Classical Mechanics (Phys 601) - October 18, 2011

* Note on notation (and summary of last lecture)

Notation F&W different from our notation:

- equilibrium q_i^0 where $\frac{\partial V}{\partial q_i} = 0$ and $q_i = q_i^0 + \eta_i$
- mass matrix $M = \underline{m}$: $M_{ij} = m_{ij} = \sum_k m_k \left.\frac{\partial x_k}{\partial q_i}\right|_{q^0} \left.\frac{\partial x_k}{\partial q_j}\right|_{q^0}$
- potential matrix $V = \underline{v}$: $V_{ij} = v_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q^0}$

Generalized eigenvalue problem : $\eta_i = \mathrm{Re}\, z_i = \mathrm{Re}\, z_i^0 e^{i\omega t}$

$$(V - \omega^2 M) z = 0 \quad \text{or} \quad (\underline{v} - \omega_s^2 \underline{m}) \rho^{(s)} = 0$$

eigenvectors $\rho^{(s)}$ with eigenvalue ω_s^2

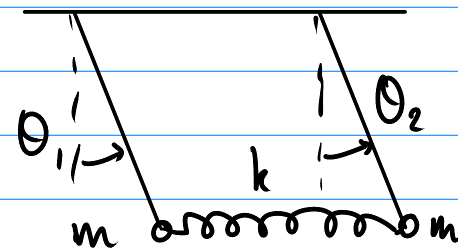
Eigenvalue ω_i^2 with associate eigenvector z_i that describes combination of η_i that oscillate in normal mode

Modal matrix $U = [z_1 \cdots z_n] = \underline{A} = [\rho^{(1)} \cdots \rho^{(n)}]$

↳ transforms new set of generalized coordinates $\xi = \underline{\zeta}$

$$\eta = U\xi \qquad \underline{\eta} = \underline{A}\,\underline{\zeta}$$

* Coupled double pendulum



$$L = \frac{1}{2}\dot{\eta}^T M \dot{\eta} - \frac{1}{2}\eta^T V \eta$$

$$M = m\ell^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad V = m\ell^2 \begin{pmatrix} \frac{k}{m} + \frac{g}{\ell} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} + \frac{g}{\ell} \end{pmatrix}$$

$\Rightarrow$ eigenvalue equation $(V - \omega^2 M) z = 0$

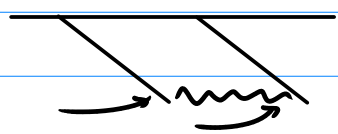
$$\omega^2 = \left(\frac{k}{m} + \frac{g}{\ell}\right) \pm \left(\frac{k}{m}\right)$$

$$\omega_1 = \sqrt{\frac{g}{\ell}} \qquad \omega_2 = \sqrt{\frac{g}{\ell} + 2\frac{k}{m}}$$

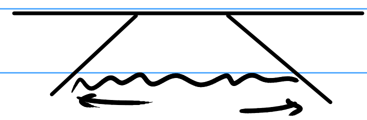
$$\tilde{z}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \tilde{z}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

z_1 and z_2 are orthogonal, but not normalized yet:

$$z_1^T M z_1 = 2m\ell^2 \Rightarrow z_1 = \frac{1}{\ell\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$z_2^T M z_2 = 2m\ell^2 \Rightarrow z_2 = \frac{1}{\ell\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$U = \mathcal{A} = [z_1 \ z_2] = \frac{1}{\ell\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

General solution is

$$\theta(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = c_1 z_1 \cos(\omega_1 t + \varphi_1) + c_2 z_2 \cos(\omega_2 t + \varphi_2)$$

$$= \frac{c_1}{\ell\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t + \varphi_1) + \frac{c_2}{\ell\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t + \varphi_2)$$

Example: initial conditions:

$$\underbrace{\theta_1(0) = \alpha \ , \ \theta_2(0) = 0}_{(1)} \ , \ \underbrace{\dot{\theta}_1(0) = 0 \ , \ \dot{\theta}_2(0) = 0}_{(2)}$$

(1) $$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \frac{c_1}{\ell\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\varphi_1 + \frac{c_2}{\ell\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\varphi_2$$

(2) $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -\frac{c_1\omega_1}{\ell\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\varphi_1 - \frac{c_2\omega_2}{\ell\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\varphi_1$$

$$\Rightarrow \varphi_1 = \varphi_2 = 0$$

(1) $$\Rightarrow \frac{c_1}{\ell\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{c_2}{\ell\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = \alpha\ell\sqrt{\frac{m}{2}}$$

$$\Rightarrow \theta(t) = \frac{\alpha}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\omega_1 t + \frac{\alpha}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\omega_2 t$$

Weak coupling $\rightarrow$ k small $\rightarrow$ $\omega_1 \approx \omega_2$

$$\Rightarrow \theta(t) = \alpha \begin{pmatrix} \cos\frac{\omega_2-\omega_1}{2}t \ \cos\frac{\omega_2+\omega_1}{2}t \\ \sin\frac{\omega_2-\omega_1}{2}t \ \sin\frac{\omega_2+\omega_1}{2}t \end{pmatrix}$$

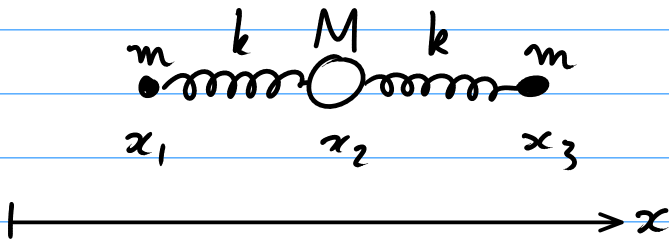
$$\downarrow$$

$\left.\begin{matrix} \cos\varepsilon t \\ \sin\varepsilon t \end{matrix}\right\}$ amplitude modulation

For larger coupling: effects are crucial to dynamics of the entire system

$\longrightarrow$ dynamic coupling

* One-dimensional tri-atomic molecule



Springs with equilibrium length

$$T = \frac{1}{2} m\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{1}{2} M \dot{x}_2^2$$

$$V = \frac{1}{2} k\left(x_1 - x_2 - L\right)^2 + \frac{1}{2} k\left(x_3 - x_2 - L\right)^2$$

$\Rightarrow$ equilibrium for $\begin{cases} x_2 = R & : \text{center of mass position} \\ x_1 = R - L & \quad (\text{assumed constant}) \\ x_3 = R + L & \end{cases}$

$\downarrow$

$R(t) = R_0 + v_{cm}(t - t_0)$ in general case

$$\Rightarrow \begin{cases} \eta_1 = x_1 - (R - L) \\ \eta_2 = x_2 - R \\ \eta_3 = x_3 - (R + L) \end{cases}$$

$$\Rightarrow T = \frac{1}{2} m\left(\dot{\eta}_1^2 + \dot{\eta}_3^2\right) + \frac{1}{2} M \dot{\eta}_2^2 = \frac{1}{2}(\dot{\eta}_1, \dot{\eta}_2, \dot{\eta}_3) M \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{pmatrix}$$

$$V = \frac{1}{2} k\left(\eta_1 - \eta_2\right)^2 + \frac{1}{2} k\left(\eta_3 - \eta_2\right)^2 = \frac{1}{2}(\eta_1, \eta_2, \eta_3) V \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \qquad V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

$$\Rightarrow \left(V - \omega^2 M\right) z = 0 \quad \text{eigenvalue problem}$$

$$\Rightarrow \det\left(V - \omega^2 M\right) = 0$$

$$\begin{vmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{vmatrix} = 0$$

$$\Leftrightarrow (k-\omega^2 m)(2k-\omega^2 M)(k-\omega^2 m) - 2k^2(k-\omega^2 m) = 0$$
$$\Leftrightarrow (k-\omega^2 m)(-k\omega^2 M - 2k\omega^2 m + \omega^4 M m) = 0$$
$$\Leftrightarrow \omega^2(k-\omega^2 m)(\omega^2 M m - k(2m+M)) = 0$$

$$\longrightarrow \quad \omega_1^2 = 0 \quad , \quad \omega_2^2 = \frac{k}{m} \quad , \quad \omega_3^2 = \frac{k}{m}\left(1 + 2\frac{m}{M}\right)$$

Zero frequency? What does that mean?

$$\ddot{\xi}_i + \omega_i^2 \xi_i = 0 \Rightarrow \ddot{\xi}_1 = 0 \Rightarrow \xi_1 = v_1 t + \xi_{0,1}$$

Eigenvectors and normal coordinates:

$$\omega_1: \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} z_1 = 0 \Rightarrow z_1 = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$z_1^T M z_1 = \frac{2m+M}{N} = 1 \Leftrightarrow N = 2m+M$$

$$\Rightarrow z_1 = \frac{1}{\sqrt{2m+M}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{collective motion of all three masses}$$

$$\omega_2: \begin{pmatrix} 0 & -k & 0 \\ -k & k(2-\frac{M}{m}) & -k \\ 0 & -k & 0 \end{pmatrix} z_2 = 0 \Rightarrow z_2 = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$z_2^T M z_2 = \frac{2m}{N} = 1 \Leftrightarrow N = 2m$$

$$\Rightarrow z_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \text{outer masses oscillating in counterphase}$$

$$\omega_3: \begin{pmatrix} -2\frac{m}{M}k & -k & 0 \\ -k & -k\frac{M}{m} & -k \\ 0 & -k & -2\frac{m}{M}k \end{pmatrix} z_3 = 0 \Rightarrow z_3 = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix}$$

$$z_3^T M z_3 = \frac{2m + 4\frac{m^2}{M}}{N} \Leftrightarrow N = 2\left(1+2\frac{m}{M}\right) m$$

$$\Rightarrow z_3 = \frac{1}{\sqrt{2m\left(1+2\frac{m}{M}\right)}} \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix}$$

Assume now $m = M$:

$$\Rightarrow U = [z_1 \; z_2 \; z_3] = \frac{1}{\sqrt{6m}} \begin{bmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{bmatrix}, \quad \eta = U\xi$$

Orthonormality: $U^T \underset{m\mathbb{1}}{\underset{\shortparallel}{M}} U = \mathbb{1}$

$$\Rightarrow m U^T U = \mathbb{1}$$

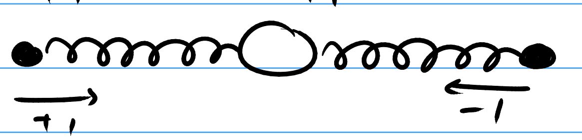
$$\Rightarrow U^{-1} = m U^T = \sqrt{\frac{m}{6}} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{bmatrix}$$

$\eta = U\xi \Rightarrow \xi = m U^T \eta$:

ξ_1 :

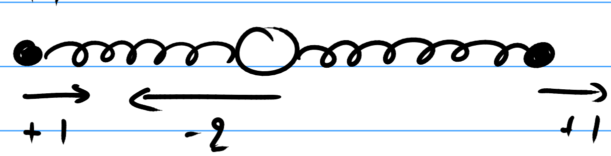
$$\xi_1 = \sqrt{\frac{m}{3}}\left(\eta_1 + \eta_2 + \eta_3\right)$$

ξ_2 :



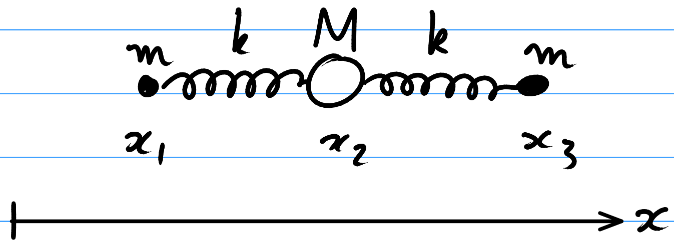
$$\xi_2 = \sqrt{\frac{m}{2}}\left(\eta_1 - \eta_3\right)$$

ξ_3 :



$$\xi_3 = \sqrt{\frac{m}{6}}\left(\eta_1 - 2\eta_2 + \eta_3\right)$$

* One-dimensional tri-atomic molecule



In center of mass system

$$m x_1 + M x_2 + m x_3 = (2m+M)R$$

$$\downarrow$$

$$x_2 = -\frac{m}{M}(x_1 + x_3)$$

$$T = \frac{1}{2} m(\dot\eta_1^2 + \dot\eta_3^2) + \underbrace{\frac{1}{2} M \dot\eta_2^2}_{\frac{1}{2}\frac{m^2}{M}(\dot\eta_1 + \dot\eta_3)^2}$$

eliminate center of mass motion

$$T = \frac{1}{2} m\left(1+\frac{m}{M}\right)\dot\eta_1^2 + \frac{1}{2} m\left(1+\frac{m}{M}\right)\dot\eta_3^2 + \frac{m^2}{M}\dot\eta_1\dot\eta_3$$

$$= \frac{1}{2}(\dot\eta_1, \dot\eta_3)\, M \begin{pmatrix}\dot\eta_1 \\ \dot\eta_3\end{pmatrix}$$

$$V = \frac{1}{2} k(\eta_1 - \eta_2)^2 + \frac{1}{2} k(\eta_3 - \eta_2)^2$$

$$= \frac{1}{2} k\left[\eta_1 + \frac{m}{M}(\eta_1 + \eta_3)\right]^2 + \frac{1}{2} k\left[\eta_3 + \frac{m}{M}(\eta_1 + \eta_3)\right]^2$$

$$= \frac{1}{2} k\left[\eta_1^2\left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2}\eta_3^2 + 2\left(1+\frac{m}{M}\right)\frac{m}{M}\eta_1\eta_3 \right.$$
$$\left. + \eta_3^2\left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2}\eta_1^2 + 2\left(1+\frac{m}{M}\right)\frac{m}{M}\eta_1\eta_3\right]$$

$$= \frac{1}{2} k\left[\eta_1^2\left(\left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2}\right) + \eta_3^2\left(\left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2}\right) \right.$$
$$\left. + 4\frac{m}{M}\left(1+\frac{m}{M}\right)\eta_1\eta_3\right]$$

$$= \frac{1}{2}(\eta_1, \eta_3)\, V \begin{pmatrix}\eta_1 \\ \eta_3\end{pmatrix} \longrightarrow$$ only 2 variables

$$M = m \begin{pmatrix} \left(1+\frac{m}{M}\right) & \frac{m}{M} \\ \frac{m}{M} & \left(1+\frac{m}{M}\right) \end{pmatrix}$$

$$V = k \begin{pmatrix} \left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2} & 2\frac{m}{M}\left(1+\frac{m}{M}\right) \\ 2\frac{m}{M}\left(1+\frac{m}{M}\right) & \left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2} \end{pmatrix}$$

Because $M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_1 \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}$

M and V commute: $MV = VM$

If $\exists U : \begin{cases} U^T M U = \text{diagonal with eigenvalues} = \Lambda_M \\ U^T V U = \text{diagonal with eigenvalues} = \Lambda_V \end{cases}$
with $U^T U = \mathbb{1}$ orthonormal

$$\hookrightarrow \Lambda_M \Lambda_V = U^T M U U^T V U = U^T M V U$$
$$= U^T V M U = U^T V U U^T M U = \Lambda_V \Lambda_M$$

Diagonalize M: $\begin{vmatrix} m\left(1+\frac{m}{M}\right) - \lambda & \frac{m^2}{M} \\ \frac{m^2}{M} & m\left(1+\frac{m}{M}\right) - \lambda \end{vmatrix} = 0$ row 1 = 1 − 2

$$\Leftrightarrow \begin{vmatrix} m - \lambda & \lambda - m \\ \frac{m^2}{M} & m\left(1+\frac{m}{M}\right) - \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (\lambda - m) \begin{vmatrix} -1 & 1 \\ \frac{m^2}{M} & m\left(1+\frac{m}{M}\right) - \lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda = m \text{ or } \lambda = m\left(1+\frac{m}{M}\right) + \frac{m^2}{M} = m + 2\frac{m^2}{M}$$

$$\Rightarrow \Lambda_M = \begin{pmatrix} m & 0 \\ 0 & m+2\frac{m^2}{M} \end{pmatrix}, \quad z_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad z_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad U^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$\hookrightarrow$ V diagonalizes to $\Lambda_V = U^T V U$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 - v_2 & v_1 + v_2 \\ v_2 - v_1 & v_1 + v_2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2(v_1 - v_2) & 0 \\ 0 & 2(v_1+v_2) \end{pmatrix} = \begin{pmatrix} v_1 - v_2 & 0 \\ 0 & v_1 + v_2 \end{pmatrix}$$

Also transform $\eta = U\xi = U\begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix}, \quad \xi = U^T \eta$

$$\Rightarrow L = \frac{1}{2}\dot{\xi}^T U^T M U \dot{\xi} - \frac{1}{2}\xi^T U^T V U \xi$$

$$= \frac{1}{2}\dot{\xi}^T \Lambda_M \dot{\xi} - \frac{1}{2}\xi^T \Lambda_V \xi$$

$\Rightarrow$ ξ_1 and ξ_3 are decoupled $\rightarrow$ normal coordinates

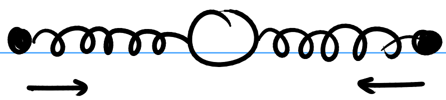
ξ_1 has frequency $\frac{v_1 - v_2}{m} = \frac{k}{m}\left[\left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2} - 2\frac{m}{M}\left(1+\frac{m}{M}\right)\right] = \frac{k}{m}$

ξ_3 has frequency $\frac{v_1 + v_2}{m+2\frac{m^2}{M}} = \frac{k}{m+2\frac{m^2}{M}}\left[\left(1+\frac{m}{M}\right)^2 + \frac{m^2}{M^2} + 2\frac{m}{M}\left(1+\frac{m}{M}\right)\right]$

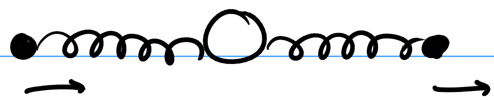
$$\Rightarrow \omega_3 = \frac{k}{m} \frac{1}{1+2\frac{m}{M}} \left(1 + 4\frac{m}{M} + 4\frac{m^2}{M^2}\right) = \frac{k}{m} \frac{\left(1+2\frac{m}{M}\right)^2}{1+2\frac{m}{M}}$$

$$= \frac{k}{m}\left(1+2\frac{m}{M}\right)$$

Normal modes are now just



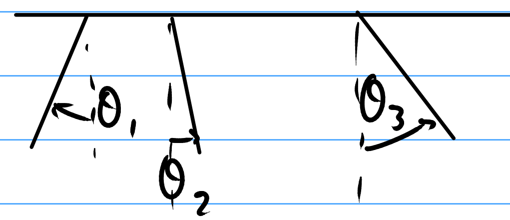
ω_1 , $z_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $\xi_1 = \frac{1}{\sqrt{2}}(\eta_1 - \eta_3)$



ω_3 , $z_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\xi_3 = \frac{1}{\sqrt{2}}(\eta_1 + \eta_3)$

+ Degenerate eigenvalues and Gram-Schmidt procedure

Three coupled pendulums :



$$T = \frac{1}{2}\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2\right)$$

$$V = \frac{1}{2}\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right) - 2\varepsilon\theta_1\theta_2 - 2\varepsilon\theta_2\theta_3 - 2\varepsilon\theta_1\theta_3$$

$$\Rightarrow M = \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix} \qquad V = \begin{pmatrix}1 & -\varepsilon & -\varepsilon \\ -\varepsilon & 1 & -\varepsilon \\ -\varepsilon & -\varepsilon & 1\end{pmatrix}$$

Eigenvalues :

$$\det(V - \omega^2 M) = \begin{vmatrix}1-\omega^2 & -\varepsilon & -\varepsilon \\ -\varepsilon & 1-\omega^2 & -\varepsilon \\ -\varepsilon & -\varepsilon & 1-\omega^2\end{vmatrix} = 0$$

Simplify first:

row 2 = 2 - 3
$$\begin{vmatrix} 1-\omega^2 & -\varepsilon & -\varepsilon \\ 0 & 1-\omega^2+\varepsilon & -\varepsilon-1+\omega^2 \\ -\varepsilon & -\varepsilon & 1-\omega^2 \end{vmatrix} = 0$$

$$\Leftrightarrow (1+\varepsilon-\omega^2)\begin{vmatrix} 1-\omega^2 & -\varepsilon & -\varepsilon \\ 0 & 1 & -1 \\ -\varepsilon & -\varepsilon & 1-\omega^2 \end{vmatrix} = 0$$

column 3 = 2 + 3
$$\Leftrightarrow (1+\varepsilon-\omega^2)\begin{vmatrix} 1-\omega^2 & -2\varepsilon & -\varepsilon \\ 0 & 0 & -1 \\ -\varepsilon & -\varepsilon+1-\omega^2 & 1-\omega^2 \end{vmatrix} = 0$$

$$\Leftrightarrow (1+\varepsilon-\omega^2)\begin{vmatrix} 1-\omega^2 & -2\varepsilon \\ -\varepsilon & 1-\varepsilon-\omega^2 \end{vmatrix} = 0$$

column 1 = 1 + 2
$$\Leftrightarrow (1+\varepsilon-\omega^2)\begin{vmatrix} 1-2\varepsilon-\omega^2 & -2\varepsilon \\ 1-2\varepsilon-\omega^2 & 1-\varepsilon-\omega^2 \end{vmatrix} = 0$$

$$\Leftrightarrow (1+\varepsilon-\omega^2)(1-2\varepsilon-\omega^2)\begin{vmatrix} 1 & -2\varepsilon \\ 1 & 1-\varepsilon-\omega^2 \end{vmatrix} = 0$$

$$\Leftrightarrow (1+\varepsilon-\omega^2)(1-2\varepsilon-\omega^2)(1+\varepsilon-\omega^2) = 0$$

$\hookrightarrow$ $\omega_1^2 = \omega_2^2 = 1+\varepsilon \longrightarrow$ degeneracy!

$\omega_3^2 = 1-2\varepsilon$

Eigenvectors:

$$z_3: \begin{pmatrix} 2\varepsilon & -\varepsilon & -\varepsilon \\ -\varepsilon & 2\varepsilon & -\varepsilon \\ -\varepsilon & -\varepsilon & 2\varepsilon \end{pmatrix} z_3 = 0 \quad \Rightarrow z_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$z_1, z_2 : \begin{pmatrix} -\varepsilon & -\varepsilon & -\varepsilon \\ -\varepsilon & -\varepsilon & -\varepsilon \\ -\varepsilon & -\varepsilon & -\varepsilon \end{pmatrix} z_1 = 0 \Rightarrow \tilde{z}_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\tilde{z}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Gram-Schmidt procedure:

- set of linear independent eigenvectors $\{\tilde{z}_1, \tilde{z}_2\}$
- normalize first eigenvector: $\tilde{z}_1^T M \tilde{z}_1 = 6$

$$z_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

- subtract projection of $\tilde{z}_2$ on z_1:

$$\tilde{z}_2 - \underbrace{\left(z_1^T M \tilde{z}_2\right)}_{\frac{3}{\sqrt{6}}} z_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$

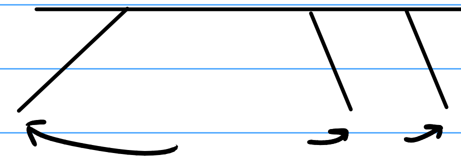
- normalize second eigenvector: $\tilde{z}_2^T M \tilde{z}_2 = 2\left(\frac{3}{2}\right)^2$

$$z_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

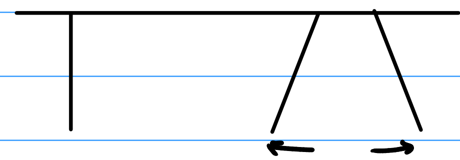
$\Rightarrow$ modal matrix $U = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & \sqrt{2} \\ -1 & \sqrt{3} & \sqrt{2} \\ -1 & -\sqrt{3} & \sqrt{2} \end{pmatrix}$

with $\xi = U^T \theta$ or $\theta = U \xi$

$$\Rightarrow \quad \xi_1 = \frac{1}{\sqrt{6}}\left(2\theta_1 - \theta_2 - \theta_3\right)$$



$$\xi_2 = \frac{1}{\sqrt{2}}\left(\theta_2 - \theta_3\right)$$



$$\xi_3 = \frac{1}{\sqrt{3}}\left(\theta_1 + \theta_2 + \theta_3\right)$$