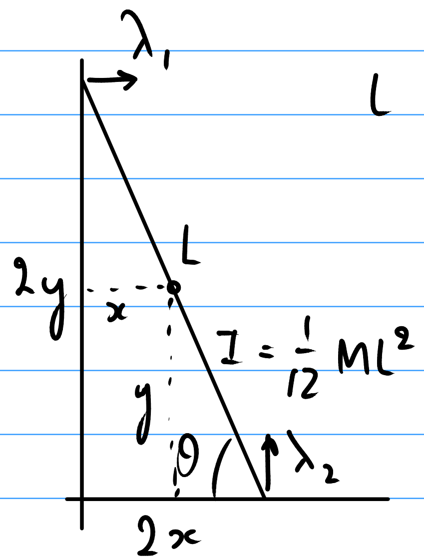
①



$$L = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta} - Mgy$$

Constraints : $\begin{cases} 2x = L \cos\theta \\ 2y = L \sin\theta \end{cases}$

$$\Downarrow$$

$$L = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta} - Mgy$$
$$+ \lambda_1 (2x - L\cos\theta) + \lambda_2 (2y - L\sin\theta)$$

Euler-Lagrange equations and constraints are :

$$\begin{cases} M\ddot{x} - 2\lambda_1 = 0 & \Rightarrow \quad 2\lambda_1 = M\ddot{x} \\ M\ddot{y} + Mg - 2\lambda_2 = 0 & \Rightarrow \quad 2\lambda_2 = M\ddot{y} + Mg \\ I\ddot{\theta} - \lambda_1 L\sin\theta + \lambda_2 L\cos\theta = 0 & \\ 2x = L\cos\theta & \\ 2y = L\sin\theta & \end{cases}$$

$$I\ddot{\theta} - \frac{1}{2} ML\ddot{x}\sin\theta + \frac{1}{2} ML(\ddot{y} + g)\cos\theta = 0$$

$$\dot{x} = -\frac{1}{2} L\dot{\theta}\sin\theta \quad \Rightarrow \quad \ddot{x} = -\frac{1}{2} L\dot{\theta}^2\cos\theta - \frac{1}{2} L\ddot{\theta}\sin\theta$$
$$\dot{y} = \frac{1}{2} L\dot{\theta}\cos\theta \quad \Rightarrow \quad \ddot{y} = -\frac{1}{2} L\dot{\theta}^2\sin\theta + \frac{1}{2} L\ddot{\theta}\cos\theta$$

$$\Rightarrow I\ddot{\theta} - \frac{1}{2} ML\left(-\frac{1}{2} L\dot{\theta}^2\cos\theta - \frac{1}{2} L\ddot{\theta}\sin\theta\right)\sin\theta$$
$$+ \frac{1}{2} ML\left(-\frac{1}{2} L\dot{\theta}^2\sin\theta + \frac{1}{2} L\ddot{\theta}\cos\theta + g\right)\cos\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{3g}{2L}\cos\theta$$

Multiply by $\dot{\theta}$: $\quad \dot{\theta}\ddot{\theta} = -\frac{3g}{2L}\dot{\theta}\cos\theta$

$$\Rightarrow \frac{\dot{\theta}^2}{2}\Big|_0^t = -\frac{3g}{2L}\sin\theta\Big|_0^t$$

$$\Rightarrow \dot{\theta}^2 = -\frac{3g}{L}(\sin\theta - \sin\theta_0)$$

Now, when $\lambda_1 = 0$, then $\ddot{x} = 0$:

$$-\frac{1}{2}L\left(-\frac{3g}{L}\right)(\sin\theta - \sin\theta_0)\cos\theta - \frac{1}{2}L\left(-\frac{3g}{2L}\right)\cos\theta\sin\theta = 0$$

$$\Leftrightarrow \frac{3g}{2}\left(\sin\theta\cos\theta - \sin\theta_0\cos\theta\right) + \frac{3g}{2}\left(\frac{1}{2}\sin\theta\cos\theta\right) = 0$$

$$\Leftrightarrow \frac{3}{2}\sin\theta\cos\theta = \sin\theta_0\cos\theta$$

$$\Leftrightarrow \sin\theta = \frac{2}{3}\sin\theta_0$$

$$\hookrightarrow H = 2y = L\sin\theta = \frac{2}{3}L\sin\theta_0$$

Subsequent motion : $\lambda_1 = 0$, constraint is gone :

② a) $L = \frac{1}{2} m |\bar{v}|^2 = \frac{1}{2} m \frac{d\bar{s}\cdot d\bar{s}}{dt^2} = \frac{1}{2} m \sum_j \sum_k g_{jk}\, \dot{q}^j \dot{q}^k$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}^i} = \frac{1}{2} m \sum_j \sum_k g_{jk} \left(\dot{q}^j \delta_{ik} + \dot{q}^k \delta_{ij} \right)$$ symmetry

$$= \frac{1}{2} m \sum_j \left(g_{ji} \dot{q}^j + g_{ij} \dot{q}^j \right) = m \sum_j g_{ij} \dot{q}^j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{1}{2} m \sum_j \sum_k \left(\frac{\partial g_{ij}}{\partial q^k} \dot{q}^k \dot{q}^j + \frac{\partial g_{ik}}{\partial q^j} \dot{q}^k \dot{q}^j \right) + m \sum_j g_{ij} \ddot{q}^j$$ swap dummy indices

$$\frac{\partial L}{\partial q^i} = \frac{1}{2} m \sum_j \sum_k \frac{\partial g_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k$$

$$\Rightarrow \sum_j g_{ij} \ddot{q}^j + \frac{1}{2} \sum_j \sum_k \left(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i} \right) \dot{q}^j \dot{q}^k = 0$$

Multiply with g^{ij}:

$$\sum_m g^{im} \sum_j g_{mj} \ddot{q}^j + \sum_m g^{im} \frac{1}{2} \sum_j \sum_k \left(\frac{\partial g_{mj}}{\partial q^k} + \frac{\partial g_{mk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^m} \right) \dot{q}^j \dot{q}^k = 0$$

$$\Leftrightarrow \sum_j \delta^i_j \ddot{q}^j + \sum_j \sum_k \underbrace{\frac{1}{2} \sum_m g^{im} \left(\frac{\partial g_{mj}}{\partial q^k} + \frac{\partial g_{mk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^m} \right)}_{\Gamma^i_{jk}} \dot{q}^j \dot{q}^k = 0$$

$$\Leftrightarrow \ddot{q}^i + \sum_j \sum_k \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0$$

b) $\bar{s}(\tau) \to ds = \sqrt{ds^2} = \sqrt{\sum_i \sum_j g_{ij}\, dq^i dq^j}$

$$= \sqrt{\sum_i \sum_j g_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau}}\, d\tau$$

$$\Rightarrow \frac{d}{d\tau}\frac{\partial}{\partial \dot q^k}\sqrt{\sum_i\sum_j g_{ij}\frac{dq^i}{d\tau}\frac{dq^j}{d\tau}} - \frac{\partial}{\partial q^k}\sqrt{\sum_i\sum_j g_{ij}\frac{dq^i}{d\tau}\frac{dq^j}{d\tau}} = 0$$

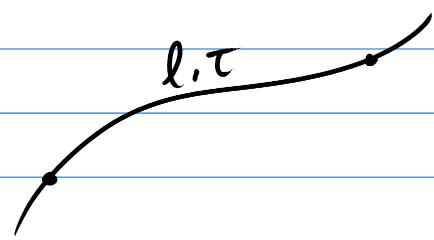
one sqrt left in denominator after derivatives

$$\Leftrightarrow \frac{d}{d\tau}\left(\frac{1}{2}\sum_i\sum_j g_{ij}\left(\delta^i_k\frac{dq^j}{d\tau} + \frac{dq^i}{d\tau}\delta^j_k\right)\right) - \frac{1}{2}\sum_i\sum_j \frac{\partial g_{ij}}{\partial q^k}\frac{dq^i}{d\tau}\frac{dq^j}{d\tau} = 0$$

$$\Leftrightarrow \sum_j g_{kj}\frac{d^2q^j}{d\tau^2} + \frac{1}{2}\sum_i\sum_j\left(\frac{\partial g_{kj}}{\partial q^i}\frac{dq^i}{d\tau}\frac{dq^j}{d\tau} + \frac{\partial g_{ik}}{\partial q^j}\frac{dq^j}{d\tau}\frac{dq^i}{d\tau}\right) - \frac{1}{2}\sum_i\sum_j\frac{\partial g_{ij}}{\partial q^k}\frac{dq^i}{d\tau}\frac{dq^j}{d\tau} = 0$$

$\Rightarrow$ same equation

c) $\frac{\partial L}{\partial t} = 0 \Rightarrow H = T + V = T = L$ is constant $\Rightarrow v^2 = \text{const}$



Since $\frac{d}{d\tau} = \frac{l}{v}\frac{d}{dt} \Rightarrow$ equations are equal

(3) $L = \frac{1}{2}m\sum_j\sum_k g_{jk}\dot q^j\dot q^k$

$$p_i = \frac{\partial L}{\partial \dot q^i} = \frac{1}{2}m\sum_j\sum_k g_{jk}\left(\delta^j_i\dot q^k + \dot q^j\delta^k_i\right) = m\sum_j g_{ij}\dot q^j$$

$$\Rightarrow \sum_k g^{ik}p_k = m\sum_j\sum_k g^{ik}g_{kj}\dot q^j = m\dot q^i$$

$$H = L = \frac{1}{2}m\sum_j\sum_k g_{jk}\frac{1}{m}\sum_l g^{jl}p_l\frac{1}{m}\sum_n g^{kn}p_n$$

$$= \frac{1}{2m}\sum_j\sum_k g^{jk}p_jp_k$$

In spherical coordinates: $ds^2 = dr^2 + r^2 \sin^2\theta \, d\varphi^2 + r^2 d\theta^2$

$$\Rightarrow \quad g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 \sin^2\theta & \\ & & r^2 \end{pmatrix}$$

$$\Rightarrow \quad g^{ij} = \begin{pmatrix} 1 & & \\ & \frac{1}{r^2 \sin^2\theta} & \\ & & \frac{1}{r^2} \end{pmatrix}$$

$$\Rightarrow \quad H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2 \sin^2\theta} + \frac{p_\theta^2}{r^2} \right)$$

(4) $$\frac{dH}{dt} = \sum_i \dot{p}_i \dot{q}_i + \cancel{\sum_i p_i \ddot{q}_i} - \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i - \cancel{\sum_i \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i} - \frac{\partial L}{\partial t}$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i - \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial t}$$

$$= \sum_i \underbrace{\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \right)} \dot{q}_i - \frac{\partial L}{\partial t} = 0$$

Euler-Lagrange equation with $\mathcal{F}(\dot{q}_i)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = - \frac{\partial \mathcal{F}}{\partial \dot{q}_i}$$

$$\Rightarrow \quad \frac{dE}{dt} = - \sum_i \frac{\partial \mathcal{F}}{\partial \dot{q}_i} \dot{q}_i = -2 \mathcal{F}$$

5 $L = \frac{1}{2}\dot{q}_1\dot{q}_2 - \frac{1}{2}\omega_0^2 q_1 q_2$

1) $\begin{cases} \ddot{q}_1 + \omega_0^2 q_1 = 0 \\ \ddot{q}_2 + \omega_0^2 q_2 = 0 \end{cases}$ $\rightarrow$ two harmonic oscillators (uncoupled)

2) $C_\lambda = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \lambda}\Big|_{\lambda=0} = \frac{1}{2}\dot{q}_2 q_1 - \frac{1}{2}\dot{q}_1 q_2 = \text{constant}$

$q_i = A \sin(\omega_0 t + \beta_i)$

$p_i = \omega_0 A \cos(\omega_0 t + \beta_i)$

$\rightarrow C_\lambda = \frac{1}{2}\omega_0 A \sin(\beta_1 - \beta_2)$ $\rightarrow$ conservation of phase difference