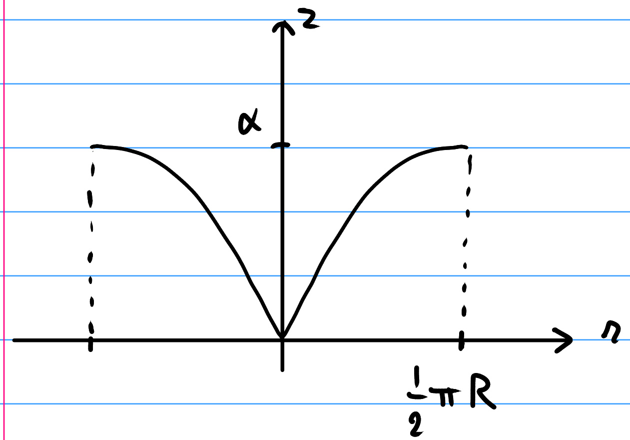
# Homework Assignment 3

(1)




a) $z = \alpha \sin\frac{r}{R} \Rightarrow T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)$

$\Downarrow$

$\dot{z} = \alpha\frac{\dot{r}}{R}\cos\frac{r}{R} \Rightarrow T = \frac{1}{2}m\left[r^2\dot{\varphi}^2 + \left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r}{R}\right)\dot{r}^2\right]$

$V = mgz = mg\alpha\sin\frac{r}{R}$

$\Rightarrow L = \frac{1}{2}m\left[r^2\dot{\varphi}^2 + \left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r}{R}\right)\dot{r}^2\right] - mg\alpha\sin\frac{r}{R}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \Leftrightarrow m\frac{d}{dt}\left[\left(1+\frac{\alpha^2}{R^2}\cos^2\frac{r}{R}\right)\dot{r}\right] - mr\dot{\varphi}^2$

$+ m\dot{r}^2\frac{\alpha^2}{R^3}\cos\frac{r}{R}\sin\frac{r}{R} + mg\frac{\alpha}{R}\cos\frac{r}{R} = 0$

$\Leftrightarrow m\ddot{r} - 2m\dot{r}^2\frac{\alpha^2}{R^3}\cos\frac{r}{R}\sin\frac{r}{R} - mr\dot{\varphi}^2$

$+ m\dot{r}^2\frac{\alpha^2}{R^3}\cos\frac{r}{R}\sin\frac{r}{R} + mg\frac{\alpha}{R}\cos\frac{r}{R} = 0$

$\Leftrightarrow m\ddot{r} - mr\dot{\varphi}^2 - m\dot{r}^2\frac{\alpha^2}{R^3}\cos\frac{r}{R}\sin\frac{r}{R} + mg\frac{\alpha}{R}\cos\frac{r}{R} = 0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \Leftrightarrow \frac{d}{dt}\left(mr^2\dot{\varphi}\right) = 0$

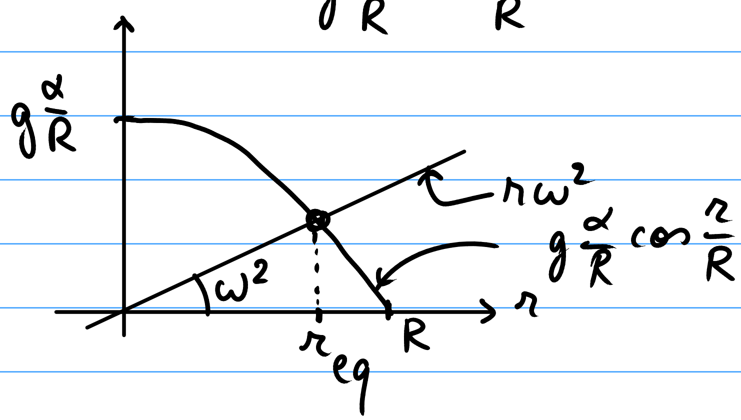
b) Horizontal orbits have $\dot z = 0 \Rightarrow \dot r = 0, \ddot r = 0$

From the equation for $\varphi$ follows then $\dot\varphi = \omega =$ constant.

The equation for $r$ becomes:

$$-mr\omega^2 + mg\frac{\alpha}{R}\cos\frac{r}{R} = 0$$

$$\Leftrightarrow \quad r\omega^2 = g\frac{\alpha}{R}\cos\frac{r}{R} \quad \Leftrightarrow \quad r = r_{eq}(\omega)$$




c) For small deviations $\Delta r$ from the equilibrium $r_{eq}$:

$$r(t) = r_{eq} + \Delta r(t), \quad \dot\varphi(t) = \omega + \Delta\omega(t)$$

$$\dot r = \Delta\dot r, \quad \ddot r = \Delta\ddot r, \quad \ddot\varphi = \Delta\dot\omega$$

$$\Rightarrow \quad m\Delta\ddot r - m(r_{eq}+\Delta r)(\omega+\Delta\omega)^2$$

$$- m\Delta\dot r^2\frac{\alpha^2}{R^3}\cos\frac{r_{eq}+\Delta r}{R}\sin\frac{r_{eq}+\Delta r}{R}$$

$$+ mg\frac{\alpha}{R}\cos\frac{r_{eq}+\Delta r}{R} = 0$$

For small $\Delta r$ and $\Delta\omega$, ignoring quadratic terms:

$$\frac{d}{dt}\left((r_{eq}+\Delta r)^2(\omega+\Delta\omega)\right) = \frac{d}{dt}\left(r_{eq}^2\omega + 2r_{eq}\Delta r\,\omega + r_{eq}^2\Delta\omega\right) = 0$$

$$\Rightarrow \quad 2\omega\Delta r + r_{eq}\Delta\omega = \text{constant}$$

$$\Delta\ddot{r} - (r_{eq}+\Delta r)(\omega+\Delta\omega)^2 - \cancelto{0}{\Delta\dot{r}^2} \frac{\alpha^2}{R^3}\cos\frac{r_{eq}+\Delta r}{R}\sin\frac{r_{eq}+\Delta r}{R}$$

$$+ \quad g\frac{\alpha}{R}\cos\frac{r_{eq}+\Delta r}{R} = 0$$

$$\Leftrightarrow 0 = \Delta\ddot{r} - r_{eq}\omega^2 - \omega^2\Delta r - 2r_{eq}\omega\Delta\omega + g\frac{\alpha}{R}\left[1-\left(\frac{r_{eq}+\Delta r}{R}\right)^2\right]$$

$$\Leftrightarrow 0 = \Delta\ddot{r} - \omega^2\Delta r - 2r_{eq}\omega\Delta\omega - g\frac{\alpha}{R}\frac{2r_{eq}\Delta r}{R^2}$$

$$\Leftrightarrow 0 = \Delta\ddot{r} - 2\omega\left(\text{constant} - 2\omega\Delta r\right) - g\frac{\alpha}{R}\frac{2r_{eq}}{R^2}\Delta r$$

$$\Leftrightarrow \quad \Delta\ddot{r} + \left(4\omega^2 - 2\frac{g\alpha r_{eq}}{R^3}\right)\Delta r = 0$$

d) $\hookrightarrow \omega'^2 = 4\omega^2 - 2\frac{g\alpha r_{eq}}{R^3}$

This is stable for $4\omega^2 > 2\frac{g\alpha r_{eq}}{R^3}$

② The velocity of $m_1$ has two components:

$$v_1^2 = a^2\dot{\theta}^2 + a^2\dot{\varphi}^2\sin^2\theta$$

$$v_2^2 = 2a^2\dot{\theta}^2\sin^2\theta$$

$$T = m_1 a^2\dot{\theta}^2 + m_1 a^2\dot{\varphi}^2\sin^2\theta + m_2 a^2\dot{\theta}^2\sin^2\theta$$

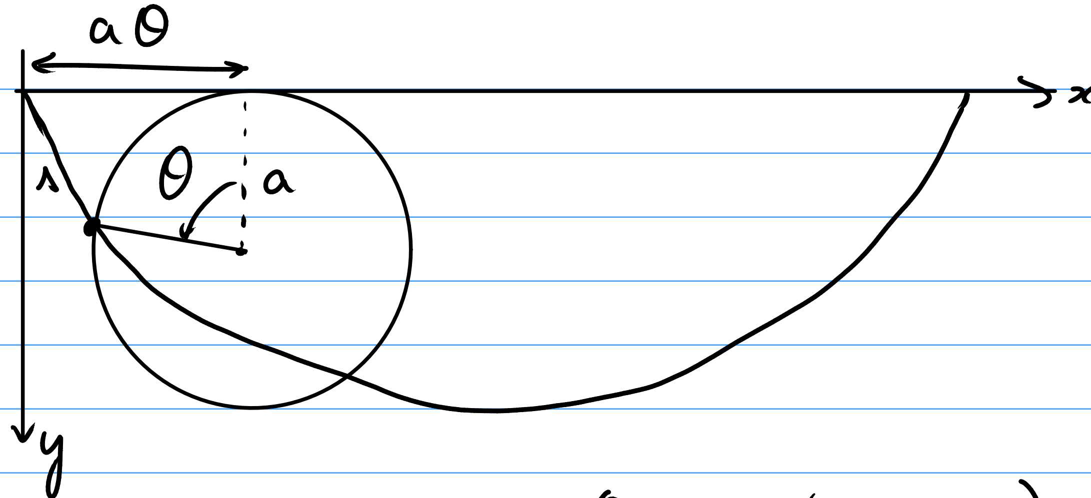
$$V = 2m_1 g a\cos\theta + 2m_2 g a\cos\theta = 2(m_1+m_2)ga\cos\theta$$

$$\Rightarrow L = m_1 a^2\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + m_2 a^2\dot{\theta}^2\sin^2\theta$$

$$- 2(m_1+m_2)ga\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \Leftrightarrow$$

③




a)
$$\begin{cases} x = a\theta - a\sin\theta = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$$

$$\begin{aligned} ds^2 = dx^2 + dy^2 &= a^2(d\theta - \cos\theta\, d\theta)^2 + a^2\sin^2\theta\, d\theta^2 \\ &= 2a^2 d\theta^2 - 2a^2\cos\theta\, d\theta^2 \\ &= 2a^2(1-\cos\theta)\, d\theta^2 \\ &= 4a^2\sin^2\frac{\theta}{2}\, d\theta^2 \end{aligned}$$

$$\Rightarrow ds = 2a\sin\frac{\theta}{2}\, d\theta$$

$$s = \int_0^s ds = \int_0^\theta 2a\sin\frac{\theta}{2}\, d\theta = -4a\cos\frac{\theta}{2}$$

c) $V = -mgy = -mga(1-\cos\theta) = -2mga\left(1-\cos^2\frac{\theta}{2}\right)$

$$\Rightarrow \quad V = -2mga + 2mga\left(\frac{s}{4a}\right)^2$$

$$L = \frac{1}{2}m\dot{s}^2 - \frac{1}{2}m\frac{g}{4a}s^2$$

$$\Rightarrow \quad m\ddot{s} + m\frac{g}{4a}s = 0$$

Harmonic oscillator with $\omega = \sqrt{\frac{g}{4a}}$, even for large $s$.

(4) $$\frac{d}{dt} L = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right) = \frac{d\dot{q}}{dt}\frac{\partial L}{\partial \dot{q}} + \dot{q}\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}}_{= \frac{\partial L}{\partial q}} = \frac{d\dot{q}}{dt}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{\partial L}{\partial q}$$

$$\Rightarrow \frac{d}{dt} L = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{\partial L}{\partial t} = \frac{d}{dt}\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right)$$

If $L$ does not depend explicitly on $t$, then $\frac{\partial L}{\partial t} = 0$

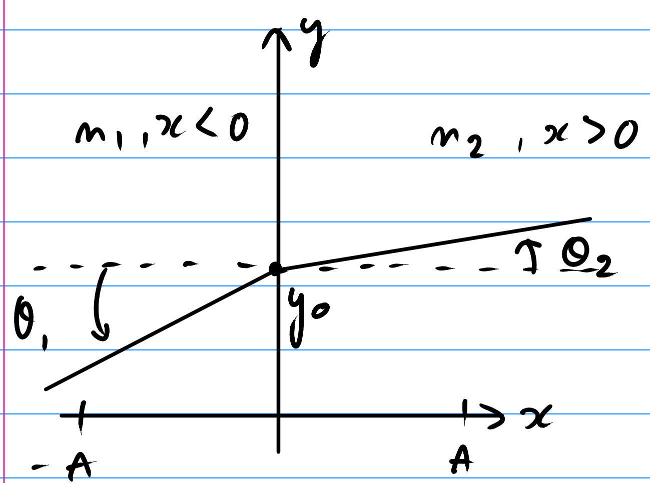
$\Rightarrow L - \dot{q}\frac{\partial L}{\partial \dot{q}} =$ constant of motion

$$L(y, y', x) \text{ over } \sqrt{1+y'^2}\, n(x,y)$$

(5) $dt = \frac{ds}{v} = \frac{n}{c} ds \rightarrow T = \int dx \underbrace{\sqrt{1+y'^2}\, n(x,y)}_{L(y,y',x)}$

$$\Rightarrow \frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} = 0$$

$$\Leftrightarrow \sqrt{1+y'^2} \frac{\partial n}{\partial y} = \frac{d}{dx}\left( n(x,y) \frac{y'}{\sqrt{1+y'^2}} \right)$$




In each medium, $n$ is constant:

$$n \frac{y'}{\sqrt{1+y'^2}} = \text{constant}$$

$\hookrightarrow$ straight line:

$$y(x) = x \tan\theta + y_0$$
$$y'(x) = \tan\theta$$
$$\frac{y'}{\sqrt{1+y'^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \sin\theta$$

For the full trajectory:

$\frac{\partial n}{\partial y} = 0$, but $n(x,y)$ depends on $x$:

$$\Rightarrow n_1 \frac{y_1'}{\sqrt{1+y_1'^2}} = n_2 \frac{y_2'}{\sqrt{1+y_2'^2}} = \text{constant}$$

$$\Leftrightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

⑥ In polar coordinates (2D):

$$ds^2 = dr^2 + r^2 d\varphi^2 = \left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\right] d\varphi^2$$

$$T = \int \frac{n}{c}\, ds = \frac{1}{c}\int n(r)\sqrt{r' + r^2}\, d\varphi$$

$$\Rightarrow L(r, r', \varphi) = n(r)\sqrt{r'^2 + r^2}$$

Using problem 4: $\frac{\partial L}{\partial \varphi} = 0 \Rightarrow L - r'\frac{\partial L}{\partial r'} = \text{constant}$

$$\Leftrightarrow \quad n\sqrt{r'^2 + r^2} - r' n \frac{r'}{\sqrt{r'^2 + r^2}} = C$$

$$\Leftrightarrow \quad n\left(r'^2 + r^2\right) - n r'^2 = C\sqrt{r'^2 + r^2}$$

$$\Leftrightarrow \quad n r^2 = C\sqrt{r'^2 + r^2}$$

$$\Leftrightarrow \quad n^2 r^4 = C^2\left(r'^2 + r^2\right)$$

$$\Leftrightarrow \quad n^2 r^2 = C^2\left(\frac{r'^2}{r^2} + 1\right)$$

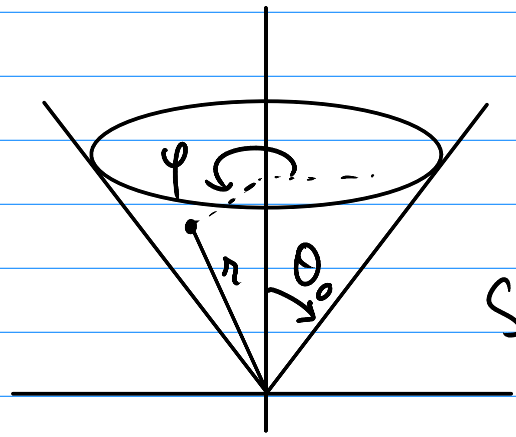
$$\Leftrightarrow \quad \left(\frac{1}{C^2} n^2 r^2 - 1\right) r^2 = r'^2$$

$$\Leftrightarrow \quad r' = r\sqrt{\frac{n^2 r^2}{C^2} - 1}$$

$$n(r) = A r^m \rightarrow r' = r\sqrt{\frac{A^2}{C^2} r^{2m+2} - 1}$$

Constant distance if $r' = 0$. If ray comes in with $r' = 0$, this determines $\frac{A^2}{C^2}$. If $m = -1$, then $r' = 0$ for all $r$ $\rightarrow n = A\frac{1}{r}$

(7)




$$ds^2 = dr^2 + r^2 \sin^2\theta \, d\varphi^2 + r^2 d\theta^2$$

Here $\theta = \theta_0 \rightarrow d\theta = 0$.

$$S = \int ds = \int \sqrt{r'^2 + r^2 \sin^2\theta_0} \, d\varphi$$

1) $$\frac{d}{d\varphi}\left(\frac{\partial L}{\partial r'}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Leftrightarrow \frac{d}{d\varphi}\left(\frac{r'}{\sqrt{r'^2 + r^2 \sin^2\theta_0}}\right) - \frac{r \sin^2\theta_0}{\sqrt{r'^2 + r^2 \sin^2\theta_0}} = 0$$

$$\Leftrightarrow r'' \frac{1}{\sqrt{r'^2 + r^2 \sin^2\theta_0}} - r' \frac{r'r'' + rr'\sin^2\theta_0}{\sqrt{(r'^2 + r^2\sin^2\theta_0)^3}} - \frac{r \sin^2\theta_0}{\sqrt{r'^2 + r^2 \sin^2\theta_0}} = 0$$

$$\Leftrightarrow r'' - r \sin^2\theta_0 - \frac{r'^2 r'' + rr'^2 \sin^2\theta_0}{(r'^2 + r^2 \sin^2\theta_0)} = 0$$

$$\Leftrightarrow r'' r'^2 + r'' r^2 \sin^2\theta_0 - rr'^2 \sin^2\theta_0 - r^3 \sin^4\theta_0 - r'^2 r'' - rr'^2 \sin^2\theta_0 = 0$$

$$\Leftrightarrow r'' r^2 \sin^2\theta_0 - 2rr'^2 \sin^2\theta_0 - r^3 \sin^2\theta_0 = 0$$

$$\Leftrightarrow r'' r - 2r'^2 - r^2 \sin^2\theta_0 = 0$$

2) $$r = r_0 \frac{1}{\cos\left[(\varphi-\varphi_0)\sin\theta_0\right]}$$

$$r' = r_0 \frac{1}{\cos^2\left[(\varphi-\varphi_0)\sin\theta_0\right]} \sin\left[(\varphi-\varphi_0)\sin\theta_0\right] \sin\theta_0$$

$$r'' = r_0 \left[\frac{\sin^2\left[(\varphi-\varphi_0)\sin\theta_0\right]}{\cos^3\left[(\varphi-\varphi_0)\sin\theta_0\right]} + \frac{1}{\cos\left[(\varphi-\varphi_0)\sin\theta_0\right]}\right] \sin^2\theta_0$$

$$\Rightarrow \quad r'' r = r_0^2 \sin^2\theta_0 \left[\frac{2\sin^2\left[(\varphi-\varphi_0)\sin\theta_0\right]}{\cos^4\left[(\varphi-\varphi_0)\sin\theta_0\right]} + \frac{1}{\cos\left[(\varphi-\varphi_0)\sin\theta_0\right]}\right]$$

$$-2r'^2 = -2r_0^2 \sin^2\theta_0 \frac{\sin^2\left[(\varphi-\varphi_0)\sin\theta_0\right]}{\cos^4\left[(\varphi-\varphi_0)\sin\theta_0\right]}$$

$$r^2\sin^2\theta_0 = r_0^2 \sin^2\theta_0 \frac{1}{\cos^2\left[(\varphi-\varphi_0)\sin\theta_0\right]}$$

$$\Rightarrow \quad r'' r - 2r'^2 - r^2\sin^2\theta_0 = 0$$