$

Now, need to express $\hat{e}_\alpha$, $\hat{e}_\beta$, $\hat{e}_\gamma$ in rigid body frame:

$$\hat{e}_\alpha = \hat{e}^0_3 \Big|_{\text{inertial}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \hat{e}^0_3 \Big|_{\text{body}} = U(\alpha, \beta, \gamma) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\beta \cos\gamma \\ \sin\beta \sin\gamma \\ \cos\beta \end{pmatrix}$$

$$\Rightarrow \quad \bar{\omega}_\alpha = \begin{pmatrix} -\dot{\alpha} \sin\beta \cos\gamma \\ \dot{\alpha} \sin\beta \sin\gamma \\ \dot{\alpha} \cos\beta \end{pmatrix}$$

$$\hat{e}_\beta = \hat{e}_2'\Big|_{\text{rotated}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

U_β : rotation around $\hat{e}_2'$

$\rightarrow$ leaves $\hat{e}_2'$ unchanged

$$\Rightarrow \hat{e}_2'\Big|_{\text{body}} = U_\gamma U_\beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = U_\gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

$$\Rightarrow \bar{\omega}_\beta = \begin{pmatrix} \dot{\beta}\sin\gamma \\ \dot{\beta}\cos\gamma \\ 0 \end{pmatrix}$$

And $\bar{\omega}_\gamma = \dot{\gamma}\,\hat{e}_\gamma = \begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix}$

$$\Rightarrow \bar{\omega} = \bar{\omega}_\alpha + \bar{\omega}_\beta + \bar{\omega}_\gamma = \begin{pmatrix} -\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma \\ \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma \\ \dot{\alpha}\cos\beta + \dot{\gamma} \end{pmatrix}$$

Note : again for φ, θ, ψ this is $\begin{pmatrix} \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\varphi}\cos\theta + \dot{\psi} \end{pmatrix}$

Kinetic energy in general case is now :

$$T = \frac{1}{2}\Big[I_1\left(-\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma\right)^2 + I_2\left(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma\right)^2 + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 \Big] + \frac{1}{2}MV^2$$

Only one cyclic coordinate : $\alpha \Rightarrow p_\alpha = \text{constant}$

and also cyclic in time $t \Rightarrow T = \text{constant}$

* Symmetric top without torque:

With only one cyclic coordinate → difficult to solve equation of motion

↓

Symmetric top leads to second cyclic coordinate : $I_1 = I_2$

↳ problem solved in previous lecture → repeat now with Lagrangian formalism & Euler angles.

$$T = \frac{1}{2}\Big[I_1\Big(\left(-\dot{\alpha}\sin\beta\cos\gamma + \dot{\beta}\sin\gamma\right)^2 + \left(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma\right)^2 \Big) + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 \Big]$$

$$= \frac{1}{2}\Big[I_1\left(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2\right) + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 \Big]$$

→ both α and γ are now cyclic:

$$\begin{cases} p_\alpha = \dfrac{\partial L}{\partial \dot{\alpha}} = I_1\dot{\alpha}\sin^2\beta + I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)\cos\beta = \text{constant} \\ p_\gamma = \dfrac{\partial L}{\partial \dot{\gamma}} = I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right) = I_3\omega_3 = L_3 = \text{constant} \end{cases}$$

↳ ω_3 = projection of $\vec{\omega}$ on principal axis $\hat{e}_3$ is constant (as before)

Other canonical momentum $p_\beta = \frac{\partial L}{\partial \dot\beta} = I_1 \dot\beta \neq$ constant

$\Rightarrow$ due to cyclic α and $\gamma \rightarrow$ one-dimensional problem

$$I_1 \ddot\beta = \frac{\partial L}{\partial \beta} = I_1 \dot\alpha^2 \sin\beta\cos\beta - I_3 (\dot\alpha\cos\beta + \dot\gamma)\dot\alpha \sin\beta$$

In principal axes : $\bar{L} = I_1(\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2) + I_3 \omega_3 \hat{e}_3$

$$\hat{e}_\alpha = \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} \qquad \hat{e}_\beta = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\bar{L}\cdot\hat{e}_\alpha &= I_1(-\dot\alpha\sin\beta\cos\gamma + \dot\beta\sin\gamma)(-\sin\beta\cos\gamma) \\
&\quad + I_1(\dot\alpha\sin\beta\sin\gamma + \dot\beta\cos\gamma)(\sin\beta\sin\gamma) \\
&\quad + I_3(\dot\alpha\cos\beta + \dot\gamma)\cos\beta \\
&= I_1 \dot\alpha^2 \sin^2\beta + I_3(\dot\alpha\cos\beta + \dot\gamma)\cos\beta = p_\alpha
\end{aligned}$$

$$\begin{aligned}
\bar{L}\cdot\hat{e}_\beta &= I_1(-\dot\alpha\sin\beta\cos\gamma + \dot\beta\sin\gamma)(\sin\gamma) \\
&\quad + I_1(\dot\alpha\sin\beta\sin\gamma + \dot\beta\cos\gamma)(\cos\gamma) \\
&= I_1\dot\beta = p_\beta
\end{aligned}$$

$$\bar{L}\cdot\hat{e}_\gamma = I_3(\dot\alpha\cos\beta + \dot\gamma) = p_\gamma$$

$\Rightarrow$ $p_\alpha, p_\beta, p_\gamma$ are projections of $\bar{L}$ on axes $\hat{e}_\alpha, \hat{e}_\beta, \hat{e}_\gamma$

Hamiltonian: express $\dot{q}$ in terms of $\dot{p}$

$$p_\alpha = I_1 \dot{\alpha} \sin^2\beta + \underbrace{I_3(\dot{\alpha}\cos\beta + \dot{\gamma})}_{I_3\omega_3 = L_3 = p_\gamma} \cos\beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2\beta + p_\gamma \cos\beta \iff \dot{\alpha} = \frac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}$$

$$p_\gamma = I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) \iff \dot{\gamma} = \frac{p_\gamma}{I_3} - \frac{(p_\alpha - p_\gamma\cos\beta)\cos\beta}{I_1 \sin^2\beta}$$

$\dot{\alpha}$ and $\dot{\gamma}$ both only depend on constant $\to$ $\dot{\alpha}$ and $\dot{\gamma}$ constant

$$L = \frac{1}{2} I_1 \Big(\underset{\overset{\shortparallel}{\frac{(p_\alpha - p_\gamma\cos\beta)^2}{I_1^2 \sin^2\beta}}}{\dot{\alpha}^2} \sin^2\beta + \underset{\overset{\shortparallel}{\frac{p_\beta^2}{I_1^2}}}{\dot{\beta}^2} \Big) + \frac{1}{2} I_3 \Big(\underbrace{\dot{\alpha}\cos\beta + \dot{\gamma}}_{\frac{p_\gamma^2}{I_3^2}} \Big)^2$$

$\Downarrow$

$$H = \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2 I_1 \sin^2\beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3}$$

$\Rightarrow$ same conclusions: α, γ cyclic

Since $\bar{L}$ is constant we can pick $\hat{e}_3^\circ$ in the inertial frame in the direction of $\bar{L}$

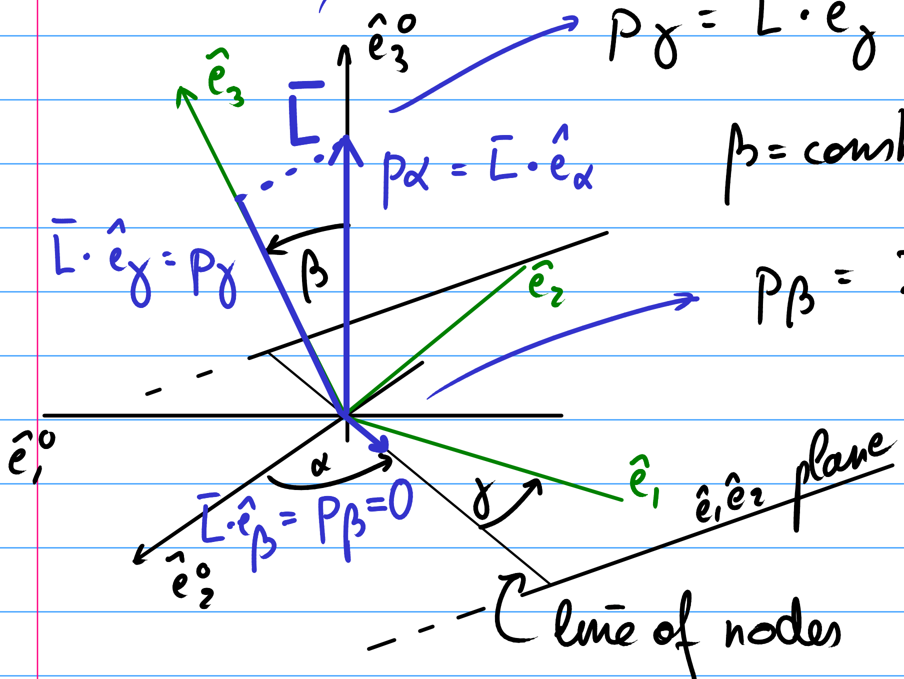
$$p_\alpha = \bar{L} \cdot \hat{e}_\alpha = |L| = \text{constant}$$

$$p_\gamma = \bar{L} \cdot \hat{e}_\gamma = \bar{L} \cdot \hat{e}_3 = |L| \cos\beta = \text{constant}$$

$$\Downarrow$$

$\beta = \text{constant}$ and $\dot{\beta} = 0$

$$p_\beta = I_1 \dot{\beta} = 0$$



Return to equation of motion for β:

$$I_1 \ddot{\beta} = I_1 \dot{\alpha}^2 \sin\beta \cos\beta - \underbrace{I_3 (\dot{\alpha} \cos\beta + \dot{\gamma})}_{= p_\gamma = I_3 \omega_3 = \text{constant}} \dot{\alpha} \sin\beta = 0$$

$$\Leftrightarrow \quad \dot{\alpha} \cos\beta = \frac{p_\gamma}{I_1} = \frac{I_3}{I_1} \omega_3$$

$$\Rightarrow \quad \dot{\gamma} = \frac{p_\gamma}{I_3} - \dot{\alpha} \cos\beta = \omega_3 - \frac{I_3}{I_1} \omega_3 = \omega_3 \frac{I_1 - I_3}{I_1}$$

$$\Rightarrow \quad \dot{\gamma} = -\Omega$$

with Ω defined before as

$$\Omega = \omega_3 \frac{I_3 - I_1}{I_1}$$

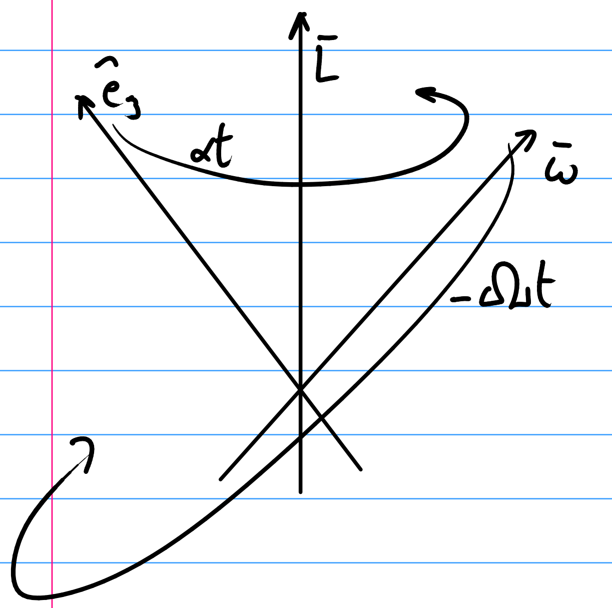
$\bar{L}$ principal axis

$$\bar{\omega} = \dot{\alpha}\,\hat{e}_3^\circ + \dot{\gamma}\,\hat{e}_3 \rightarrow \bar{\omega} \text{ is in } \hat{e}_3^\circ\,\hat{e}_3 \text{ plane}$$

$$I_3 > I_1 \Rightarrow \dot{\gamma} = -\Omega < 0$$

$\Downarrow$

$\bar{L}$ between $\bar{\omega}$ and $\hat{e}_3$



$$\dot{\alpha}\cos\beta = \frac{I_3}{I_1}\omega_3$$