Homework Assignment 10

① Principal moments $\rightarrow$ $I_{ij} = 0,\ i \neq j$

$$I_1 = \int d^3\bar{r}\ \rho(\bar{r})\ (x_2^2 + x_3^2)$$

$$= \int d^3\bar{r}\ \rho(\bar{r})\ (x_1^2 + x_2^2 + x_1^2 + x_3^2 - 2x_1^2)$$

$$= I_2 + I_3 \underbrace{- 2\int d^3\bar{r}\ \rho(\bar{r})\ x_1^2}_{\geq 0}$$

$$\Rightarrow \quad I_1 \leq I_2 + I_3$$

and similar for $I_2 \leq I_1 + I_3$ and $I_3 \leq I_1 + I_2$.

② $$I_{33} = \int d^3\bar{r}\ \rho(r,z)\ r^2 = \int_0^{+\infty} dr \int_0^{2\pi} r\,d\varphi \int_{-\infty}^{+\infty} dz\ \rho(r,z)\, r^2$$

$$= 2\pi \int_0^{+\infty} r^3\,dr \int_{-\infty}^{+\infty} dz\ \rho(r,z)$$

$$I_{13} = \int d^3\bar{r}\ \rho(r,z)\,(r\cos\varphi \cdot z)$$

$$= \int_0^{+\infty} dr \int_0^{2\pi} r\,d\varphi \int_{-\infty}^{+\infty} dz\ \rho(r,z)\, z\, r\cos\varphi$$

$$= \int_0^{+\infty} r^2\,dr \int_{-\infty}^{+\infty} z\,dz\ \rho(r,z) \underbrace{\int_0^{2\pi} \cos\varphi\, d\varphi}_{\sin\varphi \big|_0^{2\pi} = 0}$$

$$= 0$$

$I_{23} = 0$ after similar calculation

$$\Rightarrow \quad I = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

$$I_{12} = -\int d^3\bar{r}\ \rho(r,z)\ r\cos\varphi\ r\sin\varphi$$
$$= -\int_0^{+\infty} r^3\, dr \int_{-\infty}^{+\infty} dz\ \rho(r,z) \underbrace{\int_0^{2\pi} \cos\varphi \sin\varphi\, d\varphi}_{= \frac{1}{2}\sin^2\varphi \Big|_0^{2\pi}}$$
$$= 0$$

$$I_{11} = \int d^3\bar{r}\ \rho(r,z)\left(r^2\cos^2\varphi + z^2\right)$$

$$I_{22} = \int d^3\bar{r}\ \rho(r,z)\left(r^2\sin^2\varphi + z^2\right)$$
$$= \int_0^{+\infty} dr \int_{-\infty}^{+\infty} dz\ \rho(r,z) \int_0^{2\pi} r\, d\varphi \left(r^2\sin^2\varphi + z^2\right)$$
$$= \int_0^{+\infty} dr \int_{-\infty}^{+\infty} dz\ \rho(r,z)\left(r^3 \underbrace{\int_0^{2\pi} \sin^2\varphi\, d\varphi}_{= \pi} + 2\pi r z^2\right)$$
$$= \int_0^{+\infty} dr \int_{-\infty}^{+\infty} dz\ \rho(r,z)\left(\pi r^3 + 2\pi r z^2\right)$$
$$= I_{11} \quad \text{because} \int_0^{2\pi} \cos^2\varphi\, d\varphi = \pi \text{ as well}$$

$$I = \begin{pmatrix} I_{11} & & \\ & I_{11} & \\ & & I_{33} \end{pmatrix} \rightarrow \text{degenerate}$$

(3) $I_1 = I_2 = \int_0^R dr \int_{-h/2}^{h/2} dz\, \rho(\pi r^3 + 2\pi r z^2)$

$$= \rho\pi\left(\frac{R^4}{4} h + 2\frac{R^2}{2} 2 \frac{(h/2)^3}{3} \right) = \frac{1}{12} M (3R^2 + h^2)$$

$$I_3 = 2\pi \int_0^R dr \int_{-h/2}^{h/2} dz\, \rho r^3 = 2\pi h \rho \frac{R^4}{4} = \frac{1}{2} M R^2$$

$$\Rightarrow I_3 = I_1 \Leftrightarrow R^2 = \frac{1}{6}(3R^2 + h^2) \Leftrightarrow h = \sqrt{3} R$$

(4) $M = \frac{1}{3}\rho\pi R^2 h \rightarrow a = \frac{1}{M}\int_0^h dz \int_0^{R\frac{z}{h}} dr\, 2\pi \rho r z = 2\pi \frac{\rho}{M} \frac{R^2}{2h^2} \frac{h^4}{4}$

$$= \frac{3}{4} h$$

$$I_{11} = I_{22} = \int_0^h dz \int_0^{R\frac{z}{h}} dr\, \rho\left(\pi r^3 + 2\pi r z^2\right)$$

$$= \rho \int_0^h dz \left(\pi \frac{R^4 z^4}{4h^4} + 2\pi z^2 \frac{R^2 z^2}{2h^2} \right)$$

$$= \rho\left(\pi \frac{R^4}{4h^4} \frac{h^5}{5} + \pi \frac{R^2}{h^2} \frac{h^5}{5} \right) = 3M\left(\frac{R^2}{20} + \frac{h^2}{5} \right)$$

$$\Rightarrow \bar{I}_{11} = I_{11} - \frac{9}{16} h^2 M = \frac{3}{20} M\left(R^2 + \frac{1}{4} h^2\right)$$

$$\bar{I}_{33} = I_{33} = 2\pi \int_0^h dz \int_0^{R\frac{z}{h}} r^3 dr\, \rho = 2\pi\rho \int_0^h dz\, \frac{R^4}{4h^4} z^4$$

$$= 2\pi\rho \frac{R^4}{4h^4} \frac{h^5}{5} = \frac{3}{10} M R^2$$

(5) $I_1 = I_2 = I_3 = \frac{1}{3}(I_1 + I_2 + I_3) = \frac{1}{3} \int d^3 r\, \rho (2 r^2)$

$$= \frac{2}{3}\rho \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^R r^4 dr$$

$$= \frac{2}{3}\rho \cdot 2\pi \cdot 2 \cdot \frac{R^5}{5} \qquad M = \frac{4}{3}\pi R^3$$

$$= \frac{2}{5} M R^2$$

⑥ $\Delta\varphi = \omega T - \int_0^T \omega(t)\, dt$

$$\frac{d}{dt}(I\omega) = 0 \Rightarrow I\omega = I_3(t)\,\omega(t) = \text{constant}$$

$$\Leftrightarrow \quad \omega(t) = \frac{\text{constant}}{I_3(t)}$$

$$I_3(t) = \frac{2}{5}MR^2 + m\left(R\sin\frac{\pi t}{T}\right)^2$$

$$\Rightarrow \omega(t) = \omega \frac{\frac{2}{5}MR^2}{\frac{2}{5}MR^2 + mR^2\sin^2\frac{\pi t}{T}}$$

$$\Delta\varphi = \omega T - \int_0^T \omega \frac{1}{1 + \frac{5m}{2M}\sin^2\frac{\pi t}{T}}\, dt$$

$$= \omega T - \omega\frac{T}{\pi}\frac{\pi}{\sqrt{1 + \frac{5m}{2M}}}$$

$$= \omega T\left(1 - \sqrt{\frac{2M}{2M + 5m}}\right)$$