Classical Mechanics (Phys 601) – November 1, 2011

* Rigid body

"System of particles where the distance between the particles does not vary"

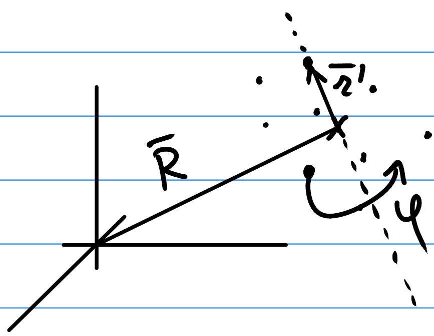
→ formulated as set of N discrete particles, but equally valid for $\underbrace{\text{continuous objects}}_{\text{allows to disregard internal structure}}$

System of N particles → 3N degrees of freedom

⇓ Charles' Theorem:

Rigid body : $\begin{cases} \text{3 degrees of freedom for translation (position of center of mass)} \\ \text{3 degrees of freedom for rotation around center of mass} \end{cases}$

→ there must be 3N − 6 constraints



→ 3 components of vector $\bar{R}$ to CM

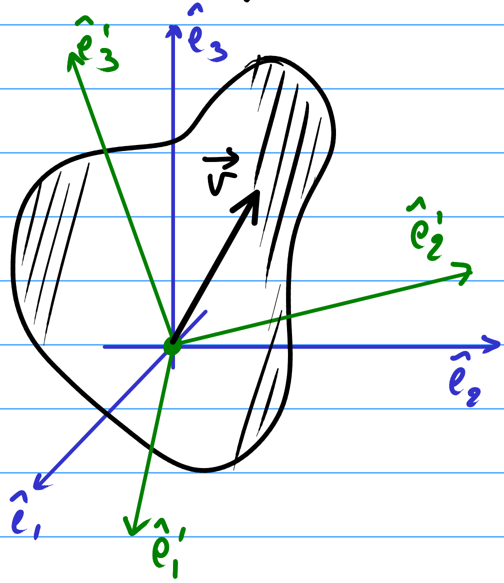
→ 2 components of one vector $\bar{r}'$ inside the rigid body (magnitude fixed)

→ 1 rotation φ around vector $\bar{r}'$

⇒ Rigid body has 6 degrees of freedom

* Rigid body with one fixed point

Consider inertial frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and rigid body with fixed point at the origin of the inertial frame:



rigid body can rotate with 3 degrees of freedom around the origin.

Consider rotating reference frame $\{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$ fixed to the rigid body with its origin at origin of inertial frame.

* Rotations: $\bar{x}(x_1, x_2, x_3) \longrightarrow \bar{x}'(x_1', x_2', x_3')$

rotation matrix U : $\bar{x}' = U\bar{x}$

$$x_i' = \sum_j U_{ij} x_j$$

Successive rotations:

$$\bar{x} \xrightarrow{U^1} \bar{x}' \xrightarrow{U^2} \bar{x}''$$

$$\underbrace{\qquad\qquad\qquad}_{U}$$

$$x_i'' = \sum_j U_{ij}^2 x_j' = \sum_{j,k} U_{ij}^2 U_{jk}^1 x_k$$

$$\Rightarrow U_{ik} = \sum_j U_{ij}^2 U_{jk}^1$$

$$\Leftrightarrow \underbrace{U = U^2 U^1}_{\text{matrix product}}$$

$\Rightarrow$ rotations do not commute!

Identity: $U = \mathbb{1} \rightarrow$ no rotation

Rotation does not change length of a vector:

$$\sum_i (x_i')^2 = \sum_i (x_i)^2 = \sum_i \left(\sum_k U_{ik} x_k\right)\left(\sum_{k'} U_{ik'} x_{k'}\right)$$

$$\Leftrightarrow \sum_i \sum_k \sum_{k'} \underbrace{U_{ik} U_{ik'}} x_k x_{k'} = \sum_i (x_i)^2$$

$$\Leftrightarrow \sum_i U_{ik} U_{ik'} = \delta_{kk'}$$

$$\Leftrightarrow U U^T = \mathbb{1}$$ → orthogonal matrices

↓

$$\Leftrightarrow U^{-1} = U^T$$ → inverse is transpose

$\Rightarrow U \in O(n)$: orthogonal group of dimension $n = 3$

special orthogonal group $SO(n)$: $\det(U) = +1$ "proper" rotations

$\det(U) = -1$: includes flip "improper" rotations → no continuous connection to $\mathbb{1}$

* Infinitesimal rotations:

$$U_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$$

↓

$$U_z(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & \\ \varepsilon & 1 & \\ & & 1 \end{pmatrix}$$

Similarly:

$$U_x(\varepsilon) = \begin{pmatrix} 1 & & \\ & 1 & -\varepsilon \\ & \varepsilon & 1 \end{pmatrix}, \quad U_y(\varepsilon) = \begin{pmatrix} 1 & & \varepsilon \\ & 1 & \\ -\varepsilon & & 1 \end{pmatrix}$$

$U(\varepsilon) = 1 + \varepsilon J$, where J is the generator of the rotation

$$U^{-1}(\varepsilon) = 1 + \varepsilon J^{-1} = 1 - \varepsilon J \rightarrow J^{-1} = -J = -J^T$$

$$J_x = \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix}, \quad J_y = \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \quad J_z = \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$$

Finite rotation is now: $U(\theta) = \lim_{N \to \infty} \left(1 + \frac{\theta}{N} J\right)^N = e^{\theta J}$

Now: $U_x(\varepsilon_x)\, U_z(\varepsilon_z) = \begin{pmatrix} 1 & -\varepsilon_z & \\ \varepsilon_z & 1 & -\varepsilon_x \\ & \varepsilon_x & 1 \end{pmatrix} = U_z(\varepsilon_z)\, U_x(\varepsilon_x)$

$= 1 + \varepsilon_x J_x + \varepsilon_z J_z \Rightarrow$ infinitesimal rotations <u>do</u> commute !

Calculate commutator of generators :

$$\left.\begin{aligned} J_x J_z &= \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix} \\ J_z J_x &= \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix} \end{aligned}\right\} \quad [J_x, J_z] = J_x J_z - J_z J_x = \begin{pmatrix} & & -1 \\ & & \\ 1 & & \end{pmatrix} = -J_y$$

$\Rightarrow [J_z, J_x] = J_y$

Similarly : $[J_x, J_y] = J_z$, $[J_y, J_z] = J_x$, $[J_z, J_x] = J_y$

$\Rightarrow [J_i, J_j] = \sum_k c_{ijk} J_k \longrightarrow c_{ijk} = \varepsilon_{ijk}$ for these generator matrices

ε_{ijk}: Levi-Civita symbol

* Lie groups: operator $[\,,\,]$: Lie bracket
such that $[A,B] = -[B,A]$
↳ here commutator, but think also of the Poisson bracket, etc ...

↳ very important in physics!

Symmetry under continuous transformation part of Lie group → Noether current is related to these generators

In both discrete and continuous case: $\left.\frac{\partial q}{\partial \varepsilon}\right|_{\varepsilon=0}$ → e.g. $q(x,y,z)$

⇒ exactly the first order term in $U = 1 + \varepsilon J$

$$q' = Uq = (1+\varepsilon J)q \rightarrow \left.\frac{\partial q}{\partial \varepsilon}\right|_{\varepsilon=0} = Jq$$

$$\Rightarrow C_\varepsilon = \sum_i \frac{\partial L}{\partial \dot{q}_i}(Jq)_i$$

$$\mathcal{L} = c^2 \partial_\mu \varphi \partial^\mu \varphi^* - m_0^2 c^2 \varphi\varphi^* \longrightarrow \begin{cases} \varphi \to e^{i\theta}\varphi \\ \varphi^* \to e^{-i\theta}\varphi^* \end{cases}$$

For complex $\varphi \to e^{i\theta}\varphi$ is phase rotation in complex plane
→ $U(1) = SU(1)$ Lie group → generator

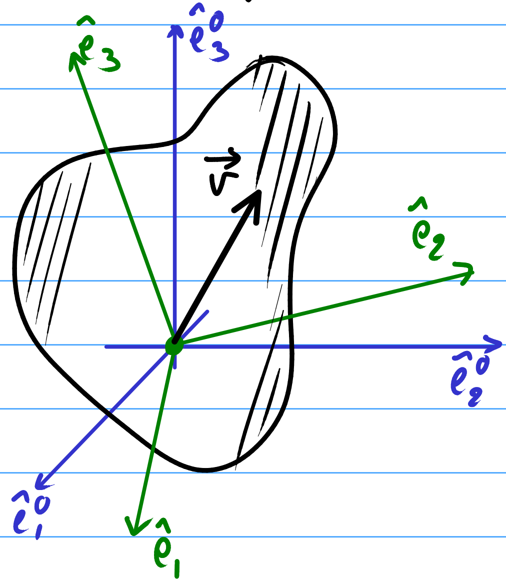
$$\begin{bmatrix} \varphi \\ \varphi^* \end{bmatrix} \to \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix} \begin{bmatrix} \varphi \\ \varphi^* \end{bmatrix} \Rightarrow J = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

Noether theorem on $\Phi = (\varphi, \varphi^*)$: $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}(J\Phi) = i(\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi)$

Now that we know more about rotations than we ever wanted to know ... back to rigid bodies ...

* Rigid body with one fixed point

Consider inertial frame $\{\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0\}$ and rigid body with fixed point at the origin of the inertial frame:



rigid body can rotate with 3 degrees of freedom around the origin.

Consider rotating reference frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ fixed to the rigid body with its origin at origin of inertial frame.

Consider a vector $\bar{v} = \underbrace{\sum_i v_i^0 \hat{e}_i^0}_{\text{inertial frame}} = \underbrace{\sum_i v_i \hat{e}_i}_{\text{rotating rigid body frame}}$

$$\left(\frac{d\bar{v}}{dt}\right)_{\text{inertial}} = \overbrace{\sum_i \frac{dv_i^0}{dt} \hat{e}_i^0}^{} = \overbrace{\sum_i \frac{dv_i}{dt} \hat{e}_i + \sum_i v_i \frac{d\hat{e}_i}{dt}}^{}$$

= change of vector $\bar{v}$ as seen by observer in inertial frame

$$\left(\frac{d\bar{v}}{dt}\right)_{\text{body}} = \sum_i \frac{dv_i}{dt} \hat{e}_i$$

= change of vector $\bar{v}$ as seen in rigid body frame

$$\Rightarrow \left(\frac{d\bar{v}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{v}}{dt}\right)_{\text{body}} + \sum_i v_i \frac{d\hat{e}_i}{dt}$$

We have to find an expression for $\frac{d\hat{e}_i}{dt}$ = change of the basis vectors of the rotating rigid body frame, expressed in the inertial frame.

* Rotating coordinate systems (F&W, ¶ 6-7)

Between t and $t+dt$:

vector in $\hat{e}_i$ basis

$$\hat{e}_i(t+dt) = \hat{e}_i(t) + d\hat{e}_i$$

To first order: $\hat{e}_i(t+dt)\cdot\hat{e}_i(t+dt) = \hat{e}_i(t)\cdot\hat{e}_i(t) + 2\,\hat{e}_i(t)\cdot d\hat{e}_i = 1$

$$\Rightarrow \hat{e}_i(t)\cdot d\hat{e}_i = 0$$

$$\Rightarrow d\hat{e}_i \perp \hat{e}_i(t)$$

Expand $d\hat{e}_i = \sum_j d\Omega_{ij}\,\hat{e}_j$, $d\Omega_{ii} = 0$ and $d\Omega_{ij} = d\hat{e}_i\cdot\hat{e}_j$

example: $d\hat{e}_1 = d\Omega_{12}\,\hat{e}_2 + d\Omega_{13}\,\hat{e}_3$

Anti-symmetry: $d(\hat{e}_i\cdot\hat{e}_j) = d(\delta_{ij}) = d\hat{e}_i\cdot\hat{e}_j + \hat{e}_i\cdot d\hat{e}_j = 0$

$$\Rightarrow d\Omega_{ij} = -d\Omega_{ji}$$

$$\Rightarrow d\Omega = \begin{pmatrix} 0 & d\Omega_{12} & -d\Omega_{13} \\ -d\Omega_{12} & 0 & d\Omega_{23} \\ d\Omega_{13} & -d\Omega_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$\Rightarrow d\hat{e}_i = d\bar{\Omega} \times \hat{e}_i \quad \text{with } d\bar{\Omega} = \sum_i d\Omega_i \hat{e}_i :$$

$$\text{example: } d\hat{e}_1 = d\Omega_3 \hat{e}_2 - d\Omega_2 \hat{e}_3 = d\bar{\Omega} \times \hat{e}_1$$

A rotation of a vector over an angle $d\theta$ around an axis $\hat{n}$ is given by :

$$d\bar{r} = d\theta \, \hat{n} \times \bar{r}$$

$\Rightarrow$ $d\bar{\Omega}$ is an infinitesimal rotation of $d\Omega_1$ around $\hat{e}_1$, $d\Omega_2$ around $\hat{e}_2$, and $d\Omega_3$ around $\hat{e}_3$.

Finally: $$\frac{d\hat{e}_i}{dt} = \frac{d\bar{\Omega}}{dt} \times \hat{e}_i = \bar{\omega} \times \hat{e}_i$$

with $\bar{\omega} = \frac{d\bar{\Omega}}{dt}$ = instantaneous angular velocity

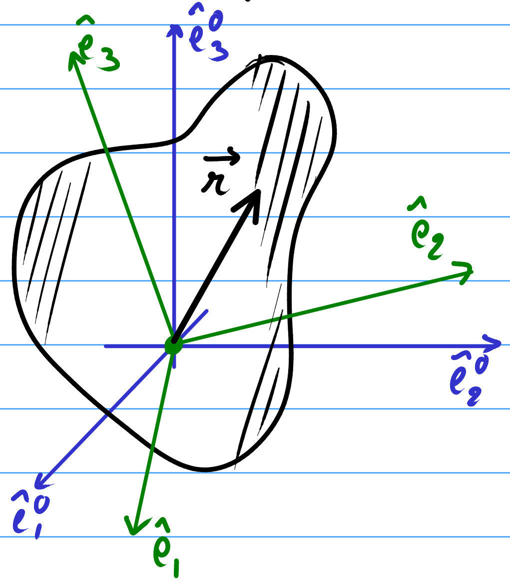
$$\Rightarrow \left(\frac{d\bar{v}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{v}}{dt}\right)_{\text{body}} + \bar{\omega} \times \bar{v}$$

Notice : $$\left(\frac{d\bar{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{\omega}}{dt}\right)_{\text{body}}$$

$\Rightarrow$ rate of change in $\bar{\omega}$ does not depend on coordinate system

* Rigid body with one fixed point

Consider inertial frame $\{\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0\}$ and rigid body with fixed point at the origin of the inertial frame:



rigid body can rotate with 3 degrees of freedom around the origin.

Consider rotating reference frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ fixed to the rigid body with its origin at origin of inertial frame.

Pick point $\bar{r}$ in rigid body: $\left(\frac{d\bar{r}}{dt}\right)_{body} = \bar{0}$

$$\left(\frac{d\bar{r}}{dt}\right)_{inertial} = \bar{\omega} \times \bar{r}$$

$$\Rightarrow T = \sum_k^N \frac{1}{2} m_k v_k^2$$

$$= \sum_k^N \frac{1}{2} m_k (\bar{\omega} \times \bar{r}_k) \cdot (\bar{\omega} \times \bar{r}_k) \qquad |\bar{u} \times \bar{v}|^2 = u^2 v^2 - (u \cdot v)^2$$

$$= \sum_k^N \frac{1}{2} m_k \left(\omega^2 r_k^2 - (\bar{\omega} \cdot \bar{r}_k)^2\right)$$

$$= \frac{1}{2} \sum_i \sum_j \omega_i \underbrace{\left(\sum_k^N m_k (r_k^2 \delta_{ij} - x_{ki} x_{kj})\right)}_{} \omega_j$$

$$= \frac{1}{2} \omega^T I \omega : \text{inertia tensor } I_{ij}$$

Inertia tensor could also be calculated for continuous mass distributions:

$$I_{ij} = \int d^3r \; \rho(\bar{r}) \left(r^2 \delta_{ij} - x_i x_j \right)$$

$\Rightarrow$ I depends only on the intrinsic mass distribution of the rigid body

Tensor of inertia:

$$\begin{bmatrix} \underbrace{\int d^3x \, \rho}\left(\underbrace{x_2^2 + x_3^2} \right) & \int d^3x \, \rho \left(-x_1 x_2 \right) & \int d^3x \, \rho \left(-x_1 x_3 \right) \\ \ddots & \vdots & \vdots \\ & & \end{bmatrix}$$

Dimensionality: mass $\times$ length^2

Symmetric: $I_{ij} = I_{ji}$

* Rigid body without fixed point

$\bar{R}$ = vector to center of mass
$\bar{r}'$ = vector in comoving rigid body frame

$$\Rightarrow \bar{r} = \bar{R} + \bar{r}'$$

$$\left(\frac{d\bar{r}}{dt}\right)_{\text{inertial}} = \dot{\bar{R}} + \left(\frac{d\bar{r}'}{dt}\right)_{\text{inertial}}$$

$$= \dot{\bar{R}} + \underbrace{\left(\frac{d\bar{r}'}{dt}\right)_{\text{body}}}_{\overset{\shortparallel}{0}} + \bar{\omega}\times\bar{r}'$$

for a point $\bar{r}'$ in the rigid body

$$\Rightarrow T = \sum_k^N \frac{1}{2} m_k v_k^2 = \sum_k^N \frac{1}{2} m_k \left(\dot{\bar{R}} + \bar{\omega}\times\bar{r}'_k\right)^2$$

$$= \frac{1}{2} M\bar{V}^2 + \frac{1}{2}\bar{\omega}^T I\bar{\omega} + \cancel{\underbrace{\sum_k^N m_k \dot{\bar{R}}\cdot(\bar{\omega}\times\bar{r}'_k)}_{\text{summation over } \bar{r}'_k}}$$

Angular momentum:

$$\sum_k^N m_k \bar{r}'_k = 0$$

$$\bar{L} = \sum_k^N m_k \bar{r}_k \times \bar{v}_k$$

$$= \sum_k^N m_k (\bar{R} + \bar{r}'_k) \times (\bar{V} + \bar{\omega}\times\bar{r}'_k)$$

$$= M\bar{R}\times\bar{V} + \sum_k^N m_k \left(\bar{r}'_k \times (\bar{\omega}\times\bar{r}'_k)\right)$$

$$= M\bar{R}\times\bar{V} + \sum_k^N m_k \underbrace{\left(r'^2_k\,\bar{\omega} - (\bar{\omega}\cdot\bar{r}'_k)\bar{r}'_k\right)}_{\text{linear in } \bar{\omega}}$$

$$= M\bar{R}\times\bar{V} + \bar{L}'$$

$$\bar{L}' = \sum_j I_{ij}\,\omega_j = I\bar{\omega}$$ = angular momentum with respect to center of mass

$$\Rightarrow T = \frac{1}{2} M\bar{V}^2 + \frac{1}{2}\bar{L}\cdot\bar{\omega}$$

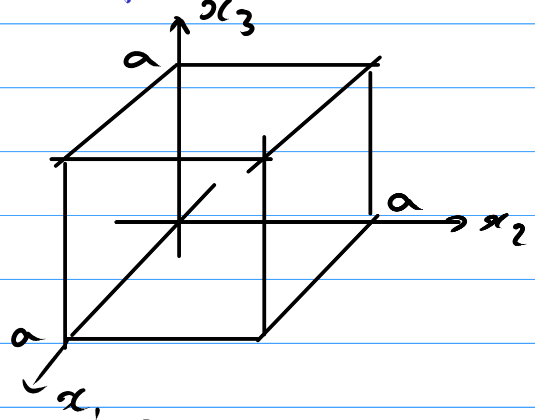
For rotation around z axis: $\bar{\omega} = (0,0,\omega)$

$\hookrightarrow T_{rot} = \frac{1}{2}\bar{L}\cdot\bar{\omega} = \frac{1}{2}L_z\omega = \frac{1}{2}I_{33}\omega^2$

In general for $\bar{L}$:

$\hookrightarrow \bar{L} = I\cdot\bar{\omega} = (I_{13}\omega, I_{23}\omega, I_{33}\omega)$

* Example: tensor of inertia of a uniform cube



$$I_{11} = \rho\int d^3x\,(x_2^2 + x_3^2)$$
$$= a\rho\int dx_2\,dx_3\,(x_2^2 + x_3^2)$$
$$= a\rho\left(\frac{a^3}{3}a + \frac{a^3}{3}a\right)$$
$$= \frac{2}{3}\rho a^5 = \frac{2}{3}Ma^2 \quad \text{with } M = \rho a^3$$

$$I_{12} = \rho\int d^3x\,(-x_1x_2)$$
$$= -\rho a\left(\frac{a^2}{2}\right)\left(\frac{a^2}{2}\right)$$
$$= -\frac{1}{4}\rho a^5 = -\frac{1}{4}Ma^2$$

$$\Rightarrow \underline{I} = Ma^2\begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$