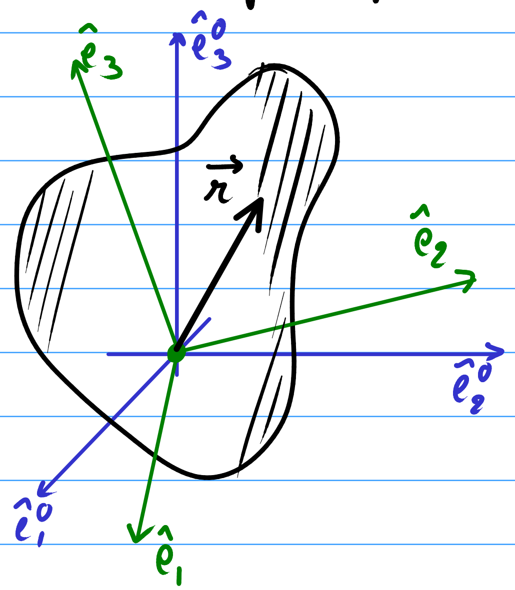
Classical Mechanics (Phys 601) – November 4, 2011

* Rigid body dynamics

$\{\hat{e}_1^0, \hat{e}_2^0, \hat{e}_3^0\}$

Consider inertial frame and rigid body with fixed point at the origin of the inertial frame:



rigid body can rotate with 3 degrees of freedom around the origin.

$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

Consider rotating reference frame fixed to the rigid body with its origin at origin of inertial frame.

$$\Rightarrow \quad \frac{d\hat{e}_i}{dt} = \bar{\omega} \times \hat{e}_i$$

instantaneous angular velocity (time-varying)

$$\Rightarrow \left(\frac{dr}{dt}\right)_{\text{inertial}} = \left(\frac{dr}{dt}\right)_{\text{body}} + \bar{\omega} \times \bar{r}$$

Kinetic energy of rigid body

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \omega^T I \omega$$

total mass

velocity of center of mass

inertia tensor

$$I_{ij} = \int_V d^3\bar{r}\ \rho(\bar{r})\ \left(r^2\delta_{ij} - x_i x_j\right)$$

$$= \sum_k m_k \left(r_k^2\,\delta_{ij} - r_{ki}\,r_{kj}\right)$$

* Angular momentum :

$$\sum_k^N m_k \bar{r}'_k = 0$$

$$\bar{L} = \sum_k^N m_k\, \bar{r}_k \times \bar{v}_k$$

$$= \sum_k^N m_k \left(\bar{R} + \bar{r}'_k\right) \times \left(\bar{V} + \bar{\omega} \times \bar{r}'_k\right)$$

$$= M\,\bar{R} \times \bar{V} + \sum_k^N m_k \left(\bar{r}'_k \times \left(\bar{\omega} \times \bar{r}'_k\right)\right)$$

$$= M\bar{R} \times \bar{V} + \sum_k^N m_k \underbrace{\left(r_k'^2\,\bar{\omega} - (\bar{\omega}\cdot\bar{r}'_k)\,\bar{r}'_k\right)}_{\text{linear in } \bar{\omega}}$$

$$= M\bar{R} \times \bar{V} + \bar{L}'$$

$$\bar{L}' = \sum_j I_{ij}\,\omega_j = I\bar{\omega}$$ = angular momentum with respect to center of mass

$$\Rightarrow T = \frac{1}{2} M \bar{V}^2 + \frac{1}{2}\bar{L}\cdot\bar{\omega}$$

For rotation around z axis : $\bar{\omega} = (0, 0, \omega)$

$$\hookrightarrow T_{rot} = \frac{1}{2}\bar{L}\cdot\bar{\omega} = \frac{1}{2} L_z\,\omega = \frac{1}{2} I_{33}\,\omega^2$$

In general for $\bar{L}$:

$$\hookrightarrow \bar{L} = I\cdot\bar{\omega} = \left(I_{13}\,\omega\,,\ I_{23}\,\omega,\ I_{33}\,\omega\right)$$

+ Parallel-axis theorem

We calculate I in the reference frame attached to the rigid body and centered on the center of mass. This is not a unique choice. If there is a physical fixed point, that is not the center of mass, that might be a better choice.

Example : inertia tensor for cube around center of mass

$$I_{11}^{(CM)} = \int d^3\bar{r}\, \rho\, (x_2^2 + x_3^2) = \rho\, \ell^2 \left[\frac{x_2^3}{3} + \frac{x_3^3}{3} \right]_{-\ell/2}^{\ell/2} = \frac{1}{6} M \ell^2$$

instead of $I_{ii}^{(corner)} = \frac{2}{3} M \ell^2$ for inertia tensor in frame centered on one of the corners

$$\Rightarrow \quad I_{ii}^{(corner)} = I_{ii}^{(CM)} + \frac{1}{2} M \ell^2$$

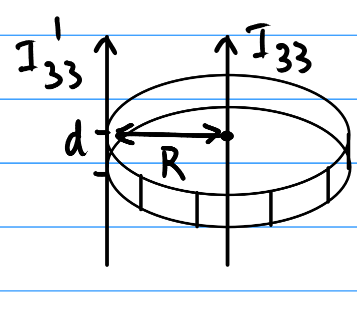
$$I_{12}^{(CM)} = \int d^3\bar{r}\, \rho\, (-x_1 x_2) = -\rho\, \ell \left[\frac{x_1^2}{2} \right]_{-\ell/2}^{\ell/2} \left[\frac{x_2^2}{2} \right]_{-\ell/2}^{\ell/2} = 0$$

instead of $I_{ij}^{(corner)} = -\frac{1}{4} M \ell^2$ around one of the corners

$$\Rightarrow \quad I_{ij}^{(corner)} = I_{ij}^{(CM)} - \frac{1}{4} M \ell^2$$

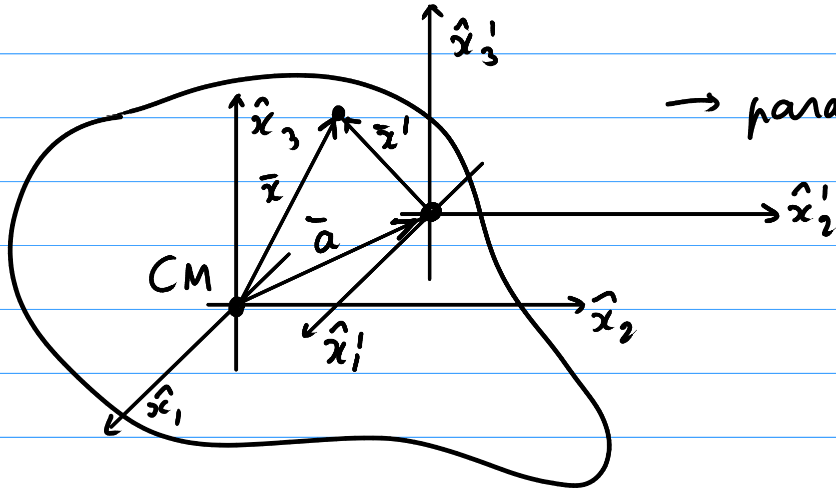
Example : inertia tensor for disk around center of mass

$$I_{33} = \int d^3\bar{r}\, \rho\, (x_1^2 + x_2^2)$$
$$= \int_0^R dr \int_0^{2\pi} r\, d\varphi \int_{-d/2}^{d/2} dz\, \rho\, r^2$$
$$= 2\pi \rho d \frac{R^4}{4} = \frac{1}{2} M R^2$$



Consider $\bar{x}' = \bar{x} - \bar{a}$ where $\bar{x}$ is centered on the center of mass

→ parallel shift in frames



$$I'_{ij} = \sum_k m_k \left[\delta_{ij} (\bar{x}'_k)^2 - x'_{ki} x'_{kj} \right]$$

$$= \sum_k m_k \left[\delta_{ij} (\bar{x}_k - \bar{a})^2 - (x_{ki} - a_i)(x_{kj} - a_j) \right]$$

$$= \sum_k m_k \Big[\delta_{ij} \left(\bar{x}_k^2 + \bar{a}^2 - \cancel{2\bar{a}\cdot\bar{x}_k} \right) - \Big(x_{ki} x_{kj} + a_i a_j$$

$$- \cancel{a_i x_{kj}} - \cancel{a_j x_{ki}} \Big) \Big]$$

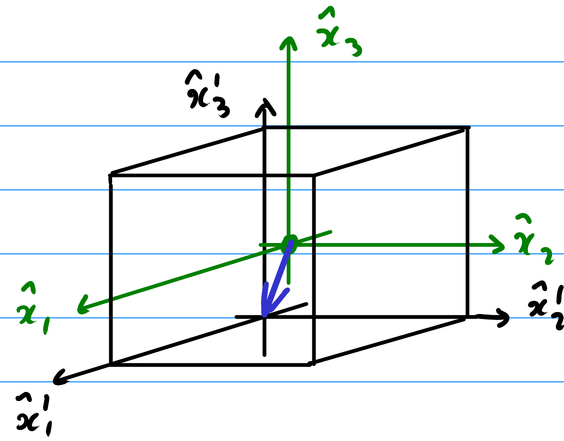
$$= I_{ij} + \sum_k m_k \left(\delta_{ij} \bar{a}^2 - a_i a_j \right) \quad \text{because } \sum_k m_k \bar{x}_k = \bar{0}$$

$$I'_{ij} = \underbrace{I_{ij}}_{\text{in center of mass: } \bar{I}_{ij}} + M \left(\delta_{ij} \bar{a}^2 - a_i a_j \right) \quad \text{(parallel-axis theorem)}$$

Note: for I'_{ii} and I_{ii} the difference is the transverse distance between the i-axis.

$$I'_{11} = I_{11} + M \left(x_1^2 + x_2^2 + x_3^2 - x_1^2 \right) = I_{11} + M \left(x_2^2 + x_3^2 \right)$$

Example: cube with edge ℓ



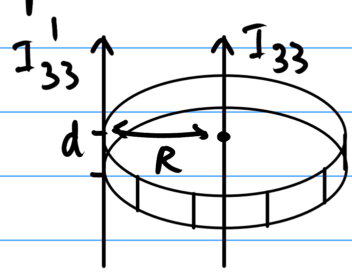
$\Rightarrow \quad \bar{a} = -\frac{\ell}{2}(1,1,1)$

$\Downarrow$

$$I'_{ij} = I_{ij} + M\left(\delta_{ij}\frac{3\ell^2}{4} - \frac{\ell^2}{4}\right)$$

$$\left.\begin{aligned} I'_{11} &= I_{11} + \frac{1}{2}M\ell^2 \\ I'_{12} &= I_{12} - \frac{1}{4}M\ell^2 \end{aligned}\right\} \text{ as found earlier}$$

Example: circular disk with radius R



↳ calculate inertia tensor around point on the edge → hard

↳ calculate inertia tensor around center of mass → easy!

Transverse distance is R → $I'_{33} = I_{33} + MR^2$

$$I_{33} = \frac{1}{2}MR^2 \Rightarrow I'_{33} = \frac{2}{3}MR^2$$

* Principal axes of a rigid body

$I_{ij} = I_{ji} \rightarrow$ real, symmetric

$\frac{1}{2} \omega^T I \omega = T_{rot} \geqslant 0$ and $\frac{1}{2}\omega^T I \omega = 0 \Leftrightarrow \omega = 0$

$\rightarrow$ positive-definite

I is a real, symmetric, positive-definite matrix

$\Downarrow$

I can be diagonalized by transforming to a different basis

$$I'_{ij} = I_i \delta_{ij} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$$

with $\begin{cases} I_i \text{ the eigenvalues of } I : \det(I_{ij} - I\delta_{ij}) = 0 \\ \hat{e}_i \text{ the orthonormal eigenvectors of } I : \\ \quad \text{directions in space : principal axes} \end{cases}$

Transformation from I to I':

$U = [\hat{e}_1 \ \hat{e}_2 \ \hat{e}_3]$ with $U^T U = U^{-1} U = \mathbb{1}$

$\Rightarrow \hat{e}_i$ are orthonormal vectors

$$\Rightarrow \quad I' = U^T I U = \mathrm{diag}(I_1, I_2, I_3)$$

We can express $\bar{\omega}$ in this new basis $\hat{e}_i$:

$$\bar{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \hat{e}_1 \cdot \bar{\omega} \\ \hat{e}_2 \cdot \bar{\omega} \\ \hat{e}_3 \cdot \bar{\omega} \end{pmatrix} = U^T \bar{\omega}$$

$$\Rightarrow \quad \bar{\omega} = U \bar{\xi}$$

$$T = \frac{1}{2} \omega^T I \omega = \frac{1}{2} \xi^T U^T I U \xi = \frac{1}{2} \sum_i I_i \xi_i^2$$

kinetic energy becomes just sum of $\frac{1}{2} I \dot{\theta}^2$-like terms

$$L' = U^T L = U^T I \omega = U^T I U \xi = \sum_i I_i \xi_i$$

angular momentum expressed in principal axes coordinates is now just sum of $I\dot{\theta}$-like terms

Classification of rigid bodies :

- $I_1 \neq I_2 \neq I_3 \rightarrow$ asymmetrical top
- $\underbrace{I_1 = I_2}_{} \neq I_3 \rightarrow$ symmetrical top
  - degenerate eigenvalues $\hat{e}_{1,2}$ in a plane perpendicular eigenvalue $\hat{e}_3$
- $I_1 = I_2 = I_3 \rightarrow$ spherical top (e.g. cube)
  - any three axes can be chosen as principal axes

Interpretation of products of inertia $I_{ij}, i \neq j$

↳ no real interpretation necessary → can always transform into diagonal system of principal axes with $I_{ij}, i\neq j = 0$

↓

if $I_{ij} \neq 0$ → not in principal axes system

Symmetry:

- plane of symmetry → { center of mass in plane; two principal axes in plane

  example: collection of particles with $x_3 = 0$

$$I_1 = \sum_k m_k x_{k2}^2 \quad , I_2 = \sum_k m_k x_{k1}^2$$

$$I_3 = \sum_k m_k \left(x_{k1}^2 + x_{k2}^2\right) = I_1 + I_2$$

- axis of symmetry → { center of mass on axis; principal axis is symmetry axis

  example: line of particles with $x_1 = x_2 = 0$

$$I_1 = \sum_k m_k x_{k3}^2 = I_2$$

$$I_3 = 0$$

→ special case because of zero eigenvalue

↳ no meaningful notion of rotation around the x_3 axis for point particles

Physical meaning of a principal axis:

- object rotating around principal axis: e.g. $\bar{\omega} = \omega \hat{e}_3$

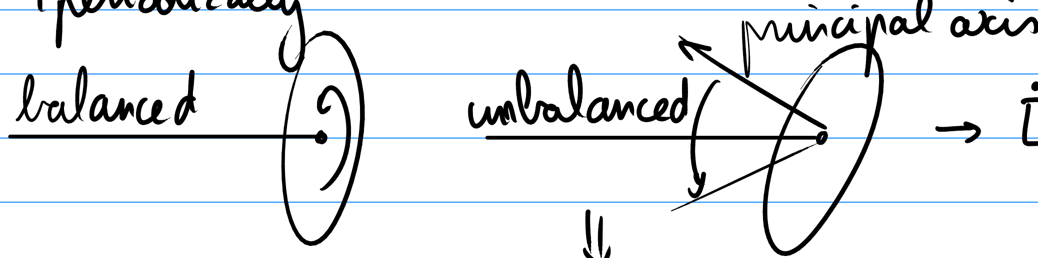
$$\bar{L} = I\bar{\omega} = I_3 \omega \hat{e}_3 = I_3 \bar{\omega}$$

$\Rightarrow$ $\bar{L}$ and $\bar{\omega}$ are parallel

- if $\dot{\bar{\omega}} = \dot{\omega}\hat{e}_3 = 0$ (uniform rotation) then $\dot{\bar{L}} = 0$

$\Rightarrow$ no torque is exerted by uniform rotation around principal axis

- if $\bar{\omega}$ is <u>not</u> along a principal axis, $I\bar{\omega}$ will be a {continuously / periodically} changing linear combination of the I_1, I_2, I_3



$\rightarrow \dot{\bar{L}} \neq 0 \rightarrow \bar{\Gamma} \neq 0$

$\Downarrow$

torque exerted/required

* Euler's Equations of Rigid Body dynamics

Newton's law : $\left(\frac{d\bar{L}}{dt}\right)_{\text{inertial}} = \bar{\Gamma}^{(e)}$ = external torque

$$\left(\frac{d\bar{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{L}}{dt}\right)_{\text{body}} + \bar{\omega} \times \bar{L}$$

Angular momentum : $\bar{L} = \underset{\substack{\uparrow \\ \text{constants in} \\ \text{rigid body frame}}}{I}\bar{\omega} \Rightarrow \left(\frac{d\bar{L}}{dt}\right)_{\text{body}} = I\frac{d\bar{\omega}}{dt}$

remember that:

$$\left(\frac{d\bar{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{\omega}}{dt}\right)_{\text{body}}$$

$$\Rightarrow \bar{\Gamma}^{(e)} = I\frac{d\bar{\omega}}{dt} + \bar{\omega} \times (I\bar{\omega})$$

In principal axes basis of rigid body :

$$I\bar{\omega} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} \Rightarrow \bar{\omega} \times (I\bar{\omega}) = \begin{pmatrix} \omega_2 I_3\omega_3 - \omega_3 I_2\omega_2 \\ \omega_3 I_1\omega_1 - \omega_1 I_3\omega_3 \\ \omega_1 I_2\omega_2 - \omega_2 I_1\omega_1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \Gamma_1^{(e)} = I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) \\ \Gamma_2^{(e)} = I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1) \\ \Gamma_3^{(e)} = I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) \end{cases}$$

$$\Gamma_i^{(e)} = I_i\dot{\omega}_i + \sum_{j,k} \varepsilon_{ijk}\,\omega_j\omega_k I_k$$