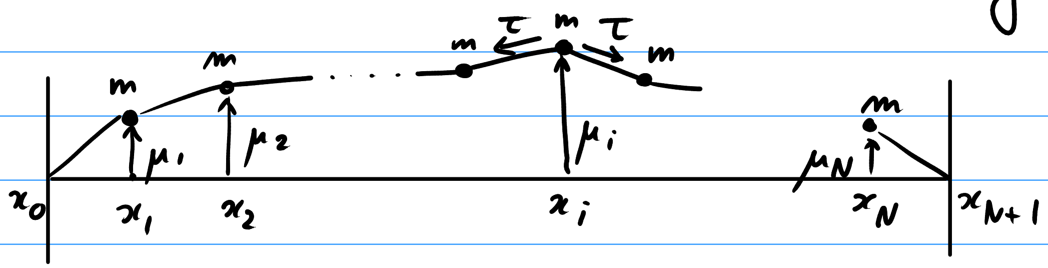
Classical Mechanics (Phys 601) - October 25, 2011

* Previous lecture: transverse oscillations on a string



$x_i - x_{i-1} = a$

tension τ

$$L = \frac{1}{2} m \sum_{i=1}^{N} \dot{\mu}_i^2 - \frac{1}{2} k \sum_{i=0}^{N} (\mu_{i+1} - \mu_i)^2 \quad , \text{ with } k = \frac{\tau}{a}$$

$\Rightarrow$ normal-mode frequencies $\omega_n^2 = 4 \frac{k}{m} \sin^2 \frac{n\pi}{2(N+1)}$

$$\ddot{\mu}(x_i) + 2\frac{k}{m}\mu(x_i) - \frac{k}{m}\left[\mu(x_{i+1}) + \mu(x_{i-1})\right] = 0$$

$\Rightarrow$ dispersion relation $\omega^2 = 4\frac{k}{m} \sin^2 \frac{ka}{2}$

and

periodic boundary $\mu(x_0) = \mu(x_N) \Rightarrow k_n = \frac{2\pi n}{Na}$, $\overbrace{n = 0, \pm 1, \ldots, \pm\frac{1}{2}(N-1)}^{\# = N}$

or

fixed ends $\quad \mu(x_0) = \mu(x_{N+1}) = 0 \Rightarrow k_n = \frac{n\pi}{(N+1)a}$, $\underbrace{n = 1, 2, \ldots, N}_{\# = N}$

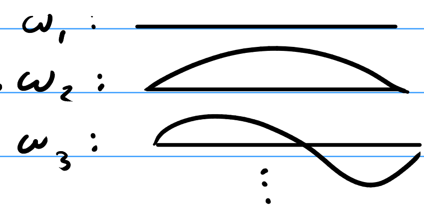
$\hookrightarrow \quad \omega_n^2 = 4\frac{k}{m} \sin^2 \frac{n\pi}{2(N+1)} \quad ✓$

$$\ddot{\mu}(x_i) - \frac{\tau}{m} a \left[\frac{\frac{\mu(x_{i+1}) - \mu(x_i)}{a} - \frac{\mu(x_i) - \mu(x_{i-1})}{a}}{a} \right] = 0$$

$\Rightarrow \quad \mu''(x) = \frac{1}{v^2} \ddot{\mu}(x) \qquad$ with $v^2 = \lim_{a \to 0} \frac{\tau}{m/a} = \frac{\tau}{\sigma}$

* Transition to the continuum limit

$$\left.\begin{array}{l} N \to \infty \\ a \to 0 \\ m \to 0 \end{array}\right\} \qquad \begin{array}{l} \ell = (N+1)\,a = \text{constant} \\ \sigma = \dfrac{m}{a} = \text{constant} \end{array}$$



$$\omega_n^2 = 4\,\frac{\tau}{ma}\,\sin^2\frac{n\pi}{2(N+1)} \quad , \quad n \ll N$$

$$\omega_n^2 = 4\,\frac{\tau}{m/a}\left[\frac{n\pi}{\underbrace{2a(N+1)}_{\ell}}\right]^2 = v^2\left(\frac{n\pi}{\ell}\right)^2 \quad \text{with } v^2 = \frac{\tau}{\sigma}$$

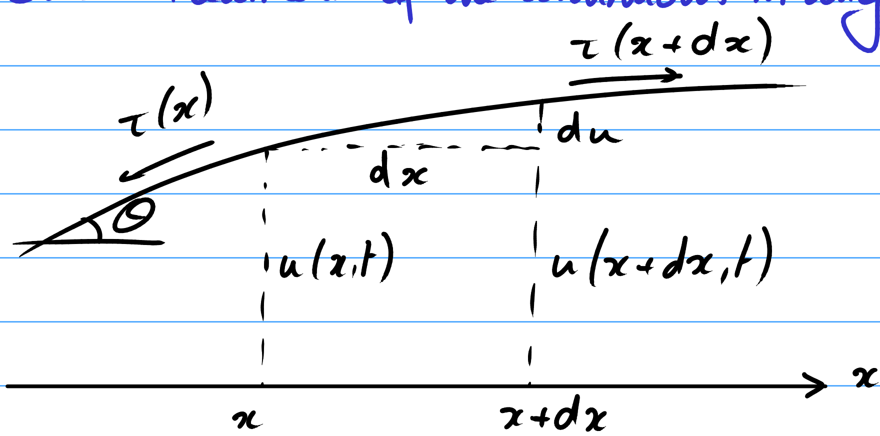
$\hookrightarrow n = 1, 2, \ldots, \infty$

$$\Rightarrow u(x,t) = A\,e^{i(kx-\omega t)}$$

$\hookrightarrow$ all ω_n are possible

$\longrightarrow \omega_n = \omega$

Direct treatment of the continuous string



density $\sigma(x)$

tension $\tau(x)$

small oscillations

$$\theta(x) \simeq \tan\theta(x) = \frac{\partial u}{\partial x} \Rightarrow \sin\theta(x) \simeq \theta(x) = \frac{\partial u}{\partial x}$$

$$\underbrace{\left[\sigma(x)\,dx\right]}_{m}\,\underbrace{\frac{\partial^2 u}{\partial t^2}}_{a} = \underbrace{\tau(x+dx)\,\frac{\partial u}{\partial x}(x+dx,t) - \tau(x)\,\frac{\partial u}{\partial x}(x,t)}_{F}$$

$$F = \tau(x+dx)\frac{\partial u}{\partial x}(x+dx,t) - \tau(x)\frac{\partial u}{\partial x}(x,t)$$

$$= \cancel{\tau(x)\frac{\partial u}{\partial x}(x,t)} + dx\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}(x,t)\right] + O(dx^2)$$

$$- \cancel{\tau(x)\frac{\partial u}{\partial x}(x,t)}$$

$$= dx\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}(x,t)\right]$$

$$\Rightarrow \quad \sigma(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left[\tau(x)\frac{\partial u}{\partial x}(x,t)\right]$$

If $\sigma(x) = \sigma =$ constant, and $\tau(x) = \tau =$ constant:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\tau}{\sigma}\frac{\partial^2 u}{\partial x^2}(x,t)$$

$$\Leftrightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \quad \text{with } v^2 = \frac{\tau}{\sigma}$$

Remember what x is: $x = x_i = ia$, $i = 0, 1, 2, 3, \dots, N$

↳ for each ω_n there is an eigenvector $\rho_n = \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} =$ vector

x continuous index $\longrightarrow$ eigenvector $\rho(x) =$ function

discrete: $u(x_i, t) = \sum_n C_n \rho_n \cos(\omega_n t + \varphi_n) = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$

continuous: $u(x,t) = \sum_n C_n \rho_n(x) \cos(\omega_n t + \varphi_n) =$ function

ω_n is an eigenvalue : from differential equation

$$\omega^2 \rho_i + \frac{\tau}{m/a}\left[\frac{1}{a}\left(\frac{\rho_{i+1}-\rho_i}{a} - \frac{\rho_i - \rho_{i-1}}{a}\right)\right] = 0$$

ω is an eigenvalue : from differential eqn or continuum limit

$$\frac{d^2\rho}{dx^2} = -\frac{\omega^2}{v^2}\rho(x) \iff \frac{d^2\rho}{dx^2} + k^2 \rho(x) = 0$$

$$\Rightarrow \rho(x) = A\cos kx + B\sin kx$$

boundary conditions : $\rho(x_0) = \rho(x_{N+1}) = 0$

$$\rho(0) = \rho(\ell) = 0 \Rightarrow A = 0$$

$$k_n = \frac{n\pi}{\ell}, \quad n = 1, 2, \dots, \infty \quad \text{eigenvalues}$$

$$\omega_n = v k_n = v\frac{n\pi}{\ell}$$

$\Rightarrow$ general solution is :

$$u(x,t) = \sum_{n=1}^{\infty} C_n \rho_n(x) \cos(\omega_n t + \varphi_n)$$

with $\rho_n(x) = \underbrace{\left(\frac{2}{\ell\sigma}\right)^{1/2}}_{\text{normalization}} \sin\frac{n\pi x}{\ell}$

normalization : $\int_0^\ell \sigma\, \rho_n(x)\rho_m(x)\, dx = \delta_{mn}$

normal mode $\xi_n(t) = C_n \cos(\omega_n t + \varphi_n)$

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n(x)\, \xi_n(t)$$

Insert this expression $u(x,t) = \sum_{n=1}^{\infty} \rho_n(x)\, \xi_n(t)$
in the Lagrangian:

$$L = \frac{1}{2}\int_0^{\ell} \sigma \left(\frac{\partial u}{\partial t}\right)^2 dx \underbrace{- \frac{1}{2}\tau \int_0^{\ell} \left(\frac{\partial u}{\partial x}\right)^2 dx}$$

integration by parts

$$= -\frac{1}{2}\tau \int_0^{\ell} u(x,t)\, \frac{\partial^2 u}{\partial x^2}\, dx$$

$$L = \frac{1}{2}\int_0^{\ell} \sigma \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^{\ell} u \underbrace{\frac{\partial^2 u}{\partial x^2}} dx$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$(\dot{\xi}_n)^2 \qquad\qquad -k_n^2\, u_n(x,t)$$

$$L = \frac{1}{2}\sum_{n=1}^{\infty}\left(\dot{\xi}_n^2 - \omega_n^2 \xi_n^2\right) \longrightarrow \ddot{\xi}_n + \omega_n^2 \xi_n = 0$$

$\rightarrow$ sum of normal modes

* Lagrangian for the continuous string

Limit of discrete system:

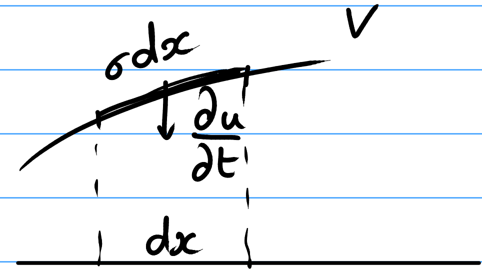
$$L = \frac{1}{2} m \sum_{i=1}^{N} \dot{\mu}_i^2 - \frac{1}{2}\frac{\tau}{a}\sum_{i=0}^{N}\left(\mu_{i+1}-\mu_i\right)^2$$

$$= \frac{1}{2}\frac{m}{a}\sum_{i=1}^{N} a\dot{\mu}_i^2 - \frac{1}{2}\tau\sum_{i=0}^{N} a\left(\frac{\mu_{i+1}-\mu_i}{a}\right)^2$$

$$N \to \infty,\ a \to 0,\ \frac{m}{a} \to \sigma,\ \sum_{i=1}^{N} a \to \int_0^{\ell} dx$$

$$L = \underbrace{\frac{1}{2}\sigma\int_0^{\ell} dx\left(\frac{\partial u}{\partial t}(x,t)\right)^2}_{T} - \underbrace{\frac{1}{2}\tau\int_0^{\ell} dx\left(\frac{\partial u}{\partial x}(x,t)\right)^2}_{V}$$

T: $\frac{\partial u}{\partial t}(x,t)$ = velocity of line element

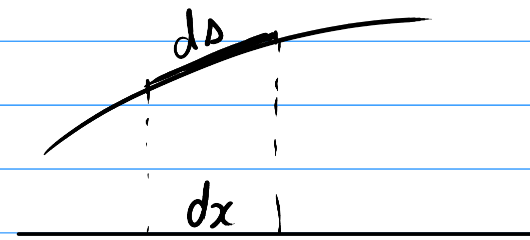


$\sigma\, dx$ = mass of line element

$$\hookrightarrow dT = \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 \cdot \sigma\, dx$$

$$T = \frac{1}{2}\int dx\, \sigma\left(\frac{\partial u}{\partial t}(x,t)\right)^2$$

V: $dV = dW = \tau\,(ds - dx)$



$$= \tau\, dx\left(\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2} - 1\right)$$

$$= \frac{1}{2}\tau\, dx\left(\frac{\partial u}{\partial x}\right)^2$$

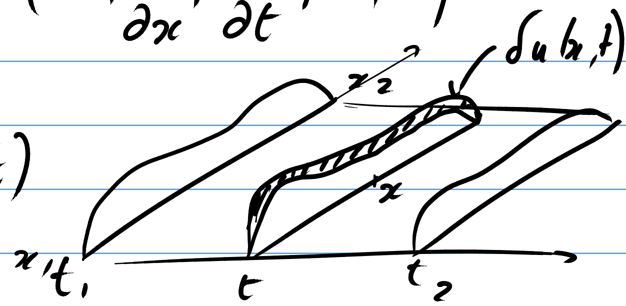
$$\hookrightarrow V = \frac{1}{2}\int dx\, \tau\left(\frac{\partial u}{\partial x}(x,t)\right)^2$$

* Lagrangian density and Lagrange's equations

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = \delta \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \, \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, x, t\right) = 0$$

Variation:

$$u(x,t) \longrightarrow u(x,t) + \delta u(x,t)$$



Fixed endpoints in time:

with $\delta u(x,t_1) = \delta u(x,t_2) = 0$

Fixed endpoints in space: < boundary conditions

with $\delta u(x_1,t) = \delta u(x_2,t) = 0$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{\partial \mathcal{L}}{\partial u} \delta u + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} \delta \frac{\partial u}{\partial x} + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} \delta \frac{\partial u}{\partial t}\right] = 0$$

$$\delta \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \delta u \quad \text{and} \quad \delta \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \delta u$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{\partial \mathcal{L}}{\partial u} + \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} \frac{\partial}{\partial x}} + \underbrace{\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} \frac{\partial}{\partial t}}\right] \delta u = 0$$

integration by parts

$$\delta S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)}\right] \delta u = 0$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial t}\right)} + \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial u}{\partial x}\right)} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

* Example of one-dimensional continuous string

$$\mathcal{L} = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2$$

↳ Lagrange's equation :

$$\sigma \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Leftrightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \quad \text{with } v^2 = \frac{\tau}{\sigma}$$

* Multi-dimensional Lagrange equation for continuous fields

$$L = \int d^D x \, \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_D}, \frac{\partial \varphi}{\partial t}, x, t\right)$$

$$\Rightarrow \sum_{\mu=0}^{D} \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_\mu}\right)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

includes time as "coordinate" $\mu = 0$

Often: $\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_\mu}\right)}\right) = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right)$

Example: $\mathcal{L} = \dot{\varphi}^2 - (\bar{\nabla}\varphi)^2 = (\partial_0\varphi)^2 - (\partial_1\varphi)^2 - (\partial_2\varphi)^2 - (\partial_3\varphi)^2$

$$\Rightarrow \sum_{\mu=0}^{3} \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) = 0$$

$$\Leftrightarrow \partial_0^2\varphi - \partial_1^2\varphi - \partial_2^2\varphi - \partial_3^2\varphi = 0$$

$$\Leftrightarrow \quad \ddot{\varphi} = \nabla^2 \varphi \quad \leftarrow \text{Laplace equation in 3D}$$

* Multi-dimensional Hamiltonian for continuous systems:

String with transverse oscillations:

$$L = \frac{1}{2}\frac{m}{a}\sum_{i=1}^{N} a\dot{\mu}_i^2 - \frac{1}{2}\tau\sum_{i=0}^{N} a\left(\frac{\mu_{i+1}-\mu_i}{a}\right)^2$$

$$= \sum_i^N a\left[\frac{1}{2}\frac{m}{a}\dot{\mu}_i^2 - \frac{1}{2}\tau\left(\frac{\mu_{i+1}-\mu_i}{a}\right)^2\right] = \sum_i a L_i$$

$$p_i = \frac{\partial L}{\partial \dot{\mu}_i} = a\frac{\partial L_i}{\partial \dot{\mu}_i}$$

$$H = \sum_i^N p_i \dot{\mu}_i - L = \sum_i^N a\left(\dot{\mu}_i \frac{\partial L_i}{\partial \dot{\mu}_i} - L_i\right)$$

$\hookrightarrow$ continuum limit : $\sum_i^N a \longrightarrow \int dx$

$$H = \int_0^\ell dx\left(\dot{\mu}\frac{\partial \mathcal{L}}{\partial \dot{\mu}} - \mathcal{L}\right) = \int_0^\ell dx\, \mathcal{H}$$

$$\mathcal{H} = \dot{\mu}\frac{\partial \mathcal{L}}{\partial \dot{\mu}} - \mathcal{L} \quad = \text{Hamiltonian density}$$

$$= \dot{\mu}\,\pi - \mathcal{L} \quad \text{with } \pi = \frac{\partial \mathcal{L}}{\partial \dot{\mu}} = \text{momentum density}$$

Not as useful as for discrete system: Lagrangian mechanism treats time and space coordinates equivalently, in Hamiltonian formalism this is lost again

* Example of complex scalar field : Klein-Gordon equation

Complex scalar field → { φ and φ^* are independent parts ; $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$ }

Summation convention : sum runs over matched indices

$$\mathcal{L} = c^2 \partial_\mu \varphi \partial^\mu \varphi^* - m_0^2 c^2 \varphi \varphi^*$$

with $\partial^\mu \varphi = g^{\mu\nu} \partial_\nu \varphi$ and $g^{\mu\nu} = \begin{bmatrix} \frac{1}{c^2} & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

$$\mathcal{L} = \dot{\varphi} \dot{\varphi}^* - c^2 \bar{\nabla}\varphi \cdot \bar{\nabla}\varphi^* - m_0^2 c^2 \varphi \varphi^*$$

Lagrange equation is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} - \frac{\partial \mathcal{L}}{\partial \varphi^*} = 0 \iff c^2 \partial_\mu \partial^\mu \varphi + m_0^2 c^2 \varphi = 0$$

$$\iff \partial_\mu \partial^\mu \varphi + m_0^2 \varphi = 0$$

$$\iff \underbrace{\frac{1}{c^2} \ddot{\varphi} - \nabla^2 \varphi}_{\Box^2 \varphi} + m_0^2 \varphi = 0$$

with $\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$

d'Alembertian

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}^* \quad , \quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^*} = \dot{\varphi}$$

$$\Rightarrow \mathcal{H} = \pi \dot{\varphi} + \pi^* \dot{\varphi}^* - \mathcal{L}$$

$$= \pi \pi^* + c^2 \bar{\nabla}\varphi \cdot \bar{\nabla}\varphi^* + m_0^2 c^2 \varphi \varphi^*$$