Homework Assignment 9

① $$V(x,y) = k\frac{q_1}{\sqrt{(x-a)^2+y^2}} + k\frac{q_1}{\sqrt{(x+a)^2+y^2}}$$

$$+ k\frac{q_2}{\sqrt{x^2+(y-a)^2}} + k\frac{q_2}{\sqrt{x^2+(y+a)^2}}$$

$$\frac{\partial V}{\partial x} = -k\frac{q_1(x-a)}{((x-a)^2+y^2)^{3/2}} - k\frac{q_1(x+a)}{((x+a)^2+y^2)^{3/2}}$$

$$- k\frac{q_2 x}{(x^2+(y-a)^2)^{3/2}} - k\frac{q_2 x}{(x^2+(y+a)^2)^{3/2}}$$

$$\frac{\partial V}{\partial y} = -k\frac{q_1 y}{((x-a)^2+y^2)^{3/2}} - k\frac{q_1 y}{((x+a)^2+y^2)^{3/2}}$$

$$- k\frac{q_2(y-a)}{(x^2+(y-a)^2)^{3/2}} - k\frac{q_2(y+a)}{(x^2+(y+a)^2)^{3/2}}$$

$\left.\frac{\partial V}{\partial x}\right|_{x=y=0} = 0$ and $\left.\frac{\partial V}{\partial y}\right|_{x=y=0} = 0 \Rightarrow$ origin is equilibrium

$$\frac{\partial^2 V}{\partial x \partial y} = 3k\frac{q_1(x-a)y}{((x-a)^2+y^2)^{5/2}} + 3k\frac{q_1(x+a)y}{((x+a)^2+y^2)^{5/2}}$$

$$+ 3k\frac{q_2 x(y-a)}{(x^2+(y-a)^2)^{5/2}} + 3k\frac{q_2 x(y+a)}{(x^2+(y+a)^2)^{5/2}}$$

$$\frac{\partial^2 V}{\partial x^2} = 3k\,\frac{q_1 (x-a)^2}{\left((x-a)^2+y^2\right)^{5/2}} - k\,\frac{q_1}{\left((x-a)^2+y^2\right)^{3/2}}$$

$$+ 3k\,\frac{q_1 (x+a)^2}{\left((x+a)^2+y^2\right)^{5/2}} - k\,\frac{q_1}{\left((x+a)^2+y^2\right)^{3/2}}$$

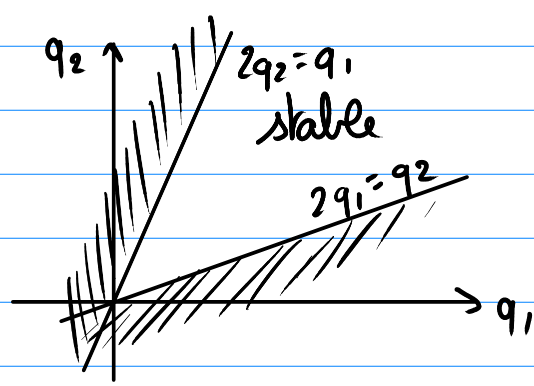
$$+ 3k\,\frac{q_2 x^2}{\left(x^2+(y-a)^2\right)^{5/2}} - k\,\frac{q_2}{\left(x^2+(y-a)^2\right)^{3/2}}$$

$$+ 3k\,\frac{q_2 x^2}{\left(x^2+(y+a)^2\right)^{5/2}} - k\,\frac{q_2}{\left(x^2+(y+a)^2\right)^{3/2}}$$

and similar for $\frac{\partial^2 V}{\partial y^2}$ with $x \leftrightarrow y$ and $q_1 \leftrightarrow q_2$.

$$\begin{cases} \left.\frac{\partial^2 V}{\partial x \partial y}\right|_{x=y=0} = 0 \\ \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=y=0} = 2\frac{k}{a^3}\left(2q_1 - q_2\right) \\ \left.\frac{\partial^2 V}{\partial y^2}\right|_{x=y=0} = 2\frac{k}{a^3}\left(2q_2 - q_1\right) \end{cases}$$

$$\left.\begin{array}{ll} \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=y=0} > 0 & \text{if } 2q_1 > q_2 \\ \left.\frac{\partial^2 V}{\partial y^2}\right|_{x=y=1} > 0 & \text{if } 2q_2 > q_1 \end{array}\right\}$$



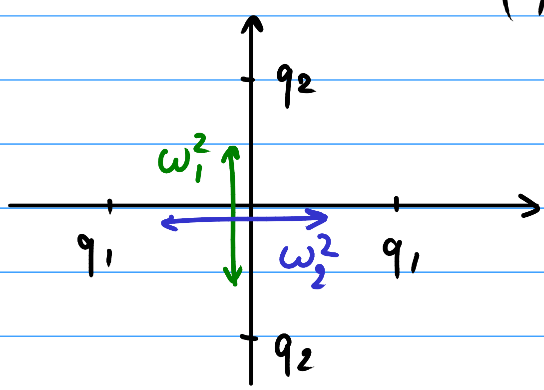
$$V = 2\frac{k}{a^3}\begin{pmatrix} 2q_1 - q_2 & 0 \\ 0 & 2q_2 - q_1 \end{pmatrix}$$

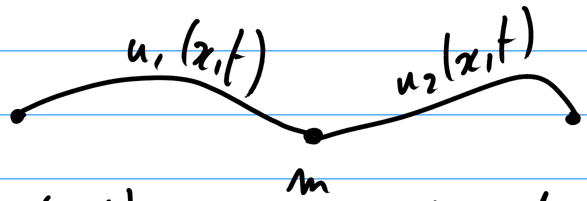
$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \Rightarrow \det\left(V - \omega^2 M\right) = 0$$

$$\Leftrightarrow \omega_1^2 = 2\frac{k}{a^3 m}\left(2q_1 - q_2\right)$$

$$\text{or } \omega_2^2 = 2\frac{k}{a^3 m}\left(2q_2 - q_1\right)$$

$$\left(V - \omega^2 M\right) z = 0 \Rightarrow z_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } z_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



② 

$u_1(x,t)$ and $u_2(x,t)$ satisfy the wave equation in region 1 and region 2:

For $x < \frac{\ell}{2}$ or $x > \frac{\ell}{2}$: $\quad \sigma \frac{\partial^2 u_{1,2}}{\partial t^2}(x,t) = \tau \frac{\partial^2 u_{1,2}}{\partial x^2}(x,t)$

For $x = \frac{\ell}{2}$: $\quad m \frac{\partial^2 u}{\partial t^2}\left(\frac{\ell}{2}, t\right) = \tau \left[\frac{\partial u_2}{\partial x}\left(\frac{\ell}{2}, t\right) - \frac{\partial u_1}{\partial x}\left(\frac{\ell}{2}, t\right) \right]$

Wave solution: $\begin{cases} u_1(x,t) = \rho_1(x) \cos(\omega t + \varphi) &, x < \frac{\ell}{2} \\ u_2(x,t) = \rho_2(x) \cos(\omega t + \varphi) &, x > \frac{\ell}{2} \end{cases}$

↳ same time behavior to ensure continuity at all times for $x = \frac{\ell}{2}$

$$\Rightarrow \begin{cases} \frac{d^2 \rho_{1,2}}{dx^2} + k^2 \rho_{1,2}(x) = 0 \,, \; k = \frac{\omega}{c} \,, \; c^2 = \frac{\tau}{\sigma} \quad \text{for } x < \frac{\ell}{2}, x > \frac{\ell}{2} \\ -m\omega^2 \rho_{1,2}\left(\frac{\ell}{2}\right) \underset{(4)}{=} \tau \left[\frac{d\rho_2}{dx}\left(\frac{\ell}{2}\right) - \frac{d\rho_1}{dx}\left(\frac{\ell}{2}\right) \right] \quad \text{at } x = \frac{\ell}{2} \end{cases}$$

Boundary conditions: $\rho_1(0) \underset{(1)}{=} \rho_2(\ell) \underset{(2)}{=} 0 \rightarrow 2$ conditions

Continuity condition: $\rho_1\left(\frac{\ell}{2}\right) \underset{(3)}{=} \rho_2\left(\frac{\ell}{2}\right) \rightarrow 1$ condition

$\left.\begin{array}{l} \rho_1(x) = A_1 \sin kx + B_1 \cos kx \\ \rho_2(x) = A_2 \sin kx + B_2 \cos kx \end{array}\right\}$ 4 unknowns

↳ will be left with one scale factor

1) $\rho_1(0) = 0 \Rightarrow B_1 = 0 \Rightarrow \rho_1(x) = A_1 \sin kx$

Easy way: $p_1(x) = A_1 \sin kx$

$\Rightarrow p_2(x) = A_1 \sin k(\ell - x)$

$\to$ immediately taken care of $p_2(\ell) = 0$ (2)

and $p_1(\frac{\ell}{2}) = p_2(\frac{\ell}{2})$ (3)

$$\Rightarrow m\omega^2 A_1 \sin k\frac{\ell}{2} = \tau\left(A_1 k \cos k\frac{\ell}{2} + A_1 k \cos k\frac{\ell}{2}\right) \quad (4)$$

$$\Leftrightarrow m\omega^2 = \tau k \cot \frac{\omega \ell}{2c}$$

$$\Leftrightarrow \frac{2c}{\omega \ell} \cot \frac{\omega \ell}{2c} = \frac{m}{\sigma \ell}$$

Hard way:

2) $p_2(\ell) \overset{!}{=} 0 = A_2 \sin k\ell + B_2 \cos k\ell \Rightarrow B_2 = -A_2 \tan k\ell$

3) $A_1 \sin k\frac{\ell}{2} = A_2 \sin k\frac{\ell}{2} + B_2 \cos k\frac{\ell}{2} = A_2\left(\sin k\frac{\ell}{2} - \cos k\frac{\ell}{2} \tan k\ell\right)$

$$\Rightarrow A_1 = A_2\left(1 - \cot k\frac{\ell}{2} \tan k\ell\right)$$

4) $m\omega^2 A_1 \sin k\frac{\ell}{2} = \tau\left[A_1 k \cos k\frac{\ell}{2} - A_2 k \cos k\frac{\ell}{2} + B_2 k \sin k\frac{\ell}{2}\right]$

$$\Leftrightarrow m\omega^2 A_1 = \tau k\left[A_1 \cot k\frac{\ell}{2} - A_2 \cot k\frac{\ell}{2} + B_2\right]$$

$$\Leftrightarrow m\omega^2 A_2\left(1 - \cot k\frac{\ell}{2} \tan k\ell\right) = A_2 \tau k\left[\left(1 - \cot k\frac{\ell}{2} \tan k\ell\right)\cot k\frac{\ell}{2} - \cot k\frac{\ell}{2} - \tan k\ell\right]$$

$$\frac{m\omega^2}{\tau k} = \frac{m\omega^2}{c^2 \sigma \frac{\omega}{c}} = \frac{m\omega}{c\sigma}$$

$$\Leftrightarrow \frac{m\omega}{c\sigma}\left(1 - \cot k\frac{\ell}{2} \tan k\ell\right) = -\tan k\ell\left(1 + \cot^2 k\frac{\ell}{2}\right)$$

$$\Leftrightarrow \quad \frac{m\omega}{c\sigma} = \frac{-\tan kl\left(1+\cot^2 k\frac{l}{2}\right)}{1-\cot k\frac{l}{2}\tan kl}$$

$$\tan kl = \frac{2\tan k\frac{l}{2}}{1-\tan^2 k\frac{l}{2}}$$

$$= \frac{-2\left(\tan k\frac{l}{2}+\cot k\frac{l}{2}\right)}{\left(1-\tan^2 k\frac{l}{2}-2\right)}$$

$$= 2\,\frac{\tan k\frac{l}{2}+\cot k\frac{l}{2}}{1+\tan^2 k\frac{l}{2}}$$

$$1+\tan^2 k\frac{l}{2} = \frac{1}{\cos^2 k\frac{l}{2}}$$

$$= 2\cos^2 k\frac{l}{2}\left(\tan k\frac{l}{2}+\cot k\frac{l}{2}\right)$$

$$= 2\cot k\frac{l}{2} = 2\cot\frac{\omega l}{2c}$$

$$\Rightarrow \quad \frac{2c}{\omega l}\cot\frac{\omega l}{2c} = \frac{m}{\sigma l}$$

$$m\to 0 \Rightarrow \frac{\omega l}{2c} = k\frac{l}{2} = \frac{\pi}{2}+n\pi$$

$$\Leftrightarrow \quad k = \frac{\pi}{l}+\frac{2n\pi}{l} \quad \to \text{ no bead at all}$$

$$m\to\infty \Rightarrow \frac{\omega l}{2c} = k\frac{l}{2} = n\pi$$

$$\Leftrightarrow \quad k = \frac{n\pi}{l/2} \quad \to \text{ two independent strings of length } \frac{l}{2}$$

$$I = \int_0^{\ell} dx\, \rho_p(x)\, m(x)\, \rho_q(x)$$

$$= \int_0^{\ell/2} dx\, \rho_p(x)\, \sigma\, \rho_q(x) + m\, \rho_p\left(\frac{\ell}{2}\right) \rho_q\left(\frac{\ell}{2}\right) + \int_{\ell/2}^{\ell} dx\, \rho_p(x)\, \sigma\, \rho_q(x)$$

$$\rho(x) = \begin{cases} \sin kx, & x < \frac{\ell}{2} \\ (\sin kx - \tan k\ell \cos kx), & x > \frac{\ell}{2} \end{cases}$$

$$= \frac{1}{\cos k\ell} \sin k(\ell - x) \longrightarrow \text{symmetric around } \frac{\ell}{2}$$

Relation : $\cot k\frac{\ell}{2} = k\frac{\ell}{2} \cdot \frac{m}{\sigma\ell} \Leftrightarrow \cos k\frac{\ell}{2} = k\frac{\ell}{2} \sin k\frac{\ell}{2} \cdot \frac{m}{\sigma\ell}$

Term at $x = \frac{\ell}{2}$:

$$m\, \rho_p\left(\frac{\ell}{2}\right) \rho_q\left(\frac{\ell}{2}\right) = m \sin k_p\frac{\ell}{2} \sin k_q\frac{\ell}{2}$$

Integral terms :

$$2\sigma \int_0^{\ell/2} dx \sin k_p x \sin k_q x \quad k_p \neq k_q$$

$$= \frac{2\sigma}{k_p^2 - k_q^2} \left(k_q \cos k_q\frac{\ell}{2} \sin k_p\frac{\ell}{2} - k_p \cos k_p\frac{\ell}{2} \sin k_q\frac{\ell}{2} \right)$$

$$= \frac{2\sigma}{k_p^2 - k_q^2} \frac{m}{2\sigma} \left(k_q^2 - k_p^2 \right) \sin k_p\frac{\ell}{2} \sin k_q\frac{\ell}{2}$$

$$= -m \sin k_p\frac{\ell}{2} \sin k_q\frac{\ell}{2}$$

$\Rightarrow$ total integral is zero for $k_p \neq k_q$

$$2\sigma \int_0^{\ell/2} dx \, \sin k_p x \, \sin k_q x \quad k_p = k_q$$

$$= 2\sigma \, \frac{1}{4}\left(\ell - \frac{2}{k_p} \sin k_p \frac{\ell}{2} \cos k_p \frac{\ell}{2}\right)$$

$$= \frac{\sigma}{2}\left(\ell - \frac{2}{k_p} \, k_p \frac{\ell}{2} \sin^2 k_p \frac{\ell}{2} \, \frac{m}{\sigma \ell}\right)$$

$$= \frac{\sigma}{2}\left(\ell - \frac{m}{\sigma} \sin^2 k_p \frac{\ell}{2}\right) = \frac{\sigma \ell}{2} - \frac{m}{2} \sin^2 k_p \frac{\ell}{2}$$

$\Rightarrow$ total integral is $\frac{\sigma \ell}{2} + \frac{m}{2} \sin^2 k_p \frac{\ell}{2} \rightarrow$ normalization

③ $$\mathcal{L} = \frac{\hbar^2}{2m}\bar{\nabla}\psi^*\cdot\bar{\nabla}\psi + V\psi^*\psi + \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{\hbar}{2i}\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = -\frac{\hbar}{2i}\psi$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)} = \frac{\hbar^2}{2m}\frac{\partial\psi^*}{\partial x_i} \quad , \quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi^*}{\partial x_i}\right)} = \frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x_i}$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = V\psi^* - \frac{\hbar}{2i}\dot{\psi}^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\psi^*} = V\psi + \frac{\hbar}{2i}\dot{\psi}$$

$$\Rightarrow \begin{cases} \frac{\hbar}{i}\dot{\psi}^* + \frac{\hbar^2}{2m}\sum_i\frac{\partial^2}{\partial x_i^2}\psi^* - V\psi^* = 0 \\ -\frac{\hbar}{i}\dot{\psi} + \frac{\hbar^2}{2m}\sum_i\frac{\partial^2}{\partial x_i^2}\psi - V\psi = 0 \end{cases}$$

$$\Leftrightarrow -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial}{\partial t}\psi \quad \text{and c.c.}$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{\hbar}{2i}\psi^* \quad , \quad \pi^* = -\frac{\hbar}{2i}\psi$$

$$\Rightarrow \mathcal{H} = \pi\dot{\psi} + \pi^*\dot{\psi}^* - \mathcal{L}$$

$$= \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right) - \frac{\hbar^2}{2m}\bar{\nabla}\psi^*\cdot\bar{\nabla}\psi$$

$$- V\psi^*\psi - \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \dot{\psi}^*\psi\right)$$

$$= -\frac{\hbar^2}{2m}\bar{\nabla}\psi^*\cdot\bar{\nabla}\psi - V\psi^*\psi$$

④ $\mathcal{L} = c^2 \partial_\mu \varphi \partial^\mu \varphi^* - m_0^2 c^2 \varphi \varphi^* + j^\lambda A_\lambda$

$= c^2 \partial_\mu \varphi \partial^\mu \varphi^* - m_0^2 c^2 \varphi \varphi^* + i A^\lambda (\varphi \partial_\lambda \varphi^* - \partial_\lambda \varphi \varphi^*)$

$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = c^2 \partial^\mu \varphi^* + i A^\mu \varphi^*$ $\qquad$ $\frac{\partial \mathcal{L}}{\partial \varphi} = - m_0^2 c^2 \varphi^*$

$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} = c^2 \partial^\mu \varphi - i A^\mu \varphi$ $\qquad$ $\frac{\partial \mathcal{L}}{\partial \varphi^*} = - m_0^2 c^2 \varphi$

$\Rightarrow \quad c^2 \partial_\mu \partial^\mu \varphi - i(\partial_\mu A^\mu)\varphi - i A^\mu (\partial_\mu \varphi) + m_0^2 c^2 \varphi = 0$

and c.c.