Classical Mechanics (Phys 601) - November 8, 2011

* Euler's equations:

$$\bar{\Gamma}^{(e)} = \left(\frac{d\bar{L}}{dt}\right)_{inertial} = \left(\frac{d\bar{L}}{dt}\right)_{body} + \bar{\omega} \times \bar{L}$$

$\left(\bar{L} = I\bar{\omega}\right.$, with I constant in body frame

$$\bar{\Gamma}^{(e)} = I \frac{d\bar{\omega}}{dt} + \bar{\omega} \times (I\bar{\omega})$$

In principal axes frame:

$$\Gamma_i^{(e)} = I_i \frac{d\omega_i}{dt} + \sum_{j,k} \varepsilon_{ijk}\, \omega_j \omega_k I_k$$

or component-wise:

$$\begin{cases} \Gamma_1^{(e)} = I_1 \dot{\omega}_1 - \omega_2\omega_3 (I_2 - I_3) \\ \Gamma_2^{(e)} = I_2 \dot{\omega}_2 - \omega_3\omega_1 (I_3 - I_1) \\ \Gamma_3^{(e)} = I_3 \dot{\omega}_3 - \omega_1\omega_2 (I_1 - I_2) \end{cases}$$

Note: if I is not constant

$$\Gamma_i^{(e)} = \frac{d}{dt}(I_i\omega_i) + \sum_{j,k} \varepsilon_{ijk}\, \omega_j \omega_k I_k$$

$\Rightarrow$ these equations require the torque in the moving rigid body principal axes frame

* Torque-free motion : $\bar{\Gamma}^{(e)} = \bar{0} \Rightarrow \left(\frac{d\bar{L}}{dt}\right)_{\text{inertial}} = \bar{0}$

Kinetic energy in principal axes frame :

$$T = \frac{1}{2}\omega^T I \omega = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Equations of motion :

$$\begin{cases} I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) \\ I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) \end{cases}$$

Spherical top :

$$\begin{cases} I_1\dot{\omega}_1 = 0 \\ I_2\dot{\omega}_2 = 0 \Rightarrow \bar{\omega} \text{ constant} \\ I_3\dot{\omega}_3 = 0 \end{cases}$$

Symmetric top : $I_1 = I_2 \neq I_3 \rightarrow$ example: $\rho(r_\perp, z)$

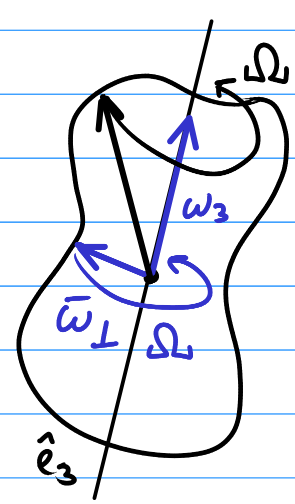
$$\Rightarrow \begin{cases} I_1\dot{\omega}_1 = \omega_2\omega_3(I_1 - I_3) \\ I_1\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 = 0 \quad \Rightarrow \omega_3 \text{ is conserved} \end{cases}$$

Introduce: $\Omega = \omega_3\left(\frac{I_3 - I_1}{I_1}\right) = \text{constant}$

$$\Rightarrow \begin{cases} \dot{\omega}_1 = -\Omega\omega_2 \\ \dot{\omega}_2 = \Omega\omega_1 \end{cases} \Leftrightarrow \begin{cases} \ddot{\omega}_1 = -\Omega^2\omega_1 \\ \ddot{\omega}_2 = -\Omega^2\omega_2 \end{cases}$$

$$\Rightarrow \begin{cases} \omega_1 = \omega_\perp \cos\Omega(t - t_0) \\ \omega_2 = \omega_\perp \sin\Omega(t - t_0) = -\frac{\dot{\omega}_1}{\Omega} \\ \omega_3 = \text{constant} \end{cases}$$

with $\omega_\perp^2 = \omega_1^2 + \omega_2^2 = \text{constant}$

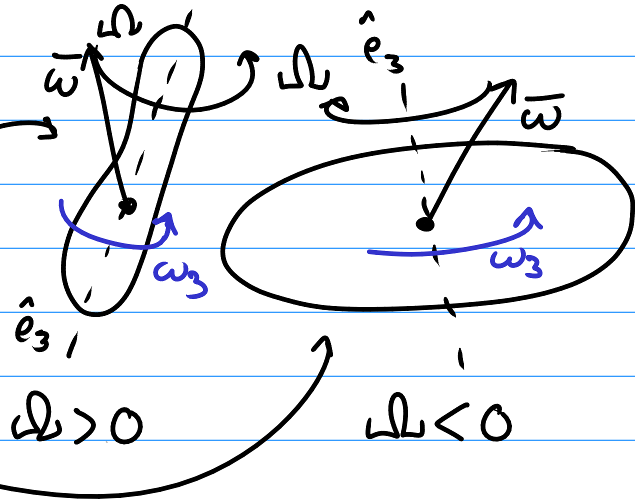


$\Rightarrow$ precession around principal axis $\hat{e}_3$

Direction of precession:

if $I_1 < I_3 \longrightarrow \Omega > 0$
↳ prolate

if $I_1 > I_3 \longrightarrow \Omega < 0$
↳ oblate



Magnitude of ω:

$\omega_1^2 + \omega_2^2 = \omega_\perp^2 = \text{constant}$

$\omega^2 = \omega_\perp^2 + \omega_3^2 = \text{constant}$ magnitude of angular velocity

At $t = t_0$, $\bar{\omega}$ in $\hat{e}_1 \hat{e}_3$ plane with angle λ between $\bar{\omega}$ and $\hat{e}_3$

$$\Rightarrow \begin{cases} \omega_3(t_0) = \omega \cos \lambda \\ \omega_1(t_0) = \omega \sin \lambda \cos \Omega(t - t_0) \\ \omega_2(t_0) = \omega \sin \lambda \sin \Omega(t - t_0) \end{cases}$$

Precession of the earth's axis:

$\frac{I_3 - I_1}{I_1} \approx \frac{1}{305}$ (i.e. $I_3 \approx I_1$) $\rightarrow$ prolate $\rightarrow$ precession opposite to earth's rotation

λ small $\rightarrow \omega_3 \approx \omega$

$$\Rightarrow \Omega = \omega_3 \frac{I_3 - I_1}{I_1} \approx \frac{\omega}{305} \Rightarrow T = 305 \text{ days}$$

In reality: $T \approx 14$ months for $\lambda = 6 \times 10^{-7}$ rad $= 4$ m at poles

Angular momentum :

$$\bar{\Gamma}^{(e)} = 0 \Rightarrow \left(\frac{d\bar{L}}{dt}\right)_{\text{inertial}} = 0 \Rightarrow \bar{L} = \text{constant of motion}$$

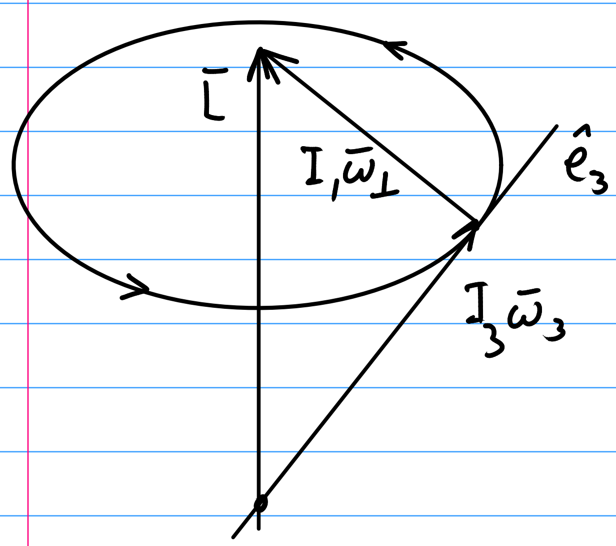
and $\bar{L} = I\bar{\omega} = I_1\bar{\omega}_\perp + I_3\bar{\omega}_3 \rightarrow \bar{L}, \hat{e}_3, \bar{\omega}_\perp$ in plane

$$I_3\dot{\omega}_3 = 0 \Rightarrow |\bar{\omega}_3| = \text{constant of motion}$$

$$|\bar{\omega}| = \sqrt{\omega_\perp^2 + \omega_3^2} = \text{constant of motion}$$

angle θ between $\bar{L}$ in inertial frame and $\hat{e}_3$ in body frame :

$$\cos\theta = \frac{\bar{L} \cdot \bar{\omega}_3}{|\bar{L}| \cdot |\bar{\omega}_3|}$$



Asymmetric top : $I_1 \neq I_2 \neq I_3 \rightarrow$ general case

1) $T = \frac{1}{2}\omega^T I \omega = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = E$ = constant of motion

2) $\Gamma^{(e)} = 0 \Rightarrow \bar{L}$ = constant in the inertial frame
Because magnitude $\bar{L}\cdot\bar{L}$ is independent of coordinate frame
$\Rightarrow \bar{L}\cdot\bar{L}$ = constant of motion in any coordinate frame

$\bar{L} = I\bar{\omega} \Rightarrow \bar{L}^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2$ = constant of motion

Explicitly : $\frac{d}{dt}\left(\bar{L}^2\right) = 2I_1^2\omega_1\dot{\omega}_1 + 2I_2^2\omega_2\dot{\omega}_2 + 2I_3^2\omega_3\dot{\omega}_3$

$$= 2I_1\omega_1\omega_2\omega_3(I_2 - I_3) + 2I_2\omega_2\omega_3\omega_1(I_3 - I_1) + 2I_3\omega_3\omega_1\omega_2(I_1 - I_2)$$

$$= 0$$

In principle : arbitrary constants from initial conditions

$$\begin{cases} I_1\omega_1^2 + I_2\omega_2^2 = 2E - I_3\omega_3^2 \\ I_1^2\omega_1^2 + I_2^2\omega_2^2 = L^2 - I_3^2\omega_3^2 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} I_1 & I_2 \\ I_1^2 & I_2^2 \end{pmatrix}\begin{pmatrix} \omega_1^2 \\ \omega_2^2 \end{pmatrix} = \begin{pmatrix} 2E - I_3\omega_3^2 \\ L^2 - I_3^2\omega_3^2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \omega_1^2 \\ \omega_2^2 \end{pmatrix} = \begin{pmatrix} I_1 & I_2 \\ I_1^2 & I_2^2 \end{pmatrix}^{-1}\begin{pmatrix} 2E - I_3\omega_3^2 \\ L^2 - I_3^2\omega_3^2 \end{pmatrix}$$

$\Rightarrow$ ω_1 and ω_2 as function of E, L, ω_3^2

Now insert ω_1 and ω_2 :

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2)$$

$\Rightarrow$ differential equation for $\omega_3(t)$ (hard to solve)

Stability of rotations around principal axes

Special case : consider $\bar{\omega} = \omega_i \hat{e}_i$ along principal axis

$$\Rightarrow \begin{cases} I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) = 0 \\ I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1) = 0 \\ I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) = 0 \end{cases} \Rightarrow \bar{\omega} \text{ is constant and remains along axis}$$

↳ equilibrium

Consider again : $\begin{cases} I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 = 2E \\ \underbrace{I_1^2 \omega_1^2}_{L_1^2} + \underbrace{I_2^2 \omega_2^2}_{L_2^2} + \underbrace{I_3^2 \omega_3^2}_{L_3^2} = L^2 \end{cases}$

$\Leftrightarrow \begin{cases} \dfrac{L_1^2}{I_1} + \dfrac{L_2^2}{I_2} + \dfrac{L_3^2}{I_3} = 2E \\ L_1^2 + L_2^2 + L_3^2 = L^2 \end{cases}$

$\rightarrow$ ellipsoid in $\bar{L}$-space with semiaxes $\sqrt{2EI_i}$

$\rightarrow$ sphere in $\bar{L}$-space with radius L

↳ L_1, L_2, L_3 are components of $\bar{L}$ in the principal axes frame $\rightarrow$ follows path on the intersection of sphere and ellipsoid

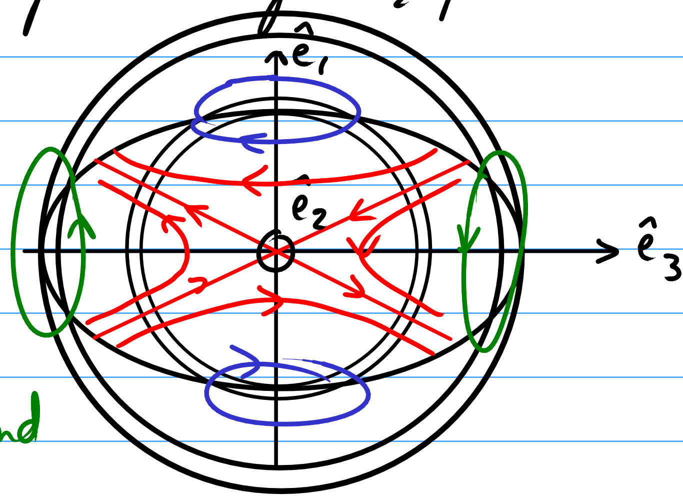
Assume $I_3 > I_2 > I_1 \Rightarrow$ shortest semiaxis along $\hat{e}_1$, etc

For any intersection to exist we must have
$$2EI_1 < L^2 < 2EI_3$$

If L^2 just larger than $2EI_1$ $\rightarrow$ path around $\hat{e}_1$ pole
If L^2 just smaller than $2EI_3$ $\rightarrow$ path around $\hat{e}_3$ pole

If $L^2 = 2EI_2$ $\rightarrow$ path through $\hat{e}_2$ pole

2D equivalent:



rotation in the vicinity of $\hat{e}_1$ and $\hat{e}_3$ is stable, but around $\hat{e}_2$ is unstable

Small oscillations:

$$\bar{\omega} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \omega_0 + \eta_3 \end{pmatrix} \Rightarrow \begin{cases} I_1 \dot{\eta}_1 = \omega_2 \omega_3 (I_2 - I_3) = \eta_2 \omega_0 (I_2 - I_3) \\ I_2 \dot{\eta}_2 = \omega_3 \omega_1 (I_3 - I_1) = \omega_0 \eta_1 (I_3 - I_1) \\ I_3 \dot{\eta}_3 = \omega_1 \omega_2 (I_1 - I_2) = 0 + O(\eta_1 \eta_2) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\eta}_1 = \omega_0 \dfrac{I_2 - I_3}{I_1} \eta_2 \\ \dot{\eta}_2 = \omega_0 \dfrac{I_3 - I_1}{I_2} \eta_1 \\ \dot{\eta}_3 = 0 \end{cases} \Rightarrow \ddot{\eta}_1 = \omega_0^2 \frac{I_2 - I_3}{I_1} \frac{I_3 - I_1}{I_2} \eta_1$$

$$\Leftrightarrow \begin{cases} \ddot{\eta}_1 = -\Omega^2 \eta_1 \\ \ddot{\eta}_2 = -\Omega^2 \eta_2 \end{cases}$$

with $\Omega^2 = \omega_0^2 \dfrac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}$

If $(I_3 - I_1)(I_3 - I_2) > 0$:

$$\begin{cases} \eta_1 = A_1 \cos(\Omega t + \varphi) \\ \eta_2 = A_2 \cos(\Omega t + \varphi) \end{cases}$$

$\Rightarrow$ stable harmonic oscillations when $I_3 > I_1$ and $I_3 > I_2$ or $I_3 < I_1$ and $I_3 < I_2$

$\Rightarrow$ largest or smallest moment

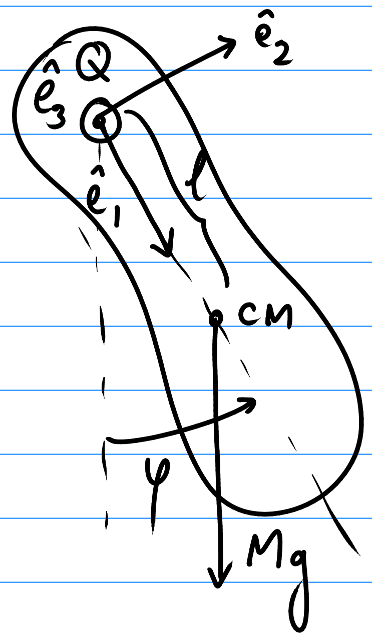
If $(I_3 - I_1)(I_3 - I_2) < 0$:

$$\begin{cases} \eta_1 = A_1 e^{|\Omega| t} + B_1 e^{-|\Omega| t} \\ \eta_2 = A_2 e^{|\Omega| t} + B_2 e^{-|\Omega| t} \end{cases}$$

$\Rightarrow$ unstable when middle moment of inertia

* Motion with torque: Compound pendulum

Rigid body rotating around a fixed axis $\hat{e}_3$ (not principal)
(not through center of mass!)



↳ create orthonormal rigid-body reference frame with $\hat{e}_1$ through center of mass $\rightarrow \bar{R} = l\hat{e}_1$

Degree of freedom: φ

Torque: $\bar{\Gamma}^{(e)} = \bar{R} \times M\bar{g}$

$$\Rightarrow \Gamma_3^{(e)} = -Mgl\sin\varphi$$

$$\frac{d\bar{L}}{dt} = \bar{\Gamma}^{(e)} \Leftrightarrow \frac{d}{dt}(I\bar{\omega}) = \bar{\Gamma}^{(e)} \qquad \bar{\omega} = (0,0,\dot{\varphi})$$

$$\Leftrightarrow I_{33}\ddot{\varphi} = -Mgl\sin\varphi$$

$$I_{33} = \int d^3\bar{r}\, \rho(\bar{r})\,(x_1^2 + x_2^2) = \int d^3\bar{r}\,\rho(\bar{r})\, r_\perp^2$$

$$\rightarrow \ddot{\varphi} = -\frac{Mgl}{I_{33}}\sin\varphi \approx -\frac{Mgl}{I_{33}}\varphi \quad \text{for small } \varphi$$

→ harmonic oscillations with frequency

$$\Omega_Q^2 = \frac{Mgl}{I_{33}} \quad \text{(around point Q)}$$

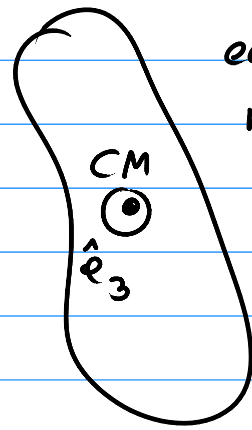
Parallel-axis theorem : $I_{33} = \bar{I}_{33} + M\ell^2$

Radius of gyration :

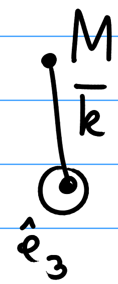
$$\bar{I}_{33} = M\bar{k}^2$$

$$I_{33} = M\bar{k}^2 + M\ell^2 = Mk^2$$

$$k^2 = \bar{k}^2 + \ell^2$$



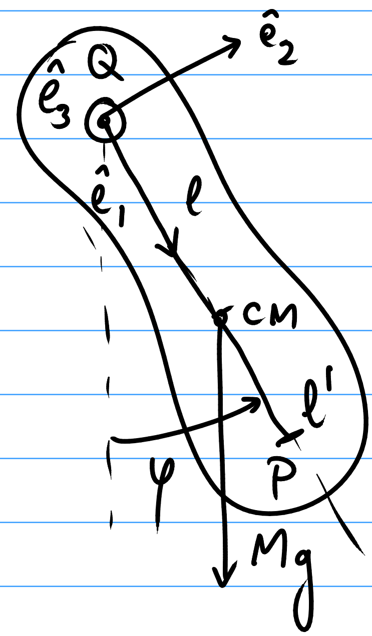
equivalent moments



$$\Rightarrow \quad \Omega_Q^2 = \frac{g\ell}{\bar{k}^2+\ell^2} = \frac{g}{\ell'}$$

with $\ell' = \dfrac{\ell^2+\bar{k}^2}{\ell} = \ell + \dfrac{\bar{k}^2}{\ell}$ = equivalent length

Consider now the axis of rotation in the point P

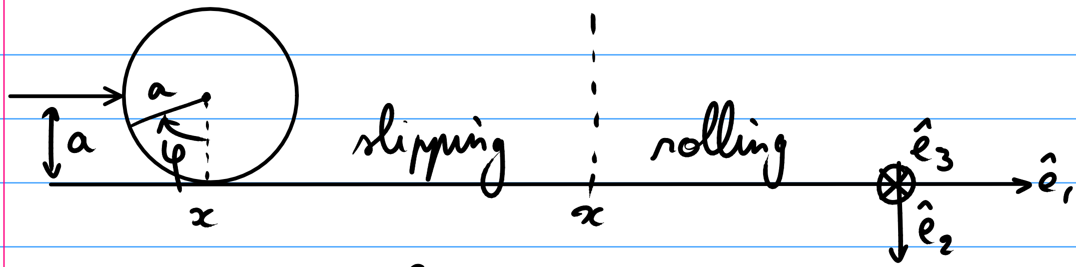


$$\Omega_P^2 = \frac{Mg(\ell'-\ell)}{M\left((\ell'-\ell)^2+\bar{k}^2\right)} \quad \text{(around point P)}$$

$$= \frac{g}{\ell'-\ell+\frac{\bar{k}^2}{\ell'-\ell}} = \frac{g}{\frac{\bar{k}^2}{\ell}+\ell} \equiv \Omega_Q^2$$

→ connection to center of percussion of baseball bat or tennis racket

* Rolling and sliding billiard ball



$$I_1 = I_2 = I_3 = \frac{2}{5} M a^2$$

friction $F_\parallel = \mu F_\perp = -\mu g M \hat{e}_1 \quad \Rightarrow \quad \ddot{x} = -\mu g \qquad (1)$

$$\frac{dL_3}{dt} = I_3 \ddot{\varphi} = F_\parallel a \quad \Rightarrow \quad a\ddot{\varphi} = \frac{5}{2}\mu g \qquad (2)$$

Hit center of ball:

→ pure sliding immediately after impulse

$(1) \Rightarrow \dot{x} = v_0 - \mu g t \qquad (x(0) = 0, \ \dot{x}(0) = v_0)$

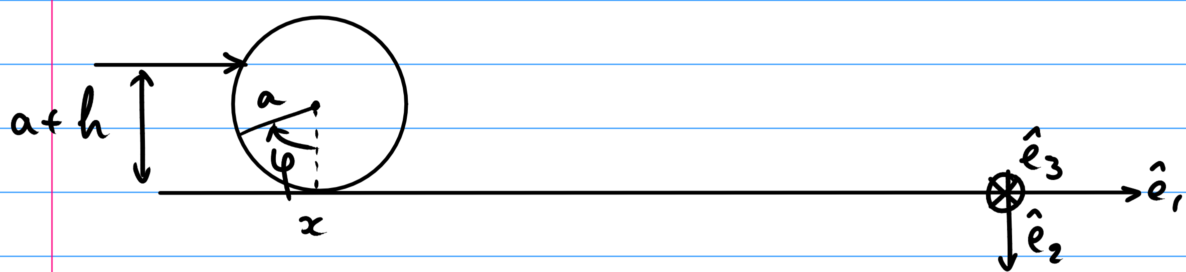
$(2) \Rightarrow a\dot{\varphi} = \frac{5}{2}\mu g t \qquad (\varphi(0) = 0, \ \dot{\varphi}(0) = 0)$

rolling without slipping when

$$\dot{x} = a\dot{\varphi} \Leftrightarrow v_0 - \mu g t = \frac{5}{2}\mu g t \Leftrightarrow t_1 = \frac{2}{7}\frac{v_0}{\mu g}$$

$$\Rightarrow \begin{cases} \dot{x}(t_1) = v_0 - \mu g t_1 = \frac{5}{7} v_0 \\ x(t_1) = v_0 t_1 - \frac{1}{2}\mu g t_1^2 = \frac{2}{7}\frac{v_0^2}{\mu g} - \frac{1}{2}\frac{4}{49}\frac{v_0^2}{\mu g} = \frac{12}{49}\frac{v_0^2}{\mu g} \end{cases}$$

Hit a height h above center:



$$\Delta p_x = \int F_x(t)\,dt = \text{impulse} = M v_0$$

$$\Delta L = \int h\, F_x(t)\,dt = h \cdot \text{impulse} = I\omega_0$$

$$\Rightarrow \quad h M v_0 = I \omega_0 \quad \Leftrightarrow \quad h v_0 = \frac{2}{5} a^2 \omega_0$$

1) $h = \frac{2}{5}a \Rightarrow v_0 = a\omega_0 \rightarrow$ no slipping because $\dot{x} = a\dot{\varphi}$ at $t=0$

2) $h < \frac{2}{5}a \Rightarrow v_0 > a\omega_0 \rightarrow$ slipping and friction retards the ball (see above for $h=0$)

3) $h > \frac{2}{5}a \Rightarrow v_0 < a\omega_0 \rightarrow$ ball rotates faster than it rolls $\rightarrow$ friction speeds up the ball