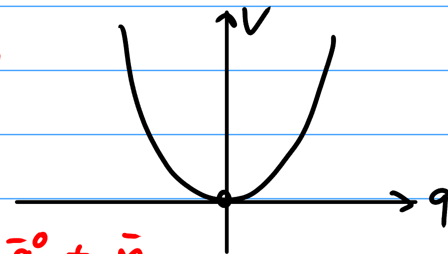
Classical Mechanics (Physics 601) - October 13, 2011

Reminder: problem was to find the behavior around the equilibrium points $\rightarrow$ linearization $\rightarrow$ frequencies

1) Equilibrium $\bar{q}_0$ with $\left(\frac{\partial V}{\partial q_i}\right)\Big|_{\bar{q}^0} = 0$



2) Linearization around $\bar{q}$: $\bar{q} = \bar{q}^0 + \bar{\eta}$

↳ keep everything up to quadratic terms in the Lagrange eqns

$$\Rightarrow \quad T = \frac{1}{2}\sum_i\sum_j m_{ij}\,\dot{\eta}_i\,\dot{\eta}_j = \frac{1}{2}\dot{\bar{\eta}}^T M \dot{\bar{\eta}}$$

with mass matrix M : $m_{ij} = \sum_k m_k \left(\frac{\partial x_k}{\partial q_i}\right)\left(\frac{\partial x_k}{\partial q_j}\right)\Big|_{q^0}$ (real and symmetric) $M^T = M$

$$\Rightarrow V = V(\bar{q}^0) + \cancel{\sum_i \eta_i \left(\frac{\partial V}{\partial q_i}\right)\Big|_{q^0}} + \frac{1}{2}\sum_i\sum_j \eta_i\,\eta_j \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)\Big|_{q^0}$$

$$= V(\bar{q}^0) + \bar{\eta}^T V \bar{\eta}$$

with potential matrix V : $v_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$ (real and symmetric) $V^T = V$

$$\Rightarrow L = \frac{1}{2}\dot{\bar{\eta}}^T M \dot{\bar{\eta}} - \frac{1}{2}\bar{\eta}^T V \bar{\eta} - V(\bar{q}^0)$$

Euler-Lagrange equations:

$$M\ddot{\bar{\eta}} + V\bar{\eta} = 0$$

linear, homogeneous
(n coupled differential equations)

3) Exponential solutions $\bar{\eta} = \mathrm{Re}\,\bar{z}$ with $\bar{z} = \bar{z}^0 e^{i\omega t}$

$\Rightarrow V\bar{z}^0 - \omega^2 M \bar{z}^0 = 0$

$\longrightarrow$ generalized eigenvalue problem

Find all eigenvalues $\omega_i^2 \rightarrow$ find eigenvectors z_i

General solution is now : $\eta = \sum_i z_i \underbrace{\left(c_i e^{+i\omega_i t} + c_i^* e^{-i\omega_i t}\right)}_{\text{real}}$

$\Downarrow$

$\eta = \sum_i z_i \rho_i \cos(\omega_i t + \varphi_i)$

* Generalized eigenvalue problems:

1) M is positive-definite if $\forall \bar{x} \neq 0,\ \bar{x}^T M \bar{x} > 0$
and $\bar{x}^T M \bar{x} = 0 \Leftrightarrow \bar{x} = 0$

Here: $T = \frac{1}{2} \dot{\bar{\eta}}^T M \dot{\bar{\eta}}$ is always positive, and only zero when there is no motion

→ positive-definite matrices are like positive numbers

2) If M is positive-definite, then there exists an orthogonal matrix U

$$\exists U, U^T U = \mathbb{1} : U^T M U = \Lambda_{diag}$$

↳ diagonalization with $\Lambda_i = \lambda_i$ eigenvalues on diagonal

For a symmetric matrix these eigenvalues are real.
For a positive-definite matrix the eigenvalues are positive.

3) Cholesky decomposition of positive-definite matrix M:

$$M = U \Lambda U^T = U \Lambda^{1/2} \Lambda^{1/2} U^T = U \Lambda^{1/2} \underbrace{\Omega^T \Omega}_{\text{orthogonal}} \Lambda^{1/2} U^T$$

$$\Rightarrow M = C^T C \quad \text{with } C = \Omega \Lambda^{1/2} U^T$$

$$C^{-1} = (\Omega \Lambda^{1/2} U^T)^{-1} = U \Lambda^{-1/2} \Omega^T \quad \text{with } \Lambda^{-1/2} = \text{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right)$$

Now rewrite: $Vz = \omega^2 M z = \omega^2 C^T \underbrace{C z}_{\tilde{z}} \rightarrow z = C^{-1}\tilde{z}$

$\Rightarrow (C^T)^{-1} V C^{-1} \tilde{z} = \omega^2 \tilde{z}$

$\hookrightarrow$ regular eigenvalue problem with eigenvectors $\tilde{z}_i$

$\tilde{z}_i^T \tilde{z}_j = \delta_{ij} \Rightarrow z_i^T C^T C z_j = z_i^T M z_j = \delta_{ij}$

generalized dot product of z_i and z_j with matrix M (positive definite)

$\Rightarrow$ There exist a general way to solve generalized eigenvalue problems, but often not needed

just evaluate $\det(V - \omega^2 M) = 0 \rightarrow$ normal mode frequencies $\omega_i \rightarrow$ eigenvectors z_i

Define $U = [z_1 \ \cdots \ z_n]$

$\hookrightarrow U^T V U = U^T [\omega_1^2 M z_1 \ \cdots \ \omega_n^2 M z_n]$

$= [z_i^T \omega_j^2 M z_j] = [\omega_i^2 \delta_{ij}] = \begin{bmatrix} \ddots \end{bmatrix} = \Omega$

$U^{-1} = U^T M \quad : \quad U^T M U = [z_i^T M z_j] = \delta_{ij} = \mathbb{1}$

* ## Normal coordinates

Is there a coordinate transformation that will make the equations of motion uncoupled ?

Remember $L = \frac{1}{2}\dot{\eta}^T M \dot{\eta} - \frac{1}{2}\eta^T V \eta$

Introduce ξ such that $\eta = U\xi$

$\hookrightarrow$ $L = \frac{1}{2}\dot{\xi}\underbrace{U^T M U}_{\mathbb{1}}\dot{\xi} - \frac{1}{2}\xi^T \underbrace{U^T V U}_{\Omega = \omega_i^2 \delta_{ij}}\xi$

$\Rightarrow$ $\ddot{\xi}_i + \omega_i^2 \xi_i = 0$ for all $i = 1, \dots, n$

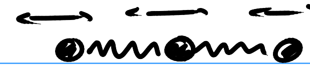
$\hookrightarrow$ normal coordinates, with normal frequencies

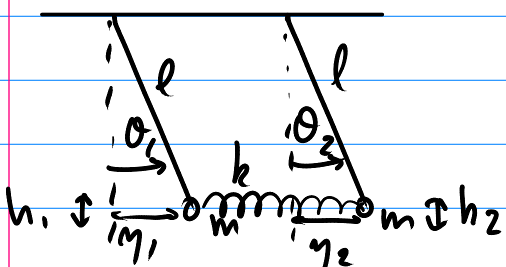
Transformation back to η:

$$\eta = U\xi \rightarrow U^T M \eta = U^T M U \xi = \xi$$

$$\Rightarrow \xi = U^T M \eta$$

* Example: double coupled pendulum

↳ general model for many systems : 



$$V = mgh_1 + mgh_2 + \frac{1}{2}k(\eta_1-\eta_2)^2$$

$$\Downarrow$$

$$V = -mg\ell\cos\theta_1 - mg\ell\cos\theta_2 + \frac{1}{2}k\ell^2(\sin\theta_1 - \sin\theta_2)^2$$

$h_1 = -\ell\cos\theta_1 \qquad \eta_1 = \ell\sin\theta_1$

$h_2 = -\ell\cos\theta_2 \qquad \eta_2 = \ell\sin\theta_2$

Equilibrium at $\theta_1 = \theta_2 = 0$ → define property of spring such that this is satisfied

$$\Rightarrow L = \frac{1}{2}m\ell^2(\dot\theta_1^2 + \dot\theta_2^2) + mg\ell(\cos\theta_1 + \cos\theta_2) - \frac{1}{2}k\ell^2(\sin\theta_1 - \sin\theta_2)^2$$

Determine matrix V :

$$\frac{\partial V}{\partial\theta_1} = k\ell^2(\sin\theta_1 - \sin\theta_2)\cos\theta_1 + mg\ell\sin\theta_1$$

$$\frac{\partial V}{\partial\theta_2} = -k\ell^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2 + mg\ell\sin\theta_2$$

$$\frac{\partial^2 V}{\partial\theta_1^2} = k\ell^2\cos^2\theta_1 - k\ell^2(\sin\theta_1 - \sin\theta_2)\sin\theta_1\cos\theta_1 + mg\ell\cos\theta_1$$

$$\frac{\partial^2 V}{\partial\theta_2^2} = k\ell^2\cos^2\theta_2 + k\ell^2(\sin\theta_1 - \sin\theta_2)\sin\theta_2\cos\theta_2 + mg\ell\cos\theta_2$$

$$\frac{\partial^2 V}{\partial\theta_1\partial\theta_2} = -k\ell^2\cos\theta_1\cos\theta_2$$

↳ evaluate all these at $\theta_1 = \theta_2 = 0$

$$V = \begin{pmatrix} kl^2 + mgl & -kl^2 \\ -kl^2 & kl^2 + mgl \end{pmatrix} = l^2 \begin{pmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{pmatrix}$$

$$M = ml^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Determine eigenvalues:

scaling of determinant $\rightarrow a^{\#\text{rows}}$

$$\det(V - \omega^2 M) = 0 \iff (ml^2)^2 \begin{vmatrix} \left(\frac{k}{m} + \frac{g}{l}\right) - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \left(\frac{k}{m} + \frac{g}{l}\right) - \omega^2 \end{vmatrix} = 0$$

$$\iff \left(\frac{k}{m} + \frac{g}{l} - \omega^2\right)^2 = \left(\frac{k}{m}\right)^2$$

$$\iff \omega^2 = \left(\frac{k}{m} + \frac{g}{l}\right) \pm \frac{k}{m} = \begin{cases} \omega_1 = \sqrt{\frac{g}{l}} \\ \omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}} \end{cases}$$

$\Downarrow$

normal mode frequencies ω_1, ω_2 $\quad$ ↳ only consider positive eigenvalues

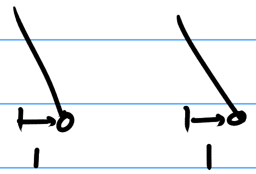
Determine eigenvectors:

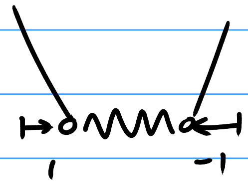
$$(V - \omega^2 M) z = 0 \Rightarrow \omega_1 : \frac{k}{m} z_1 - \frac{k}{m} z_2 = 0 \Rightarrow z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_2 : -\frac{k}{m} z_1 - \frac{k}{m} z_2 = 0 \Rightarrow z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Indeed $z_1 \perp z_2$

Eigenvectors indicate how normal modes oscillate:

ω_1 : 

ω_2 : 

no effect from spring
$\rightarrow$ ω_1 does not depend on k