Classical Mechanics (Phys 601) – September 15, 2011

* Generalized momentum and cyclic coordinates

Lagrangian $L(\{q_j\}, \{\dot{q}_j\}, t)$, $j = 1, \dots, n$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad q_j = \begin{cases} \text{generalized coordinate} \\ \text{canonical} \end{cases}$$

$$\hookrightarrow \frac{\partial L}{\partial \dot{q}_j} = p_j = \begin{cases} \text{generalized momentum} \\ \text{canonical} \end{cases}$$

$$\Rightarrow \quad \dot{p}_j = \frac{\partial L}{\partial q_j}$$

$\hookrightarrow$ (p_j, q_j) are canonical variables

If $\frac{\partial L}{\partial q_j} = 0 \rightarrow q_j$ is cyclic

$\Rightarrow$ p_j is constant for cyclic coordinates q_j

Relation to symmetry : if $\frac{\partial L}{\partial q_j} = 0$, then L is the same for all q_j

$\Downarrow$

symmetric under change of q_j

"A continuous symmetry operation implies a conserved generalized momentum"

* Examples of symmetry in the coordinates :

$\rightarrow L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$ (cartesian)

$\hookrightarrow \frac{\partial L}{\partial x} = 0 \Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ conserved

$\dots \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$ conserved

$\frac{\partial L}{\partial z} \neq 0 \Rightarrow p_z$ not conserved

$\rightarrow L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + G \frac{mM}{r}$ (2D-polar)

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \underbrace{m r^2 \dot{\varphi}}_{\text{angular momentum}}$ conserved

$\frac{\partial L}{\partial r} \neq 0 \Rightarrow \frac{\partial L}{\partial r} = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}}$

$\Leftrightarrow m r \dot{\varphi}^2 - G \frac{mM}{r^2} = m\ddot{r}$

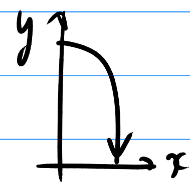
* Noether's Theorem

" For any continuous symmetry there exists a constant of motion. "

Assume that $q_i(t)$ is a solution of the Lagrange equation.

If the Lagrangian is cyclic in a parameter s, then we can consider the family of trajectories

$$q_i(s,t), \quad \text{with} \quad q_i(0,t) = q_i(t)$$



Example: if $\frac{\partial L}{\partial x} = 0$ and $(v_0 t, -\frac{1}{2} g t^2)$ is a solution

$\rightarrow (v_0 t + s, -\frac{1}{2} g t^2)$ is a family of solutions

$$\left.\frac{\partial L}{\partial s}\right|_{s=0} = 0 = \sum_i^n \left(\frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial s} + \frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial s} \right)\Bigg|_{s=0}$$

$$\frac{\partial}{\partial s} \frac{d}{dt} q_i = \frac{d}{dt} \frac{\partial}{\partial s} q_i \quad \text{and} \quad \frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$$

$$\Rightarrow \left.\frac{\partial L}{\partial s}\right|_{s=0} = \sum_i^n \left(\frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \frac{\partial q_i}{\partial s} + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s} \right)\Bigg|_{s=0}$$

$$= \frac{d}{dt} \left(\sum_i^n \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s} \right)\Bigg|_{s=0} = \frac{d}{dt} C_s = 0$$

$\Rightarrow$ constant is $\sum_i^n \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s} = C_s$

Example : $q_1 = x = v_0 t + s$, $q_2 = y = -\frac{1}{2} g t^2$

$\Rightarrow \frac{\partial q_1}{\partial s} = 1$, $\frac{\partial q_2}{\partial s} = 0$

$\Rightarrow C_s = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ = horizontal momentum

* Hamiltonian and symmetry in the time

Introduce Hamiltonian H : $p_j = \frac{\partial L}{\partial \dot{q}_j}$

$$H(\{q_j\},\{\dot{q}_j\},t) = \sum_j^n \dot{q}_j p_j - L(\{q_j\},\{\dot{q}_j\},t)$$

Total time derivative:

$$\frac{dH}{dt} = \sum_j^n \left(\cancel{\ddot{q}_j p_j} + \cancel{\dot{q}_j \dot{p}_j}\right) - \sum_j^n \left(\cancel{\underbrace{\frac{\partial L}{\partial q_j}}_{} \dot{q}_j} - \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{} \ddot{q}_j}\right) - \frac{\partial L}{\partial t}$$

(Lagrange) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \dot{p}_j$ $\quad$ p_j (definition)

$$\Rightarrow \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

If the Lagrangian does not depend explicitly on t, then H is a conserved quantity of motion.

We will show that it is true for systems with time-independent constraints and potential energy, and additionally in this case H = total energy

Special case:

If T is a $\left\{\begin{matrix}\text{symmetric}\\ \text{quadratic}\end{matrix}\right.$ form of the $\dot{q}_j$: $T = \sum_{ij}^n \frac{1}{2}\dot{q}_i a_{ij} \dot{q}_j$,

$a_{ij} = a_{ji}$

and V a function of only q_j ,

then $H = E = T + V$

$$H = \sum_j^m p_j \dot q_j - L \quad \text{with} \quad p_j = \frac{\partial L}{\partial \dot q_j} = \frac{\partial T}{\partial \dot q_j} = \sum_i \frac{1}{2} \dot q_i (a_{ij} + a_{ji})$$

$$\Rightarrow H = \sum_j^m \sum_i^m \frac{1}{2} \dot q_i (a_{ij} + a_{ji}) \dot q_j - L$$

$$= 2T - L = T + V$$

For a general holonomic constraint:

cartesian coordinates $x_i = f_i(\{q_j\}, t)$, $i = 1, \dots, 3N$

$$\Rightarrow \dot x_i = \sum_j^m \frac{\partial f_i}{\partial q_j} \dot q_j + \frac{\partial f_i}{\partial t}$$

$$T = \sum_i^{3N} \frac{1}{2} m_i \dot x_i^2 = \frac{1}{2} \sum_i^{3N} m_i \left(\sum_j^m \frac{\partial f_i}{\partial q_j} \dot q_j + \frac{\partial f_i}{\partial t} \right) \left(\sum_k^m \frac{\partial f_i}{\partial q_k} \dot q_k + \frac{\partial f_i}{\partial t} \right)$$

$$= \frac{1}{2} \sum_j^m \sum_k^m \left(\sum_i^{3N} m_i \frac{\partial f_i}{\partial q_j} \frac{\partial f_i}{\partial q_k} \right) \dot q_j \dot q_k$$

$$+ \sum_j^m \left(\sum_i^{3N} m_i \frac{\partial f_i}{\partial q_j} \frac{\partial f_i}{\partial t} \right) \dot q_j$$

$$+ \left(\sum_i^{3N} m_i \frac{\partial f_i}{\partial t} \frac{\partial f_i}{\partial t} \right)$$

This will be a quadratic form for all $\dot q_j$ if $\frac{\partial f_i}{\partial t} = 0$

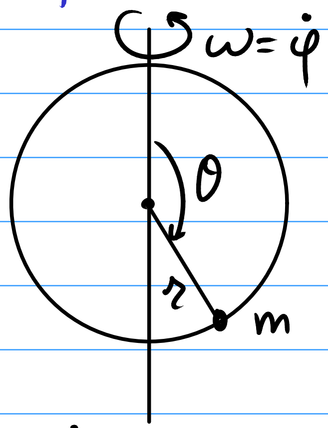
$\Rightarrow$ scleronomic system, time-independent constraints

Even more generally: if $V(\{q_j\}, t)$ this leads to

$H = E = T - V$

even in many cases with time-dependent constraints

* Examples of hamiltonian as conserved quantity :



$$\begin{cases} x = r \sin\theta \cos \omega t \;\varphi \\ y = r \sin\theta \sin \omega t \;\varphi \\ z = r \cos\theta \end{cases}$$

(r constant)
↳ $\dot r = 0$

explicit time-dependence of constraint
$\Downarrow$
$H \neq E = T + V$

$$\Rightarrow \begin{cases} \dot x = r\dot\theta\cos\theta\cos\omega t \\ \qquad - r\omega\sin\theta\sin\omega t \\ \dot y = r\dot\theta\cos\theta\sin\omega t \\ \qquad + r\omega\sin\theta\cos\omega t \\ \dot z = -r\dot\theta\sin\theta \end{cases}$$

$$\Rightarrow T = \frac{1}{2}m\left(\dot x^2 + \dot y^2 + \dot z^2\right) = \frac{1}{2}mr^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right)$$

$$V = mgr\cos\theta$$

$$\Rightarrow L = \frac{1}{2}mr^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right) - mgr\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = mr^2\dot\theta$$

$$\Rightarrow H = p_\theta\dot\theta - L = mr^2\dot\theta^2 - \frac{1}{2}mr^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right) + mgr\cos\theta$$

$$= \frac{1}{2}mr^2\left(\dot\theta^2 - \omega^2\sin^2\theta\right) + mgr\cos\theta$$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \Rightarrow H \text{ is constant}$$

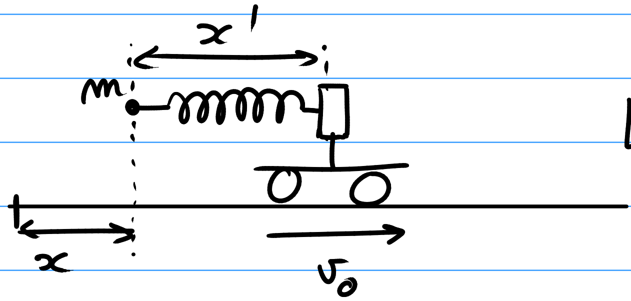
But : $H \neq E = T + V$ (minus sign!)

* Changes of coordinates in the Hamiltonian :

In the Lagrangian : $L = T - V$ is independent of the system of coordinates. Different coordinates will lead to the same magnitude of L at the same physical point.

Not so for the Hamiltonian !

Hamiltonian → constant of motion if $\frac{\partial L}{\partial t} \neq 0$
→ total energy of system if

1) 

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k (x - v_0 t)^2$$

at $t = 0$ and $x = 0$ m is in equilibrium position

$$\Rightarrow m \ddot{x} = - k (x - v_0 t)$$

$x' = x - v_0 t$

$$m \ddot{x}' = - k x'$$

⇒ harmonic motion in cart's coordinate system

Determine Hamiltonian : $p_x = \frac{\partial L}{\partial \dot{x}}$, $H = \dot{x} p_x - L$

$$H = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k (x - v_0 t)^2 = E = T + V$$

Since $\frac{\partial L}{\partial t} \neq 0 \rightarrow \frac{dH}{dt} \neq 0$, but $H = E$

2) If we had started off with x' :

$$L = T - V = \frac{1}{2} m (\dot{x}' - v_0)^2 - \frac{1}{2} k x'^2$$

$$\Rightarrow \frac{d}{dt}\left(m (\dot{x}' - v_0)\right) + k x' = 0$$

$$\Leftrightarrow \quad m \ddot{x}' + k x' = 0$$

$\Rightarrow$ same harmonic motion

Determine Hamiltonian : $p_{x'} = \frac{\partial L}{\partial \dot{x}'} = m(\dot{x}' - v_0)$

$$H = m(\dot{x}' - v_0)\dot{x}' - \frac{1}{2} m (\dot{x}' - v_0)^2 + \frac{1}{2} k x'^2$$

$$H = \frac{1}{2} m (\dot{x}' - v_0)^2 + \frac{1}{2} k x'^2 - \frac{1}{2} m v_0^2$$

Now $\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0$, but $H \neq E$

Hamiltonian conserved

$\updownarrow$

Hamiltonian = total energy

* Hamiltonian as function of $H(\{q_j\},\{p_j\},t)$

Cartesian : $T = \frac{1}{2} m \dot{x}^2 \Rightarrow p_x = m\dot{x}$

$$\Rightarrow H = m\dot{x}^2 - \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \dot{x}^2 = \frac{p_x^2}{2m}$$

3D - polar : $q_j\ (r, \varphi, \theta)$, $V(r)$

$$\Rightarrow T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2 + r^2 \dot{\theta}^2 \right)$$

extra term, previous problem had $\dot{r} = 0$

$$\Rightarrow L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2 + r^2 \dot{\theta}^2 \right) - V(r)$$

$$\begin{cases} p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} & \Rightarrow \dot{p}_r = \frac{\partial L}{\partial r} = -\frac{dV}{dr} \\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} & \Rightarrow \dot{p}_\theta = \frac{\partial L}{\partial \theta} = m r^2 \dot{\varphi}^2 \sin\theta \cos\theta \\ p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\theta\, \dot{\varphi} & \Rightarrow \dot{p}_\varphi = 0 \rightarrow \text{conserved} \end{cases}$$

$$\Rightarrow H = m\dot{r}^2 + m r^2 \dot{\theta}^2 + m r^2 \sin^2\theta\, \dot{\varphi}^2 - L$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2 \right) + V(r)$$

$$= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta} \right)$$

$\hookrightarrow$ p_φ is conserved because $\frac{\partial H}{\partial \varphi} = 0$

* Cyclic coordinates of the Lagrangian and Hamiltonian:

Remember: if $\frac{\partial L}{\partial q_j} = 0 \rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j}$ is conserved

By definition, when $H(\{q_j\},\{p_j\},t)$

$$\frac{\partial H}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\sum_i p_i \dot{q}_i - L \right) = -\frac{\partial L}{\partial q_j}$$

$\Rightarrow$ if $\frac{\partial H}{\partial q_j} = 0 \longrightarrow p_j$ is conserved

equivalent

Lagrangian mechanics $\longleftrightarrow$ Hamiltonian mechanics

Lagrangian mechanics	Hamiltonian mechanics
direct solution of problems	theoretical extensions to other areas of physics
$(q, \dot{q}, t)$	(q, p, t)
n functions of time	$2n$ functions of time