Classical Mechanics (Phys 601) - September 20, 2011

Canonical variables (introduced in previous lecture):

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \rightarrow \quad (q_i, p_i) \text{ are canonical variables}$$

Lagrangian mechanics:

$q_i(t)$ is only independent function

↳ second order differential eqn

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Hamiltonian mechanics:

$q_i(t)$ <u>and</u> $p_i(t)$ are independent functions

↳ find differential equations for both $q_i(t)$ and $p_i(t)$

Remember how we got to the (Euler) Lagrange equation:

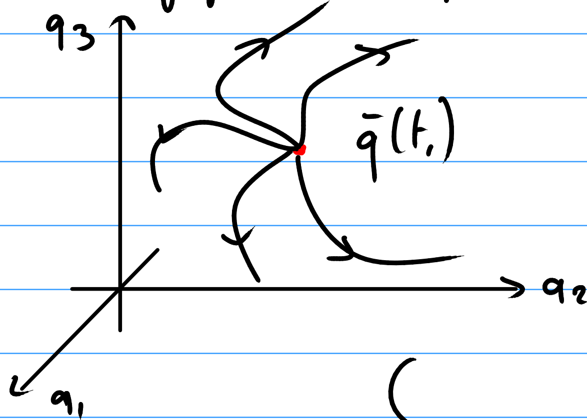
$$S[q_i(t)] = \int_{t_1}^{t_2} L \, dt$$

$$\delta S = 0 \iff \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

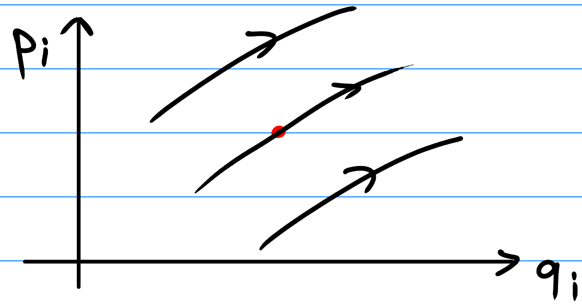
This second order differential equation requires that we specify not only $q_i(t_1)$, but also $\dot{q}_i(t_1)$ to determine how the system will evolve in time!

$\Rightarrow$ use p_i as independent "coordinate"

In configuration space : many possible trajectories go through one point



In phase space :



only one trajectory through each point in phase space

paths do NOT intersect either

* Hamilton's equations

Recall : $H = \sum_j p_j \dot{q}_j - L$

and $p_j = \dfrac{\partial L}{\partial \dot{q}_j} = p_j(q_i, \dot{q}_i, t)$

$\dot{q}_j = \dot{q}_j(q_i, p_i, t)$

To find the Hamiltonian, we determined the generalized momenta p_i and inverted those expressions to find $\dot{q}$ as a function of q_i, p_i and t.

$$\Rightarrow \quad H = \sum_j p_j \, \dot{q}_j(q_i, p_i, t) - L\left(q_i, \dot{q}_i(q_j, p_j, t), t\right)$$

$$= H(q_i, p_i, t)$$

Total differential in terms of $q_i, \dot{q}_i, t$:

$$dH = \sum_i^n \left(\dot{q}_i \, dp_i + \cancel{p_i \, d\dot{q}_i} \right)$$

$$- \sum_i \frac{\partial L}{\partial q_i} dq_i - \cancel{\sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{= p_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i = \frac{\partial L}{\partial q_i}$

$$dH = \sum_i \left(\dot{q}_i \, dp_i - \dot{p}_i \, dq_i \right) - \frac{\partial L}{\partial t} dt$$

Total differential in terms of q_i, p_i, t:

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = - \frac{\partial H}{\partial q_i}$$

Hamilton's equations of motion

Two first order differential equations for the independent functions q_i and p_i of time.

Notice also $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, and even

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} \right) + \frac{\partial H}{\partial t}$$

$$= \sum_i \left(\dot{p}_i \dot{q}_i - \dot{q}_i \dot{p}_i \right) + \frac{\partial H}{\partial t}$$

$$\Rightarrow \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

All time dependence in the Hamiltonian is explicit, even though q_i and p_i vary with time.

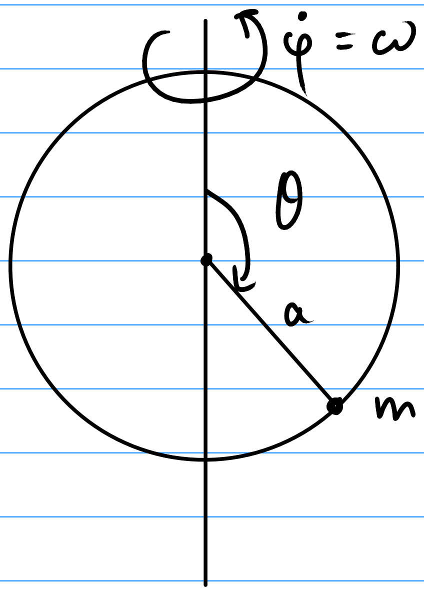
By construction $\left(H = \sum p_i \dot{q}_i - L \right)$ the $\frac{\partial}{\partial t}$ of L and H are equal and opposite.

Again, if $\frac{\partial L}{\partial t} = 0$ or $\frac{\partial H}{\partial t} = 0 \Rightarrow$ H is a constant of motion.

In the special case of conservative systems with $V(q_i)$ and time-independent constraints, we have shown that

$$\sum p_i \dot{q}_i = 2T \quad \Rightarrow \quad H = T + V = E = \text{constant}$$

* Example (but, really, Hamilton's equations are more for theoretical applications)



$\rightarrow$ previously we have shown:

$$L = T - V = \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \omega^2 \sin^2 \theta \right) - a m g \cos \theta$$

and $$H = \frac{1}{2} m a^2 \left(\dot{\theta}^2 - \omega^2 \sin^2 \theta \right) + a m g \cos \theta$$

$\hookrightarrow \neq E$ (constraint time-dependent)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m a^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m a^2}$$

$$\Rightarrow H = \frac{1}{2} m a^2 \left(\frac{p_\theta^2}{(m a^2)^2} - \omega^2 \sin^2 (\theta) \right) + a m g \cos \theta$$

$$H = \frac{p_\theta^2}{2 m a^2} - \frac{1}{2} m a^2 \omega^2 \sin^2 \theta + a m g \cos \theta$$

$$\begin{cases} \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = m a^2 \omega^2 \sin\theta \cos\theta + a m g \sin \theta \\ \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{m a^2} \end{cases}$$

$\Rightarrow$ two coupled first order differential equations

* Cyclic variables in Hamiltonian mechanics:

Hamilton's equations: $\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial H}{\partial p_i} \end{cases}, i = 1, \ldots, n$

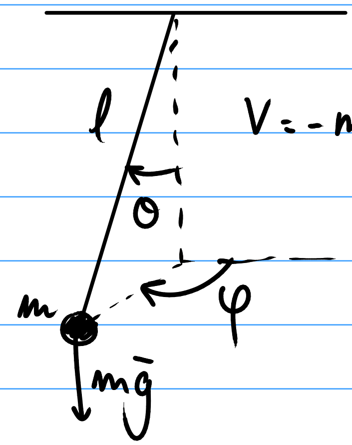
If $\dfrac{\partial H}{\partial q_j} = 0$, then p_j = constant $= \omega_j$

$$H \to \tilde{H}(q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n, \; p_1, \ldots, p_{j-1}, \omega_j, p_{j+1}, \ldots, p_n, t)$$

$$\Rightarrow \quad \dot{q}_j = \frac{\partial \tilde{H}}{\partial \omega_j} \quad \Rightarrow \quad q_j = \int dt \, \frac{\partial \tilde{H}}{\partial \omega_j} + q_{j,0}$$

and $\begin{cases} \dot{p}_i = -\dfrac{\partial \tilde{H}}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial \tilde{H}}{\partial p_i} \end{cases}, i = 1, \ldots, j-1, j+1, \ldots, n$ $\Rightarrow$ two variables eliminated!

Example: spherical pendulum



$$L = \frac{1}{2} m \ell^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + mg\ell\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta}, \quad p_\varphi = m\ell^2 \sin^2\theta \, \dot{\varphi}$$

$$\Rightarrow H = \frac{1}{2}\left(\frac{p_\theta^2}{m\ell^2} + \frac{p_\varphi^2}{m\ell^2 \sin^2\theta} \right) - mg\ell\cos\theta$$

$$\frac{\partial H}{\partial \varphi} = 0 \to \begin{cases} p_\varphi = \omega \Rightarrow \tilde{H} = \dfrac{1}{2}\left(\dfrac{p_\theta^2}{m\ell^2} + \dfrac{\omega^2}{m\ell^2\sin^2\theta} \right) - mg\ell\cos\theta \\ \varphi = \displaystyle\int dt \, \frac{\omega}{m\ell^2 \sin^2\theta} + \varphi_0 \end{cases}$$

$\Rightarrow \varphi$

* Variational derivation of Hamilton's equations :

Hamilton's principle stated $\delta S = 0$
with action $S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$

Because $L = \sum_i p_i \dot{q}_i - H(q_i, p_i, t)$

$$\delta S = \delta \int_{t_1}^{t_2} dt \left[\sum_i p_i \dot{q}_i - H(q_i, p_i, t) \right]$$

$$= \int_{t}^{t_2} dt \sum_i \left(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right)$$

↳ now we allow q_i and p_i to vary independently

Consider that $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$

and $p_i \frac{d}{dt} \delta q_i = \frac{d}{dt}(p_i \delta q_i) - \dot{p}_i \delta q_i$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} dt \sum_i \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right]$$

$$+ \cancel{\sum_i p_i \delta q_i \Big|_{t_1}^{t_2}}$$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

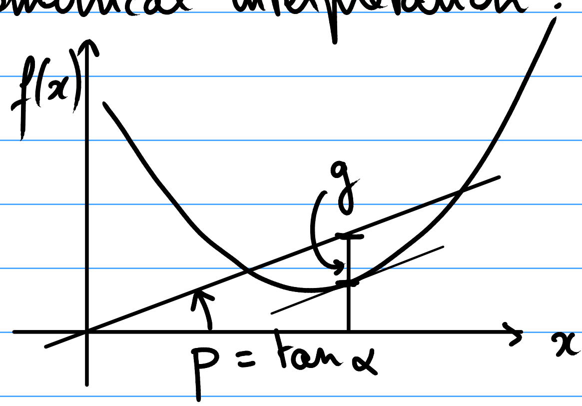
* Relation between L and H :

Legendre transformation of $f(x)$ to $g(p)$

$$g(p) = \mathcal{L}(f(x)) = x(p)\,p - f(x(p))$$

with $p = \dfrac{\partial f}{\partial x}$

Geometrical interpretation :



For $f(x)$ convex $(f''(x) > 0)$ find $x(p)$ such that the distance g between $f(x)$ and the line is maximal.

$$g = xp - f(x)$$

$$\frac{dg}{dx} = p - f'(x) = 0$$

$$\Leftrightarrow \quad p = f'(x)$$

Example : $f(x) = x^2 \rightarrow$ find $g(p) = \mathcal{L}(f(x))$

$$p = f'(x) = 2x \Rightarrow x(p) = \frac{p}{2}$$

$$g(p) = x(p) \cdot p - f(x(p)) = \frac{p^2}{2} - \left(\frac{p}{2}\right)^2 = \frac{1}{4}p^2$$

Property : if $f(x)$ is convex then also $g(p)$ convex

$$g'(p) = \frac{dx}{dp} \cdot p + x(p) - f'(x(p)) \frac{dx}{dp} = x(p)$$

$$g''(p) = \frac{dx}{dp}$$

If $f''(x) > 0$ with $p = f'(x)$, then $\frac{dp}{dx} > 0 \Rightarrow \frac{dx}{dp} > 0$

Property : $\mathcal{L}^2 = 1$ → Legendre transform of Legendre transform is identity operation

$$f(x) \xrightarrow{\mathcal{L}} g(p) = x(p) \cdot p - f(x(p)) \quad \text{with } p = f'(x)$$
$$\hookrightarrow x = x(p)$$

$$g(p) \xrightarrow{\mathcal{L}} h(x) = x \cdot p(x) - g(p(x)) \quad \text{with } x = g'(p) = x(p)$$

$$\Rightarrow h(x) = x \cdot p(x) - x(p(x)) \cdot p(x) + f(x(p(x)))$$
$$= f(x(p(x))) = f(x)$$

→ needs some more justification but is intuitively clear

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$$\mathcal{L}^2(f(x)) = f(x)$$

and $\mathcal{L}^{-1}(f(x)) = \mathcal{L}(f(x))$ → self-inverse

+ Legendre transform in thermodynamics:

U = internal energy of a system $= U(S,V)$

S = entropy

V = volume

$$\Rightarrow \quad dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV = T dS - P dV$$

$$T = \frac{\partial U}{\partial S} \qquad P = -\frac{\partial U}{\partial V}$$

Introduce Helmholtz free energy using Legendre transform:

$$F(T,V) = U(S,V) - TS \qquad \text{with } T = \frac{\partial U}{\partial S}$$

$$\Rightarrow dF = \cancel{\frac{\partial U}{\partial S} dS} + \frac{\partial U}{\partial V} dV \cancel{- T dS} - S dT$$

$$= \frac{\partial F}{\partial T} dT + \frac{\partial F}{\partial V} dV \qquad \text{with } \begin{cases} \frac{\partial F}{\partial V} = \frac{\partial U}{\partial V} = -P \\ \frac{\partial F}{\partial T} = -S \end{cases}$$

$$= -S dT - P dV$$

Introduce enthalpy $H(S,P)$ (or Gibbs free energy $G(T,P)$)

$$H(S,P) = U(S,V) + PV \qquad \text{with } P = -\frac{\partial U}{\partial V}$$

$$\Rightarrow dH = \frac{\partial U}{\partial S} dS + \cancel{\frac{\partial U}{\partial V} dV} + \cancel{P dV} + V dP$$

$$= T dS + V dP \qquad \text{with } \begin{cases} T = \frac{\partial H}{\partial S} \\ V = \frac{\partial H}{\partial P} \end{cases}$$