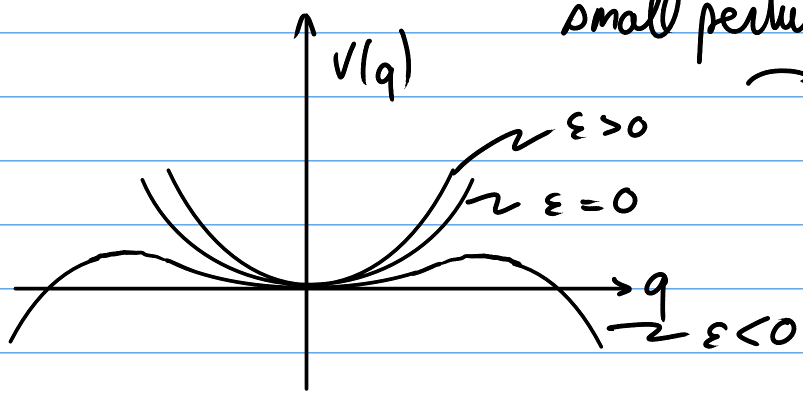
Classical Mechanics (Phys 601) - November 22, 2011

* Duffing oscillator:

$$V(q) = \frac{1}{2} m\omega_0^2 q^2 + \underbrace{\frac{1}{4} m\varepsilon q^4}_{\text{small perturbation}}$$



→ we will focus on small region around $q = 0$

→ no problems with $q \to \infty$

$$L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega_0^2 q^2 - \frac{1}{4} m\varepsilon q^4$$

$$\ddot{q} + \omega_0^2 q + \varepsilon q^3 = 0$$

* Naive perturbation theory:

$$q(t) = q_0(t) + \varepsilon q_1(t) + O(\varepsilon^2)$$

$$\begin{cases} \text{1st order: } \ddot{q}_0 + \omega_0^2 q_0 = 0 \\ \text{2nd order: } \ddot{q}_1 + \omega_0^2 q_1 = -q_0^3 \\ \vdots \end{cases} \quad \text{with} \quad \begin{cases} q_0(0) = a \\ \dot{q}_0(0) = 0 \\ \vdots \end{cases}$$

$$q(t) = a\cos\omega_0 t - \frac{\varepsilon a^3}{8\omega_0^2}\left(3\omega_0 t \sin\omega_0 t + \frac{1}{4}(\cos\omega_0 t - \cos 3\omega_0 t)\right)$$

↳ unbounded for $t \to \infty$

* Perturbation theory with frequency shift:

$$\begin{cases} q(t) = q_0(t) + \varepsilon q_1(t) + O(\varepsilon^2) \\ \omega = \omega_0 + \varepsilon \omega_1 + O(\varepsilon^2) \end{cases}$$

$$\begin{cases} \text{1st order: } \omega_0^2 \left(\dfrac{d^2 q_0}{d\tau^2} + q_0 \right) = 0 \qquad \text{with } \tau = \omega t \\ \text{2nd order: } \omega_0^2 \left(\dfrac{d^2 q_1}{d\tau^2} + q_1 \right) = -2\omega_0 \omega_1 \dfrac{d^2 q_0}{d\tau^2} - q_0^3 \\ \vdots \end{cases}$$

Choose $\omega_1 = \dfrac{3a^2}{8\omega_0}$ $\Rightarrow$ no secular term

$$q(t) = a \cos\left[\left(\omega_0 + \varepsilon \frac{3a^2}{8\omega_0} \right) t \right] - \varepsilon \frac{a^3}{32\omega_0^2} \left(\cos\omega_0 t - \cos 3\omega_0 t \right)$$

Expansion of first term $\rightarrow$ same linear term as in naive perturbation theory

* Action-angle variables for periodic systems:

Hamilton-Jacobi equation for conserved Hamiltonian:

$$\begin{cases} S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t) = W(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n) - E(\alpha_1, \dots, \alpha_n)\, t \\ H\left(\dfrac{\partial W}{\partial q_1}, \dots, \dfrac{\partial W}{\partial q_n}, q_1, \dots, q_n \right) = E(\alpha_1, \dots, \alpha_n) \end{cases}$$

Seperable systems :

$$W(q_1,\dots,q_m,\alpha_1,\dots,\alpha_m) = W_1(q_1,\alpha_1,\dots,\alpha_m) + \dots + W_m(q_m,\alpha_1,\dots,\alpha_m)$$

$$(p_1,\dots,p_m,q_1,\dots,q_m) \xrightarrow{S(q_1,\dots,q_m,P_1,\dots,P_m,t)} \underset{\alpha_1,\ \dots,\ \alpha_m,\ \beta_1\ \dots\ \beta_m}{(P_1,\dots,P_m,Q_1,\dots,Q_m)}$$

$(P_1,\dots,P_m,Q_1,\dots,Q_m) = (\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m)$ constant because $\tilde{H}(P,Q,t)\equiv 0$

with
$$p_i = \frac{\partial S}{\partial q_i}(q_1,\dots,q_m,P_1,\dots,P_m,t) = \frac{\partial W_i}{\partial q_i}(q_i,\alpha_1,\dots,\alpha_m)$$

and
$$Q_i = \frac{\partial S}{\partial P_i}(q_1,\dots,q_m,P_1,\dots,P_m,t) = \text{constant}$$

For cyclic variables $q_i \to W_i(q_i,\alpha_1,\dots,\alpha_m) = q_i\alpha_i$

$$\Rightarrow p_i = \frac{\partial W_i}{\partial q_i}(q_i,\alpha_1,\dots,\alpha_m) = \alpha_i$$

Action variable:

$$J_i = \frac{1}{2\pi} \oint_{\text{cycle}} p_i \, dq_i = \frac{1}{2\pi} \oint_{\text{cycle}} \frac{\partial W_i}{\partial q_i}(q_i, \alpha_1, \dots, \alpha_m) \, dq_i = J_i(\alpha_1, \dots, \alpha_m) = \text{constant}$$

Invert to obtain $\alpha_i = \alpha_i(J_1, \dots, J_m)$
and determine new generating function

$$\bar{S}(q_1, \dots, q_m, J_1, \dots, J_m, t) = \bar{W}_1(q_1, J_1, \dots, J_m) + \dots + \bar{W}_m(q_m, J_1, \dots, J_m) - E(J_1, \dots, J_m) . t$$

$$(p_1, \dots, p_m, q_1, \dots, q_m) \xrightarrow{\bar{S}(q_1, \dots, q_m, J_1, \dots, J_m, t)} (J_1, \dots, J_m, \underset{\bar{\beta}_1}{\underset{\parallel}{\bar{Q}_1}}, \dots, \underset{\bar{\beta}_m}{\underset{\parallel}{\bar{Q}_m}}) \text{ constant}$$

with
$$p_i = \frac{\partial \bar{S}}{\partial q_i}(q_1, \dots, q_m, J_1, \dots, J_m, t) = \frac{\partial \bar{W}_i}{\partial q_i}(q_i, J_1, \dots, J_m)$$

and
$$\bar{\beta}_i = \frac{\partial \bar{S}}{\partial J_i}(q_1, \dots, q_m, J_1, \dots, J_m, t) = \text{constant}$$
$$= \frac{\partial \bar{W}}{\partial J_i}(q_1, \dots, q_m, J_1, \dots, J_m) - \frac{\partial E}{\partial J_i}(J_1, \dots, J_m) . t$$
$$= \varphi_i - \omega_i(J_1, \dots, J_m) t$$

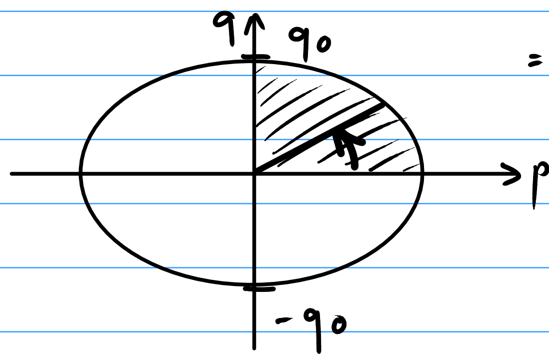
$$\Leftrightarrow \varphi_i = \omega_i t + \bar{\beta}_i$$

Harmonic oscillator in one dimension:

$$H = E = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 \Rightarrow p = \pm \sqrt{2m} \sqrt{E - \frac{1}{2} m \omega_0^2 q^2}$$

At $t=0$: $q(0) = q_0$ and $\dot{q}(0) = 0 \rightarrow E = \frac{1}{2} m \omega_0^2 q_0^2$

$$\Rightarrow J = \frac{1}{2\pi} \int_{\text{cycle}} p \, dq = \frac{4\sqrt{2m}}{2\pi} \int_{\text{quadrant}} \sqrt{\frac{1}{2} m \omega_0^2 (q_0^2 - q^2)} \qquad x = \frac{q}{q_0}$$



$$= \frac{2 q_0 \sqrt{2mE}}{\pi} \int_0^1 dx \sqrt{1 - x^2}$$

$$J = \frac{E}{\omega_0} \Leftrightarrow E(J) = \omega_0 J$$

$$\Rightarrow \omega = \frac{\partial E}{\partial J} = \omega_0$$

Solution of Hamilton-Jacobi equation is

$$H\left(\frac{\partial W}{\partial q}, q, t\right) = E$$

$$\Leftrightarrow W(q, E) = \sqrt{2m} \int dq \sqrt{E - \frac{1}{2} m \omega_0^2 q^2} + W_0(E)$$

$$\Leftrightarrow \bar{W}(q, E(J)) = \sqrt{2m} \int dq \sqrt{\omega_0 J - \frac{1}{2} m \omega_0^2 q^2} + \bar{W}_0(J)$$

angle $\varphi = \frac{\partial \bar{W}}{\partial J}(q, J) = \frac{\omega_0}{2} \sqrt{2m} \int dq \frac{1}{\sqrt{\omega_0 J - \frac{1}{2} m \omega_0^2 q^2}} + \varphi_0(J)$

$$= \sin^{-1} \frac{q}{q_0} = \bar{\beta} + \omega_0 t$$

$\Rightarrow q = \sqrt{\frac{2J}{m\omega_0}} \sin \varphi$ and $p = \sqrt{2 m \omega_0 J} \cos \varphi \Rightarrow$ ellipse

Harmonic oscillators in multiple dimensions :

$$H = E = \sum_{i=1}^{n} \underbrace{\left(\frac{1}{2m_i} p_i^2 + \frac{1}{2} m_i \omega_i^2 q_i^2 \right)}_{= E_i = \alpha_i}$$

$$\Rightarrow \quad E(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i$$

Hamilton-Jacobi equation (Hamiltonian conserved, seperable)

$$\frac{1}{2m_i} \left(\frac{\partial W_i}{\partial q_i} \right)^2 + \frac{1}{2} m_i \omega_i^2 q_i^2 = E_i$$

$\Rightarrow$ same as one dimensional harmonic oscillator :

$$\begin{cases} J_i = \dfrac{E_i}{\omega_i} \Rightarrow E_i = \omega_i J_i \Rightarrow E = \sum_{i=1}^{n} \omega_i J_i \\ \omega_i = \dfrac{\partial E}{\partial J_i} (J_1, \ldots, J_n) \end{cases}$$

$$\begin{cases} p_i = \sqrt{2 m_i \omega_i J_i} \cos \varphi_i \\ q_i = \sqrt{\dfrac{2 J_i}{m_i \omega_i}} \sin \varphi_i \\ \varphi_i = \omega_i t + \bar{\beta}_i \end{cases}$$

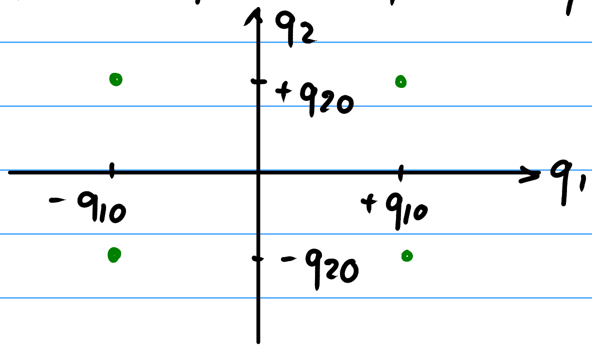
One dimensional system $\rightarrow$ phase diagram (p, q) is ellipse

Two dimensional system $\rightarrow$ phase diagram (p_1, p_2, q_1, q_2) is torus in 4D space

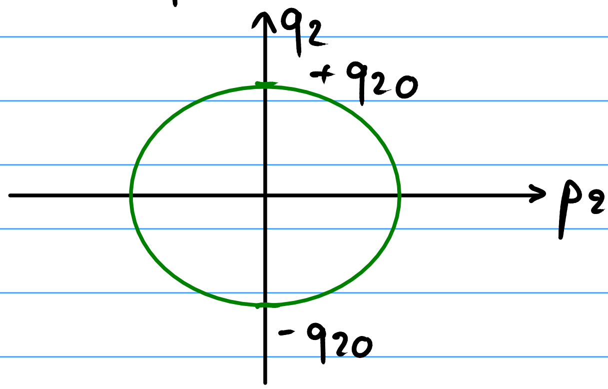
Poincaré sections in phase space

Record intersections with plane in phase space
→ planar intersection for 4D space

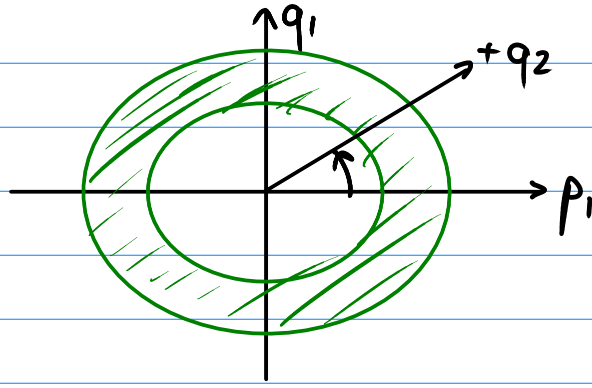
Example : intersection with $p_1 = 0$, $p_2 = 0$ plane
$q_1 = \pm q_{10}$, $q_2 = \pm q_{20}$ = turning points



Example : intersection with $p_1 = 0$, $q_1 = 0$ plane
p_2 and q_2 not constrained → regular ellipse



Example : intersection with $p_2 = 0$ plane



Perturbation of periodic Hamiltonian systems

$$H(p,q,t) = H_0(p,q,t) + \varepsilon V(p,q,t)$$

Canonical transformation defined by $S_0(q,P,t)$ does <u>not</u> make $\tilde{H}(P,Q,t) \equiv 0$ anymore:

$$p = \frac{\partial S_0}{\partial q}$$

$$\tilde{H}(P,Q,t) = H_0\left(\frac{\partial S_0}{\partial q}, q, t\right) + \varepsilon V(P,Q,t) + \frac{\partial S_0}{\partial t}$$

$$\tilde{H}(P,Q,t) = \varepsilon V(P,Q,t)$$

$$\Rightarrow \begin{cases} \dot{Q} = \varepsilon \dfrac{\partial V}{\partial P} \neq 0 & \rightarrow Q \neq \text{constant} \\ \dot{P} = -\varepsilon \dfrac{\partial V}{\partial Q} \neq 0 & \rightarrow P \neq \text{constant} \end{cases}$$

We can now pick $(J, \bar{Q})$ instead of (P,Q):

$$\Rightarrow \begin{cases} \dot{\bar{Q}} = \varepsilon \dfrac{\partial V}{\partial J}(J,\bar{Q},t) \\ \dot{J} = -\varepsilon \dfrac{\partial V}{\partial \bar{Q}}(J,\bar{Q},t) \end{cases}$$

We can even pick (J, φ) instead of (P,Q):

$$\begin{cases} \dot{\varphi} = \dfrac{\partial H}{\partial J}(J,\varphi,t) \\ \dot{J} = -\dfrac{\partial H}{\partial \varphi}(J,\varphi,t) \end{cases} \quad \text{with } H(J,\varphi,t) = E_0(J) + \varepsilon V(J,\varphi,t)$$

In $(J_1,\dots,J_m,\bar{Q}_1,\dots,\bar{Q}_m)$ Hamilton's equations are

$$\begin{cases} \dot{\bar{Q}}_i = \dfrac{\partial \tilde{H}}{\partial J_i}(J_1,\dots,J_m,\bar{Q}_1,\dots,\bar{Q}_m) \\[2ex] \dot{J}_i = -\dfrac{\partial \tilde{H}}{\partial \bar{Q}_i}(J_1,\dots,J_m,\bar{Q}_1,\dots,\bar{Q}_m) \end{cases}$$

with $\bar{Q}_i = \varphi_i - \omega_i(J_1,\dots,J_m)\,t$

Note that: $$\dot{\bar{Q}}_i = \dot{\varphi}_i - \omega_i(J_1,\dots,J_m) - \sum_{j=1}^{m} \frac{\partial \omega_i}{\partial J_j}(J_1,\dots,J_m)\frac{dJ_j}{dt}\cdot t$$

Also: $$\dot{\bar{Q}}_i = \frac{\partial \tilde{H}}{\partial J_i}(J_1,\dots,J_m,\varphi_1-\omega_1(J_1,\dots,J_m)t,\dots,\varphi_m-\omega_m(J_1,\dots,J_m)t)$$
$$= \frac{\partial \tilde{H}}{\partial J_i}(J_1,\dots,J_m,\varphi_1,\dots,\varphi_m,t) - \sum_{j=1}^{m}\frac{\partial \tilde{H}}{\partial \varphi_j}(J_1,\dots,J_m,\varphi_1,\dots,\varphi_m,t)\,\frac{\partial \omega_j}{\partial J_i}(J_1,\dots,J_m)\,t$$

$$\Rightarrow \dot{\varphi}_i = \omega_i(J_1,\dots,J_m) + \frac{\partial \tilde{H}}{\partial J_i}(J_1,\dots,J_m,\varphi_1,\dots,\varphi_m,t) + \sum_{j=1}^{m}\Big(\dot{J}_j - \underbrace{\frac{\partial \tilde{H}}{\partial \varphi_j}(J_1,\dots,J_m,\varphi_1,\dots,\varphi_m,t)}_{=\frac{\partial \tilde{H}}{\partial \bar{Q}_j}(J_1,\dots,J_m,\varphi_1,\dots,\ {}_m,t)\,\cancel{\frac{d\bar{Q}_j}{d\varphi_j}}^{=1}}\Big)\frac{d\omega_j}{dJ_i}\,t$$

$$\Rightarrow \dot{\varphi}_i = \frac{\partial}{\partial J_i}\Big(E_0(J_1,\dots,J_m) + \tilde{H}(J_1,\dots,J_m,\varphi_1,\dots,\varphi_m,t)\Big)$$

Duffing oscillator:

$$H(p,q,t) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2 + \frac{1}{4} m \varepsilon q^4$$

$$p = \sqrt{2m\omega_0 J}\cos\varphi$$

$$q = \sqrt{\frac{2J}{m\omega_0}}\sin\varphi$$

$$E_0(J) = \omega_0 J$$

$$\Rightarrow H(J,\varphi) = \omega_0 J + \frac{1}{4} m\varepsilon \left(\frac{2J}{m\omega_0}\right)^2 \sin^4\varphi$$

$$= \omega_0 J + \varepsilon \frac{J^2}{m\omega_0^2}\sin^4\varphi$$

$$\begin{cases} \dot{J} = -\dfrac{\partial H}{\partial \varphi}(J,\varphi) = -\varepsilon \dfrac{4J^2}{m\omega_0^2}\sin^3\varphi\cos\varphi \\ \dot{\varphi} = \dfrac{\partial H}{\partial J}(J,\varphi) = \omega_0 + \varepsilon \dfrac{2J}{m\omega_0^2}\sin^4\varphi \end{cases}$$

Average over one period 2π: $\langle f(\varphi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi$

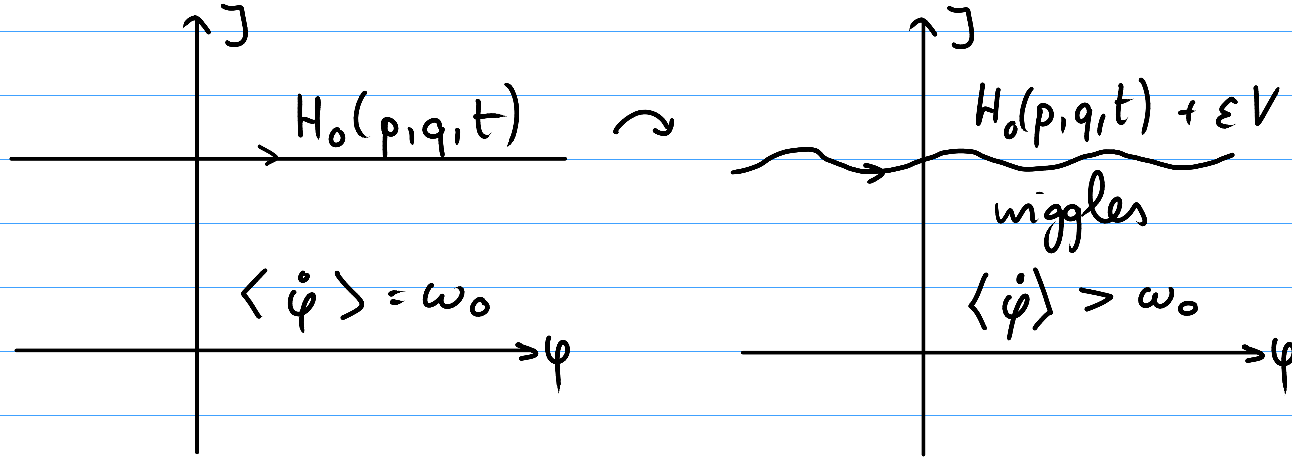
$$\langle \sin^3\varphi\cos\varphi\rangle = 0 \quad (\text{odd in } \varphi)$$

$$\langle \sin^4\varphi\rangle = \frac{1}{2\pi}\int_0^{2\pi}\sin^4\varphi\,d\varphi = \frac{3}{8}$$

$$\Rightarrow \langle\dot{\varphi}\rangle = \omega_0 + \varepsilon\frac{3J}{4m\omega_0^2}$$

$$J = \frac{E_0}{\omega_0} = \frac{1}{2} m\omega_0 a^2$$

$$= \omega_0 + \varepsilon\frac{3a^2}{8\omega_0}$$



J
$H_0(p,q,t)$
$\langle \dot{\varphi} \rangle = \omega_0$
φ
J
$H_0(p,q,t) + \varepsilon V$
wiggles
$\langle \dot{\varphi} \rangle > \omega_0$
φ