Homework Assignment 2

① $$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}\right)$$

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\frac{dF}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt}\right) + \frac{\partial}{\partial q_i}\frac{dF}{dt}$$

$$\frac{dF}{dt} = \underbrace{\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}}_{\text{only depends on } q_i, t} \quad \text{only depends on } q_i, t$$

1) $$\frac{\partial}{\partial \dot{q}_i}\frac{dF}{dt} = \frac{\partial F}{\partial q_i} \Rightarrow \frac{d}{dt}\frac{\partial F}{\partial q_i} = \sum_j \frac{\partial}{\partial q_j}\frac{\partial F}{\partial q_i}\dot{q}_j + \frac{\partial}{\partial t}\frac{\partial F}{\partial q_i}$$

2) $$\frac{\partial}{\partial q_i}\frac{dF}{dt} = \sum_j \frac{\partial}{\partial q_i}\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial}{\partial q_i}\frac{\partial F}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = -\sum_j \frac{\partial}{\partial q_j}\frac{\partial F}{\partial q_i}\dot{q}_j - \frac{\partial}{\partial t}\frac{\partial F}{\partial q_i} + \sum_j \frac{\partial}{\partial q_i}\frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial}{\partial q_i}\frac{\partial F}{\partial t}$$

$$= 0$$

② $L = T - U = T - e\varphi + e\bar{v}\cdot\bar{A}$

$L' = T - e\varphi' + e\bar{v}\cdot\bar{A}'$

$= T - e\varphi + e\frac{\partial\psi}{\partial t} + e\bar{v}\cdot\bar{A} + e\bar{v}\cdot\bar{\nabla}\psi$

$= L + e\frac{\partial\psi}{\partial t} + e\bar{v}\cdot\bar{\nabla}\psi$

$= L + e\frac{d\psi}{dt}$

because $\frac{d\psi}{dt} = \sum_i \frac{\partial\psi}{\partial x_i}\dot{x}_i + \frac{\partial\psi}{\partial t} = \bar{\nabla}\psi\cdot\bar{v} + \frac{\partial\psi}{\partial t}$

③ $T = \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}m\left((\dot{q}_2 + \dot{q}_1\cos\alpha)^2 + \dot{q}_1^2\sin^2\alpha\right)$

$V = -mgq_1\sin\alpha$

$L = T - V = \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2}m\dot{q}_1^2 + m\dot{q}_1\dot{q}_2\cos\alpha + mgq_1\sin\alpha$

q_1: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_1}\right) = m\ddot{q}_1 + m\ddot{q}_2\cos\alpha$

$\frac{\partial L}{\partial q_1} = mg\sin\alpha$

$\Rightarrow \quad m\ddot{q}_1 + m\ddot{q}_2\cos\alpha - mg\sin\alpha = 0$

q_2: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_2}\right) = (M+m)\ddot{q}_2 + m\ddot{q}_1\cos\alpha$

$\Rightarrow \quad (M+m)\ddot{q}_2 + m\ddot{q}_1\cos\alpha = 0 \Leftrightarrow \ddot{q}_2 = -\frac{m}{M+m}\ddot{q}_1\cos\alpha$

Eliminate $q_2 \Rightarrow \left(1 - \frac{m\cos\alpha}{M+m}\right)\ddot{q}_1 = g\sin\alpha$

$$\Rightarrow \quad q_1 = \frac{1}{2}\frac{g\sin\alpha}{1 - \frac{m\cos\alpha}{M+m}}t^2$$

$$\Rightarrow \quad t = \sqrt{\frac{2\ell}{g}\frac{1 - \frac{m\cos\alpha}{M+m}}{\sin\alpha}}$$