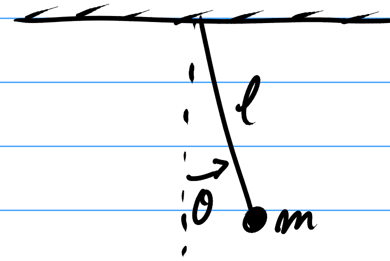
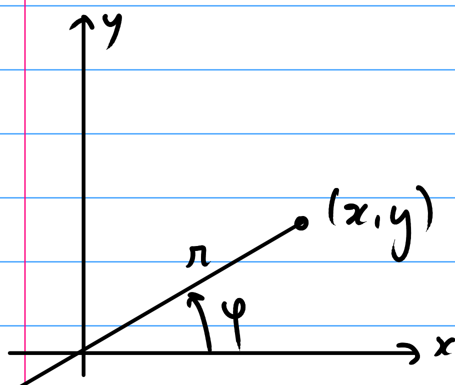
Classical mechanics (Phys 601) - September 6, 2011

* **Pendulum:** $T = \frac{1}{2} m (\ell\dot{\theta})^2$



$$V = -mg\ell\cos\theta \Rightarrow \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

* **Cylindrical coordinates in plane**



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{cases}$$

$$\Rightarrow T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\left(\dot{r}^2 + (r\dot{\varphi})^2\right)$$

$$\bar{r} = x\hat{x} + y\hat{y} = r\hat{r}$$

$$\hookrightarrow d\bar{r} = dr\,\hat{r} + r\,d\varphi\,\hat{\varphi}$$

$$\hookrightarrow \bar{v} = \frac{d\bar{r}}{dt} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$$

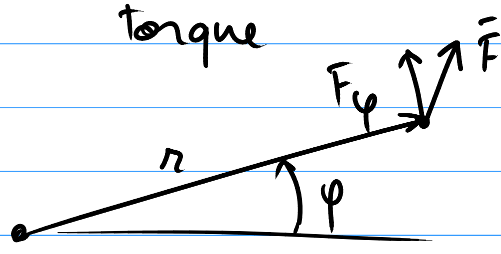
generalized forces $\quad Q_r = \sum_i F_i \frac{\partial x_i}{\partial r} = \bar{F}\cdot\frac{\partial\bar{r}}{\partial r} = \bar{F}\cdot\hat{r} = F_r$

$$Q_\varphi = \bar{F}\cdot\frac{\partial\bar{r}}{\partial\varphi} = \bar{F}\cdot r\hat{\varphi} = rF_\varphi$$

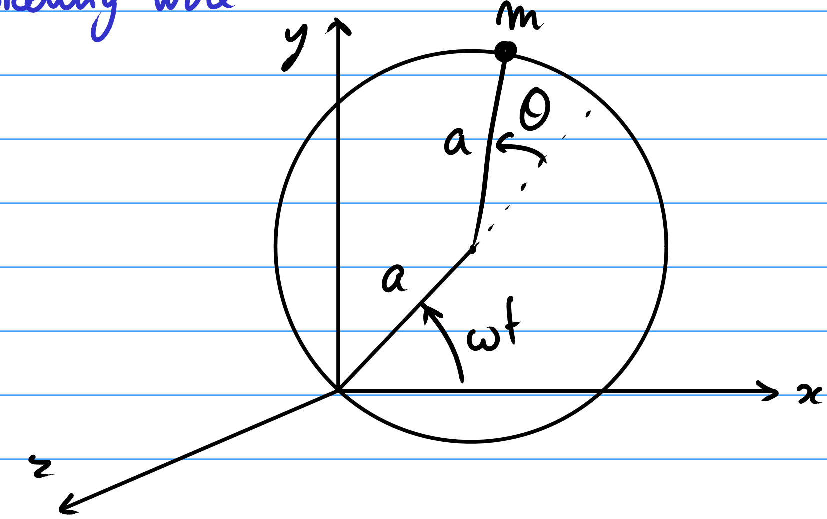
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = Q_r \Rightarrow m\ddot{r} - mr\dot{\theta}^2 = F_r$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) - \frac{\partial L}{\partial\varphi} = Q_\varphi \Rightarrow \frac{d}{dt}\underbrace{(mr^2\dot{\varphi})}_{\text{angular momentum}} = \underbrace{rF_\varphi}_{\text{torque}}$$

$$\bar{L} = \bar{r}\times m\bar{v}$$
$$= mr^2\dot{\varphi}$$



* Bead on a rotating wire



$$\begin{cases} x = a\cos\omega t + a\cos(\omega t+\theta) \\ y = a\sin\omega t + a\sin(\omega t+\theta) \end{cases}$$

total time-derivative $\dot\theta \neq 0$

$$\Rightarrow \begin{cases} \dot x = -a\omega\sin\omega t - a(\omega+\dot\theta)\sin(\omega t+\theta) \\ \dot y = a\omega\cos\omega t + a(\omega+\dot\theta)\cos(\omega t+\theta) \end{cases}$$

$$\Rightarrow T = \frac{1}{2}m(\dot x^2+\dot y^2) = \frac{1}{2}m\Big[a^2\omega^2 + a^2(\omega+\dot\theta)^2 + 2a^2\omega(\omega+\dot\theta)\big(\sin\omega t\sin(\omega t+\theta) + \cos\omega t\cos(\omega t+\theta)\big)\Big]$$

$$= \frac{1}{2}ma^2\left[\omega^2 + (\omega+\dot\theta)^2 + 2\omega(\omega+\dot\theta)\cos\theta\right]$$

→ scale factor will drop out in Lagrange eqn

$$\Rightarrow L = T - V = T = L(\theta,\dot\theta)$$

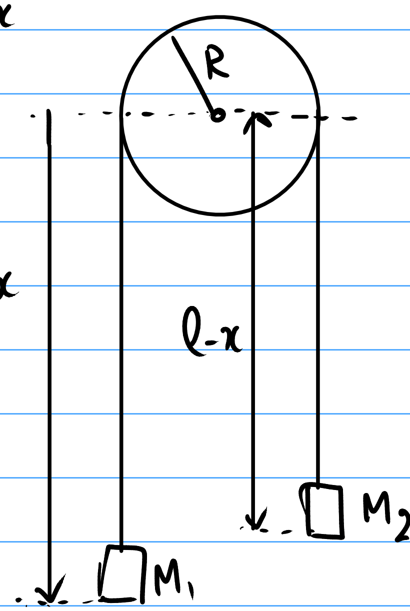
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow \left(2\ddot\theta - 2\omega\sin\theta\,\dot\theta\right) - \left(-2\omega(\omega+\dot\theta)\sin\theta\right) = 0$$

$$\Leftrightarrow \ddot\theta + \omega^2\sin\theta = 0$$

→ equivalence between gravitational force and acceleration

* Atwood Machine

$$T = \frac{1}{2} M_1 \dot{x}^2 + \frac{1}{2} M_2 \dot{x}^2$$



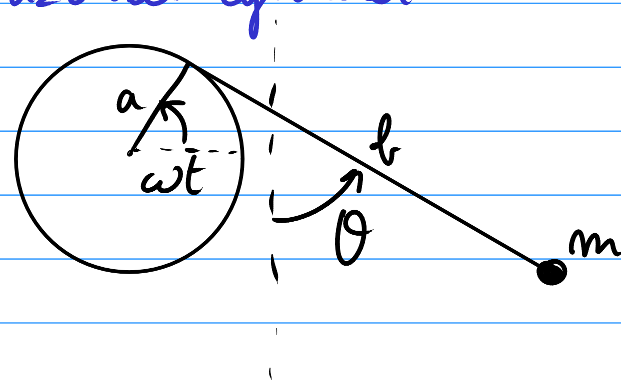
$$V = - M_1 g x - M_2 g (l + R\pi - x)$$

$$\hookrightarrow L = T - V = \frac{1}{2}(M_1 + M_2)\dot{x}^2$$
$$+ (M_1 - M_2) g x + M_2 g (l + R\pi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow (M_1 + M_2)\ddot{x} - (M_1 - M_2) g = 0 \Rightarrow \ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g$$

* Mass attached to rotating horizontal cylinder



$$\begin{cases} x = a \cos \omega t + b \sin \theta \\ y = a \sin \omega t - b \cos \theta \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = - a\omega \sin \omega t + b\dot{\theta} \cos \theta \\ \dot{y} = a\omega \underbrace{\cos \omega t + b\dot{\theta} \sin \theta} \end{cases}$$

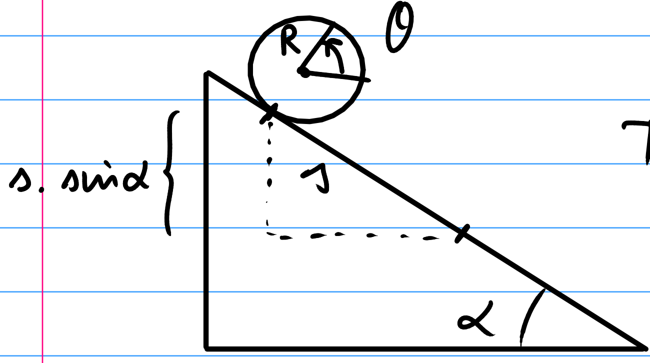
$$V = mgy \qquad \sin \omega t \cos \theta - \cos \omega t \sin \theta = \sin(\omega t - \theta)$$

$$L = \frac{1}{2} m \left(a^2\omega^2 + b^2\dot{\theta}^2 - 2ab\omega\dot{\theta} \sin(\omega t - \theta)\right) - mg(a \sin \omega t - b \cos \theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow m b^2 \ddot{\theta} - mab\omega \cos(\omega t - \theta)(\omega - \dot{\theta})$$
$$- mab\omega\dot{\theta} \cos(\omega t - \theta) + mgb \sin \theta = 0$$

$$\Rightarrow \ddot{\theta} - \frac{a\omega^2}{b} \cos(\omega t - \theta) + \frac{g}{b} \sin \theta = 0$$

+ Hoop rolling on a slope :



generalized coordinate s

constraint $R\theta = s$

$$T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} I \dot{\theta}^2$$

with moment of inertia $I = \frac{1}{2} m R^2$

T should only depend on the coordinate s, not θ

$$\Rightarrow \quad T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} I \left(\frac{\dot{s}}{R}\right)^2$$

$$V = - mg\, s \sin\alpha + \text{constant}$$

$$\Rightarrow L = T - V = \frac{1}{2} m \dot{s}^2 + \frac{1}{4} m \dot{s}^2 + mg\, s \sin\alpha$$

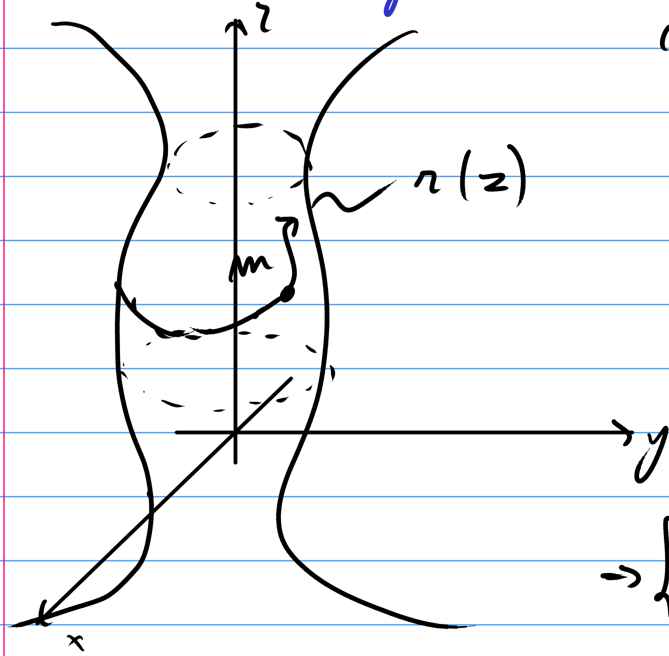
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \Rightarrow \frac{3}{2} m \ddot{s} - mg \sin\alpha = 0$$

$$\Leftrightarrow \quad \ddot{s} = \frac{2}{3} g \sin\alpha$$

$$\Rightarrow \quad \dot{s} = \left(\frac{2}{3} g \sin\alpha\right) t + \dot{s}_0$$

$$\Rightarrow \quad s = \frac{1}{2}\left(\frac{2}{3} g \sin\alpha\right) t^2 + \dot{s}_0 t + s_0$$

* Particle on cylindrical surface



cylindrical coordinates r, φ, z

constraint: $r = r(z)$

$\downarrow$

generalized coordinates φ, z

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{cases}$$

$$\Rightarrow T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\varphi})^2 + \dot{z}^2\right)$$

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{dz}\frac{dz}{dt} = \frac{dr}{dz}\dot{z}$$

$$\Rightarrow T = \frac{1}{2}m\left[\left(\frac{dr}{dz}^2 + 1\right)\dot{z}^2 + r^2\dot{\varphi}^2\right]$$

- In absence of potential: $V = 0 \Rightarrow L = T$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{d}{dt}\left(m r^2 \dot{\varphi}\right) = 0 \Rightarrow m r^2\dot{\varphi} = \ell$$

angular mom. constant

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)} - \frac{\partial L}{\partial z} = 0 \ : \quad \frac{\partial L}{\partial z} = m\dot{z}^2\frac{dr}{dz}\frac{d^2r}{dz^2} + m\dot{\varphi}^2 r\frac{dr}{dz}$$

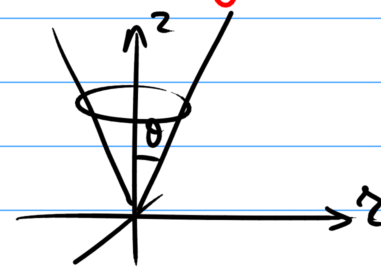
$$\hookrightarrow m\frac{d}{dt}\left(\left(\frac{dr}{dz}^2 + 1\right)\dot{z}\right) = m\ddot{z}\left(\frac{dr}{dz}^2 + 1\right) + 2m\dot{z}\frac{dr}{dz}\frac{d^2r}{dz^2}\dot{z}$$

$$\Rightarrow m\ddot{z}\left(\frac{dr}{dz}^2 + 1\right) + m\dot{z}^2\frac{dr}{dz}\frac{d^2r}{dz^2} - m\dot{\varphi}^2 r\frac{dr}{dz} = 0$$

- Gravitational potential : $V(z) = mgz$

$$\Rightarrow m\ddot{z}\left(\frac{dr}{dz}^2+1\right) + m\dot{z}^2\frac{dr}{dz}\frac{d^2r}{dz^2} - m\dot{\varphi}^2 r\frac{dr}{dz} = -mg$$

Equilibrium orbits: $\dot{\varphi} = \omega = \text{constant}$



Cone with opening angle $z = r\tan\theta$

$\Rightarrow$ circular orbits : $\dot{z} = 0 \Rightarrow m\omega^2\frac{z}{\tan^2\theta} = mg$

$$\Rightarrow z = g\frac{\tan^2\theta}{\omega^2} = z_0$$

(note: $mr^2\omega = \ell = \text{constant}$)

$\Rightarrow$ perturbations around circular orbit : $z = z_0 + \eta(t)$
$\eta \ll z_0$

$$\cancel{m}\ddot{\eta}\left(\frac{1}{\tan^2\theta}+1\right) - \cancel{m}\omega^2\frac{z_0+\eta}{\tan^2\theta} = -\cancel{m}g$$

$$\ddot{\eta}\left(\frac{1}{\tan^2\theta}+1\right) - \frac{\omega^2}{\tan\theta}\eta = -g + z_0\frac{\omega^2}{\tan^2\theta} = 0$$

$$\Rightarrow \ddot{\eta} - \frac{\omega^2}{\tan\theta}\left(\frac{\tan^2\theta}{1+\tan^2\theta}\right)\eta = 0$$

$\Rightarrow$ η is harmonic with frequency $\omega'^2 = \omega^2\frac{\tan\theta}{1+\tan^2\theta}$

* Non-holonomic constraints :

In the derivation of the Lagrange equation from d'Alembert principle, we used

$$\sum_i (F_i - \dot{p}_i)\,\delta x_i = 0$$

$$\Rightarrow \sum_j \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) \delta q_j = 0 \qquad (1)$$

This lead to $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$ for independent δq_j.

For non-holonomic constraints this is not satisfied !

Assume that we can still write the constraint as

$$\sum_j^n a_{\ell j}\, dq_j + b_\ell\, dt = 0 \quad , \quad \ell = 1, \dots, k$$

↳ constraint is $\sum_j A_{\ell j}\, \dot{q}_j + B_\ell = 0$

Virtual displacements are instantaneous and satisfy the constraints

$$\sum_j^n a_{\ell j}\, \delta q_j = 0 \quad , \quad \ell = 1, \dots, k$$

Multiply by Lagrange multipliers $\lambda_\ell(q,t)$ and sum

$$\sum_\ell^k \lambda_\ell(q,t) \sum_j^n a_{\ell j}\, \delta q_j = 0$$

Add this expression to (1)

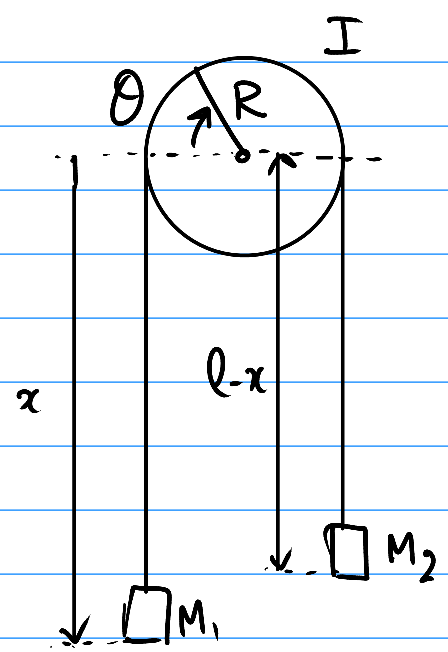
$$\sum_j^n \left(\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} \right] + \sum_\ell^k \lambda_\ell a_{\ell_j} \right) \delta q_j = 0$$

With the k arbitrary functions λ_ℓ we can set the coefficients of $\delta q_{n-k+1}, \dots, \delta q_n$ to zero. The remaining δq_j are now independent.

$$\Rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot q_j}\right) - \dfrac{\partial L}{\partial q_j} + \sum_\ell^k \lambda_\ell a_{\ell_j} = 0 & (n \text{ equations}) \\ \sum_j^n a_{\ell_j} \dot q_j + b_\ell = 0 & (k \text{ equations}) \end{cases}$$

$(n+k$ unknowns$)$

+ Atwood Machine with inertia



$T = \frac{1}{2}(M_1 + M_2)\dot x^2 + \frac{1}{2} I \dot\theta^2$

$V = -M_1 g x - M_2 g(\ell + R\pi - x)$

constraint: no slipping (holonomic)

$$\underset{\substack{\uparrow \\ q_1}}{\dot x} = -R \underset{\substack{\uparrow \\ q_2}}{\dot\theta} \qquad (k=1)$$

$a_1 = 1$, $a_2 = R$, $b = 0$

↳ arbitrary Lagrange multiplier $\lambda(x, \theta, t)$

$$L = \frac{1}{2}(M_1+M_2)\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + (M_1-M_2)gx + \text{constants}$$

$$\begin{cases} x: \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) - \dfrac{\partial L}{\partial x} + \lambda = 0 \\ \theta: \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} + \lambda R = 0 \\ \text{constraint}: \dot{x} + R\dot{\theta} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (M_1+M_2)\ddot{x} - (M_1-M_2)g + \lambda = 0 \\ I\ddot{\theta} + \lambda R = 0 \quad \Rightarrow \quad \lambda = -\dfrac{I}{R}\ddot{\theta} \\ \dot{x} + R\dot{\theta} = 0 \quad \Rightarrow \quad \ddot{x} = -R\ddot{\theta} \end{cases} \Rightarrow \lambda = \frac{I}{R^2}\ddot{x}$$

$$\Leftrightarrow (M_1+M_2)\ddot{x} - (M_1-M_2)g + \frac{I}{R^2}\ddot{x} = 0$$

$$\Leftrightarrow \begin{cases} \ddot{x} = \dfrac{(M_1-M_2)g}{(M_1+M_2) + \dfrac{I}{R^2}} \\ \ddot{\theta} = \dfrac{-R(M_1-M_2)g}{(M_1+M_2) + \dfrac{I}{R^2}} \\ \lambda = \dfrac{I(M_1-M_2)g}{(M_1+M_2)R^2 + I} \end{cases}$$