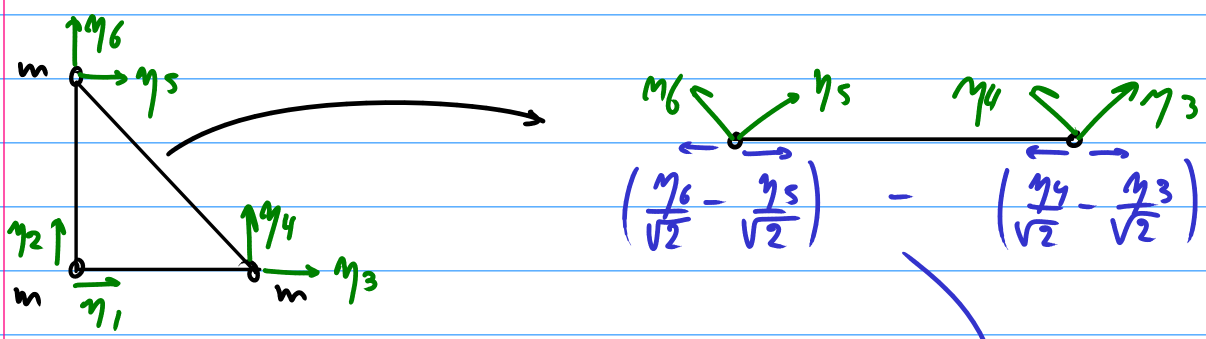
# Homework Assignment 8

(1)



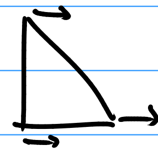

$$V = \frac{1}{2}k(\eta_3-\eta_1)^2 + \frac{1}{2}k(\eta_6-\eta_2)^2$$
$$+ \frac{1}{2}k\left(\frac{(\eta_5-\eta_3)}{\sqrt{2}} - \frac{(\eta_6-\eta_4)}{\sqrt{2}}\right)^2$$
$$= \frac{1}{2}k(\eta_3-\eta_1)^2 + \frac{1}{2}k(\eta_6-\eta_2)^2$$
$$+ \frac{1}{4}k(\eta_5-\eta_3)^2 + \frac{1}{4}k(\eta_6-\eta_4)$$
$$- \frac{1}{2}k(\eta_5-\eta_3)(\eta_6-\eta_4)$$

$$V = \frac{1}{2}k\begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 \\ -2 & 0 & 3 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -2 & 1 & -1 & -1 & 3 \end{bmatrix} \qquad M = m\mathbb{1}$$

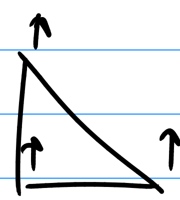
$$\det(V - \omega^2 M) = 0 \Leftrightarrow \det\left(\frac{2}{k}V - 2\omega^2\frac{m}{k}\mathbb{1}\right) = 0$$
$$\Leftrightarrow (\omega^2)^3\left(\omega^2 - \frac{k}{m}\right)\left(\omega^2 - \frac{2k}{m}\right)\left(\omega^2 - \frac{3k}{m}\right) = 0$$

Eigenvectors of $\omega^2 = 0$: (not asked for, use Gram-Schmidt)

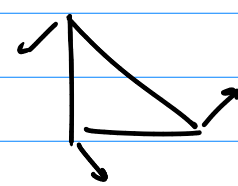
$$\frac{1}{\sqrt{3m}}\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix} \rightarrow \text{uniform motion in } x \text{ direction}$$



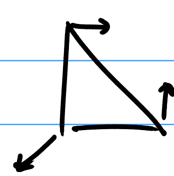
$$\frac{1}{\sqrt{3m}}\begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix} \rightarrow \text{uniform motion in } y \text{ direction}$$



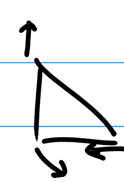
$$\frac{1}{\sqrt{6m}}\begin{bmatrix}1\\-1\\1\\1\\-1\\-1\end{bmatrix} \rightarrow \text{uniform rotation}$$



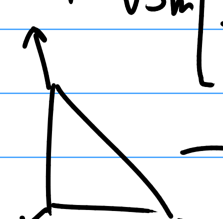
$$\omega^2 = \frac{k}{m} : \frac{1}{\sqrt{4m}}\begin{bmatrix}-1\\-1\\0\\1\\1\\0\end{bmatrix}$$



$$\omega^2 = \frac{2k}{m} : \frac{1}{\sqrt{4m}}\begin{bmatrix}1\\-1\\-1\\0\\0\\1\end{bmatrix}$$



$$\omega^2 = \frac{3k}{m} : \frac{1}{\sqrt{3m}}\begin{bmatrix}-1/2\\-1/2\\1\\-1/2\\-1/2\\1\end{bmatrix}$$



→ "breathing" mode

③ $$L = \frac{1}{2}LI_1^2 + \frac{1}{2}LI_2^2 - \frac{1}{2}\frac{Q_1^2}{C} - \frac{1}{2}\frac{Q_2^2}{C} - \frac{1}{2}\frac{(Q_1+Q_2)^2}{C}$$

$$\hookrightarrow \begin{cases} L\dot{I}_1 + 2\frac{Q_1}{c} + \frac{Q_2}{c} = 0 \\ L\dot{I}_2 + 2\frac{Q_2}{c} + \frac{Q_1}{c} = 0 \end{cases}$$

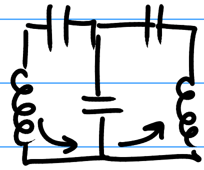
$$\Rightarrow M = L\,\mathbb{1} \quad , \quad V = \frac{1}{C}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det(V - \omega^2 M) = 0 \iff \frac{1}{C^2}(2 - LC\omega^2)^2 = \frac{1}{C^2}$$

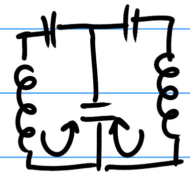
$$\iff 2 - LC\omega^2 = \pm 1$$

$$\iff \omega^2 = \frac{1}{LC}(2 \pm 1)$$

$$\omega^2 = \frac{1}{LC} : \frac{1}{\sqrt{2L}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$$\omega^2 = \frac{3}{LC} : \frac{1}{\sqrt{2L}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



④ $L = \frac{1}{2} m\ell^2 \sum_{j=1}^{N} \dot{\theta}_j^2 - \frac{1}{2} mg\ell \sum_{j=1}^{N} \theta_j^2 - \frac{1}{2} k\ell^2 \sum_{j=0}^{N} (\theta_{j+1} - \theta_j)$

Euler-Lagrange equations:

$$m\ell^2 \ddot{\theta}_j + mg\ell\,\theta_j - k\ell^2(\theta_{j-1} - 2\theta_j + \theta_{j+1}) = 0$$

Assume solutions $\theta_j = A_j\, e^{i(kja - \omega t)}$

$$\Leftrightarrow \quad -m\ell^2\omega^2 + mg\ell - k\ell^2(e^{-iak} - 2 + e^{iak})$$

$$\Leftrightarrow \quad \omega^2 = \frac{g}{\ell} + 2\frac{k}{m}(1 - \cos ka) = \frac{g}{\ell} + 4\frac{k}{m}\sin^2\frac{ka}{2}$$

Periodic boundary conditions: $\theta_j = \theta_{j+}$

$$\Leftrightarrow \quad Nka = 2n\pi \quad , n = 0, \pm 1, \ldots, \pm\tfrac{1}{2}(N-1), (+\underset{\text{even}}{\tfrac{1}{2}N})$$

$$\Leftrightarrow \quad k_n = \frac{2n\pi}{Na} \quad , n = 0, \pm 1, \ldots, \pm\tfrac{1}{2}(N-1), (+\underset{\text{even}}{\tfrac{1}{2}N})$$

$$\Rightarrow \quad \omega_n^2 = \frac{g}{\ell} + 4\frac{k}{m}\sin^2\frac{k_n a}{2}$$

$$= \frac{g}{\ell} + 4\frac{k}{m}\sin^2\frac{n\pi}{N}$$

$\Rightarrow \omega_0^2 = \frac{g}{\ell} \longrightarrow$ all pendulums swing in phase