Classical Mechanics (Phys 601) - November 15, 2011

* ## Top without torque :

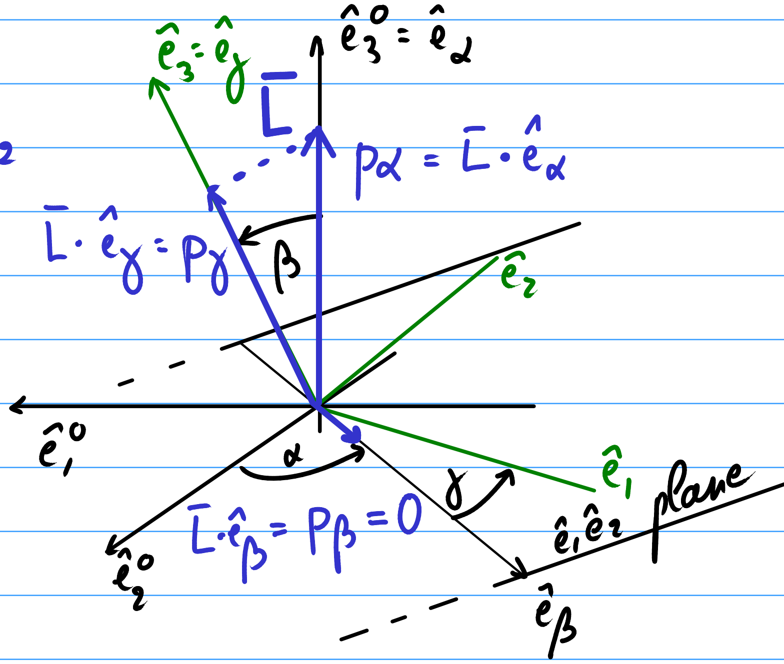
In general case $I_1 \neq I_2 \neq I_3 \neq I_1$

Euler angles α, β, γ
↓
angular velocity $\vec{\omega}$ in principal axes frame centered at center of mass

$$\Rightarrow T = \frac{1}{2} I_1 (-\dot{\alpha} \sin\beta \cos\gamma + \dot{\beta} \sin\gamma)^2 + \frac{1}{2} I_2 (\dot{\alpha} \sin\beta \sin\gamma + \dot{\beta} \cos\gamma)^2 + \frac{1}{2} I_3 (\dot{\alpha} \cos\beta + \dot{\gamma})^2 + \frac{1}{2} M V^2$$

For symmetric top $I_1 = I_2$

$$T = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos\beta + \dot{\gamma})^2 + \frac{1}{2} M V^2$$



↳ α and γ are cyclic
$\Rightarrow$ p_α and p_γ are constant

$$p_\alpha = \bar{L} \cdot \hat{e}_\alpha \,, \quad p_\beta = \bar{L} \cdot \hat{e}_\beta \,, \quad p_\gamma = \bar{L} \cdot \hat{e}_\gamma = I_3 \omega_3$$

$$\bar{L} = p_\alpha \hat{e}_\alpha + p_\beta \hat{e}_\beta + p_\gamma \hat{e}_\gamma = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

When $\bar{L}$ along $\hat{e}_3^0 = \hat{e}_\alpha$ in the inertial frame :

$p_\alpha = \bar{L} \cdot \hat{e}_\alpha = |\bar{L}| = \text{constant}$

$p_\gamma = \bar{L} \cdot \hat{e}_\gamma = |L| \cos\beta = \text{constant} \Rightarrow \beta$ itself is constant

$\Rightarrow$ $\hat{e}_3$ precesses around $\hat{e}_3^0 \sim \bar{L}$ at rate $\dot{\alpha}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = I_1 \ddot{\beta} = 0 \Leftrightarrow \frac{\partial L}{\partial \beta} = 0 \Leftrightarrow \dot{\alpha}\cos\beta = \frac{p_\gamma}{I_3} = \text{constant}$$

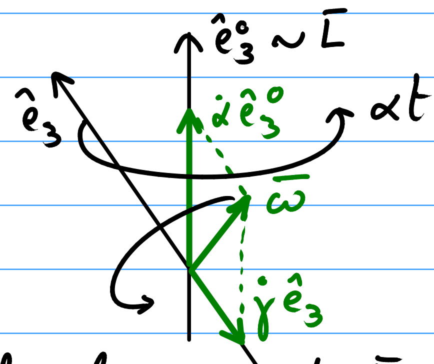
$\Rightarrow$ rate of precession $\dot{\alpha}$ is constant : $\dot{\alpha} = \dfrac{\omega_3}{\cos\beta}$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) \Rightarrow \dot{\gamma} = -\Omega$$

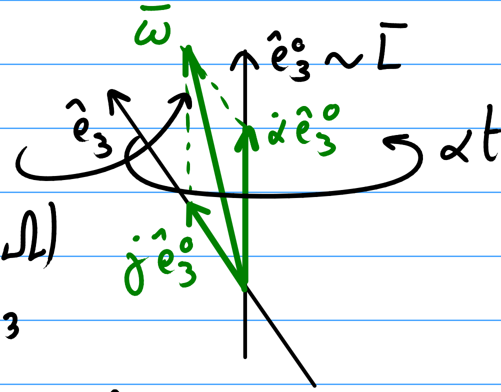
$\Rightarrow$ body rotates around $\hat{e}_3$ with $\Omega = \omega_3 \dfrac{I_3 - I_1}{I_1}$

Angular velocity :

$\bar{\omega} = \dot{\alpha}\,\hat{e}_3^0 + \dot{\gamma}\,\hat{e}_3$:



* $I_3 > I_1 \rightarrow \Omega > 0 \rightarrow \dot{\gamma} < 0$
  (oblate)
  - axis of symmetry $\hat{e}_3$ rotates uniformly around $\bar{L}$
  - body rotates uniformly (with $-\Omega$) in opposite direction around $\hat{e}_3$

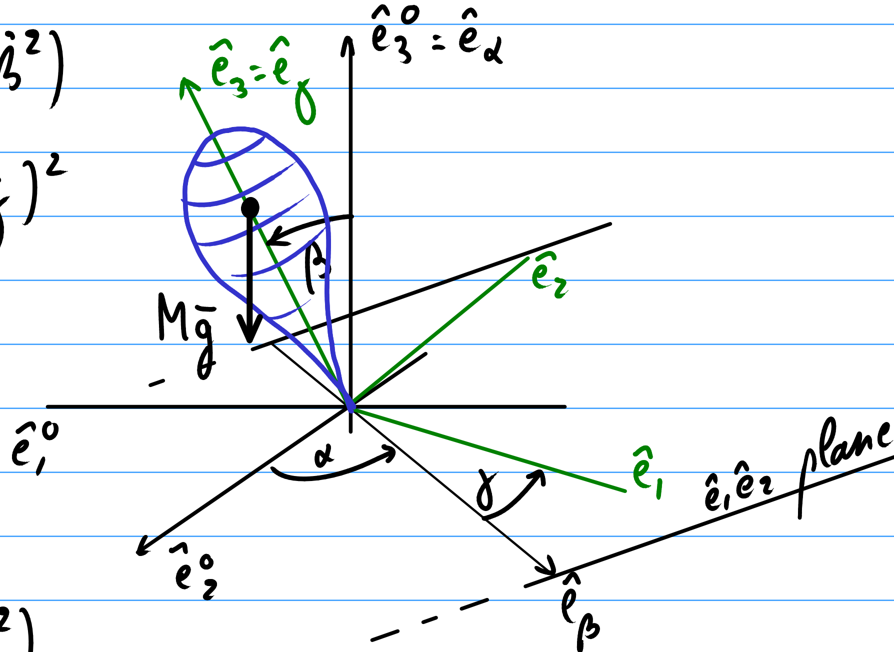


* $I_3 < I_1 \rightarrow \Omega < 0 \rightarrow \dot{\gamma} > 0$
  (prolate)
  - body rotates uniformly (with Ω) in same direction around $\hat{e}_3$

$$\Rightarrow \begin{cases} \dot{\alpha} \text{ is rate of precession of principal axis } \hat{e}_3 \text{ around } \hat{e}_3^0 \sim \bar{L} \\ \dot{\gamma} \text{ is rate of rotation around principal axis } \hat{e}_3 \end{cases}$$

* Symmetric top in gravitational field:

Take tip of top fixed and origin of body frame $\Rightarrow V=0$



$$T = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})^2$$

$$V = Mg\, l \cos\beta$$

$$\Downarrow$$

$$L = T - V$$

$$= \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})^2 - Mg\, l\cos\beta$$

$\Rightarrow$ α and γ are cyclic $\Rightarrow$ p_α and p_γ constant

$$\begin{cases} p_\alpha = I_1 \dot{\alpha} \sin^2\beta + I_3(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta = \text{constant} \\ p_\gamma = I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) = I_3\omega_3 = \text{constant} \end{cases}$$

Elimination gives:

$$\begin{cases} \dot{\alpha} = \dfrac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta} = \text{rate of precession of } \hat{e}_3 \\ \dot{\gamma} = \dfrac{p_\gamma}{I_3} - \dfrac{(p_\alpha - p_\gamma\cos\beta)\cos\beta}{I_1\sin^2\beta} = \text{angular velocity of rotation around } \hat{e}_3 \end{cases}$$

Hamiltonian is conserved $\left(\frac{\partial L}{\partial t} = 0\right)$:

$$H = E = \frac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2\right) + \frac{p_\gamma^2}{2 I_3} + Mgl\cos\beta$$

$$= \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2 I_1 \sin^2\beta} + \frac{p_\gamma^2}{2I_3} + Mgl\cos\beta$$

↳ p_α, p_γ are constants

$$= \frac{1}{2} I_1 \dot{\beta}^2 + V_{eff}(\beta) = \frac{p_\beta^2}{2I_1} + V_{eff}(\beta)$$

with the effective potential energy $V_{eff}(\beta)$

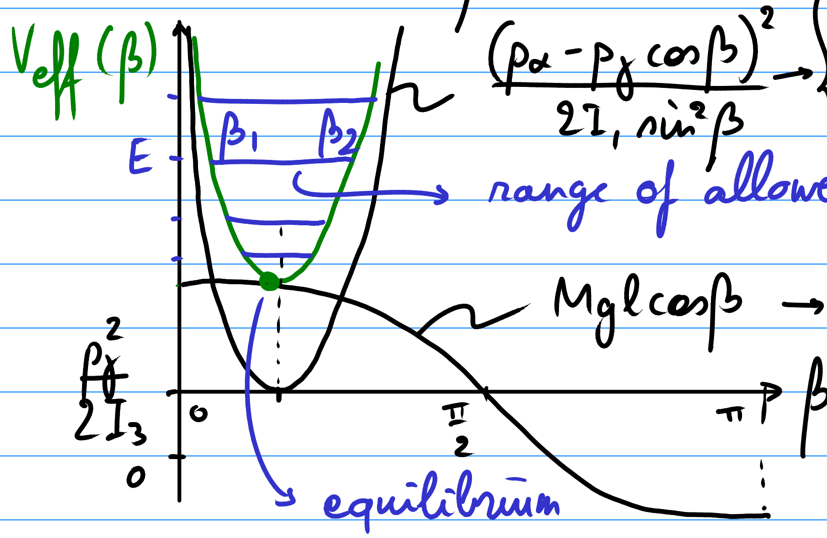
$$V_{eff}(\beta) = \underbrace{Mgl\cos\beta}_{\text{gravity}} + \underbrace{\frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3}}_{\text{centrifugal force}}$$

Hamilton's equations for β are now :

$$\dot{\beta} = \frac{\partial H}{\partial p_\beta} = \frac{p_\beta}{I_1} \quad \text{and} \quad \dot{p}_\beta = -\frac{\partial H}{\partial \beta} = -\frac{\partial V_{eff}}{\partial \beta}$$

$$\Leftrightarrow \quad I_1 \ddot{\beta} = -\frac{\partial V_{eff}}{\partial \beta}$$

$$\frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} \rightarrow \begin{cases} +\infty \text{ when } \beta \to 0 \\ \text{zero when } \cos\beta = \frac{p_\alpha}{p_\gamma} \\ +\infty \text{ when } \beta \to \pi \end{cases}$$



range of allowed β for fixed E

$Mgl\cos\beta \rightarrow$ shifts equilibrium to $\beta < \text{acos}\frac{p_\alpha}{p_\gamma}$

Qualitative description of motion

Motion in β constrained between two "turning points" where

$$V_{eff}(\beta_1) = E = V_{eff}(\beta_2)$$

Solution for arbitrary energy:

$$E = \frac{1}{2} I_1 \dot\beta^2 + V_{eff}(\beta) \iff \frac{d\beta}{dt} = \dot\beta = \frac{2}{I_1}\sqrt{E - V_{eff}(\beta)}$$

$$\Rightarrow \quad t - t_0 = \frac{2}{I_1}\int \frac{d\beta}{\sqrt{E - V_{eff}(\beta)}} \quad \Rightarrow \quad \beta(t)$$

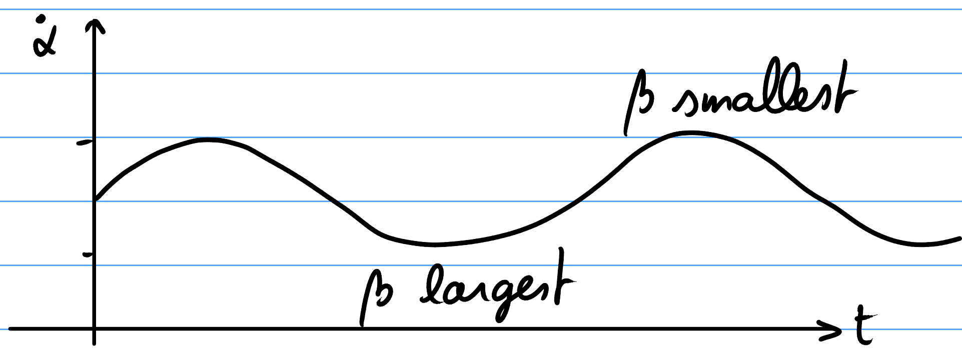
For $E - V_{eff}(\beta) = \text{constant}$ (equilibrium, minimum of V_{eff})

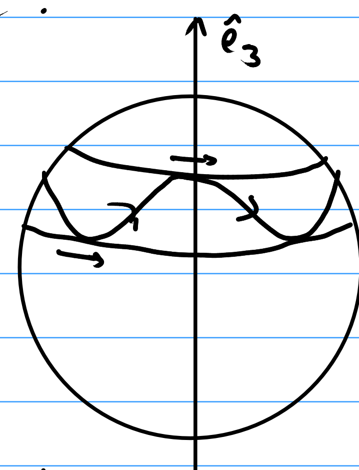
$\Rightarrow \dot\beta = 0 \Rightarrow \beta$ is constant

Because $\dot\alpha = \dfrac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}$ and β varies between β_1 and β_2

$\Rightarrow \dot\alpha$ varies with time $\Rightarrow$ nutation

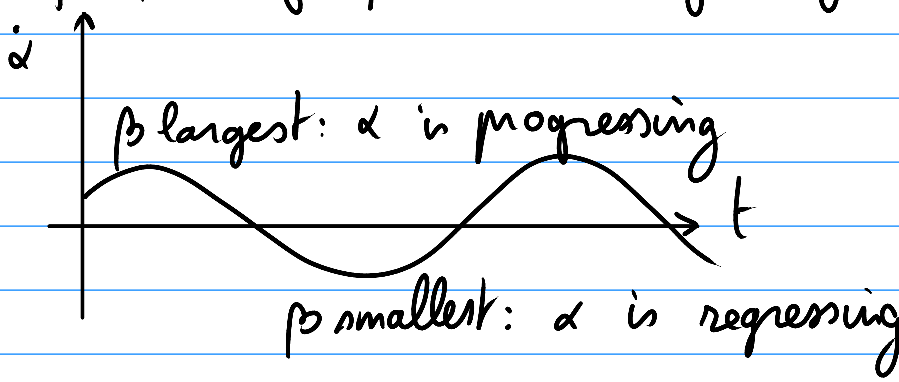
If $(p_\alpha - p_\gamma \cos\beta)$ doesn't change sign:

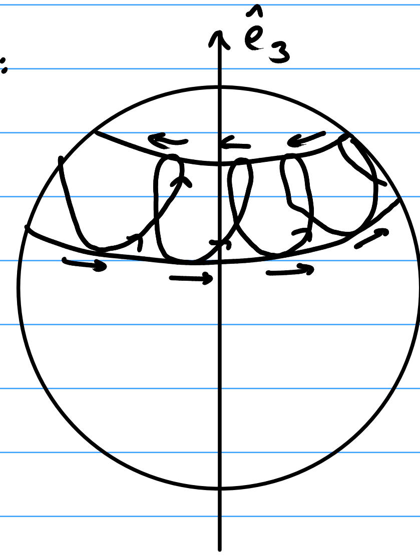




↳ if axis gets closer to I → rotation speeds up

If $(p_\alpha - p_\gamma \cos\beta)$ does change sign:





Equilibrium of effective potential:

$$V_{eff}(\beta) = Mgl\cos\beta + \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3}$$

$$\Rightarrow \frac{\partial V_{eff}}{\partial\beta} = -Mgl\sin\beta + \frac{(p_\alpha - p_\gamma\cos\beta)p_\gamma}{I_1\sin\beta} + \frac{(p_\alpha - p_\gamma\cos\beta)^2\cos\beta}{I_1\sin^3\beta}$$

$$\Rightarrow (p_\alpha - p_\gamma\cos\beta)p_\gamma\sin^2\beta + (p_\alpha - p_\gamma\cos\beta)^2\cos\beta + MglI_1\sin^4\beta = 0$$

Solution for $u = p_\alpha - p_\gamma\cos\beta$ assuming $\beta \neq 0$:

$$u = \frac{p_\gamma\sin^2\beta \pm \sqrt{p_\gamma^2\sin^4\beta - 4MglI_1\sin^4\beta\cos\beta}}{2\cos\beta}$$

$$u = \frac{p_\gamma\sin^2\beta}{2\cos\beta}\left(1 \pm \sqrt{1 - \frac{4MglI_1\cos\beta}{p_\gamma^2}}\right)$$

To have a real solution, this implies:

$$p_\gamma = I_3\omega_3 \geq 2\sqrt{Mg\ell I_1\cos\beta}$$

$\Rightarrow$ for a stable equilibrium to exist, the top has to spin at an angular velocity ω_3 faster than

$$\omega_3 \geqslant \frac{2}{I_3}\sqrt{Mg\ell I_1\cos\beta} = \omega_3^c$$

When the top has a stable equilibrium, it has two solutions u_+ and u_- : $u_\pm = p_\alpha - p_\gamma\cos\beta_0^\pm$

$$\dot\alpha = \frac{p_\alpha - p_\gamma\cos\beta_0^\pm}{I_1\sin^2\beta_0^\pm} = \frac{u^\pm}{I_1\sin^2\beta_0^\pm}$$

For a fast spinning top ($p_\gamma = L_3 = I_3\omega_3$ large) :

$$u = \frac{p_\gamma\sin^2\beta_0}{2\cos\beta_0}\left(1 \pm \sqrt{1 - \frac{4Mg\ell I_1\cos\beta_0}{p_\gamma^2}}\right)$$

$$\simeq \frac{p_\gamma\sin^2\beta_0}{2\cos\beta_0}\left(1 \pm \left(1 - \frac{2Mg\ell I_1\cos\beta_0}{p_\gamma^2}\right)\right)$$

$$\begin{cases} u_+ = \dfrac{p_\gamma\sin^2\beta_0}{\cos\beta_0} \rightarrow \dot\alpha = \dfrac{p_\gamma}{I_1\cos\beta_0} = \dfrac{I_3\omega_3}{I_1\cos\beta_0} \quad \text{(fast)} \\ u_- = \dfrac{Mg\ell I_1\sin^2\beta_0}{p_\gamma} \rightarrow \dot\alpha = \dfrac{Mg\ell}{p_\gamma} = \dfrac{Mg\ell}{I_3\omega_3} \quad \text{(slow)} \end{cases}$$

* Oscillations around equilibrium: general treatment

$$V_{eff}(\beta) = Mgl\cos\beta + \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3}$$

$$\frac{\partial V_{eff}}{\partial\beta} = -Mgl\sin\beta + \frac{(p_\alpha - p_\gamma\cos\beta)\,p_\gamma}{I_1\sin\beta} + \frac{(p_\alpha - p_\gamma\cos\beta)^2\cos\beta}{I_1\sin^3\beta}$$

Take now: $\beta = \beta_0 + \eta(t)$

$$\Rightarrow \quad V_{eff}(\beta) = \underbrace{V_{eff}(\beta_0)}_{\text{constant}} + \underbrace{\cancel{\eta(t)\left.\frac{\partial V_{eff}}{\partial\beta}\right|_{\beta_0}}}_{\text{zero at equilibrium}} + \frac{\eta^2(t)}{2}\left.\frac{\partial^2 V_{eff}}{\partial\beta^2}\right|_{\beta_0} + \cdots$$

After some work:

$$\left.\frac{\partial^2 V_{eff}}{\partial\beta^2}\right|_{\beta_0} = \frac{p_\alpha p_\gamma - MglI_1(4-3\sin^2\beta_0)}{I_1\cos\beta_0}$$

$$\Rightarrow H = \frac{1}{2}I_1\dot{\eta}^2 + V_{eff}(\beta_0) + \frac{1}{2}\eta^2\left.\frac{\partial^2 V_{eff}}{\partial\beta^2}\right|_{\beta_0}$$

$\Rightarrow$ harmonic oscillator with frequency Ω:

$$\Omega^2 = \frac{1}{I_1}\left.\frac{\partial^2 V_{eff}}{\partial^2\beta}\right|_{\beta_0} = \frac{p_\alpha p_\gamma - MglI_1(4-3\sin^2\beta_0)}{I_1^2\cos\beta_0}$$

$$\beta(t) = \beta_0 + \eta_0\cos(\Omega t + \varphi_0)$$

Now use $\beta(t)$ to expand $\dot{\alpha}(t)$ in $\eta(t)$

$$\dot{\alpha}(t) = \frac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}$$

$$= \frac{p_\alpha - p_\gamma \cos\beta_0}{I_1 \sin^2\beta_0} + \eta(t) \frac{\partial}{\partial\beta}\left(\frac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}\right)_{\beta_0} + \dots$$

$$= (\dot{\alpha})_0 + \eta(t)(\dot{\alpha})_1 + \dots$$

$\Rightarrow$ if $\eta(t)$ changes harmonically $\rightarrow$ also $\dot{\alpha}(t)$ will change harmonically

Rate of rotation of body around $\hat{e}_3$:

$\dot{\gamma}(t) = \frac{p_\gamma}{I_3} - \frac{(p_\alpha - p_\gamma \cos\beta)\cos\beta}{I_1 \sin^2\beta}$ also changes with β

$$\dot{\gamma}(t) = \underbrace{(\dot{\gamma})_0}_{\text{constant rate of rotation}} + \eta(t) \underbrace{(\dot{\gamma})_1}_{\text{harmonic amplitude}} + \dots$$

For a fast top: $p_\gamma \gg p_\alpha \Rightarrow \dot{\gamma}(t)$ is always positive.

* Vertical axis of symmetry ($\beta = 0$)

If $\beta = 0 \to$ rotation around vertical symmetry axis

$\hookrightarrow \hat{e}_3^0 = \hat{e}_3 \longrightarrow p_\alpha = \bar{L} \cdot \hat{e}_3^0 = \bar{L} \cdot \hat{e}_3 = p_\gamma$

When is this rotation stable?

1) using formula obtained for $\beta \neq 0$:

$$p_\gamma^2 > 4 M g \ell I_1 \cos\beta \quad \Rightarrow \quad \omega_3 > \frac{2}{I_3} \sqrt{M g \ell I_1}$$

2) positive second derivative of effective potential energy:

$$\left.\frac{\partial^2 V_{eff}}{\partial \beta^2}\right|_{\beta_0} = \frac{p_\alpha p_\gamma - M g \ell I_1 (4 - 3 \sin^2\beta_0)}{I_1 \cos\beta_0} > 0$$

$$\Rightarrow \quad p_\gamma^2 > 4 M g \ell I_1$$

3) expand $V_{eff}(\beta)$ for small β:

$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2 I_1 \sin^2\beta} + \frac{p_\gamma^2}{2 I_3} + M g \ell \cos\beta$$

$$\simeq \frac{\left(p_\alpha - p_\gamma \left(1 - \frac{\beta^2}{2}\right)\right)^2}{2 I_1 \beta^2} + \frac{p_\gamma^2}{2 I_3} + M g \ell \left(1 - \frac{\beta^2}{2}\right)$$

$$= \frac{p_\gamma^2}{2 I_1 \beta^2} \left(\frac{\beta^2}{2}\right)^2 + \frac{p_\gamma^2}{2 I_3} + M g \ell - \frac{1}{2} M g \ell \beta^2$$

$$= M g \ell + \frac{p_\gamma^2}{2 I_3} + \left(\frac{1}{8} \frac{p_\gamma^2}{I_1} - \frac{1}{2} M g \ell\right) \beta^2$$

$$\Rightarrow \quad p_\gamma^2 > 4 M g \ell I_1$$

Top will slow down gradually until $\omega_3^c \to$ unstable