Classical Mechanics (Phys 601) – November 29, 2011

* **Duffing oscillator:**

$$V(q) = \frac{1}{2} m \omega_0^2 q^2 + \frac{1}{4} m \varepsilon q^4$$

$$\hookrightarrow \quad H(p,q) = H_0(p,q) + \varepsilon V(p,q)$$

with $H_0(p,q) = \dfrac{p^2}{2m} + \dfrac{1}{2} m \omega_0^2 q^2$ (unperturbed)

and $V(p,q) = \dfrac{1}{4} m q^4$ (perturbation)

$\hookrightarrow$ action-angle variables defined by $H_0(p,q)$

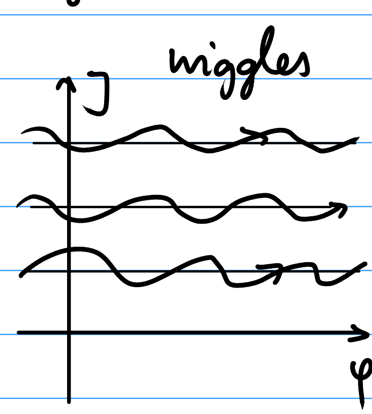
$$H(J,\varphi) = E_0(J) + \varepsilon V(J,\varphi)$$

with $E_0(J) = \omega_0 J$

$$\begin{cases} p = \sqrt{2 m \omega_0 J}\, \cos\varphi \\ q = \sqrt{\dfrac{2J}{m\omega_0}}\, \sin\varphi \end{cases}$$

and $V(J,\varphi) = \dfrac{J^2}{m\omega_0^2} \sin^4\varphi$

$$\Rightarrow \begin{cases} \dot{J} = 0 - \varepsilon \dfrac{4J^2}{m\omega_0^2} \sin^3\varphi \cos\varphi \\ \dot{\varphi} = \omega_0 + \varepsilon \dfrac{2J}{m\omega_0^2} \sin^4\varphi \end{cases}$$



average: $\langle \dot{\varphi} \rangle = \omega_0 + \varepsilon \dfrac{3a^2}{8\omega_0} \rightarrow$ frequency shift

* Periodic forcing of the Duffing oscillator :

From the Lagrangian $L(q,\dot q) = \frac{1}{2} m \dot q^2 - \frac{1}{2} m\omega_0^2 q^2 - \frac{1}{4} m\varepsilon q^4$ we find the equation of motion:

$$\ddot q + \omega_0^2 q + \varepsilon q^3 = 0$$

Add damping term :

$$\ddot q + 2\delta\omega_0 \dot q + \omega_0^2 q + \varepsilon q^3 = 0$$

Add periodic forcing term :

$$\ddot q + 2\delta\omega_0\dot q + \omega_0^2 q + \varepsilon q^3 = f \cos \omega t$$

* Linear system : $\varepsilon = 0$

steady-state solution of form $A\cos(\omega t - \varphi)$:

$$A\left[(\omega_0^2 - \omega^2)\cos(\omega t - \varphi) - 2\delta\omega\omega_0 \sin(\omega t - \varphi)\right] = f\cos\omega t$$

$$\hookrightarrow \begin{cases} A\left[(\omega_0^2 - \omega^2)\cos\varphi + 2\delta\omega\omega_0\sin\varphi\right]\cos\omega t = f\cos\omega t \\ A\left[(\omega_0^2-\omega^2)\sin\varphi - 2\delta\omega\omega_0\cos\varphi\right]\sin\omega t = 0 \end{cases}$$

$$\hookrightarrow \begin{cases} \sin\varphi = \dfrac{2\delta\omega\omega_0}{\omega_0^2 - \omega^2}\cos\varphi \\ A^2\left[(\omega_0^2-\omega^2)^2\cos^2\varphi + 4\delta^2\omega^2\omega_0^2\sin^2\varphi + 4\delta\omega\omega_0(\omega_0^2-\omega^2)\cos\varphi\sin\varphi\right] = f^2 \end{cases}$$

$$\Leftrightarrow \quad A^2\Big[(\omega_0^2-\omega^2)^2\cos^2\varphi + 4\delta^2\omega^2\omega_0^2\,(1-\cos^2\varphi)$$
$$+\,2\left(4\delta^2\omega^2\omega_0^2\right)\cos^2\varphi\Big] = f^2$$

$$\Leftrightarrow \quad A^2\left[(\omega_0^2-\omega^2)^2 + 4\delta^2\omega^2\omega_0^2\right] = f^2$$

$$\Leftrightarrow \quad \begin{cases} \tan\varphi = \dfrac{2\delta\omega\omega_0}{\omega_0^2-\omega^2} \\ A = \dfrac{f}{\sqrt{(\omega_0^2-\omega^2)+4\delta^2\omega_0^2\omega^2}} \end{cases}$$

* Non-linear system : $\varepsilon \neq 0$, and for definiteness $\varepsilon > 0$

Consider small forcing term f → additional term based on steady-state solution for $\varepsilon = 0$

$$\Rightarrow \quad \ddot{q} + 2\delta\omega_0\dot{q} + \omega_0^2 q = f\cos\omega t - \varepsilon A^3 \underbrace{\cos^3(\omega t-\varphi)}_{\frac{1}{4}\left(3\cos(\omega t-\varphi)+\cos 3(\omega t-\varphi)\right)}$$

↓

$\cos\omega t$ term gets additional $\frac{3}{4}\varepsilon A^3$ contribution

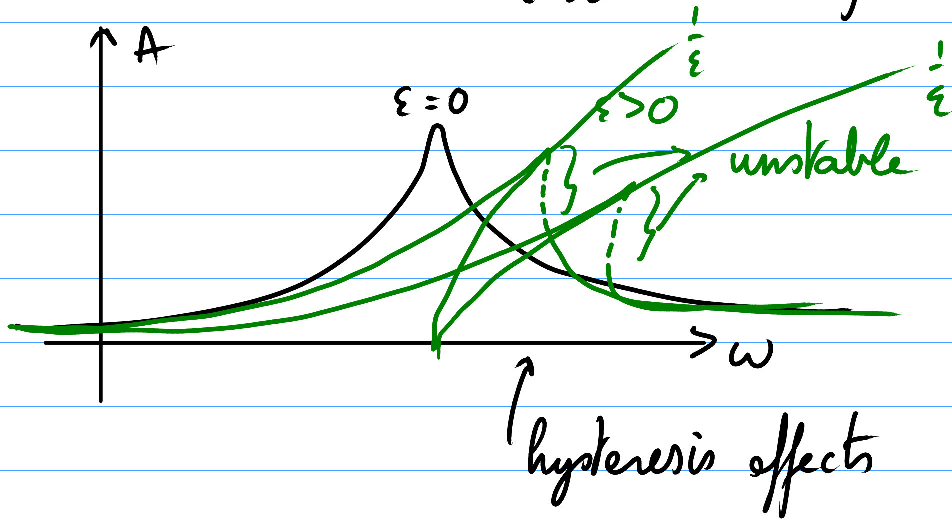
$$\Rightarrow \quad \left(\left[(\omega_0^2-\omega^2) + \tfrac{3}{4}\varepsilon A^2\right]^2 + 4\delta^2\omega^2\omega_0^2\right)A^2 = f^2$$

3rd order equation in A^2 → 1 or 3 real roots

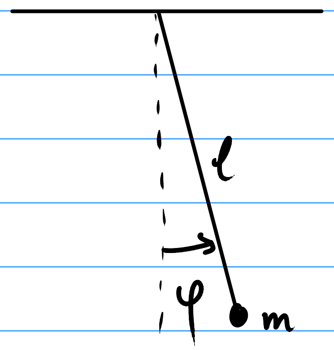
If $f=0$ → $A^2 = A_0^2 = \dfrac{4}{3\varepsilon}(\omega^2-\omega_0^2)$

Small $f \neq 0 \rightarrow A^2 = A_0^2 \pm \tilde{A}$

with $\tilde{A} = f \left[\frac{4}{3\varepsilon} (\omega^2 - \omega_0^2) \right]^{-1/2}$



* Connection of Duffing oscillator to pendulum :



pendulum : $\ddot{\varphi} + \frac{g}{\ell} \sin\varphi = 0$

$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0$ ⤸ $\omega_0^2 = \frac{g}{\ell}$

damped pendulum : $F_f = -bv = -b\ell\dot{\varphi}$

$\ddot{\varphi} + \frac{b}{m}\dot{\varphi} + \omega_0^2 \sin\varphi = 0$

$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \sin\varphi = 0$ ⤸ $\beta = \frac{b}{2m}$

driven damped pendulum :

$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \sin\varphi = \frac{F}{m\ell}\cos\omega t$

$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \sin\varphi = \gamma\omega_0^2 \cos\omega t$ ⤸ $\gamma = \frac{F}{mg}$

γ : dimensionless, comparison of magnitude of driving force and weight

$$\begin{cases} \gamma < 1 : \text{ small driving force} \rightarrow \varphi \text{ small} \\ \gamma > 1 : \text{ large driving force} \rightarrow \varphi \text{ large} \end{cases}$$

Expansion in φ : $\sin\varphi \simeq \varphi - \frac{1}{6}\varphi^3$

$$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\varphi - \frac{1}{6}\omega_0^2\varphi^3 = \gamma\omega_0^2\cos\omega t$$

* Requirements for chaos:

Chaos in Hamiltonian systems of n dimensions:

$$H(q_1, \dots, q_n, p_1, \dots, p_n, t)$$

$q_1, \dots, q_n, p_1, \dots, p_n$ treated as independent coordinates $\xi_i, i \leq N$

note: if no cyclic coordinates $\rightarrow N = 2n$

if k cyclic coordinates $\rightarrow N = 2n - k < 2n$

$$\Rightarrow \begin{cases} \dot{\xi}_1 = f_1(\xi_1, \dots, \xi_N) \\ \vdots \\ \dot{\xi}_N = f_N(\xi_1, \dots, \xi_N) \end{cases}$$

$\rightarrow$ system of N first-order differential equations

If system is dissipative $\left(\frac{\partial H}{\partial t} \neq 0\right)$:

chaos if system is non-linear and $N \geq 3$

If system is conservative $\left(\frac{\partial H}{\partial t} = 0\right)$:

chaos if system is non-linear and $N \geq 4$

Simple pendulum: $\begin{cases} \dot{\varphi} = \frac{p_\varphi}{m\ell^2} = \bar{p}_\varphi \\ \dot{\bar{p}}_\varphi = -\omega_0^2 \sin\varphi \end{cases}$ $\Rightarrow N = 2$

$\rightarrow$ no chaos

Damped pendulum: $\begin{cases} \dot{\varphi} = \bar{p}_\varphi \\ \dot{\bar{p}}_\varphi = -2\beta\,\bar{p}_\varphi - \omega_0^2 \sin\varphi \end{cases}$ $\Rightarrow N = 2$

$\rightarrow$ no chaos

Driven damped pendulum :

$$\begin{cases} \dot{\varphi} = \bar{p}_\varphi \\ \dot{\bar{p}}_\varphi = -2\beta \bar{p}_\varphi - \omega_0^2 \sin\varphi + \gamma \omega_0^2 \cos\psi \\ \dot{\psi} = \bar{p}_\psi = \omega \end{cases} \Rightarrow N = 3 \rightarrow \text{chaos!}$$

↳ explicit energy added/removed

Double pendulum (conservative, no driving)

$$(\varphi_1, p_{\varphi_1}, \varphi_2, p_{\varphi_2}) \Rightarrow N = 4 \rightarrow \text{chaos}$$

* Driven damped pendulum :

$$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\left(\varphi - \frac{1}{6}\varphi^3\right) = \gamma\,\omega_0^2 \cos\omega t$$

↳ $\varphi(t) \simeq A\cos(\omega t - \delta)$ for small non-linear term
small values of φ

↳ $\varphi(t) \simeq A\cos(\omega t - \delta) + B\cos 3(\omega t - \delta)$
for larger values of φ

↳ $\varphi(t)$ = sum of all harmonics $n\omega$

→ demos of period-doubling in driven damped pendulum

1) simple pendulum with $\varphi(0) \simeq \pi$
2) damped pendulum
3) driven damped pendulum with $\omega = 1.5\,\omega_0$
$\beta = \frac{\omega_0}{4}$

$\gamma = 1.060$ → period 1 in units of $\tau = \frac{2\pi}{\omega}$

4) $\gamma = 1.073$ → period 2

5) $\gamma = 1.077$ → period 3

↓

period-doubling to period 4, 6, 8, ...

6) $\gamma = 1.5$ → chaotic motion

Bifurcation points: γ_n = threshold where period changes from 2^{n-1} to 2^n

$\hookrightarrow$ $\gamma_1 = 1.0663$: $1 \to 2$
$\gamma_2 = 1.0793$: $2 \to 4$
$\gamma_3 = 1.0821$: $4 \to 8$
$\gamma_4 = 1.0827$: $8 \to 16$
$\vdots$

Universality: many different systems have similar sets of bifurcation points

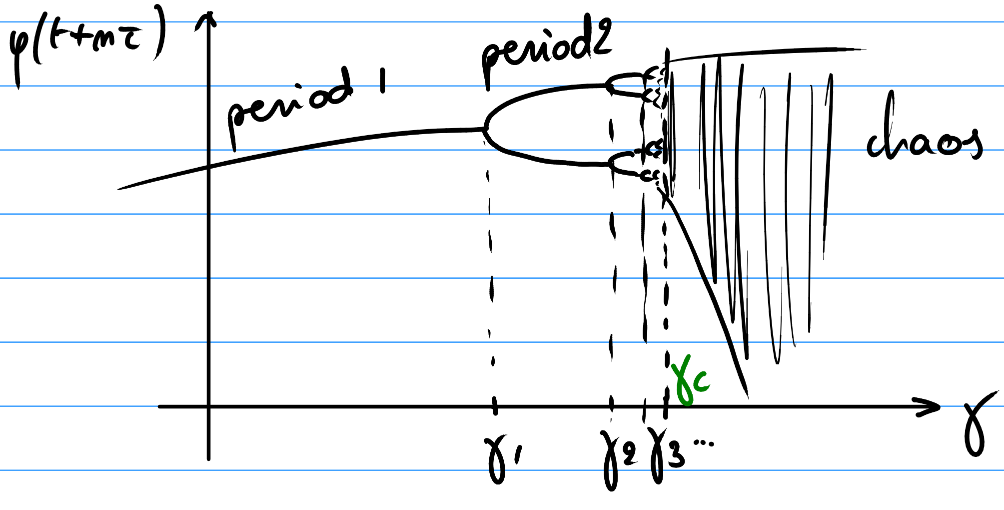
$$\gamma_{n+1} - \gamma_n \simeq \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$$

with δ = Feigenbaum number = 4.6692

As $n \to \infty$: $\gamma_n \to \gamma_c = 1.0829$: chaos

Bifurcation diagram:

Plot $\varphi(t), \varphi(t+\tau), \varphi(t+2\tau), \dots$ for steady-state

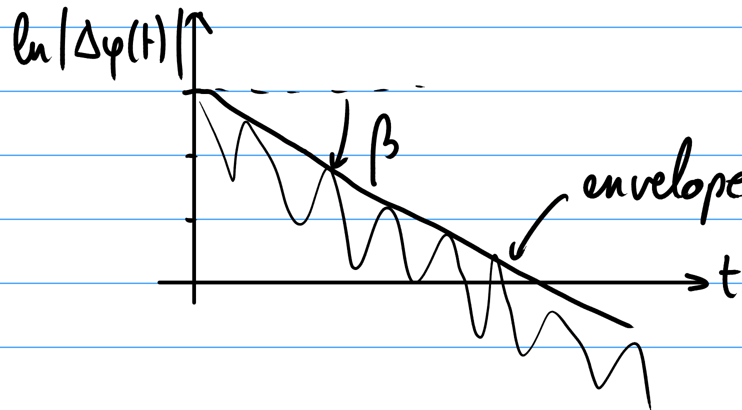


* Lyapunov exponent :

Sensitivity to initial conditions : $\Delta\varphi(t) = \varphi_2(t) - \varphi_1(t)$

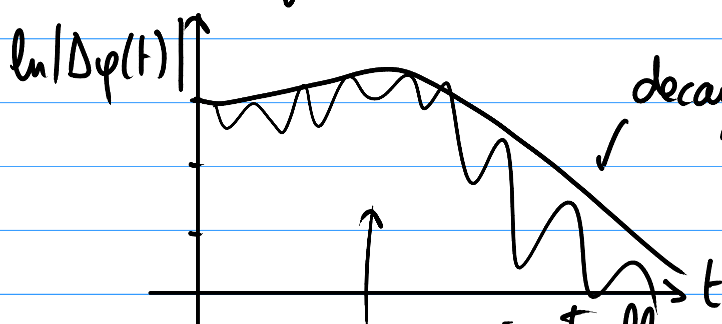
* For small oscillations (linear) : $\Delta\varphi(t) = C \underbrace{e^{-\beta t}}_{\text{exponential}} \cos(\omega t - \delta)$

$$\ln|\Delta\varphi(t)| = \ln C - \underbrace{\beta t}_{\text{decreases with time}} + \ln|\cos(\omega t - \delta)|$$



envelope with variations due to $\ln|\cos(\omega t - \delta)|$

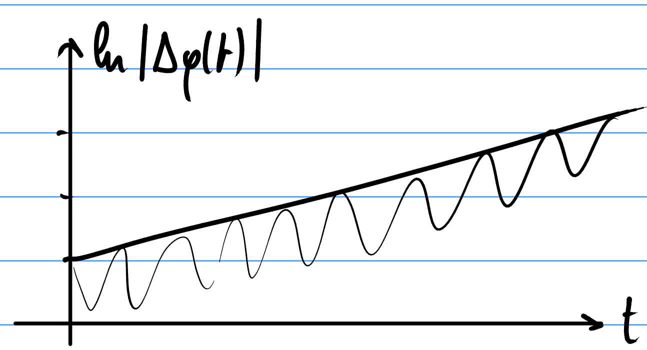
* For larger oscillations, larger γ ($\gamma > \gamma_1$ but $\gamma < \gamma_c$)



decays eventually

can increase initially

* For $\gamma > \gamma_c$:

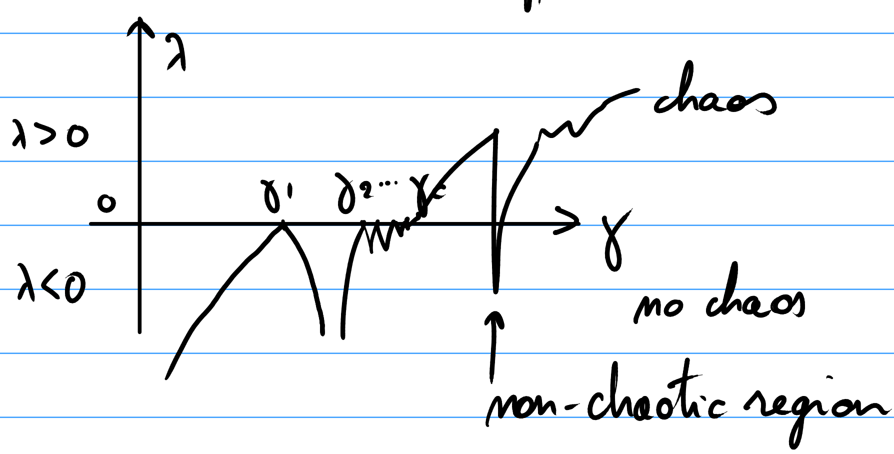


⇒ Lyapunov exponent λ

if $|\Delta\varphi(t)| \sim k\, e^{\lambda t}$ for some $k>0$

$$\lim_{t\to\infty}\left(\ln|\Delta\varphi(t)| \right) = \ln k + \lambda t$$

$$\begin{cases} \lambda<0 \rightarrow \text{differences decay away} \\ \lambda>0 \rightarrow \text{differences increase} \rightarrow \text{chaos} \end{cases}$$

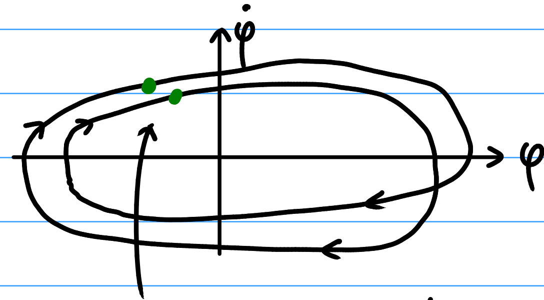


* Poincaré sections

$$\begin{cases} \dot\varphi = \bar p_\varphi \\ \dot{\bar p}_\varphi = -2\beta\, \bar p_\varphi - \omega_0^2 \sin\varphi + \gamma\, \omega_0^2 \cos\psi \\ \dot\psi = \bar p_\psi = \omega \end{cases}$$

⇒ section for $\psi = 0$ ⇒ fixed phase of driving term

Example : phase-space trajectory

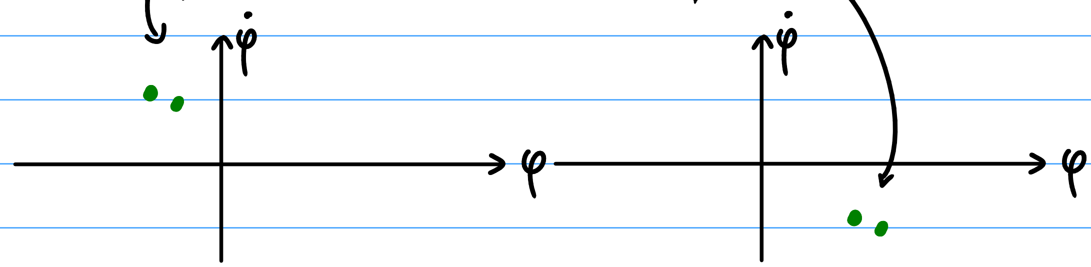


period 2 cycle

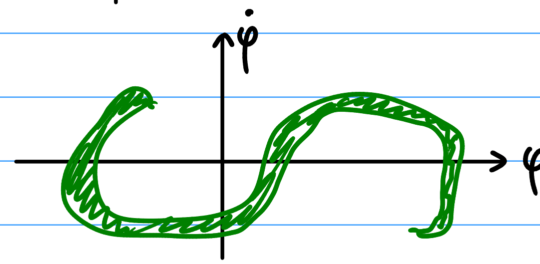
$\psi = 0 \rightarrow$ 2 points

$\psi = \pi$

Poincaré section



Chaotic motion :



$\psi = 0$

animation for varying ψ