Classical Mechanics (Phys 601) - October 4, 2011

* Recap of Hamilton-Jacobi theory :

$p, q \xrightarrow[\text{canonical}]{} P, Q$

canonical transformation : $\begin{cases} q_i = q_i(Q_j, P_j, t) \\ p_i = p_i(Q_j, P_j, t) \end{cases}$

↳ derives from generating function $F(q_i, Q_j, t)$

Hamiltonian becomes $K = H + \frac{\partial F}{\partial t}$

Properties :

transformation is canonical $\Longleftrightarrow$
- there exists a generating function F
- Hamilton's equations are equivalent (and therefore different by only $\frac{\partial F}{\partial t}$)
- Poisson brackets are equal

Generating function → multiple forms : $F_1(q, Q, t)$, $F_2(q, P, t)$, $F_3(Q, p, t)$, $F_4(p, P, t)$

↓

no physical meaning, multiple generating functions for the same transformation

↳ if not required, prefer to ignore and focus on physics

Now, try to find canonical transformation such that $K \equiv 0$

$$\Rightarrow \quad H(q_i, p_i, t) + \frac{\partial F}{\partial t}(q_i, Q_j, t) = 0$$

Instead of $F(q_i, Q_j, t)$, use $S(q_i, P_j, t)$ where

$$F(q_i, Q_j, t) = -\sum_j P_j Q_j + S(q_i, P_j, t)$$

$$\text{with} \begin{cases} Q_j = \dfrac{\partial S}{\partial P_j} \\ p_i = \dfrac{\partial S}{\partial q_i} \end{cases}$$

$$\Rightarrow \quad H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t}(q_i, P_j, t) = 0$$

Solutions of this Hamilton-Jacobi equation will have cyclic P_j and Q_j:

$$P_j = \alpha_j \quad \text{and} \quad Q_j = \beta_j$$

$$\Rightarrow \quad Q_j = \beta_j = \frac{\partial S}{\partial P_j} = \frac{\partial S}{\partial \alpha_j}(q_i, \alpha_j, t)$$

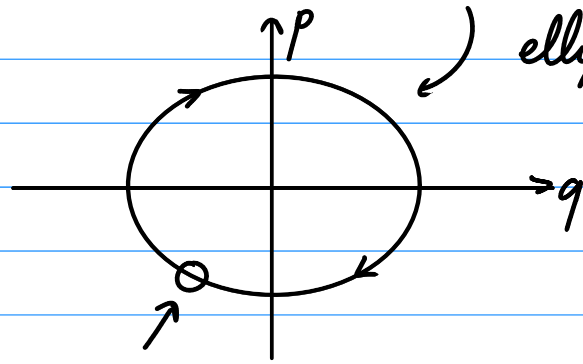
$$\Rightarrow \quad q_i = q_i(\alpha_j, \beta_j, t)$$

$$\Rightarrow \quad p_i = \frac{\partial S}{\partial q_i}(q_i, \alpha_j, t) = p_i(\alpha_j, \beta_j, t)$$

* Action - angle variables

Consider the pendulum:

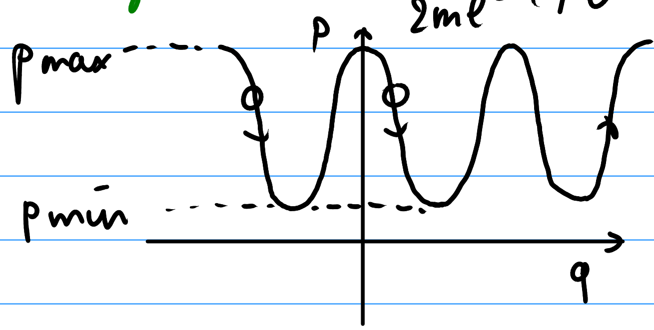
- small angles : $H = \frac{1}{2ml^2}\left(p_\theta^2 + m^2\omega^2\theta^2\right)$



ellipse in phase space

the motion brings the system back to the same point in phase space $\longrightarrow$ { libration / oscillation

- larger angles : $H = \frac{1}{2ml^2}\left(p_\theta^2 + m^2\omega^2 \sin^2\theta\right)$



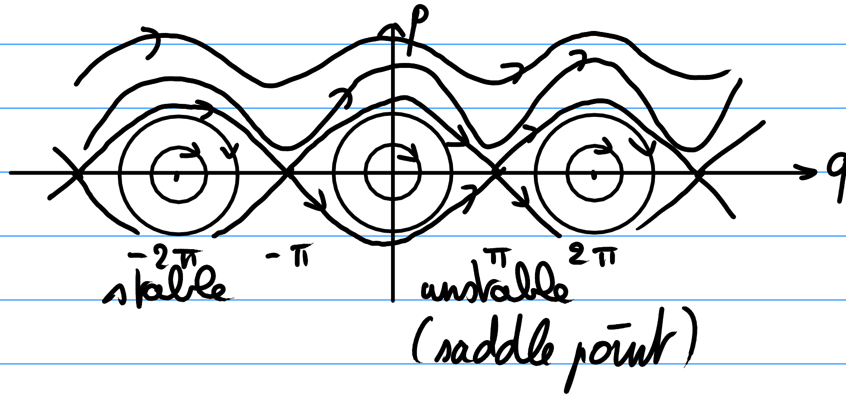
not a closed orbit in phase space

θ grows forever when large enough initial momentum p_θ

$\downarrow$

not the same point in phase space, but p is periodic in q $\longrightarrow$ rotation

→ general phase space diagram



⇒ consider only periodic systems as above

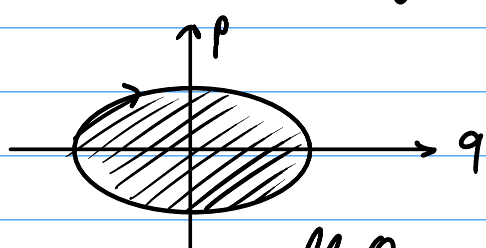
Also: conservative $\left(\frac{dH}{dt}=0\right)$ and separable:

$$S = W_1(q_1, \alpha_1, \dots \alpha_n) + \dots + W_n(q_n, \alpha_1, \dots \alpha_n) - \alpha t$$

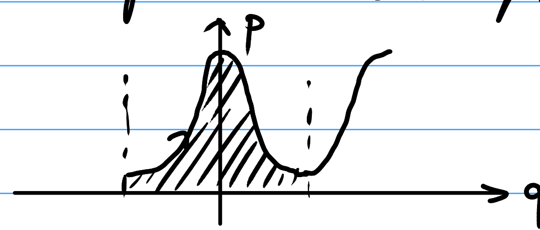
How can we calculate the period without solving the full equations of motion?

Define: action variables $J_i = \oint p_i \, dq_i$

→ integral of cycle in phase space = area in phase space



small θ
(libration)



large θ
(rotation)

Notice J_i has dimensions of $p_i q_i$ = dimensions of S
action

Hamilton-Jacobi theory: $p_i = \frac{\partial W}{\partial q_i} = \frac{\partial W_i}{\partial q_i}(q_i, \alpha_1 \ldots \alpha_n)$

$$\Rightarrow J_i = \int \frac{\partial W_i}{\partial q_i}(q_i, \alpha_1 \ldots \alpha_n)\, dq_i = J_i(\alpha_1, \ldots, \alpha_n)$$

(only depends on the momenta α_i) / constants of motion

Now invert this set of relations:

$$\alpha_i = \alpha_i(J_1, \ldots, J_n)$$

For conservative systems: $\alpha_1 = E = E(J_1, \ldots, J_n)$

The α_i were just constants of integration $\rightarrow$ could also use J_i as constants of integration:

$$W = W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n) = \bar{W}(q_1, \ldots, q_n, J_1, \ldots, J_n)$$

$$\alpha_1(J_1, \ldots, J_n) \cdots \alpha_n(J_1, \ldots, J_n)$$

$$\Downarrow$$

$$S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n) = \bar{S}(q_1, \ldots, q_n, J_1, \ldots, J_n)$$

$$= \bar{W}(q_1, \ldots, q_n, J_1, \ldots, J_n) - \alpha_1(J_1, \ldots, J_n)\, t$$

then: $p_i = \frac{\partial \bar{S}}{\partial q_i}$ and $\bar{Q}_i = \frac{\partial \bar{S}}{\partial J_i} = \beta_i$

$$\bar{P}_i = J_i$$

Now define the angle variables $w_i = \frac{\partial \bar{W}}{\partial J_i}(q, J)$

Then $\bar{Q}_i = \beta_i = \frac{\partial \bar{S}}{\partial J_i} = \frac{\partial}{\partial J_i}\left(\bar{W}(q,J) - \alpha_1(J)\, t \right)$

$$\Rightarrow \beta_i = w_i - \frac{\partial \alpha_1}{\partial J_i}(J) \cdot t$$

a priori not frequency as we know it!

Define: frequency $\nu_i = \frac{\partial \alpha_1}{\partial J_i}(J) = \frac{\partial E}{\partial J_i}(J)$

$$\Rightarrow w_i = \nu_i t + \beta_i = \text{linear function of time}$$

Notice for (w, J) : $H = H(J) \rightarrow \begin{cases} \dot{w}_i = \frac{\partial H}{\partial J_i} = \nu_i \\ \dot{J}_i = -\frac{\partial H}{\partial w_i} = 0 \Rightarrow \text{constant} \end{cases}$

$\sim (q, p)$

For simple harmonic oscillator: one 'cycle' is always equal to the period of the oscillation or rotation:

$$\Delta w = \nu \Delta t = \nu \tau$$

τ: period of oscillation or rotation

For multi-dimensional systems ($>$ SHO): more general case will have multiple cycles in each of the sets of canonical coordinates.

Example: 2HO $\rightarrow \frac{\omega_1}{\omega_2} = \frac{n}{m} \rightarrow n\tau_1 = m\tau_2 = \Delta t$

$\Rightarrow \Delta t = n_i \tau_i$ (for all i there is an n_i such that this holds for periodic motion)

$$\Rightarrow \Delta w_i = \nu_i \Delta t = \nu_i n_i \tau_i$$

Consider infinitesimal change in w_i :

$$\delta w_i = \sum_j \frac{\partial w_i}{\partial q_j} \delta q_j = \sum_j \frac{\partial}{\partial q_j} \frac{\partial \overline{W}}{\partial J_i} \delta q_j$$

$$= \frac{\partial}{\partial J_i} \sum_j \underbrace{\frac{\partial \overline{W}}{\partial q_j}}_{= p_j} \delta q_j = \frac{\partial}{\partial J_i} \sum_j p_j \delta q_j$$

For a full period of all coordinates :

$$\Delta w_i = \frac{\partial}{\partial J_i} \sum_j \underbrace{\oint p_j \delta q_j}_{m_j J_j} = m_i$$

↳ w_i increases by 1 for each periodic cycle

Compare this with $\Delta w_i = m_i \nu_i \tau_i$

$$\Rightarrow \quad \nu_i \tau_i = 1$$

$\Rightarrow$ Indeed, $\nu_i = \frac{1}{\tau_i}$ is the frequency of the coordinate i

and $\nu_i = \frac{\partial}{\partial J_i} H(J_1, \ldots, J_m)$ = fundamental frequency

* Example : simple harmonic oscillator :

1) Start from Hamiltonian : $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$

2) Determine p as function of q : $H = E = \alpha$
$\Rightarrow p = \pm\sqrt{2m(\alpha - \frac{1}{2}kq^2)}$

3) Evaluate action integral : $J = \oint p\,dq = \oint \sqrt{2m(\alpha - \frac{1}{2}kq^2)}\,dq$
$= 2\pi\alpha\sqrt{\frac{m}{k}}$ (see later)

4) Write H in terms of J :

$$H = \alpha = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\,J$$

5) Determine frequencies :

$$\nu = \frac{\partial H}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega}{2\pi}$$

$\Rightarrow$ frequencies determined without ever solving the system

* Example : uncoupled double harmonic oscillator

$$H = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2q_2^2$$

$$\frac{\partial H}{\partial t} = 0 \rightarrow H = E = \alpha \rightarrow S = W - \alpha t = W_1 + W_2 - \alpha t$$

Hamilton - Jacobi equation:

$$\left.\begin{aligned} \frac{1}{2m}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \frac{1}{2}k_1 q_1^2 &= \alpha_1 \\ \frac{1}{2m}\left(\frac{\partial W_2}{\partial q_2}\right)^2 + \frac{1}{2}k_2 q_2^2 &= \alpha_2 \end{aligned}\right\} \alpha_1 + \alpha_2 = \alpha$$

↳ To solve this completely we would find $W_i(q_i, \alpha_i)$ and determine $q_i = \frac{\partial W_i}{\partial \alpha_i}$ and $p_i = \frac{\partial W_i}{\partial q_i}$

Here we only need $p_i = \frac{\partial W_i}{\partial q_i}$, which is much easier.

↳ $$p_i = \frac{\partial W_i}{\partial q_i} = \pm\sqrt{2m\left(\alpha_i - \frac{1}{2}k_i q_i^2\right)}$$

$$J_i = \oint p_i \, dq_i = \oint \sqrt{2m\alpha_i}\sqrt{1 - \underbrace{\frac{k_i}{2\alpha_i}q_i^2}_{0 < x < 1}}\, dq_i$$

$0 < x < 1 \Rightarrow \frac{k_i}{2\alpha_i}q_i^2 = \sin^2\theta$

↳ $dq_i = \sqrt{\frac{2\alpha_i}{k_i}}\cos\theta \, d\theta$

$$= \int_0^{2\pi} 2\alpha_i \sqrt{\frac{m}{k_i}} \overbrace{\cos^2\theta \, d\theta}^{\pi}$$

$$= 2\pi\alpha_i\sqrt{\frac{m}{k_i}} \Rightarrow \alpha_i = \frac{1}{2\pi}\sqrt{\frac{k_i}{m}}\, J_i$$

$$H(J_1, J_2) = \alpha_1 + \alpha_2 = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}}\, J_1 + \frac{1}{2\pi}\sqrt{\frac{k_2}{m}}\, J_2$$

Frequencies are now:

$$\nu_i = \frac{\partial H}{\partial J_i} = \frac{1}{2\pi}\sqrt{\frac{k_i}{m}} = \frac{\omega_i}{2\pi}$$

* Sommerfeld - Wilson quantization revisited :

In original treatment on Sommerfeld - Wilson quantization, postulated that area was quantized.

Now, more accurate formulation :

$$J_i = n_i h$$

For SHO : $H = E = \alpha = \frac{J}{2\pi}\sqrt{\frac{k}{m}}$

$$\hookrightarrow \quad E = \frac{nh}{2\pi}\sqrt{\frac{k}{m}} = n\hbar\omega$$

* Frequencies of the hydrogen atom :

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2\sin^2\theta} + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r}$$

$\hookrightarrow$ Hamilton - Jacobi equation with $S = W_r + W_\varphi + W_\theta - Et$

φ is cyclic $\rightarrow W_\varphi = \underbrace{\alpha_\varphi \varphi}_{\text{form of } W \text{ for cyclic variables}} \rightarrow \frac{\partial W_\varphi}{\partial \varphi} = \alpha_\varphi = p_\varphi$

$$\Rightarrow \frac{1}{2m}\left[\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{r^2}\underbrace{\left(\frac{\alpha_\varphi^2}{\sin^2\theta} + \left(\frac{\partial W_\theta}{\partial \theta}\right)^2\right)}_{\alpha_\theta^2 \text{ (separation)}}\right] - \frac{k}{r} = E$$

$$\begin{cases} \dfrac{1}{2m}\left[\left(\dfrac{\partial W_r}{\partial r}\right)^2 + \dfrac{\alpha_\theta^2}{r^2}\right] - \dfrac{k}{r} = E \\ \text{and } \dfrac{\alpha_\varphi^2}{\sin^2\theta} + \left(\dfrac{\partial W_\theta}{\partial \theta}\right)^2 = \alpha_\theta^2 \end{cases}$$

$$\Rightarrow \begin{cases} J_\varphi = \oint p_\varphi \, d\varphi = \int_0^{2\pi} \alpha_\varphi \, d\varphi = 2\pi \alpha_\varphi \\ J_\theta = \oint p_\theta \, d\theta = \oint \sqrt{\alpha_\theta^2 - \dfrac{\alpha_\varphi^2}{\sin^2\theta}} \, d\theta \\ \quad \overset{\text{(trig. subst.)}}{=} \dots = 2\pi(\alpha_\theta - \alpha_\varphi) \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_\varphi = \dfrac{J_\varphi}{2\pi} \\ \alpha_\theta = \dfrac{J_\varphi + J_\theta}{2\pi} \end{cases}$$

$$J_r = \oint p_r \, dr = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{(J_\varphi + J_\theta)^2}{(2\pi r)^2}} \, dr$$

$$\overset{\text{(contour int.)}}{=} \dots = -(J_\varphi + J_\theta) + \pi k \sqrt{\frac{2m}{-E}}$$

$$\Rightarrow H = E = -\frac{2\pi^2 m k^2}{(J_r + J_\varphi + J_\theta)^2}$$

Frequencies are all degenerate :

$$\nu = \frac{\partial H}{\partial J} = \frac{4\pi^2 m k^2}{(J_r + J_\varphi + J_\theta)^3} = \frac{1}{\pi k}\sqrt{\frac{-2E^3}{m}}$$

Quantization :

$$J_r + J_\varphi + J_\theta = (n_r + n_\varphi + n_\theta)\, h = n\, h$$

$$\Rightarrow \quad E = \frac{-2\pi^2 m k^2}{J^2} = \frac{-2\pi^2 m k^2}{n^2 h^2} \sim \frac{1}{n^2}$$

This is exactly what Bohr and Sommerfeld proposed for the hydrogen atom, but this quantization turned out to be not the right transition to quantum mechanics.