Homework assignment 7

① $L = \left(\frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \omega_0^2 q_1^2\right) + \left(\frac{1}{2} m \dot{q}_2^2 + \frac{1}{2} m \omega_0^2 q_2^2\right) - \alpha q_1 q_2$

Equilibrium $q_1 = q_2 = 0$

$M = m \mathbb{1}$

$$V = m \begin{pmatrix} \omega_0^2 & -\frac{\alpha}{m} \\ -\frac{\alpha}{m} & \omega_0^2 \end{pmatrix}$$

$\det(V - \omega^2 M) = 0$

$$\Leftrightarrow \quad m^2 \begin{vmatrix} \omega_0^2 - \omega^2 & -\frac{\alpha}{m} \\ -\frac{\alpha}{m} & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (\omega_0^2 - \omega^2)^2 - \left(\frac{\alpha}{m}\right)^2 = 0$$

$$\Leftrightarrow \quad \left(\omega_0^2 - \omega^2 + \frac{\alpha}{m}\right)\left(\omega_0^2 - \omega^2 - \frac{\alpha}{m}\right) = 0$$

$$\Leftrightarrow \quad \omega_\pm^2 = \omega_0^2 \pm \frac{\alpha}{m}$$

$$z_+ : \begin{pmatrix} -\frac{\alpha}{m} & -\frac{\alpha}{m} \\ -\frac{\alpha}{m} & -\frac{\alpha}{m} \end{pmatrix} z_+ = 0 \Leftrightarrow z_+ = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

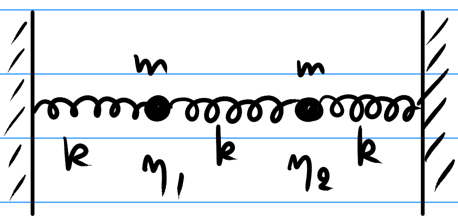
$$z_- : \begin{pmatrix} \frac{\alpha}{m} & -\frac{\alpha}{m} \\ -\frac{\alpha}{m} & \frac{\alpha}{m} \end{pmatrix} z_- = 0 \Leftrightarrow z_- = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When $\alpha \ll \frac{\omega_0^2}{m} \Rightarrow \omega_\pm^2 = \omega_0^2(1+\varepsilon)$

$$q = c_+ z_+ \cos\omega_+ t + c_- z_- \cos\omega_- t$$

$$= \frac{c_+ + c_-}{2}\begin{pmatrix} \cos\frac{\omega_+ - \omega_-}{2}t \cos\frac{\omega_+ + \omega_-}{2}t \\ \sin\underbrace{\frac{\omega_+ - \omega_-}{2}}t \sin\frac{\omega_+ + \omega_-}{2}t \end{pmatrix}$$

$\cos\varepsilon t$ term $\rightarrow$ amplitude modulation

② 

$$T = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2$$

$$V = \frac{1}{2}k\eta_1^2 + \frac{1}{2}k(\eta_2-\eta_1)^2 + \frac{1}{2}k\eta_2^2$$

$$\Rightarrow L = \frac{1}{2}(\dot{\eta}_1, \dot{\eta}_2)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} - \frac{1}{2}(\eta_1, \eta_2)\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$\begin{cases} m\ddot{\eta}_1 + 2k\eta_1 - k\eta_2 = 0 \\ m\ddot{\eta}_2 + 2k\eta_2 - k\eta_1 = 0 \end{cases}$$

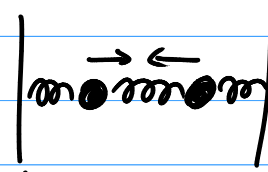
$$\det(V - m\omega^2) = 0$$

$$\Leftrightarrow \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$$

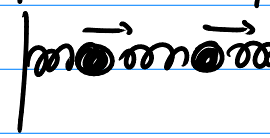
$$\Leftrightarrow (2k - m\omega^2 + k)(2k - m\omega^2 - k) = 0$$

$$\Leftrightarrow \omega_\pm^2 = \frac{k}{m}(2 \pm 1) \quad \begin{cases} \omega_+ = 3\frac{k}{m} \\ \omega_- = \frac{k}{m} \end{cases}$$

$$z_+ : \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} z_+ = 0 \Rightarrow z_+ = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$z_- : \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} z_- = 0 \Rightarrow z_- = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$U = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \longrightarrow U^T M U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\eta = U \xi \Rightarrow \xi = U^T M \eta = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{cases} \xi_+ = \sqrt{\frac{m}{2}} (\eta_1 - \eta_2) \\ \xi_- = \sqrt{\frac{m}{2}} (\eta_1 + \eta_2) \end{cases}$$

$$U^T V U = \frac{1}{2m} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2m} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3k & k \\ -3k & k \end{pmatrix}$$

$$= \begin{pmatrix} 3\frac{k}{m} & 0 \\ 0 & \frac{k}{m} \end{pmatrix} = \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix}$$

$$\Rightarrow L = \frac{1}{2} \dot{\xi}^T \dot{\xi} - \frac{1}{2} \xi^T \begin{pmatrix} \omega_+ & 0 \\ 0 & \omega_- \end{pmatrix} \xi$$

$$= \left(\frac{1}{2} \dot{\xi}_+^2 - \frac{1}{2} \omega_+ \xi_+^2 \right) + \left(\frac{1}{2} \dot{\xi}_-^2 - \frac{1}{2} \omega_-^2 \xi_-^2 \right)$$

General solution:

$$\eta = \frac{c_+}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_+ t + \varphi_+) + \frac{c_-}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \varphi_-)$$

Initial condition: $\eta_1(0) = \alpha, \quad \eta_2(0) = 0$

$\dot{\eta}_1(0) = 0, \quad \dot{\eta}_2(0) = 0$

$\hookrightarrow \varphi_1 = \varphi_2 = 0$

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \frac{c_+}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{c_-}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow c_+ = c_- = \alpha \sqrt{\frac{m}{2}}$$

$$\Rightarrow \quad \eta = \frac{\alpha}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos \omega_+ t + \frac{\alpha}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \omega_- t$$

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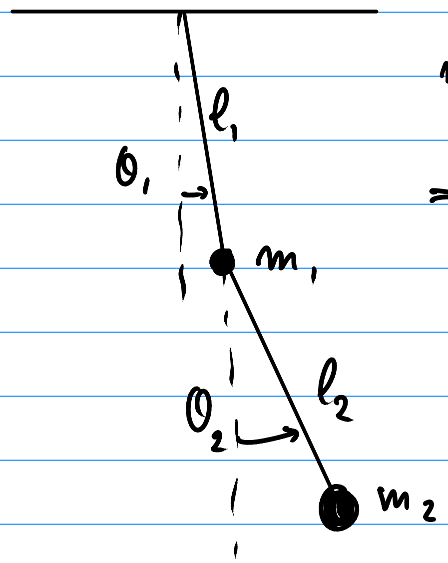


$$\eta_1 \cong \ell_1\theta_1 \quad , \quad \eta_2 \cong \ell\theta_2$$

$$\Rightarrow T = \frac{1}{2} m_1 \ell_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 (\ell_1\dot\theta_1 + \ell_2\dot\theta_2)^2$$

$$\cong \frac{1}{2} m_1 \dot\eta_1^2 + \frac{1}{2} m_2 (\dot\eta_1 + \dot\eta_2)^2$$

$$\hookrightarrow M = \begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$$

$$V = m_1 g \ell_1 (1-\cos\theta_1) + m_2 g \ell_1 (1-\cos\theta_1) + m_2 g \ell_2 (1-\cos\theta_2)$$

$$\cong \frac{1}{2} m_1 g \ell_1 \theta_1^2 + \frac{1}{2} m_2 g \ell_1 \theta_1^2 + \frac{1}{2} m_2 g \ell_2 \theta_2^2$$

$$\cong \frac{1}{2} g \left(\frac{m_1}{\ell_1}\eta_1^2 + \frac{m_2}{\ell_1}\eta_1^2 + \frac{m_2}{\ell_2}\eta_2^2 \right)$$

$$\hookrightarrow V = g \begin{pmatrix} \frac{m_1+m_2}{\ell_1} & 0 \\ 0 & \frac{m_2}{\ell_2} \end{pmatrix}$$

$$\Rightarrow L = \frac{1}{2} m_1 \dot\eta_1^2 + \frac{1}{2} m_2 (\dot\eta_1 - \dot\eta_2)^2 - \frac{1}{2} g \left(\frac{m_1+m_2}{\ell_1}\eta_1^2 + \frac{m_2}{\ell_2}\eta_2^2 \right)$$

$$\det(V - \omega^2 M) = 0$$

$$\Leftrightarrow \begin{vmatrix} \frac{g}{\ell_1}(m_1+m_2) - \omega^2(m_1+m_2) & -\omega^2 m_2 \\ -\omega^2 m_2 & \frac{g}{\ell_2} m_2 - \omega^2 m_2 \end{vmatrix} = 0$$

$$\Leftrightarrow \left(m_2(m_1+m_2) - m_2^2\right)\omega^4 - \left(\frac{g}{\ell_2} + \frac{g}{\ell_1}\right) m_2(m_1+m_2)\omega^2$$

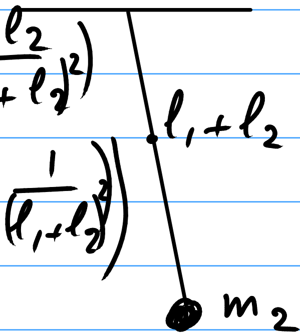
$$+ \frac{g^2}{\ell_1\ell_2} m_2(m_1+m_2) = 0$$

$$\Leftrightarrow \quad \frac{m_1}{m_1+m_2}\omega^4 - g\,\frac{l_1+l_2}{l_1 l_2}\,\omega^2 + \frac{g^2}{l_1 l_2} = 0$$

$$D = g^2\left(\frac{l_1+l_2}{l_1 l_2}\right)^2 - 4g^2\,\frac{1}{l_1 l_2}\,\frac{m_1}{m_1+m_2}$$

$$\omega_\pm^2 = \frac{m_1+m_2}{2m_1}\left(g\,\frac{l_1+l_2}{l_1 l_2} \pm \sqrt{D} \right)$$

$\frac{m_1}{m_2}$ small : $\sqrt{D} \cong g\left(\frac{l_1+l_2}{l_1 l_2}\right)\left(1 + \frac{1}{2}\,2\,\frac{m_1}{m_2}\,\frac{l_1 l_2}{(l_1+l_2)^2}\right)$

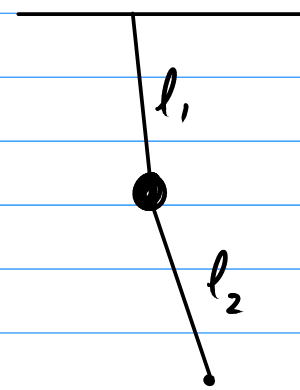


$$\omega_\pm^2 = \frac{m_2}{2m_1}\left(g\,\frac{l_1+l_2}{l_1 l_2} \pm g\,\frac{l_1+l_2}{l_1 l_2}\left(1 + \frac{m_1}{m_2}\,\frac{1}{(l_1+l_2)^2}\right)\right)$$

$$\Rightarrow \begin{cases} \omega_-^2 = \dfrac{g}{l_1+l_2} \\ \omega_+^2 = g\,\dfrac{m_2}{m_1}\,\dfrac{l_1+l_2}{l_1 l_2} \end{cases}$$

$\rightarrow$ one long pendulum

$\frac{m_2}{m_1}$ small : $\sqrt{D} \cong \frac{g}{l_1 l_2}(l_1 - l_2)$



$$\omega_\pm^2 = \frac{1}{2}\,\frac{g}{l_1 l_2}\Big((l_1+l_2) \pm (l_1 - l_2)\Big)$$

$$\Rightarrow \begin{cases} \omega_-^2 = \dfrac{g}{l_1} \\ \omega_+^2 = \dfrac{g}{l_2} \end{cases}$$

$\hookrightarrow$ two independent pendulums

Equal lengths :

$$D = 4\frac{g^2}{\ell^2} - 4\frac{g^2}{\ell^2}\frac{m_1}{m_1+m_2} = 4\frac{g^2}{\ell^2}\frac{m_2}{m_1+m_2}$$

$$\hookrightarrow \quad \omega_\pm^2 = \frac{m_1+m_2}{m_1}\left(\frac{g}{\ell} \pm \frac{g}{\ell}\underbrace{\sqrt{\frac{m_2}{m_1+m_2}}}_{=\gamma}\right)$$

$$= \frac{g}{\ell}(1\pm\gamma)\underbrace{\frac{m_1+m_2}{m_1}}_{=\frac{1}{1-\gamma^2}}$$

$$= \frac{g}{\ell}\frac{1}{1\mp\gamma}$$

Eigenvectors :

$$(V - \omega_\pm^2 M) z_\pm = 0$$

$$\Leftrightarrow \begin{pmatrix} \frac{g}{\ell}(m_1+m_2) - \omega_\pm^2(m_1+m_2) & -m_2\omega_\pm^2 \\ -m_2\omega_\pm^2 & \frac{g}{\ell}m_2 - \omega_\pm^2 m_2 \end{pmatrix} z_\pm = 0$$

$$\Leftrightarrow \begin{pmatrix} \frac{g}{\ell}(m_1+m_2)\left(1-\frac{1}{1\mp\gamma}\right) & -\frac{g}{\ell}m_2\frac{1}{1\mp\gamma} \\ -\frac{g}{\ell}m_2\frac{1}{1\mp\gamma} & \frac{g}{\ell}m_2\left(1-\frac{1}{1\mp\gamma}\right) \end{pmatrix} z_\pm = 0$$

$$\Leftrightarrow \quad \tilde{z}_\pm = \begin{pmatrix} \pm\gamma \\ 1 \end{pmatrix}$$

Normalization : $\tilde{z}_\pm^T M \tilde{z}_\pm = (m_1+m_2) + \gamma^2 m_2 \pm 2\gamma m_2$

$$= 2m_2 \pm 2\gamma m_2$$

$$= 2m_2(1\pm\gamma)$$

$$\Rightarrow \quad U = \frac{1}{\sqrt{2m_2}} \begin{pmatrix} \frac{\gamma}{\sqrt{1+\gamma}} & \frac{-\gamma}{\sqrt{1-\gamma}} \\ \frac{1}{\sqrt{1+\gamma}} & \frac{1}{\sqrt{1-\gamma}} \end{pmatrix}$$

$$\Rightarrow \quad \xi = U^T M \eta = \frac{1}{\sqrt{2m_2}} \begin{pmatrix} \frac{-1}{\sqrt{1-\gamma}} & \frac{\gamma}{\sqrt{1-\gamma}} \\ \frac{1}{\sqrt{1+\gamma}} & \frac{\gamma}{\sqrt{1+\gamma}} \end{pmatrix} \begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix} \eta$$

$$= \sqrt{\frac{m_1}{2}} \begin{pmatrix} \frac{1}{\sqrt{1-\gamma}} & \frac{\gamma}{\sqrt{1-\gamma}} \\ \frac{-1}{\sqrt{1+\gamma}} & \frac{\gamma}{\sqrt{1+\gamma}} \end{pmatrix}$$

$$\xi_1 = \sqrt{\frac{m_1}{2}} \frac{\eta_1 + \gamma\eta_2}{\sqrt{1-\gamma}} \quad , \quad \xi_2 = \sqrt{\frac{m_1}{2}} \frac{-\eta_1 + \gamma\eta_2}{\sqrt{1+\gamma}}$$

General solution :

$$\eta = c_+ z_+ \cos(\omega_+ t + \varphi_+) + c_- z_- \cos(\omega_- t + \varphi_-)$$

Initial conditions ,

$$\eta_1(0) = \delta \, , \; \eta_2(0) = 0$$
$$\dot{\eta}_1(0) = 0 \quad , \; \dot{\eta}_2(0) = 0$$

$$\dot{\eta} = - c_+ \omega_+ z_+ \sin(\omega_+ t + \varphi_+) - c_- \omega_- z_- \sin(\omega_- t + \varphi_-)$$
$$\dot{\eta}(0) = 0 \Rightarrow \varphi_+ = \varphi_- = 0$$

$$\eta(0) = c_+ z_+ + c_- z_-$$

$$= \frac{1}{\sqrt{2m_2}} \frac{1}{\sqrt{1+\gamma}} \begin{pmatrix} \gamma \\ 1 \end{pmatrix} c_+ + \frac{1}{\sqrt{2m_2}} \frac{1}{\sqrt{1-\gamma}} \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} c_- = \begin{pmatrix} \delta \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad \frac{c_+}{\sqrt{1+\gamma}} + \frac{c_-}{\sqrt{1-\gamma}} = 0$$

$$\Leftrightarrow \quad \begin{cases} c_+ = \frac{\delta}{2}\sqrt{1+\gamma} \\ c_- = -\frac{\delta}{2}\sqrt{1-\gamma} \end{cases}$$