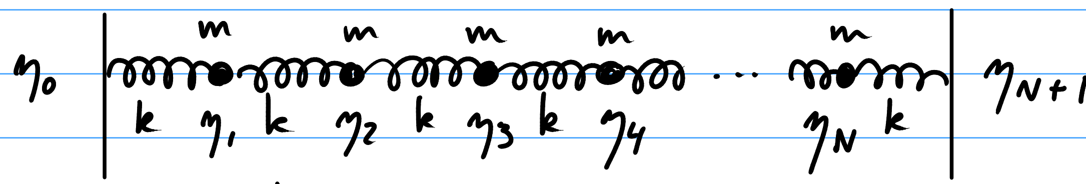
Classical Mechanics (Phys 601) - October 20, 2011

Few degrees of freedom $\rightarrow$ ω_i^2 can be determined

For large number of degrees of freedom N $\rightarrow$ difficult

Special symmetries can make the N-body problems tractable.

* Longitudinal oscillations in crystal lattices



$$L = \underbrace{\frac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i^2}_{T} - \underbrace{\frac{1}{2} k \sum_{i=0}^{N} \left(\eta_{i+1} - \eta_i\right)^2}_{V}$$

$\eta_0 = \eta_{N+1} = 0$ are the boundary conditions

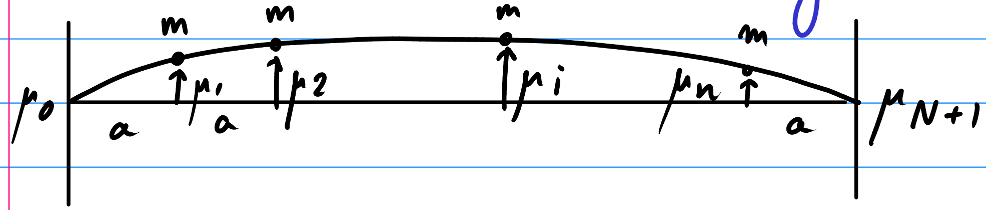
Euler - Lagrange equations :

$$m\ddot{\eta}_i - k\left(\eta_{i+1} - \eta_i\right) + k\left(\eta_i - \eta_{i-1}\right) = 0$$

$$\Leftrightarrow m\ddot{\eta}_i + 2k\eta_i - k\left(\eta_{i+1} + \eta_{i-1}\right) = 0$$

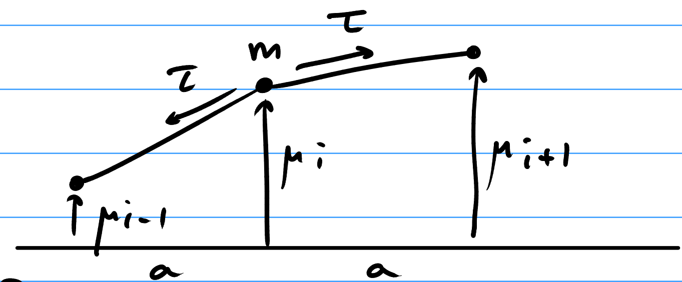
$$M = m\,\mathbb{1} \qquad V = k\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

* Transverse oscillations on a string



$$m\ddot{\mu}_i = F_i$$

Force diagram around mass i :



$$F_i = \tau \frac{\mu_{i+1} - \mu_i}{a} - \tau \frac{\mu_i - \mu_{i-1}}{a}$$

$$= \frac{\tau}{a}\left[(\mu_{i+1} - \mu_i) - (\mu_i - \mu_{i-1})\right]$$

$$= \frac{\tau}{a}\left[(\mu_{i+1} + \mu_{i-1}) - 2\mu_i\right]$$

$$\Rightarrow m\ddot{\mu}_i + 2\frac{\tau}{a}\mu_i - \frac{\tau}{a}(\mu_{i+1} + \mu_{i-1}) = 0$$

Boundary conditions are now $\mu_0 = \mu_{N+1} = 0$.

This is the same Lagrange equation as for longitudinal oscillations, with $k = \frac{\tau}{a}$

$$\Rightarrow L = \frac{1}{2} m \sum_{i=1}^{N} \dot{\mu}_i^2 - \frac{1}{2}\frac{\tau}{a}\sum_{i=0}^{N}(\mu_{i+1} - \mu_i)^2$$

with same form for M and V matrices

To calculate the normal modes, we have to solve this eigenvalue problem: $\det(V - \omega^2 M) = 0$
and $(V - \omega^2 M) z = 0$

* Eigenvalues for this N-body problem:

$$\begin{vmatrix} 2k-m\omega^2 & -k & & & \\ -k & 2k-m\omega^2 & -k & & \\ & -k & \ddots & -k & \\ & & -k & 2k-m\omega^2 & -k \\ & & & -k & 2k-m\omega^2 \end{vmatrix} = 0$$

Introduce $\lambda = 2 - \omega^2 \frac{m}{k}$ and D_N of the $N \times N$ matrix:

$$D_N = \begin{vmatrix} \lambda & -1 & & \\ -1 & \lambda & \ddots & \\ & \ddots & \lambda & -1 \\ & & -1 & \lambda \end{vmatrix} = \lambda D_{N-1} - (-1)(-1) D_{N-2}$$

$$\Rightarrow D_N = \lambda D_{N-1} - D_{N-2}$$

Now: $D_1 = |\lambda| = \lambda$ and $D_2 = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1$

$\rightarrow$ can calculate every D_N recursively and solve $D_N = 0$

To find explicit solution: remember second order differential equation with constant coefficients:

$$y''(x) + a y'(x) + b y(x) = 0$$

$$\hookrightarrow \quad y(x) = A_1 e^{ik_1 x} + A_2 e^{ik_2 x}$$

with k_1 and k_2 solutions of equation after plugging e^{ikx} in the differential equation.

For an explicit expression for D_N, assume that D_N can be written as:

$$D_N = A e^{iBN} \quad , \text{ for constant } A \text{ and } B$$

$$\Leftrightarrow \quad A e^{iBN} = \lambda A e^{iB(N-1)} - A e^{iB(N-2)}$$

$$\Leftrightarrow \quad 1 = \lambda e^{-iB} - e^{-2iB}$$

$$\Leftrightarrow \quad \lambda = e^{iB} + e^{-iB} = 2\cos B \Rightarrow B = \pm \operatorname{acos}\frac{\lambda}{2} = \pm\psi$$

definition of ψ

Two solutions $e^{iN\psi}$ and $e^{-iN\psi}$ with arbitrary constant A_+ and A_-

$$D_N = A_+ e^{iN\psi} + A_- e^{-iN\psi}$$

Determine A_+ and A_- from D_1 and D_2:

$$\begin{cases} D_1 = A_+ e^{i\psi} + A_- e^{-i\psi} = \lambda = 2\cos\psi = e^{i\psi} + e^{-i\psi} \\ D_2 = A_+ e^{2i\psi} + A_- e^{-2i\psi} = \lambda^2 - 1 = 4\cos^2\psi - 1 \end{cases}$$

Cramer's rule:

$$A_+ = \frac{\begin{vmatrix} \lambda & e^{-i\psi} \\ \lambda^2-1 & e^{-2i\psi} \end{vmatrix}}{\begin{vmatrix} e^{i\psi} & e^{-i\psi} \\ e^{2i\psi} & e^{-2i\psi} \end{vmatrix}} = \frac{\lambda e^{-2i\psi} - \lambda^2 e^{-i\psi} + e^{-i\psi}}{e^{-i\psi} - e^{i\psi}}$$

$$= \frac{1}{-2i\sin\psi}\left[\cancel{e^{-i\psi}} + \cancel{e^{-3i\psi}} - e^{i\psi} - \cancel{e^{-3i\psi}} - \cancel{2e^{-i\psi}} + \cancel{e^{-i\psi}}\right]$$

$$= \frac{e^{i\psi}}{2i\sin\psi}$$

$$A_- = \frac{\begin{vmatrix} e^{i\varphi} & \lambda \\ e^{2i\varphi} & \lambda^2-1 \end{vmatrix}}{-2i\sin\varphi} = \frac{(\lambda^2-1)e^{i\varphi} - \lambda e^{2i\varphi}}{-2i\sin\varphi}$$

$$= \frac{\cancel{e^{3i\varphi}} + e^{-i\varphi} + \cancel{2e^{i\varphi}} - \cancel{e^{i\varphi}} - \cancel{e^{3i\varphi}} - \cancel{e^{i\varphi}}}{-2i\sin\varphi}$$

$$= \frac{-e^{-i\varphi}}{2i\sin\varphi}$$

$$D_N = A_+ e^{iN\varphi} + A_- e^{-iN\varphi}$$

$$= \frac{1}{2i\sin\varphi}\left(e^{i(N+1)\varphi} - e^{-i(N+1)\varphi}\right)$$

$$D_N = \frac{\sin(N+1)\varphi}{\sin\varphi} \quad \text{with} \quad \lambda = 2 - \frac{m}{k}\omega^2 = 2\cos\varphi$$

Back to the original problem: the eigenvalues ω^2 are the solutions of the equation $D_N = 0$.

$n = 0 \to \sin\varphi = 0$

$$D_N = \frac{\sin(N+1)\varphi}{\sin\varphi} = 0 \iff (N+1)\varphi = n\pi, \quad \text{for } n = 1, \dots, N$$

$$\Rightarrow \omega^2 = \frac{k}{m}(2-\lambda) = 2\frac{k}{m}(1-\cos\varphi) = 4\frac{k}{m}\sin^2\frac{\varphi}{2}$$

$$\omega_n^2 = 2\frac{k}{m}\left(1 - \cos\frac{n\pi}{N+1}\right) = 4\frac{k}{m}\sin^2\frac{n\pi}{2(N+1)}$$

for $n = 1, \dots, N$

Note: for $n = 0$ we find $\omega^2 = 0 \to$ no oscillations

Next, one could try to find the eigenvectors, but that is cumbersome : ↳ normal-mode amplitudes

diagonal: $2k - m\omega_n^2 = 2k\left(\cos\frac{n\pi}{N+1}\right)$

off-diagonal: $-k$

$$\Rightarrow \quad 2\left(\cos\frac{n\pi}{N+1}\right) z_i = k\left(z_{i-1} + z_{i+1}\right)$$

* Dispersion relations

$$m\ddot{\mu}_i + 2\frac{\tau}{a}\mu_i - \frac{\tau}{a}\left(\mu_{i-1} + \mu_{i+1}\right) = 0$$

Assume that there is a solution:

notation $\mu_i(t) = \mu(x_i, t)$

$$\mu(x_i, t) = Ae^{i(kx_i - \omega t)}$$

→ plug into equation :

$$-m\omega^2 + 2\frac{\tau}{a} - \frac{\tau}{a}\left(e^{ik\overbrace{(x_{i-1} - x_i)}^{a}} + e^{ik\overbrace{(x_{i+1} - x_i)}^{a}}\right) = 0$$

$$\Leftrightarrow \quad -m\omega^2 + 2\frac{\tau}{a} - \frac{\tau}{a}\left(e^{-ika} + e^{ika}\right) = 0$$

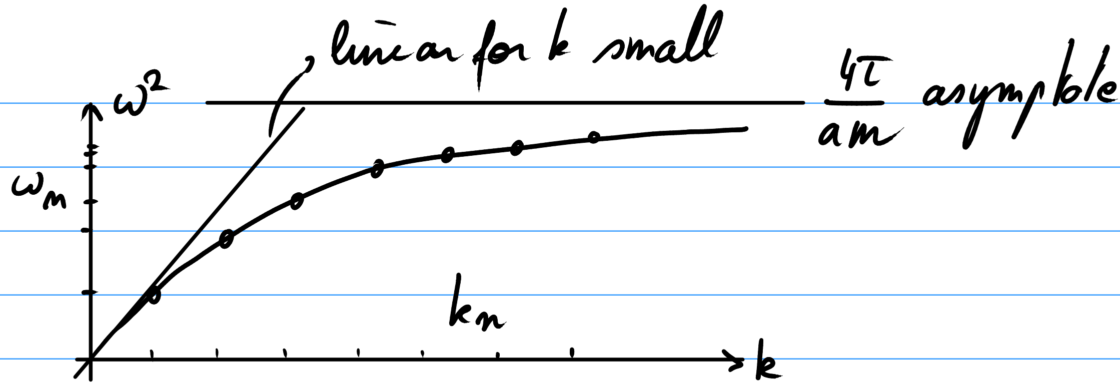
$$\Leftrightarrow \quad -m\omega^2 + 2\frac{\tau}{a}\left(1 - \cos ka\right) = 0$$

$$\omega^2 = 2\frac{\tau}{am}\left(1 - \cos ka\right) = 4\frac{\tau}{am}\sin^2\frac{ka}{2}$$

↳ dispersion relation between k and ω

$\omega_n \leftrightarrow k_n$

Dispersion:



General solution to the transverse oscillations on a string:

$$\mu(x_i) = \sum_{n=1}^{N} c_m z_m e^{i(k_m x_i - \omega_m t)}$$

eigenvectors

Dispersion relations still assume continuous k and ω.

* Boundary conditions

If we introduce periodic boundary conditions:

$$\mu(x_i) = \mu(x_{i+N}) = \mu(x_i + aN)$$

$$\Rightarrow \quad e^{ikNa} = 1 \quad \Leftrightarrow \quad kNa = 2n\pi$$

$$\Rightarrow \quad k_m = \frac{2\pi}{Na} m, \quad m = 0, \dots, N-1$$

or $m = 0, \pm 1, \dots, \pm \frac{1}{2}(N-1)$ (odd)

$m = 0, \pm 1, \dots, \pm \frac{1}{2}N - 1, \frac{1}{2}N$ (even)

If we introduce fixed end boundary conditions (above):
e.g. combine $+k_m$ and $-k_m$ above

$$\mu(x_i, t) = A\left(e^{ikx_i - i\omega t} - e^{-ikx_i - i\omega t}\right)$$

At $x_0 = 0$: $\mu(x_0, t) = 0$ by construction

At $x_{N+1} = (N+1)a$: $\mu(x_{N+1}, t) = 0$

$$\Leftrightarrow \sin k(N+1)a = 0$$
$$\Leftrightarrow k(N+1)a = m\pi, \quad m = 1, .., N$$

$$\Leftrightarrow k_m = \frac{m\pi}{a(N+1)}, \quad m = 1, ..., N$$

→ with dispersion relation this gives again the expression for the eigenvalues ω_m^2

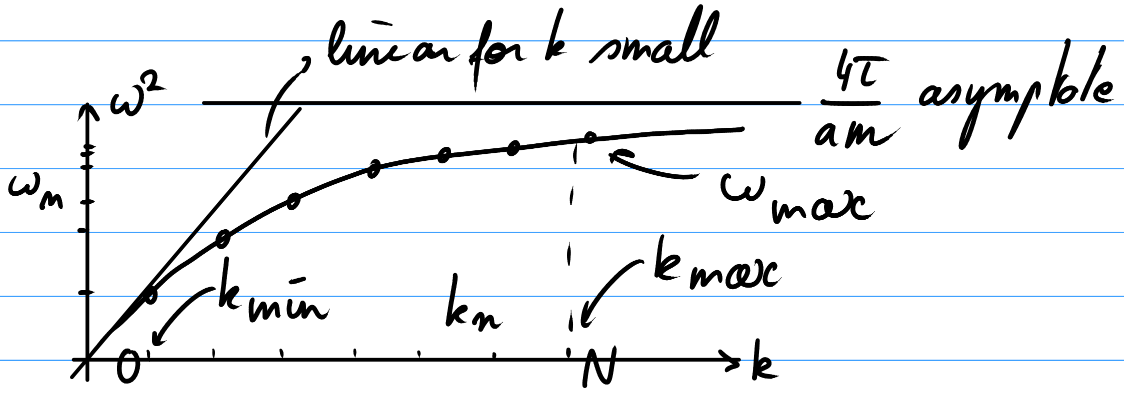
$$\Rightarrow \mu(x_i, t) = Re\left(A \sin \frac{m\pi x_i}{a(N+1)} e^{-i\omega_m t}\right)$$

Now, look back at dispersion relation :

$$\omega^2 = 4\frac{\tau}{am}\sin^2\frac{ka}{2} \Rightarrow \omega = 2\sqrt{\frac{\tau}{am}}\sin\frac{m\pi a}{2\ell}$$

↳ maximum $\omega_{max} \cong 2\sqrt{\frac{\tau}{am}}$ for maximum $k_{max} =$

wavelength $\lambda = \frac{2\pi}{k}$



* Wave equation :

$$m\ddot{\mu}(x_i) + 2\frac{\tau}{a}\mu(x_i) - \frac{\tau}{a}\Big(\mu(x_{i-1}) + \mu(x_{i+1})\Big) = 0$$

$$\Leftrightarrow \quad m\ddot{\mu}(x_i) + \tau\left[\underbrace{\frac{\mu(x_i) - \mu(x_{i-1})}{a}}_{\mu'(x_i)} - \underbrace{\frac{\mu(x_{i+1}) - \mu(x_i)}{a}}_{\mu'(x_{i+1})}\right] = 0$$

$$\Leftrightarrow \quad \ddot{\mu}(x_i) + \frac{\tau a}{m}\left[\frac{\mu'(x_i) - \mu'(x_{i+1})}{a}\right] = 0$$

$$\Leftrightarrow \quad \ddot{\mu}(x_i) + \frac{\tau a}{m}\mu''(x_i) = 0$$

$\sigma = \frac{m}{a}$ = specific weight

$$\Leftrightarrow \quad \underbrace{\ddot{\mu}(x_i) + \frac{\tau}{\sigma}\mu''(x_i) = 0}_{\text{Wave equation}}$$

* External forcing and damping :

$$L = \frac{1}{2}\dot{\eta}^T M \dot{\eta} - \frac{1}{2}\eta^T V \eta + F(t)\,\eta$$

$\hookrightarrow$ diagonalization : $\eta = U\xi$

$$L = \frac{1}{2}\dot{\xi}^T\dot{\xi} - \frac{1}{2}\xi^T \Omega\, \xi + \underbrace{F(t)\,U}_{Q(t)}\,\xi$$

$$\downarrow$$

$$\ddot{\xi}_i + \omega_i^2 \xi_i = Q_i(t)$$

$$\hookrightarrow \xi = \sum_i c_i z_i \cos(\omega_i t + \varphi_i) + \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{Q_i(\omega)\, e^{i\omega t}}{\omega_i^2 - \omega^2}}$$

$$-\omega^2 \mathcal{F}[\xi_i] + \omega_i^2 \mathcal{F}[\xi_i] = Q_i(\omega) = \mathcal{F}[Q_i(t)]$$

$$\mathcal{F}[\xi_i] = \frac{Q_i(\omega)}{\omega_i^2 - \omega^2}$$

$$\xi_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{Q_i(\omega)}{\omega_i^2 - \omega^2}\, e^{i\omega t}\, d\omega$$

Damping term : dissipation function $\mathcal{D} = \frac{1}{2}\dot{\eta}^T D \dot{\eta}$

$$\hookrightarrow M\ddot{\eta} + D\dot{\eta} + V\eta = F(t)$$

(diagonalization not generally possible

$$\ddot{\xi}_i + \gamma_i \dot{\xi}_i + \omega_i^2 \xi_i = Q_i(t)$$