Classical Mechanics (Phys 601) — October 6, 2011

* Small oscillations

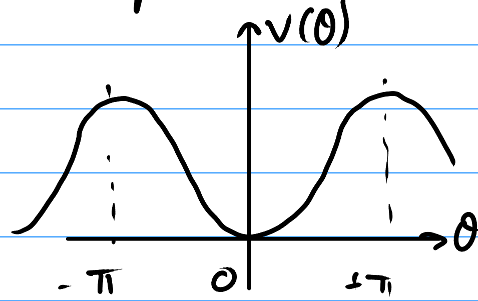
Assume $T = \frac{1}{2}\sum_{i,j} m_{ij}(q)\,\dot{q}_i\dot{q}_j$ ← $m_{ij} = \sum_k m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}$

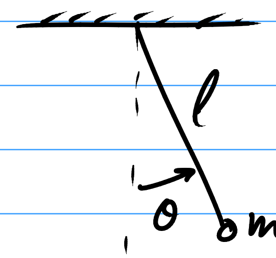
↳ $L = T - V(q) = \frac{1}{2}\sum_{i,j} m_{ij}(q)\,\dot{q}_i\dot{q}_j$

At equilibrium: $\dot{q}_i = 0$ and $\ddot{q}_i = 0$

$$\Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \Leftrightarrow \frac{\partial L}{\partial q_i} = 0 \Leftrightarrow \frac{\partial V}{\partial q_i} = 0$$

Example: pendulum $V(\theta) = -mg\cos\theta$



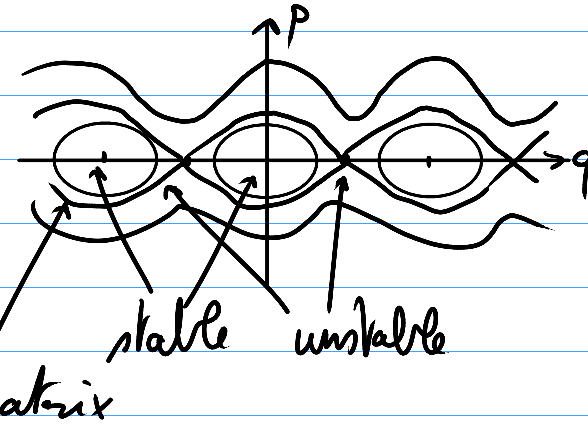


↳ equilibrium: stable and unstable

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$\begin{cases} \dot{p} = -mgl\sin\theta \\ \dot{\theta} = \frac{p}{ml^2} \end{cases}$$



* Stability of equilibrium

Let A be a map (operator) from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

A has a stable fixed point x_0 if

1) $A x_0 = x_0$
2) $\forall \varepsilon > 0, \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |A^n x - A^n x_0| < \varepsilon, \forall n$

Time-evolution $\sim$ Hamiltonian flow:

$$A x = x(t_0 + \delta t)$$
$$\downarrow$$
$$A^n x = x(t_0 + \Delta t)$$

$\Rightarrow$ stable fixed point if $\forall \varepsilon > 0, \exists \delta > 0$:

$|x - x_0| < \delta \Rightarrow |x(t) - x(t_0)| < \varepsilon$

* Linearized systems

Assume $q_i^{(0)}$ is a stable fixed point (e.g. $\theta = 0$ for pendulum)

Let $q_i = q_i^{(0)} + \eta_i$ with η_i small $\hookrightarrow \dot{q}_i = \dot{\eta}_i$

$\downarrow$ expand L to $O(\eta^2)$

$$T = \frac{1}{2} \sum_{i,j} m_{ij}(q)\, \dot{\eta}_i \dot{\eta}_j \simeq \frac{1}{2} \sum_{i,j} m_{ij}(q^{(0)})\, \dot{\eta}_i \dot{\eta}_j + O(\eta^3)$$

$$V = \underbrace{V(q^{(0)})}_{=\text{constant}} + \underbrace{\sum_i \frac{\partial V}{\partial q_i}(q^{(0)})}_{=0 \text{ at fixed point } q^{(0)}}\, \eta_i + \sum_{i,j} \frac{\partial^2 V}{\partial q_i \partial q_j}(q^{(0)})\, \eta_i \eta_j + O(\eta^3)$$

$$\Rightarrow \quad L = \frac{1}{2}\sum_{i,j}\left[m_{ij}(q^{(0)})\dot\eta_i\dot\eta_j - \frac{\partial^2 V}{\partial q_i \partial q_j}(q^{(0)})\eta_i\eta_j \right]$$

Now define the matrices and vector

$$M \quad : \quad M_{ij} = m_{ij}(q^{(0)}) \qquad (M^T = M)$$

$$V \quad : \quad V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}(q^{(0)}) \qquad (V^T = V)$$

$$\eta \quad : \quad \eta_i = \eta_i$$

$$\Rightarrow \quad L = \frac{1}{2}\dot\eta^T M \dot\eta - \frac{1}{2}\eta^T V \eta$$

Equations of motion for η

$$\frac{\partial L}{\partial \eta_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\eta_i}\right) = 0 \Leftrightarrow -\frac{1}{2}\sum_j V_{ij}\eta_j - \frac{1}{2}\sum_j \eta_j V_{ji}$$

$$- \frac{d}{dt}\left(\sum_j M_{ij}\dot\eta_j + \sum_j \dot\eta_j M_{ji} \right) = 0$$

$$\Leftrightarrow \quad M\ddot\eta + V\eta = 0$$

$$\Leftrightarrow \quad M\ddot\eta = -V\eta$$

Solutions to this equation will result in eigenvalue problem:

$$\det(-M\omega^2 + V) = 0$$

* Example for one-dimensional system

Lagrange equation is $m\ddot{\eta} + k\eta = 0$

Assume periodic solution $\eta = Ae^{i\omega t}$ or $Be^{-i\omega t}$

$\Rightarrow \ddot{\eta} = -A\omega^2 e^{i\omega t}$

$\Leftrightarrow -m\omega^2 + k = 0$

$\Leftrightarrow \omega = \pm\sqrt{\frac{k}{m}}$

$\Rightarrow \eta = Ae^{i\omega t} + Be^{-i\omega t}$
with A, B determined by initial conditions
such that η is real

General case :

$M\ddot{\eta} + V\eta$

Assume $\eta = z e^{i\omega t}$ (i.e. $\eta_i = z_i e^{i\omega t}$)

$\Downarrow$

solve $-\omega^2 Mz + Vz = 0$

* Linear Algebra

Symmetric matrices: $A^T = A$, $A_{ij} = A_{ji}$

Orthogonal matrices: $A^{-1} = A^T \Rightarrow A^T A = \mathbb{1}$

Identity matrix: $\mathbb{1} A = A \mathbb{1} = A$, $\mathbb{1}_{ij} = \delta_{ij}$

Multiplication: $A = B \cdot C$, $A_{ij} = \sum_k B_{ik} C_{kj}$

Consider M: set of matrices $(n \times n)$ of real numbers A_{ij} with $\det(A) \neq 0$

1) $A(BC) = (AB)C$ (associativity)
2) $\exists \, \mathbb{1} : \mathbb{1} A = A \mathbb{1} = A$ (identity)
3) $\exists \, A^{-1} : A^{-1} A = A A^{-1} = \mathbb{1}$ (inverse)

$\Rightarrow (M, \times)$ is a group

non-abelian : $AB \neq BA$
abelian : $AB = BA$

Proofs: 1) $\sum_l A_{il} \left(\sum_k B_{lk} C_{kj} \right) = \sum_k \left(\sum_l A_{il} B_{lk} \right) C_{kj}$

2) $\mathbb{1}_{ij} = \delta_{ij} \Rightarrow \sum_k A_{ik} \delta_{kj} = A_{ij} = \sum \delta_{ik} A_{kj}$

Inner product (dot product) with vectors

Let V: set of $(n \times 1)$ vectors with v_i real

Operations $+$, $*$ $\Rightarrow$ $(V, +, *)$ is a vector space on $\mathbb{R}$

$(V, +)$ is a group (abelian!)
$(V, *)$ is a group
$a \cdot (b+c) = a \cdot b + a \cdot c$ (distributivity)

$\Big\}$ $(V, +, *)$ is a field

Properties of vectors space on $\mathbb{R}$

1) $a \cdot (v_1 + v_2) = a \cdot v_1 + a \cdot v_2$
2) $(a+b) \cdot v = a \cdot v + b \cdot v$
3) $a\,(b \cdot v) = (ab) \cdot v$
4) $1\, v = v\, 1 = 1$

identity and multiplication in $\mathbb{R}$

Properties of dot product:

$V \times V \longrightarrow \mathbb{R}$: vector space $\times$ vector space to real numbers

1) $x \cdot y = \overline{y \cdot x}$ (but for real numbers $\bar{x} = x$)
2) $(a\,x) \cdot y = a\,(x \cdot y)$
3) $(x+y) \cdot z = x \cdot z + y \cdot z$

$\Big\}$ linearity

4) $x \cdot x \geq 0$
5) $x \cdot x = 0 \Leftrightarrow x = 0$

$\Big\}$ positive definite

$$x \cdot y = x^T y = \sum_k x_k y_k$$

* ## Eigenvalues

$$A v = \lambda v$$

λ eigenvalue in $\mathbb{R}$

v eigenvector in V

A: matrix, λ: scalar

$$\Rightarrow (A - \lambda \mathbb{1}) v = 0$$

$$\Leftrightarrow \det(A - \lambda \mathbb{1}) = 0 \longrightarrow \text{polynomial of order } n$$

$\longrightarrow$ n complex roots

If $A^T = A$ (symmetric) then only real eigenvalues.
Assume complex field on $\mathbb{C}$:

$$v^+ A v = \lambda$$

$$(A v) = \lambda v \Rightarrow (A v)^+ = \lambda^* v^+$$

$$(v^+ A v)^+ = v^+ A^+ v = \lambda$$

Uniqueness: If $A v_1 = \lambda_1 v_1$ and $A v_2 = \lambda_2 v_2$, $\lambda_1 \neq \lambda_2$
then $v_1^+ v_2 = 0$

$$\left.\begin{aligned} v_1^T A v_2 &= \lambda_2 v_1^T v_2 \\ v_2^T A v_1 &= \lambda_1 v_2^T v_1 \end{aligned}\right\}$$

but $v_1^T v_2 = v_2^T v_1$

$\Downarrow$

$$\underbrace{(v_1^T A v_2)^T}_{= v_1^T A v_2} = v_2^T A^T v_1 = v_2^T A v_1$$

$\leftarrow$ this is a number

$$\Rightarrow 0 = (\lambda_1 - \lambda_2) v_1^T v_2 \Rightarrow v_1 \cdot v_2 = 0$$

$\hookrightarrow$ eigenvectors orthogonal