Classical Mechanics (Phys 601) - October 27, 2011

Continuous systems:

* scalar field $u(x,t)$ in one dimension x

$\hookrightarrow$ $$L = \int_0^\ell dx\, \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, x, t\right)$$

Examples: transverse position, longitudinal pressure in air

* scalar field $\varphi(x^\mu)$, $\mu = 0, \dots, D$ in D dimensions with $x^0 = ct$

$\hookrightarrow$ $$L = \int d^D x\, \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$$

Raising/lowering of indices: $a_\mu = \sum_{\nu=0}^{D} g_{\mu\nu} a^\nu$, $g = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

Summation convention: $a^\mu b_\mu = \sum_{\mu=0}^{D} a^\mu b_\mu \Rightarrow a_\mu = g_{\mu\nu} a^\nu$

Derivatives: $v_{\mu,\nu} = \frac{\partial}{\partial x^\nu} v_\mu$, $v_{\mu,\kappa\lambda} = \frac{\partial}{\partial x^\kappa}\frac{\partial}{\partial x^\lambda} v_\mu$

$\Rightarrow$ continuous Euler-Lagrange equations:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

note: this is vector norm!

Example: $\mathcal{L} = \dot{\varphi}^2 - (\bar{\nabla}\varphi)^2$ $\left(\text{compare: } \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2\right)$

$$\Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0 \Leftrightarrow (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\varphi = 0 \Leftrightarrow \Box^2 \varphi = 0$$

$$\Leftrightarrow \ddot{\varphi} = \nabla^2 \varphi \quad \text{(Laplace equation)}$$

* Multi-dimensional Hamiltonian for continuous systems:

String with transverse oscillations:

$$L = \frac{1}{2}\frac{m}{a}\sum_{i=1}^{N} a\dot{\mu}_i^2 - \frac{1}{2}\tau\sum_{i=0}^{N} a\left(\frac{\mu_{i+1}-\mu_i}{a}\right)^2$$

$$= \sum_i^N a\left[\frac{1}{2}\frac{m}{a}\dot{\mu}_i^2 - \frac{1}{2}\tau\left(\frac{\mu_{i+1}-\mu_i}{a}\right)^2\right] = \sum_i a L_i$$

$$p_i = \frac{\partial L}{\partial \dot{\mu}_i} = a\frac{\partial L_i}{\partial \dot{\mu}_i}$$

$$H = \sum_i^N p_i\dot{\mu}_i - L = \sum_i^N a\left(\dot{\mu}_i\frac{\partial L_i}{\partial \dot{\mu}_i} - L_i\right)$$

↳ continuum limit : $\sum_i^N a \rightarrow \int dx$

$$H = \int_0^l dx\left(\dot{\mu}\frac{\partial \mathcal{L}}{\partial \dot{\mu}} - \mathcal{L}\right) = \int_0^l dx\,\mathcal{H}$$

$$\mathcal{H} = \dot{\mu}\frac{\partial \mathcal{L}}{\partial \dot{\mu}} - \mathcal{L} \quad = \text{Hamiltonian density}$$

$$= \dot{\mu}\,\pi - \mathcal{L} \quad \text{with } \pi = \frac{\partial \mathcal{L}}{\partial \dot{\mu}} = \text{momentum density}$$

Not as useful as for discrete system: Lagrangian mechanism treats time and space coordinates equivalently, in Hamiltonian formalism this is lost again

* 1-forms, metrics, tensors, and all that

If $v^\mu(v^0, v^1, v^2, v^3)$ is a vector (= four-vector)
then $v_\mu = g_{\mu\nu} v^\nu$ is a 1-form or covariant vector.

$$\left\{\begin{array}{l} v^\mu : \text{contravariant vector} \\ v_\mu : \text{covariant vector} \end{array}\right. \text{ metric tensor } g_{\mu\nu}$$

Dot product $u \cdot v = u^\mu v_\mu = u_\mu v^\mu = u^\mu v^\nu g_{\mu\nu}$

In Minkowski space (special relativity):

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \Rightarrow \left\{\begin{array}{l} v_\mu(v^0, -v^1, -v^2, -v^3) \\ u \cdot v = u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3 \end{array}\right.$$

In Euclidean space : $g_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \Rightarrow \left\{\begin{array}{l} v_\mu(v^1, v^2, v^3) \\ u \cdot v = u^1 v^1 + u^2 v^2 + u^3 v^3 \end{array}\right.$

From original vector space, transform to dual vector space.

vector basis: $\hat{e}_1, \hat{e}_2, \ldots$ 1-form basis : $\hat{\omega}^1, \hat{\omega}^2, \ldots$

such that $\hat{e}_\mu \cdot \hat{\omega}^\nu = \delta_\mu^\nu$ is orthonormal

$\Rightarrow$ $v = v^\mu \hat{e}_\mu = v_\mu \hat{\omega}^\mu$: two descriptions of one object

Examples of covariant vectors : $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{1}{c}\dfrac{\partial}{\partial t}, \bar{\nabla}\right)$ = gradient

$\Box = \partial_\mu \hat{\omega}^\mu$ (d'Alembertian)

$j_\mu = (\rho c, -\bar{J})$ = charge-current

	metric	contravariant vector	covariant 1-form
D = 3 Euclidean :	$g_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	(dx, dy, dz)	(dx, dy, dz)

$\longrightarrow dx^2 + dy^2 + dz^2$

Spherical : $g_{ij} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{bmatrix}$ $(dr, d\varphi, d\theta)$ $(dr, r^2 \sin^2\theta\, d\varphi, r^2 d\theta)$

$\longrightarrow dr^2 + r^2 \sin^2\theta\, d\varphi^2 + r^2 d\theta^2$

Quantum : $|i\rangle\langle j|$ $\quad |i\rangle \quad \langle j|$

basis states

contraction $\langle j|i\rangle$

D = 4

Minkowski : $g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ $(c\,dt, d\bar{r})$ $(c\,dt, -d\bar{r})$

$\longrightarrow c^2 dt^2 - dr^2$

$\left.\begin{array}{l}\text{If we have vector } V\ (V^1, V^2, V^3) \\ \text{and its 1-form } v\ (v_1, v_2, v_3)\end{array}\right\} \Rightarrow$ the magnitude is now always $|V|^2 = V^1 v_1 + V^2 v_2 + V^3 v_3$

$\Rightarrow$ dot product as operation on element in vector space V and element in dual vector space $\tilde{V}$

Tensors are general extension of vectors and 1-forms:

examples: $g_{\mu\nu}, T^{\alpha\beta}$: tensors of rank $\binom{2}{0}$

$S^{\alpha\beta}{}_{\gamma}$: tensor of rank $\binom{2}{1}$

$\Rightarrow$ tensor of rank $\binom{n}{p}$ defines an operator on p vectors and n 1-forms:

$$T, \text{rank} \binom{n}{p}: \quad V^p \times \tilde{V}^n \to \mathbb{R}$$

Example: $g_{\mu\nu}$ of rank $\binom{0}{2}$ is operator on 2 vectors:

$$g_{\mu\nu}: \quad V \times V \to \mathbb{R} \quad \equiv \text{dot product}$$

Tensor product: $T = u \otimes v$ is tensor with $T^{\mu\nu} = u^\mu v^\nu$

Wedge product: $T = u \wedge v$ is tensor with $T^{\mu\nu} = u^\mu v^\nu - u^\nu v^\mu$

Tensors and vector fields $\rightarrow \quad \mathcal{L}(\varphi^\alpha, \partial_\beta \varphi^\alpha)$

Need to introduce different derivative: covariant derivative

$$\text{vector}: \quad \partial_\beta V = \partial_\beta (V^\alpha \hat{e}_\alpha) = (\partial_\beta V^\alpha)\hat{e}_\alpha + V^\alpha \underbrace{(\partial_\beta \hat{e}_\alpha)}_{\Gamma^\gamma_{\alpha\beta} \hat{e}_\gamma}$$

$$= (\partial_\beta V^\alpha)\hat{e}_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta} \hat{e}_\gamma$$

Christoffel symbol

$$= (\partial_\beta V^\alpha)\hat{e}_\alpha + (V^\gamma \Gamma^\alpha_{\gamma\beta})\hat{e}_\alpha$$

$$= (\partial_\beta V^\alpha + \Gamma^\alpha_{\gamma\beta} V^\gamma)\hat{e}_\alpha = (D_\beta V^\alpha)\hat{e}_\alpha$$

$\Rightarrow$ covariant derivative D_β

In Euclidean space: $\Gamma^{\alpha}_{\gamma\beta} = 0 \Rightarrow D_\beta \equiv \partial_\beta$

For vector and tensor fields $\mathcal{L}(\psi^\alpha, \partial_\beta \psi^\alpha)$

↳ Euler-Lagrange equation uses this covariant derivative:

$$D_\mu \frac{\partial \mathcal{L}}{\partial(D_\mu \psi^\alpha)} - \frac{\partial \mathcal{L}}{\partial \psi^\alpha} = 0 \Bigg\} \Rightarrow \text{multiple equations of motion}$$

Most of what we talked about/will talk about is valid for vector and tensor field with $\partial_\mu \to D_\mu$

Example: electromagnetic tensor $F^{\alpha\beta}$

scalar and vector potential: $A = \left(\frac{\varphi}{c}, \bar{A}\right)$

$$\Box \cdot A = \partial_\mu A^\mu = \mu_0 \varepsilon_0 \frac{\partial \varphi}{\partial t} + \bar{\nabla} \cdot \bar{A} = 0$$

$$\Box^2 A = \partial_\mu \partial^\mu A = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) A = 0$$

$$\left\{\begin{array}{l} \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\varepsilon_0} \\ \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} = \mu_0 \bar{j} \end{array}\right. \qquad \left(\mu_0 \varepsilon_0 = \frac{1}{c^2}\right)$$

source terms $\mu_0 j = (\mu_0 \rho c, \mu_0 \bar{j})$

Lorentz force: $\frac{d\bar{p}}{dt} = e(\bar{E} + \bar{v} \times \bar{B})$

↳ can be written in tensor notation as
$\dot{p}^\mu = e F^\mu{}_\nu u^\nu$ with u^ν the four-velocity

$$F_{\mu\nu} = \frac{1}{c}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & +cB_z & -cB_y \\ E_y & -cB_z & 0 & +cB_x \\ E_z & +cB_y & -cB_x & 0 \end{pmatrix} = \partial_\mu A_\nu - \partial_\nu A_\mu = \Box \wedge A$$

F = electromagnetic field tensor

Lagrangian density with F becomes:

$$\mathcal{L} = \frac{1}{8\mu_0} F_{\mu\nu} F^{\mu\nu}$$

Euler-Lagrange equation for electromagnetic now become

$$D_\mu F^{\mu\nu} = 0 \xrightarrow{\text{Euclidean space}} \partial_\mu F^{\mu\nu} \rightarrow 4 \text{ equations}$$

$\Rightarrow$ Maxwell equations : $\begin{cases} \bar{\nabla} \cdot \bar{E} = 0 , & \bar{\nabla} \cdot \bar{B} = 0 \\ \bar{\nabla} \times \bar{E} = -\dfrac{\partial \bar{B}}{\partial t} , & \bar{\nabla} \times \bar{B} = \dfrac{1}{c^2} \dfrac{\partial \bar{E}}{\partial t} \end{cases}$

* ## Noether theorem for continuous fields

Remember that for discrete systems we constructed a conserved quantity for each continuous symmetry of the Lagrangian:

if $q_i(s,t)$ with $q_i(0,t) = q_i(t)$ leads to a Lagrangian cyclic in s: $\frac{\partial L}{\partial s} = 0$
then there exists a conserved quantity:

$$C_s = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s} \right) \Big|_{s=0}$$

↳ example: if L is cyclic in $s = x$, then $C_s = p_x$

$\Rightarrow$ conserved "charge" for each continuous symmetry in discrete systems.

Now, for continuous systems, there will be a conserved current density for each continuous symmetry:

$$\frac{\partial \mathcal{L}}{\partial s} = \frac{\partial \mathcal{L}}{\partial \varphi} \frac{\partial \varphi}{\partial s} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial(\partial_\mu \varphi)}{\partial s} = 0$$

$$= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \varphi}{\partial s} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu \frac{\partial \varphi}{\partial s} = 0$$

$$\Leftrightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \varphi}{\partial s} \right) = 0$$

$$\Leftrightarrow \quad j_s^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \varphi}{\partial s}\Big|_{s=0} = \pi^\mu \frac{\partial \varphi}{\partial s}\Big|_{s=0}$$

Example for Klein-Gordon complex scalar field:

$$\mathcal{L} = c^2 (\partial_\mu \varphi)(\partial^\mu \varphi^*) - m_0^2 c^2 \varphi \varphi^*$$

$$= \dot{\varphi}\dot{\varphi}^* - c^2 (\bar{\nabla}\varphi)\cdot(\bar{\nabla}\varphi^*) - m_0^2 c^2 \varphi\varphi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = c^2 \partial^\mu \varphi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial \varphi} = -m_0^2 c^2 \varphi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} = c^2 \partial^\mu \varphi \quad , \quad \frac{\partial \mathcal{L}}{\partial \varphi^*} = -m_0^2 c^2 \varphi$$

$\Rightarrow$ Lagrange equation: $\partial_\mu \partial^\mu \varphi + m_0^2 c^2 \varphi = 0$

$\mathcal{L}$ is invariant under $\varphi \to \varphi e^{is}$, $\varphi^* \to \varphi^* e^{-is}$

$$\Rightarrow j^\mu_s = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}}_{c^2 \partial^\mu \varphi^*} \underbrace{\left.\frac{\partial \varphi}{\partial s}\right|_{s=0}}_{i\varphi} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)}}_{c^2 \partial^\mu \varphi} \underbrace{\left.\frac{\partial \varphi^*}{\partial s}\right|_{s=0}}_{-i\varphi^*}$$

$$= ic^2 \left(\varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi\right)$$