Classical Mechanics (Phys 601) - September 8, 2011

September 20 and 22

options
- have someone else teach these two days
- study on your own with additional office hours
- make up for this later in semester by going faster or adding extra lecture

* Hamilton's Principle

Previously: derived Lagrange's equation from d'Alembert's principle of virtual displacements:

$$\delta W = \sum_i F_i \, \delta x_i = 0 \rightarrow (\text{differential principle})$$

Through change of coordinates: $Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}$

$$\delta W = \sum_i F_i \sum_j \frac{\partial x_i}{\partial q_j} \delta q_j = \sum_j Q_j \, \delta q_j$$

$\Rightarrow$ principle is invariant under change of coordinates

Still, there is a more fundamental approach we can take: Hamilton's principle of stationary action

$$\delta I = \delta \int_1^2 L(q, \dot{q}, t) \, dt = 0$$

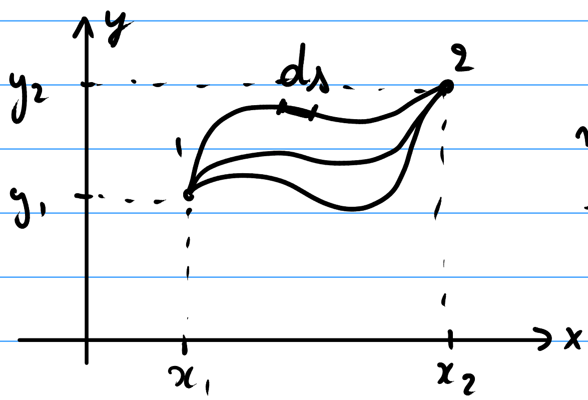
Advantages:
- only relies on $L(q, \dot{q}, t)$, not on virtual work
- integral principle: entire motion is involved

Problem now becomes:

determine $q(t)$, $\dot{q}(t)$ such that $\delta I = 0$, or the action is minimized (or maximized).

* Calculus of variations: one dimensional case

Consider 2D



many curves exist between 1 and 2

Let $y(x)$ describe these curves (single-valued, differentiable) satisfying $y(x_1) = y_1$ and $y(x_2) = y_2$

$$ds^2 = dx^2 + dy^2 = dx^2 + (y')^2 dx^2 = (1+y'^2)\,dx^2$$

$$\Rightarrow ds = \sqrt{1+y'^2}\,dx$$

$\Rightarrow$ length of the curve is $S = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$

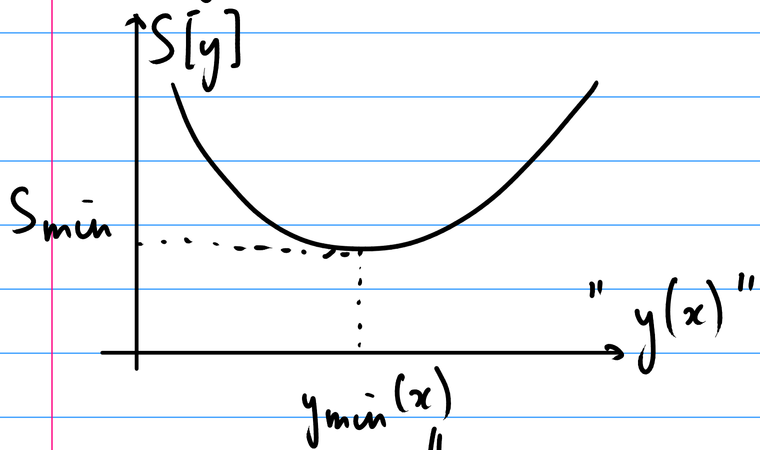
$S[y(x)]$ is a **functional**: function $y(x) \xrightarrow{S[y]}$ real numbers ($\mathcal{Y} \to \mathbb{R}$)

Now: determine the function $y(x)$ such that $S[y(x)]$ is a minimum.

Variational Calculus similar to Calculus

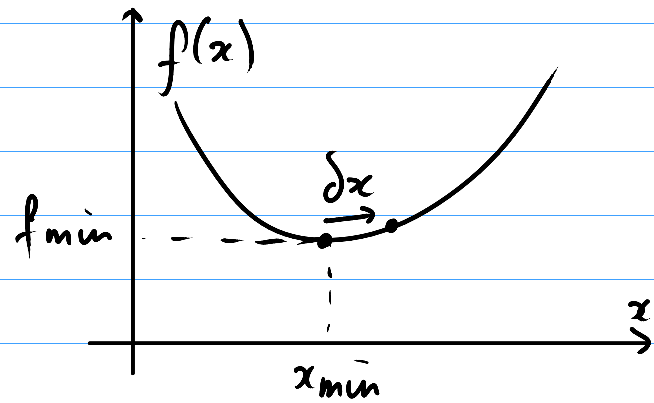
$S[y] : \mathcal{Y} \to \mathbb{R}$

$f(x) : R \to R$



$\Downarrow$

?



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$\frac{df}{dx} = 0$

What is actually happening? Compare neighboring points:

$\delta f = f(x+\delta x) - f(x)$, then let δx go to zero

$\Rightarrow$ minimum when variation $\delta f = 0$

Extremum when $\delta S = 0$ for arbitrary δy

Extremum when $\delta f = 0$ for arbitrary δx

infinitesimal changes in the function $y(x)$ $\Rightarrow$ family $y(x,\varepsilon) = y(x) + \varepsilon\,\eta(x)$

arbitrary function $\eta(x)$, but $\eta(x_1) = \eta(x_2) = 0$

$\downarrow$

Find $y(x)$ such that

$L[y(x,\varepsilon)] > L[y(x)]$, for all ε

Consider : $S[y(x)] = \int_{x_1}^{x_2} F[y, y', x]\, dx$

$$S[y(x,\varepsilon)] = \int_{x_1}^{x_2} F[y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x), x]\, dx$$

$$= \int_{x_1}^{x_2} \left(F[y(x), y'(x), x] + \varepsilon \frac{\partial F}{\partial y}\eta(x) + \varepsilon \frac{\partial F}{\partial y'}\eta'(x) \right) dx + O(\varepsilon^2)$$

For one function $y(x)$ and $\eta(x)$, this is a function of ε: $S(\varepsilon)$

If $y(x)$ is the solution, then $S(\varepsilon)$ must be an extremum for $\varepsilon = 0$ for all $\eta(x)$:

$$\left.\frac{dS}{d\varepsilon}\right|_{\varepsilon=0} = 0 \iff \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta(x) + \underbrace{\frac{\partial F}{\partial y'}\eta'(x)}_{} \right) dx = 0$$

partial integration:

$$\underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\eta(x) \right)}_{\text{total derivative}} - \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)$$

and $\eta(x_1) = \eta(x_2) = 0$

$$\Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \eta(x)\, dx = 0 \quad , \text{ for all } \eta(x)$$

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0 \qquad \text{(Euler-Lagrange equation)}$$

Remember that for the discrete case we did something similar:

$$\sum_j \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) \delta q_j = 0 \ , \ \delta q_j \text{ independent}$$

$$\Longrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Let $I = \int_{x_1}^{x_2} f(x)\, \eta(x)\, dx$,

where $f(x)$ is continuous on $[x_1, x_2]$

If $I = 0$ for all $\eta(x)$ $\Rightarrow$ then $f(x) = 0$, for all x.

Proof by contradiction:

- suppose there exists an $x_0 \in]x_1, x_2[$ for which $f(x_0) > 0$ (same for $f(x_0) < 0$)
- because $f(x)$ is continuous around x_0

$$\exists\, a, b \in]x_1, x_2[: f(x) > 0 \text{ for } a < x < b$$

- we can pick the arbitrary $\eta(x)$:

$$\eta(x) \begin{cases} > 0 & , \ a < x < b \\ = 0 & , \ x < a \text{ or } b < x \end{cases}$$

$$\Rightarrow I = \int_{x_1}^{x_2} f(x)\, \eta(x)\, dx = \int_a^b \overset{>0}{f(x)}\, \overset{>0}{\eta(x)}\, dx > 0$$

$$\longrightarrow f(x) = 0, \ \forall x \ \blacksquare$$

* Calculus of variations : multidimensional case

Introduce notation in one dimensional case :

δS = variation of the functional
$\delta y(x)$ = variation in function $y(x)$ $\sim \varepsilon \eta(x)$

$$\Rightarrow \delta S = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y(x) \, dx = 0$$

Now on to multidimensional functions :

$$S\left[\{y_i(x)\}, \{y_i'(x)\} \right] = \int_{x_1}^{x_2} F\left(\{y_i(x)\}, \{y_i'(x)\}, x \right) dx$$

$\hookrightarrow i = 1, \dots, n$

$$\Rightarrow \delta S = \int_{x_1}^{x_2} dx \sum_i^n \left(\frac{\partial F}{\partial y_i} \delta y_i + \frac{\partial F}{\partial y_i'} \delta y_i' \right) dx$$

$$\overset{\text{parts}}{=} \int_{x_1}^{x_2} dx \sum_i^n \left(\frac{\partial F}{\partial y_i} - \frac{d}{dx} \frac{\partial F}{\partial y_i'} \right) \delta y_i = 0$$

IF all δy_i are independent (holonomic constraints), then we can set all of them zero, except for δy_k

$$\Rightarrow \delta S = \int_{x_1}^{x_2} dx \left(\frac{\partial F}{\partial y_k} - \frac{d}{dx} \frac{\partial F}{\partial y_k'} \right) \delta y_k = 0$$

$$\Leftrightarrow \frac{\partial F}{\partial y_k} - \frac{d}{dx} \frac{\partial F}{\partial y_k} = 0 , \quad k = 1, \dots, n$$

* Shortest distance between two points :

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\, dx$$

$$\Rightarrow S[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

$$\frac{\partial F}{\partial y} = 0 \quad , \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

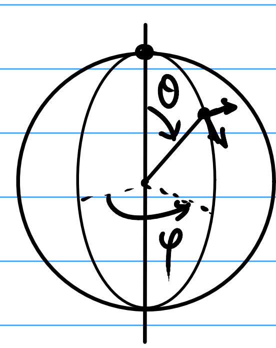
$$\Leftrightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{constant} = c$$

$$\Leftrightarrow y' = \frac{c}{\sqrt{1-c^2}} = a$$

$$\Leftrightarrow y = ax + b$$

* ... on a unit sphere

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$



Rotate system such that point 1 is at the north pole $(\theta = 0)$.

$$\Rightarrow S[\varphi(\theta)] = \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \frac{d\varphi}{d\theta}^2}\, d\theta$$

$$\Rightarrow \quad \frac{\partial F}{\partial \varphi} = 0 \quad , \quad \frac{\partial F}{\partial \varphi'} = \frac{\sin^2\theta \frac{d\varphi}{d\theta}}{\sqrt{1+\sin^2\theta \frac{d\varphi}{d\theta}^2}} = \text{constant}$$

$$\Leftrightarrow \quad \sin^4\theta \, \frac{d\varphi}{d\theta}^2 = c\left(1 + \sin^2\theta \frac{d\varphi}{d\theta}^2\right)$$

- This has to satisfy the initial condition ($\theta_0 = 0$)

$$\Rightarrow \quad c = 0$$

The geodesic $\varphi(\theta)$ now has to satisfy $\sin^2\theta \frac{d\varphi}{d\theta} = 0$

$$\Rightarrow \quad \varphi = \text{constant} = \varphi_0$$

We find the expected solution = great circle

- In general ($\theta_0 \neq 0$) :

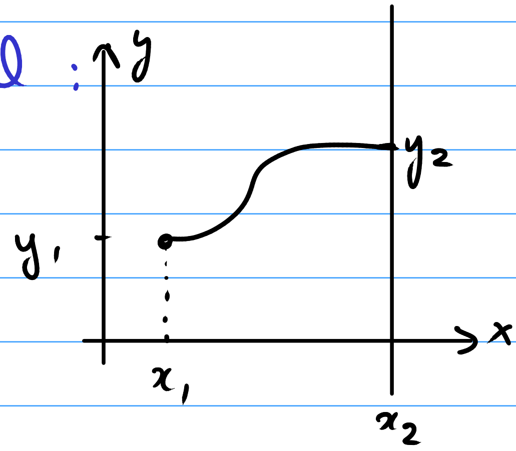
$$\frac{d\varphi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

$$\Rightarrow \quad \varphi = \int \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}} \, d\theta \overset{u = \frac{1}{\tan\theta}}{=} \cos^{-1}\left(\frac{c}{\tan\theta\sqrt{1-c^2}}\right) + \varphi_0$$

$$\Rightarrow \quad \frac{1}{\tan\theta} = \frac{\sqrt{1-c^2}}{c}\cos(\varphi - \varphi_0) \quad = \text{great circle}$$

↳ determine c from initial conditions

* Distance between point and wall :



Boundary condition at x_2
is now $y'(x_2) = 0$

$\Downarrow$

Solve Euler-Lagrange equation
with $y(x_1) = y_1$ and $y'(x_2) = 0$

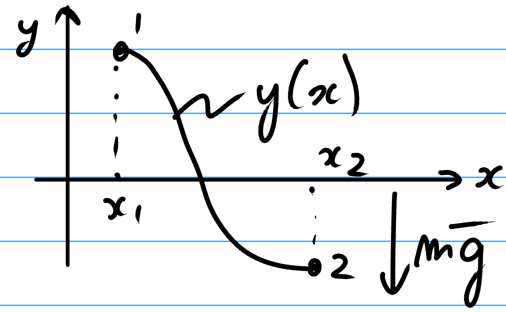
We found $y(x) = Ax + B \rightarrow y'(x_2) = A = 0$

$y(x) = B \rightarrow y(x_1) = y_1$

$\Rightarrow y(x) = y_1$

* Brachistochrone : (motivation for variational calculus)

A bead slides on a wire (frictionless) under the influence of gravity starting from rest. What shape of the wire will lead to the shortest time for the bead to reach the end point ?



Energy : $\frac{1}{2} m v^2 = mgy$

$\Rightarrow v = \sqrt{2gy}$

Time : $dt = \frac{ds}{v} \Rightarrow T[y(x)] = \int_{x_1}^{x_2} dx \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$

$$\Rightarrow \frac{\partial F}{\partial y} = -g\sqrt{\frac{1+y'^2}{2gy^3}}, \qquad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{2gy(1+y'^2)}}$$

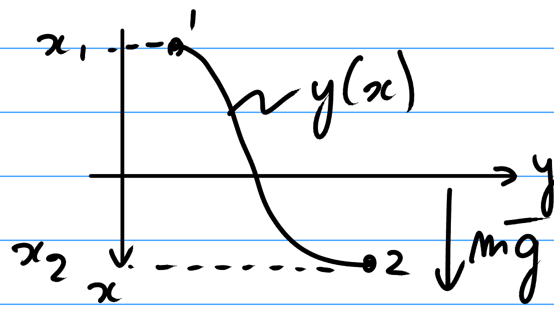
$\hookrightarrow \frac{d}{dt}\frac{\partial F}{\partial y'} \quad \dots$ but gets complicated.

Problems become much simpler if $\frac{\partial F}{\partial y} = 0$, then $\frac{\partial F}{\partial y'} = c$

Here, we picked $m\bar{g}$ in the y direction.

$\Rightarrow$ rotate coordinate system.

$$T[y(x)] = \int_{x_1}^{x_2} dx \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}$$



$$\Rightarrow \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{2gx(1+y'^2)}} = \text{constant} = c$$

$$\hookrightarrow \frac{y'^2}{2gx(1+y'^2)} = c^2 \longrightarrow a > 0$$

$$\Leftrightarrow \quad y'^2 = \underbrace{(2gc^2)}_{=\frac{1}{2a}}\, x(1+y'^2) \qquad 0 < x < 2a$$

$$\Leftrightarrow (2a - x)\, y'^2 = x \Leftrightarrow y'^2 = \frac{x}{2a-x} \;\Big|\; \frac{\cdot x}{\cdot x}$$

$$y(x) = \int dx \frac{x}{\sqrt{2ax - x^2}} \longrightarrow \text{substitute } x = a(1-\cos\theta)$$

$\hookrightarrow 0 < x < 2a$

$dx = a\sin\theta\, d\theta$

$$= \int \frac{a^2 \sin\theta(1-\cos\theta)\, d\theta}{\sqrt{2a^2(1-\cos\theta) - a^2(1-\cos\theta)^2}}$$

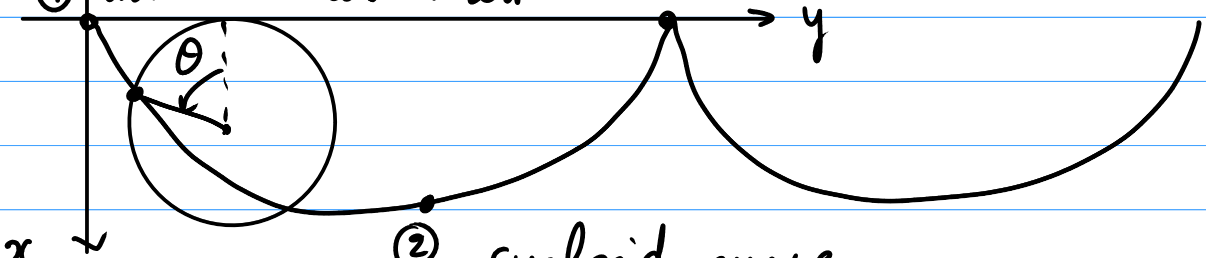
$$= \int \frac{a^2 \sin\theta \,(1-\cos\theta)\, d\theta}{\sqrt{2a^2(1-\cos\theta) - a^2(1+\cos^2\theta - 2\cos\theta)}}$$

$$= \int \frac{a^2 \sin\theta \,(1-\cos\theta)\, d\theta}{\sqrt{a^2 - a^2\cos^2\theta}}$$

$$y = \int a(1-\cos\theta)\, d\theta = a(\theta - \sin\theta)$$

$$\Rightarrow \begin{cases} x = a(1-\cos\theta) \\ y = a(\theta - \sin\theta) \end{cases}$$

① instantaneous at rest



② cycloid curve