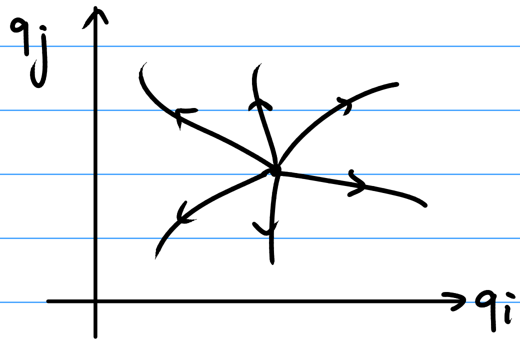
Classical Mechanics (Phys 601) - September 27, 2011

* **Phase space** : (already used last lecture : quantization)

Remember : coordinate space (n-dimensional)

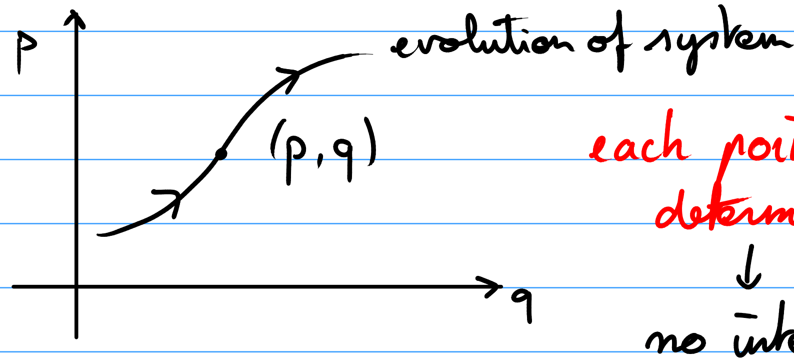


multiple trajectories from one point

↓

trajectories can intersect

$2n$-dimensional space of $\{q_i, p_i\}$ for n degrees of freedom

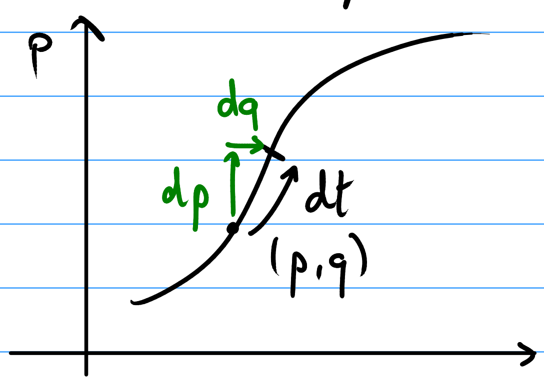


each point uniquely & completely determines trajectory

↓

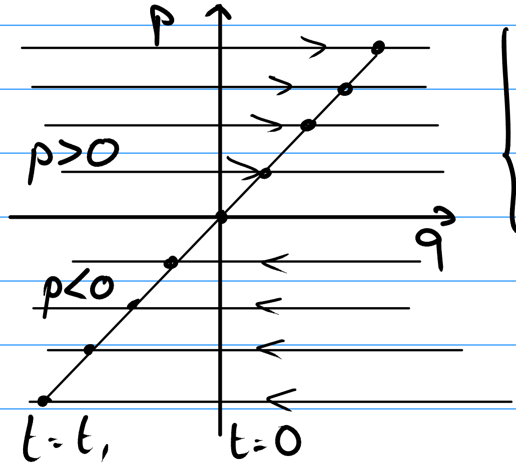
no intersections of trajectories

Evolution in phase space given by Hamiltonian :



$$\begin{cases} \dot{p} = \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} \\[2ex] \dot{q} = \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} \end{cases}$$

Example: free particle $\quad H = \frac{1}{2m} p^2$



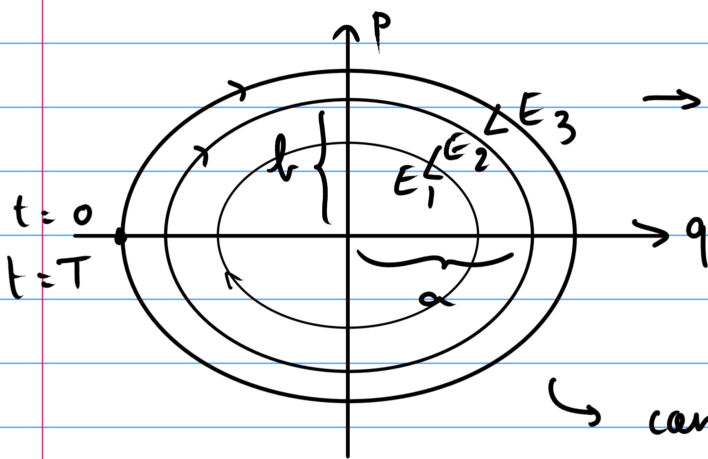
$$\begin{cases} \dot{p} = 0 \longrightarrow \text{horizontal lines} \\ \dot{q} = \frac{\partial H}{\partial p} = \frac{1}{m} p \end{cases}$$

$$\Rightarrow \quad q = \frac{1}{m} p_0 t$$

Example: harmonic oscillator $\quad H = \frac{1}{2m} p^2 + \frac{1}{2} k q^2$

$$\begin{cases} \dot{p} = -kq \\ \dot{q} = \frac{p}{m} \end{cases} \qquad \frac{\partial H}{\partial t} = 0 \longrightarrow H = E$$

$$\Rightarrow \quad \frac{q^2}{2E/k} + \frac{p^2}{2mE} = 1$$



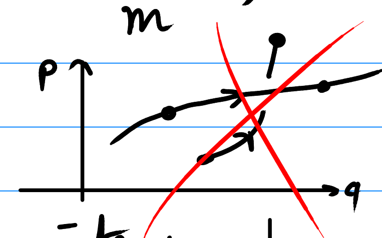
$\longrightarrow$ ellipse with $\begin{cases} a = \sqrt{2E/k} \\ b = \sqrt{2mE} \end{cases}$

$a, b \sim \sqrt{E}$

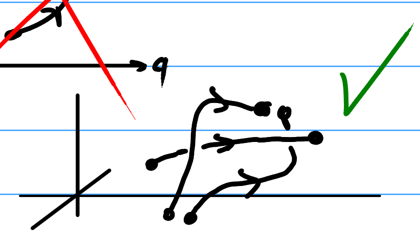
$\hookrightarrow$ can renormalize p, q to get circle

One revolution per period : $\omega^2 = \frac{k}{m} \Rightarrow T = 2\pi \sqrt{\frac{m}{k}}$

Brief note on intersections: in 2D :



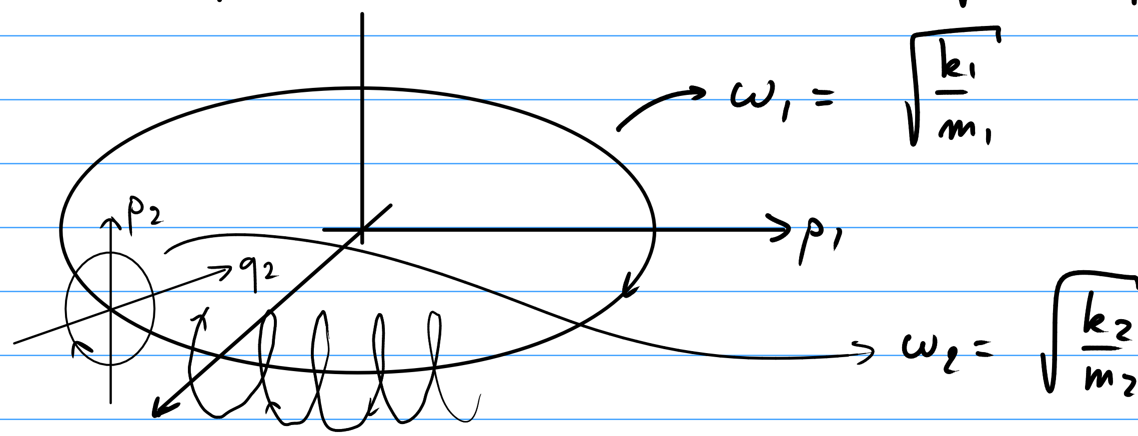
in >2 D : much easier to reach points :



Example: Double uncoupled harmonic oscillator

$$H = \frac{1}{2m_1} p_1^2 + \frac{1}{2m_2} p_2^2 + \frac{1}{2} k_1 q_1^2 + \frac{1}{2} k_2 q_2^2$$

$\rightarrow$ 2 ellipses (circles) in 4-dimensional phase space



$\rightarrow$ trajectory is on 2-torus in 4-dimensional space
$\hookrightarrow$ 3-dimensional surface
(NOT a 3-dimensional torus!)
$\hookrightarrow$ constant energy surface or manifold

If $\frac{\omega_2}{\omega_1} = n$ integer, then orbit closes after $T_1 = \frac{2\pi}{\omega_1}$

If $\omega_2 \neq n\omega_1$, but $m\omega_2 = n\omega_1$, $\frac{\omega_2}{\omega_1} = \frac{n}{m}$ = rational,

then orbit closes after $mT_1 = 2\pi \frac{m}{\omega_1} = 2\pi \frac{n}{\omega_2} = nT_2$

If $\frac{\omega_2}{\omega_1} \neq$ rational, then orbit will not close but come arbitrarily close to any point.

General case: N uncoupled harmonic oscillators $\rightarrow$ N-torus d-dimensional object

Attractor: manifold to which a trajectory evolves for $t \to \infty$ after transients have died out (for a variety of initial conditions)

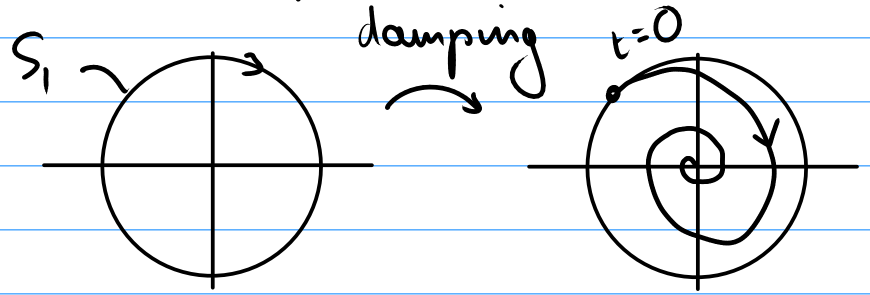
Regular attractor: d = number of phase space dimensions $- 1$

Example: simple harmonic oscillator $\to q, p$
$\longrightarrow$ 1-dimensional manifold $= S_1$
circle

Example: double uncoupled harmonic oscillator
$\downarrow$
4 phase space dimensions: q_1, q_2, p_1, p_2
$\hookrightarrow$ 3-dimensional manifold $=$ 2-torus

Fixed point: (0-dimensional) point in phase space

Example: damped harmonic oscillator



Limit cycle: (1-dimensional) cycle in phase space

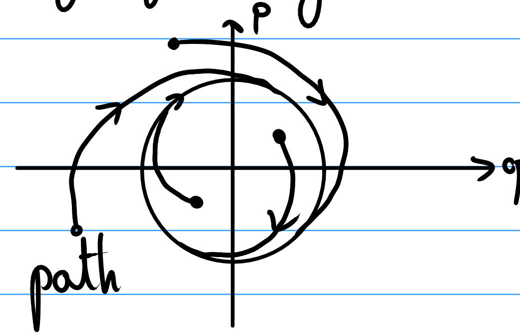
Example: van der Pol equation:

$$m\ddot{x} - \varepsilon(1 - x^2)\dot{x} + m\omega_0 x = F\cos\omega_D t$$

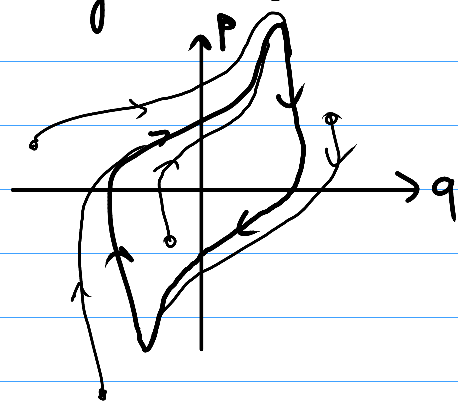
For $\varepsilon = 0 \rightarrow$ simple harmonic oscillator
with driving for $F \cos \omega_D t$

If ε small $\rightarrow$ damping with sign given by x^2

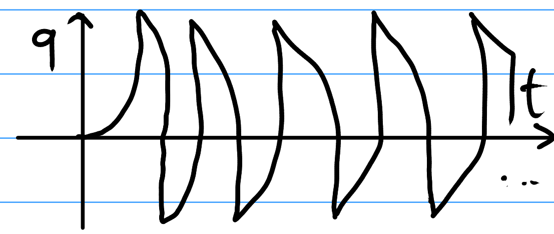
$x^2 > 1 \rightarrow$ damping
$x^2 < 1 \rightarrow$ increase



If ε large $\rightarrow$ distortion of path



$\rightarrow$



↳ hearth rythm

Strange attractors: fractional dimension (chaos)

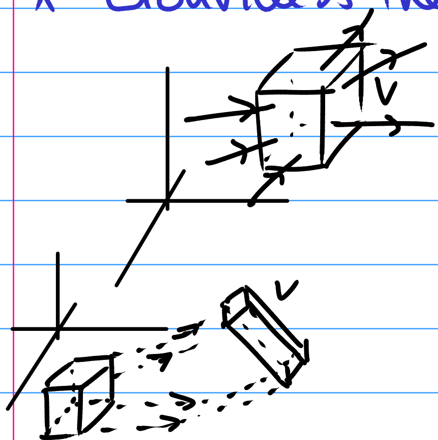
↳ looks like a limit cycle, but a real limit cycle is never reached (not even asymptotically)

- the orbit starting in one region I_1 will pass through the region I_2 eventually, for all I_2.
- the orbit behaves quasi-periodic, without ever closing, without definite period.
- the orbit will come arbitrarily close to any point.

↳ ergodic divertors in fusion plasmas $\rightarrow$ mixing properties

D N

* **Liouville's theorem:** density (and number) of states in given volume of phase space is constant under Hamiltonian flow



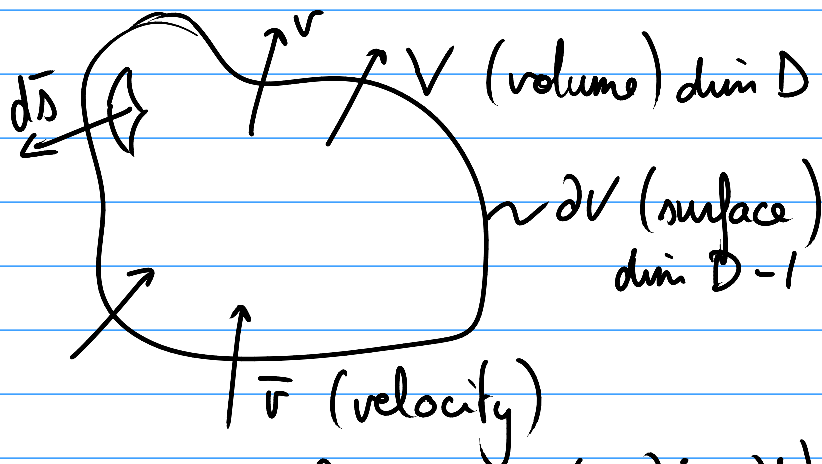
Also: a given volume of states in phase space will continue to occupy the same volume under Hamiltonian flow

$$V = \int_V \prod_i dp_i dq_i$$ = volume in phase space for coordinate volume V

A flow preserves volume (= incompressible) if

$$\int_{\partial V} d\bar{s} \cdot \bar{v} = 0$$

$$\bar{v} = (\dot{q}_i, \dot{p}_i)$$



$$\int_{\partial V} d\bar{s} \cdot \bar{v} = \int_V \prod_i dp_i dq_i \, \mathrm{div}(\bar{v}) = \int_V \prod_i dp_i dq_i \left(\sum_j \frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} \right)$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\int_{\partial V} d\bar{s} \cdot \bar{v} = \int_V \prod_i dp_i dq_i \left(\sum_i \frac{\partial^2 H}{\partial p_j \partial q_j} - \frac{\partial^2 H}{\partial p_j \partial q_j} \right) = 0$$

Application to density of states $D = \frac{dN}{dV}$

$\rightarrow$ $\dot{D} = 0$ because of Liouville's theorem

$\Rightarrow$ $\frac{\partial D}{\partial t} + [D, H] = 0$

If D does not depend on t explicitly (e.g. equilibrium), then D will be a function of only constants of motion (e.g. E) which commute with H.

$\hookrightarrow$ statistical mechanics & ensemble theory

* Canonical transformations:

We discussed previously how the Lagrangian is invariant under point transformations in configuration space:

$$Q_j = Q_j(q_i, t)$$

But the Hamiltonian is not, because $H = \sum_i p_i \dot{q}_i - L$ depends explicitly on p_i and q_i.

How does H behave under coordinate transformations?

Consider combined coordinate transformations of both q and p in phase space:

$$\begin{cases} q_i = q_i(Q_j, P_j, t) \\ p_i = p_i(Q_j, P_j, t) \end{cases} \xrightarrow{\text{inverse}} \begin{cases} Q_j = Q_j(q_i, p_i, t) \\ P_j = P_j(q_i, p_i, t) \end{cases}$$

assumed non-singular

point transformation in phase space

$$L(q_i, p_i, t) = L'(Q_j, P_j, t) + \frac{dF}{dt}(q_i, Q_j, t)$$

(we can always add a total time derivative)

$$\Rightarrow \sum_i p_i \dot{q}_i - H(q_i, p_i, t) = \sum_j P_j \dot{Q}_j - K(Q_j, P_j, t) + \frac{dF}{dt}(q_i, Q_j, t)$$

where we assume that there is a $K(Q_j, P_j, t)$ which is the transformed Hamiltonian = Kamiltonian for some choice of generating function $F(q_i, Q_j, t)$.

$\Rightarrow$ If we can find a Kamiltonian $K(Q_j, P_j, t)$ for some choice of generating function $F(q_i, Q_j, t)$ and Hamilton's equations retain their form

then the transformation is canonical.

Why would you want to use a transformation?
↳ coordinates or momenta become cyclic

Example: polar coordinates

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + V\left(\sqrt{x^2+y^2}\right)$$

$$\hookrightarrow \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p_x}, & \dot{y} = \dfrac{\partial H}{\partial p_y} \\ \dot{p}_x = -\dfrac{\partial H}{\partial x}, & \dot{p}_y = -\dfrac{\partial H}{\partial y} \end{cases} \Rightarrow \text{4 coupled equations}$$

Transformation to r, φ : $K = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r)$

↳ φ is cyclic $\rightarrow p_\varphi =$ constant

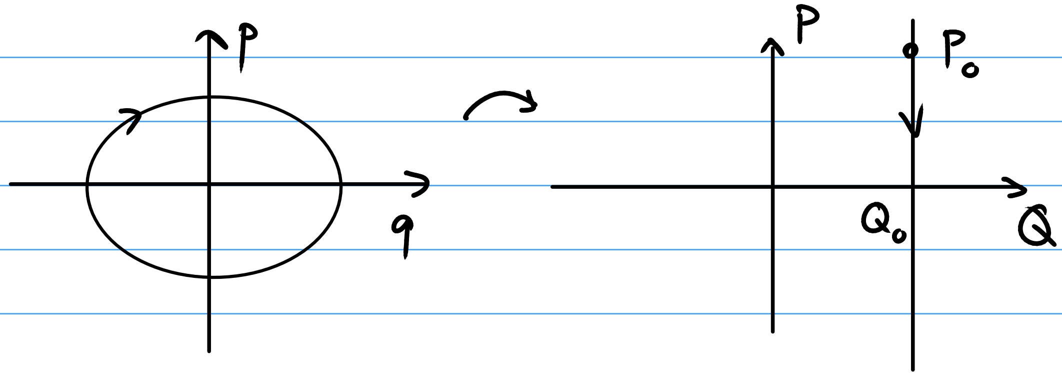
$$\begin{cases} \dot{r} = \dfrac{\partial K}{\partial p_r}, & \dot{p}_r = -\dfrac{\partial K}{\partial r} \\ \dot{\varphi} = \dfrac{\partial K}{\partial p_\varphi} \end{cases} \Rightarrow \text{3 coupled equations}$$

Example : simple harmonic oscillator :

$$H = \frac{1}{2m} p^2 + \frac{1}{2} k q^2 \rightarrow \text{take } Q = \sqrt{\frac{p^2}{m} + k q^2} \;, P = ?$$

$$\Rightarrow K = \frac{1}{2} Q^2$$

$$\hookrightarrow \begin{cases} \dot{Q} = \frac{\partial K}{\partial P} = 0 & \rightarrow Q = \text{constant} = Q_0 \\ \dot{P} = -\frac{\partial K}{\partial Q} = Q & \rightarrow P = -Q_0 t + P_0 \end{cases}$$



$\hookrightarrow$ But how can we find these transformations for more complicated cases ?

Substitute

$$\frac{dF}{dt}(q_i, Q_j, t) = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \sum_j \frac{\partial F}{\partial Q_j}\dot{Q}_j + \frac{\partial F}{\partial t}$$

in

$$\sum_i p_i \dot{q}_i - H(q_i, p_i, t) = \sum_j P_j \dot{Q}_j - K(Q_j, P_j, t) + \frac{dF}{dt}(q_i, Q_j, t)$$

$$\Rightarrow \sum_i \left(p_i - \frac{\partial F}{\partial q_i}\right)\dot{q}_i - H(q_i, p_i, t) = \sum_j \left(P_j + \frac{\partial F}{\partial Q_j}\right)\dot{Q}_j - K(Q_j, P_j, t) + \frac{\partial F}{\partial t}(q_i, Q_j, t)$$

this is satisfied for:

$$p_i = \frac{\partial F}{\partial q_i}(q_i, Q_j, t)^{(1)}, \quad P_j = -\frac{\partial F}{\partial Q_j}(q_i, Q_j, t)^{(2)}$$

and $K(Q_j, P_j, t) = H(q_i, p_i, t) + \frac{\partial F}{\partial t}(q_i, Q_j, t)^{(3)}$

(2) all q_i can be expressed as $q_i(Q_j, P_j, t)$

(1) p_i can be expressed as $p_i(Q_j, P_j, t)$

$\Rightarrow F(q_i, Q_j, t)$ generates the canonical transformation

Note: for every $F(q_i, Q_j, t)$ there will be a canonical transformation!

Note: under canonical transformation:

$$[F,G]_{pq} = [F,G]_{PQ}$$ (by direct calculation)

$\Rightarrow$ transformation is canonical if:

- there exist a generating function $F(q_i, Q_j, t)$

OR

- Poisson brackets are preserved

Note: Also other forms of $F(q_i, Q_j, t)$ exist

↳ Legendre transformation:

$$F = F_1(q_i, Q_j, t)$$

↳ $F = -\sum_j P_j Q_j + F_2(q_i, P_j, t)$

with $Q_j = \frac{\partial F_2}{\partial P_j}$, $p_i = \frac{\partial F_2}{\partial q_i}$

↳ $F = \sum_i p_i q_i + F_3(Q_j, p_i, t)$

with $q_i = -\frac{\partial F_3}{\partial p_i}$, $P_j = -\frac{\partial F_3}{\partial Q_j}$

↳ $F = \sum_i p_i q_i - \sum_j P_j Q_j + F_4(p_i, q_i, t)$

with $q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_j = \frac{\partial F_4}{\partial P_j}$

Example : simple harmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 \qquad \text{with } Q = \sqrt{p^2+q^2}$$

$\hookrightarrow Q^2 = p^2 + q^2$

$$p = \frac{\partial F}{\partial q} = \sqrt{Q^2 - q^2}$$

$$\hookrightarrow F(q,Q) = \int dq \sqrt{Q^2 - q^2}$$

$$= \frac{1}{2} Q^2 \left(\sin^{-1}\frac{q}{Q} + \frac{q}{Q}\sqrt{1 - \frac{q^2}{Q^2}} \right)$$

$$\Rightarrow P = -\frac{\partial F}{\partial Q} = -Q \sin^{-1}\frac{q}{Q}$$

Let's do that again, with some foresight:

$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 \Rightarrow$ we want to write

$$\begin{cases} p = f(P)\cos Q \\ q = f(P)\sin Q \end{cases}$$

$$\Rightarrow K = \frac{1}{2} f(P)^2$$

$\hookrightarrow Q$ is cyclic