Classical Mechanics (Phys 601) - September 22, 2011

* Particle in an electromagnetic field :

Remember that electromagnetic field can be described using a velocity-dependent potential $U(\dot{q},t)$

$$U = e\phi(\bar{r},t) - \frac{e}{c}\,\dot{\bar{r}}\cdot\bar{A}(\bar{r},t)$$

with $\bar{A}(\bar{r},t)$ the vector potential :

$$\begin{cases} \bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \bar{B} = \bar{\nabla}\times\bar{A} \\ \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} \Rightarrow \bar{\nabla}\times\left(\bar{E} + \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right) = 0 \end{cases}$$

$$\Rightarrow \bar{E} + \frac{1}{c}\frac{\partial\bar{A}}{\partial t} = -\bar{\nabla}\phi \Leftrightarrow \bar{E} = -\bar{\nabla}\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}$$

Lagrangian :

$$L = \frac{1}{2}m\dot{\bar{r}}^2 - e\phi + \frac{e}{c}\dot{\bar{r}}\cdot\bar{A}$$

$$p_i = \frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i$$

$$\dot{p}_i = \frac{d}{dt}\left(m\dot{x}_i + \frac{e}{c}A_i\right) = m\ddot{x}_i + \frac{e}{c}\frac{\partial A_i}{\partial t} + \frac{e}{c}\sum_j\frac{\partial A_i}{\partial x_j}\dot{x}_j$$

$$\dot{x}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right) \Rightarrow \dot{\bar{r}} = \frac{1}{m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)$$

$$\frac{\partial L}{\partial x_i} = -e\frac{\partial\phi}{\partial x_i} + \frac{e}{c}\sum_j\dot{x}_j\frac{\partial A_j}{\partial x_i}$$

Lagrange equation :

$$m\ddot{x}_i + \underbrace{\frac{e}{c}\frac{\partial A_i}{\partial t}} + \underbrace{\frac{e}{c}\sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j} + \underbrace{e\frac{\partial \phi}{\partial x_i}} - \underbrace{\frac{e}{c}\sum_j \frac{\partial A_j}{\partial x_i}\dot{x}_j} = 0$$

$$\Leftrightarrow \quad m\ddot{x}_i - eE_i - \frac{e}{c}\left[\dot{x} \times (\bar{\nabla} \times \bar{A})\right]_i = 0$$

$$\Leftrightarrow \quad m\ddot{\bar{r}} = e\left(\bar{E} + \frac{1}{c}\dot{\bar{r}} \times \bar{B}\right) \qquad \text{(Lorentz force)}$$

Hamiltonian :

$$H(\bar{r}, \dot{\bar{r}}, t) = \bar{p} \cdot \dot{\bar{r}} - L$$

$$= \left(m\dot{\bar{r}} + \cancel{\frac{e}{c}\bar{A}}\right) \cdot \dot{\bar{r}} - \frac{1}{2}m\dot{\bar{r}}^2 + e\phi - \cancel{\frac{e}{c}\dot{\bar{r}}} \cdot \bar{A}$$

$$= \frac{1}{2}m\dot{\bar{r}}^2 + e\phi(\bar{r}, t) \quad = \text{total energy (kinetic + potential)}$$

$$H(\bar{r}, \bar{p}, t) = \bar{p} \cdot \dot{\bar{r}} - \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2 + e\phi - \frac{e}{c}\frac{1}{m}\left(\bar{p} - \frac{e}{c}\bar{A}\right) \cdot \bar{A}$$

$$= \bar{p} \cdot \frac{1}{m}\left(\bar{p} - \frac{e}{c}\bar{A}\right) - \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2 + e\phi - \frac{e}{c}\frac{1}{m}\left(\bar{p} - \frac{e}{c}\bar{A}\right) \cdot \bar{A}$$

$$= \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2 + e\phi$$

$\Rightarrow$ Hamilton's equations

$$\dot{\bar{p}} = -\frac{\partial H}{\partial \bar{r}}, \quad \dot{\bar{r}} = \frac{\partial H}{\partial \bar{p}}$$

$\Rightarrow$ $\dot{\bar{p}}$ will lead to same Lorentz equation

* Poisson Brackets :

Consider function $F(q_i, p_i, t)$

↳ total time-derivative :

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) + \frac{\partial F}{\partial t}$$

But we have Hamilton's equations :

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

$$\Rightarrow \frac{dF}{dt} = \sum_i \underbrace{\left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right)}_{\text{new notation: } [F, H]} + \frac{\partial F}{\partial t}$$

Poisson Bracket $[F, G] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$

$\Rightarrow \dfrac{dF}{dt} = [F, H] + \dfrac{\partial F}{\partial t}$ ⇒ Poisson bracket of F with Hamiltonian describes time-evolution of F !

$$\Rightarrow \begin{cases} \dot{p}_i = \dfrac{dp_i}{dt} = [p_i, H] \\ \dot{q}_i = \dfrac{dq_i}{dt} = [q_i, H] \end{cases}$$

⇒ new formulation of Hamilton's equations

* Properties of Poisson brackets :

* $[aA + bB, C] = a[A,C] + b[B,C]$ (linearity)

* $[AB, C] = [A,C]B + A[B,C]$ (distributivity)

* $[A,B] = -[B,A]$ (anti-symmetry)

non-commutative

* $[A,A] = 0$

* Jacobi's identity : $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

From the definition $[F,G] = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right)$:

* $$[p_i, q_j] = \sum_k \left(\cancel{\frac{\partial p_i}{\partial q_k}\frac{\partial q_j}{\partial p_k}} - \frac{\partial q_j}{\partial q_k}\frac{\partial p_i}{\partial p_k} \right)$$
$$= -\sum_k \delta_{kj}\delta_{ki} = -\delta_{ij}$$

* $$[p_i, p_j] = \sum_k \left(\cancel{\frac{\partial p_i}{\partial q_k}}\frac{\partial p_j}{\partial p_k} - \cancel{\frac{\partial p_j}{\partial q_k}}\frac{\partial p_i}{\partial p_k} \right) = 0 = [q_i, q_j]$$

Poisson brackets satisfy $[u_i, u_j] = \sum_k c_{ij}^k u_k$ for $u = \{q_i, p_i\}$

basis

$\Rightarrow$ they form a non-commutative Lie algebra.

Other examples of Lie algebra : vector product $\bar{A} \times \bar{B}$, matrices

Because $[H,H]=0 \Rightarrow \dfrac{dH}{dt} = [H,H] + \dfrac{\partial H}{\partial t} = \dfrac{\partial H}{\partial t}$,
as found earlier by explicit evaluation

* If u is a constant of motion : $\dfrac{du}{dt} = 0 = [u,H] + \dfrac{\partial u}{\partial t}$

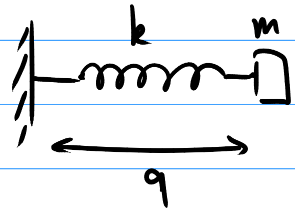
$\Rightarrow$ if u also does not depend explicitly on $t \rightarrow [u,H]=0$

* If u and v are constants of motion $\Rightarrow$ Jacobi's identity :

$$[u,\cancel{[v,H]}] + [v,\cancel{[H,u]}] + [H,[u,v]] = 0$$

$\Rightarrow [H,[u,v]] = 0 \Rightarrow [u,v]$ is a constant of motion.

* Linear harmonic oscillator :



$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 \quad \text{with } \omega = \sqrt{\frac{k}{m}}$$

$$p = \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$

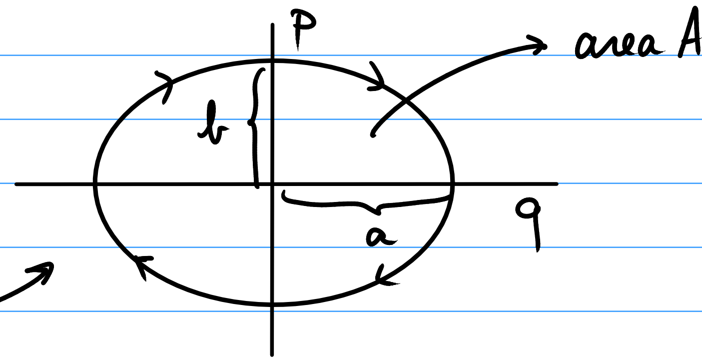
$\Rightarrow H = \dfrac{p^2}{2m} + \dfrac{1}{2} m \omega^2 q^2 \longrightarrow \dfrac{\partial H}{\partial t} = 0 \Rightarrow H = E =$ total energy

Hamilton's equations :

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \dfrac{1}{m} p & \left(\text{equivalent with } p = \dfrac{\partial L}{\partial \dot{q}}\right) \\ \dot{p} = -\dfrac{\partial H}{\partial q} = m\omega^2 q & \rightarrow \text{take time-derivative and use } \dot{q} \text{ above} \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{p} + \omega^2 p = 0 \\ \ddot{q} + \omega^2 q = 0 \end{cases}$$

$$\Rightarrow \begin{cases} q = a \sin(\omega t + \alpha) \\ p = b \cos(\omega t + \alpha) \end{cases}$$



with $a = \sqrt{\dfrac{2E}{m\omega^2}}$, $b = \sqrt{2mE}$

Area A in phase space : $A = \oint p\,dq = ab \int_0^{2\pi} \cos^2\theta \, d\theta = \underbrace{\pi a b}_{\text{area ellipse}}$

$$A = \pi \sqrt{\frac{2E}{m\omega^2}} \sqrt{2mE} = 2\pi \frac{E}{\omega}$$

(We will come back to this in context of canonical transformations)

* Transition to Quantum Mechanics : Sommerfeld-Wilson

Sommerfeld-Wilson quantization (naive quantization)

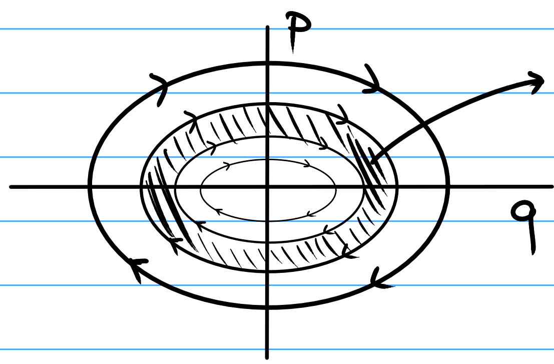
action variable $J_i = \oint p_i dq_i = n h$, $n = 1, 2, \ldots$

↳ integration over one period in phase space

For harmonic oscillator : $J = \oint p\,dq = 2\pi \dfrac{E}{\omega} = n h$

$$\Rightarrow E = n \frac{h}{2\pi} \omega = n \hbar \omega$$

$\Rightarrow$



each band has area $\hbar\omega$

↳ band between allowed orbits in phase space.

For central forces $V(r)$

$$\Rightarrow L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2 + r^2 \dot{\theta}^2 \right) - V(r)$$

$\frac{\partial L}{\partial \varphi} = 0 \rightarrow p_\varphi$ conserved $\rightarrow$ angular momentum l_z

$$J_\varphi = \int_0^{2\pi} p_\varphi \, d\varphi = 2\pi \, p_\varphi = 2\pi \, l_z$$

Sommerfeld - Wilson quantization : $J_\varphi = n h$

$$\Rightarrow \quad l_z = n \hbar$$

But: problems with Sommerfeld - Wilson quantization :
we know that $l_z = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ (central forces)

and that $E = \frac{1}{2}\hbar\omega, \frac{3}{2}\hbar\omega, \frac{5}{2}\hbar\omega, \ldots$ (HO)

* Transition to Quantum Mechanics : Canonical Quantization

Canonical quantization :

Replace $[A,B]$ with $\frac{1}{i\hbar}[\hat{A},\hat{B}]$

Poisson bracket   commutator $\hat{A}\hat{B} - \hat{B}\hat{A}$

Now : $\hat{A}$ and $\hat{B}$ are operators working on vectors in Hilbert spaces .

Relations between p_i and q_i:

$$[\hat{p}_i, \hat{p}_j] = 0 = [\hat{q}_i, \hat{q}_j] \quad , \quad \frac{1}{i\hbar}[\hat{p}_i, \hat{q}_j] = -\delta_{ij}$$

$$\hookrightarrow [\hat{p}_i, \hat{q}_j] = \frac{\hbar}{i}\delta_{ij}$$

But all results are transposed:

$$\frac{d\hat{F}}{dt} = \frac{1}{i\hbar}[\hat{F}, \hat{H}] + \frac{\partial \hat{F}}{\partial t}$$

Example: quantum-mechanical wave function $\psi(q_i)$

Define $\hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial q_i} \Rightarrow [\hat{p}_i, \hat{q}_j] = \frac{\hbar}{i}\frac{\partial}{\partial q_i} q_j - \frac{\hbar}{i} q_j \frac{\partial}{\partial q_i}$

remember: operators! $\longrightarrow$ $= \frac{\hbar}{i}\left(q_j \frac{\partial}{\partial q_i} + \delta_{ij}\right) - \frac{\hbar}{i} q_j \frac{\partial}{\partial q_i}$

$= \frac{\hbar}{i}\delta_{ij}$

$\Rightarrow$ classical mechanics $\longrightarrow$ quantum mechanics

classical mechanics	quantum mechanics
$F(q,p,t)$	$\hat{F}(\hat{q},\hat{p},t)$
$\hookrightarrow$ function	$\hookrightarrow$ operator on $\lvert\psi\rangle$
$\frac{dF}{dt} = [F,H] + \frac{\partial F}{\partial t}$	$\frac{d\hat{F}}{dt} = [\hat{F},\hat{H}] + \frac{\partial \hat{F}}{\partial t}$
$\hookrightarrow$ Poisson bracket	$\hookrightarrow$ commutator

$$[q_i, p_j] = -\delta_{ij} \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0 \qquad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

$\Rightarrow$ this is necessarily incomplete because we haven't covered continuous classical fields yet...