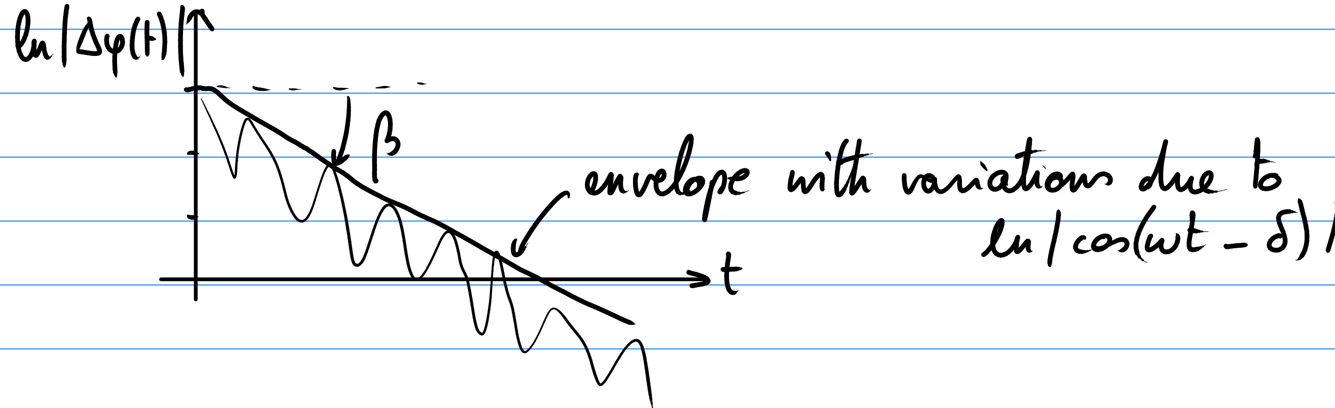
Classical Mechanics (Phys 601) - December 1, 2011

* Lyapunov exponent:

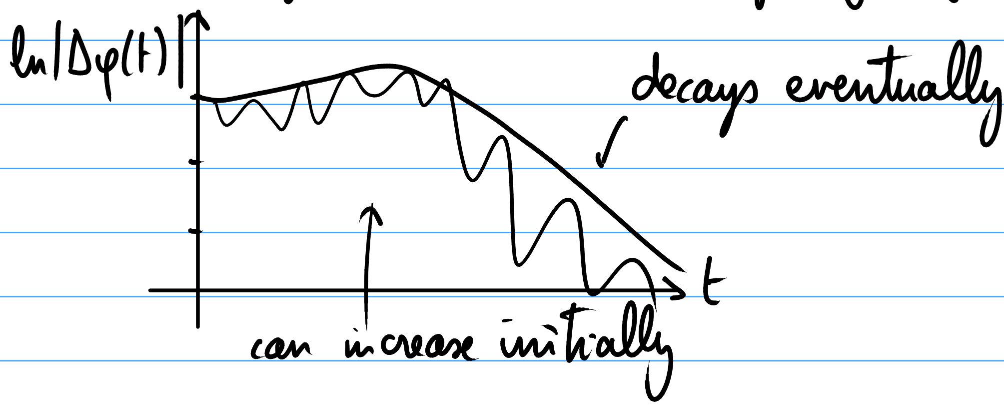
Sensitivity to initial conditions: $\Delta\varphi(t) = \varphi_2(t) - \varphi_1(t)$

* For small oscillations (linear): $\Delta\varphi(t) = C\underbrace{e^{-\beta t}}_{\text{exponential}}\cos(\omega t - \delta)$

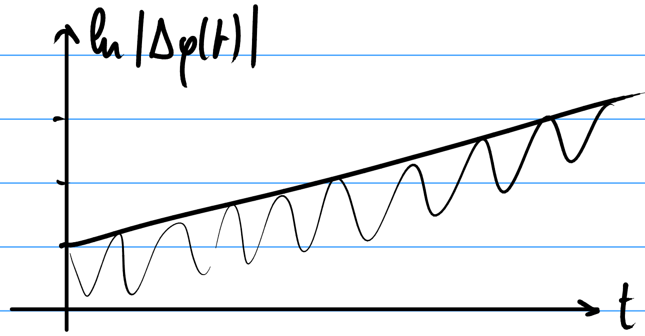
$$\ln|\Delta\varphi(t)| = \ln C \underbrace{- \beta t}_{\text{decreases with time}} + \ln|\cos(\omega t - \delta)|$$



* For larger oscillations, larger γ ($\gamma > \gamma_1$ but $\gamma < \gamma_c$)



* For $\gamma > \gamma_c$:

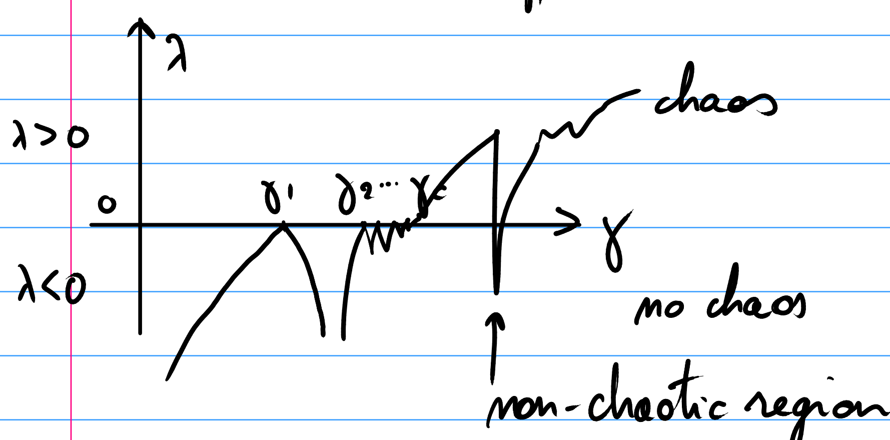


$\Rightarrow$ Lyapunov exponent λ

if $|\Delta\varphi(t)| \sim k\, e^{\lambda t}$ for some $k > 0$

$$\lim_{t \to \infty} \Big(\ln|\Delta\varphi(t)| \Big) = \ln k + \lambda t$$

$$\begin{cases} \lambda < 0 \rightarrow \text{differences decay away} \\ \lambda > 0 \rightarrow \text{differences increase} \rightarrow \text{chaos} \end{cases}$$



* Review:

d'Alembert's principle:
"no work by forces of constraint under virtual displacement"
(at fixed time)

$$\sum_i R_i \, dx_i = 0$$

$$\downarrow$$

$$\sum_i \left(F_i^{(e)} - \dot{p}_i \right) = 0$$

Hamilton's principle:
" actual dynamical trajectory makes the action S stationary"

$$\delta S = 0$$

with $S = \int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$

and $\delta q(t_1) = \delta q(t_2) = 0$

$\searrow$ **Euler-Lagrange equations** $\swarrow$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i$$

with generalized force Q_i

For conservative systems:

$$Q_i = -\frac{\partial}{\partial q_i} V(q_1, \dots, q_n, t)$$

and $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Dissipation function: $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i - \frac{\partial \mathcal{F}}{\partial \dot{q}_i}$$

with $\mathcal{F} = \frac{1}{2}\sum_i k_i \dot{q}_i^2$

Constraints: $f_j(q_1, \dots, q_n, t) = c_j \quad , j = 1, \dots, k$

Introduce Lagrange multipliers $\lambda_j \, , j = 1, \dots, k$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} - \dfrac{\partial L}{\partial q_i} = \displaystyle\sum_{j=1}^{k} \lambda_j \dfrac{\partial f_j}{\partial q_i} & : n \text{ equations} \\ f_j(q_1, \dots, q_n, t) = c_j & : k \text{ equations} \end{cases}$$

in $n+k$ independent unknowns $q_1, \dots, q_n, \lambda_1, \dots, \lambda_k$

$\rightarrow$ generalized reaction force $Q^r_i = \sum_j \lambda_j \dfrac{\partial f_j}{\partial q_i}$

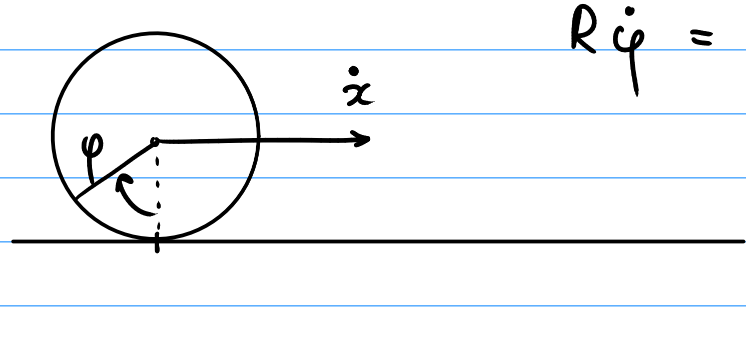
Note: non-holonomic constraints of non-integrable form

$$\sum_{i=1}^{n} \alpha_{ij} \, dq_i + \alpha_{0j} \, dt = 0$$

$$\hookrightarrow \begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} - \dfrac{\partial L}{\partial q_i} = \displaystyle\sum_{j=1}^{k} \lambda_j \alpha_{ij} \, , & i = 1, \dots, n \\ \displaystyle\sum_{i=1}^{n} \alpha_{ij} \dot{q}_i + \alpha_{0j} = 0 \, , & j = 1, \dots, k \end{cases}$$

Example: rolling without slipping

$$R\dot{\varphi} = \dot{x}$$



Continuous systems :

index i $\longrightarrow$ coordinate x

$q_i(t) \longrightarrow \varphi(x,t)$

$L = \sum_i L_i \longrightarrow L = \int dx\, \mathcal{L}(\varphi, \partial_\mu \varphi, x, t)$

$$\hookrightarrow \quad \sum_\mu^D \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \mu} = 0$$

Noether's theorem :

$q_i(t)$ is a solution $\rightarrow q_i(s,t)$ with $q_i(0,t) = q_i(t)$

$$C_s = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s} \right)_{s=0}$$ is constant of motion

$$\left(\frac{\partial L}{\partial s} \right)_{s=0} = \frac{d}{dt} C_s = 0$$

Pair of canonical variables:

generalized coordinate $q_i \rightarrow$ generalized momentum p_i

with definition $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ becomes $\dot{p}_i = \frac{\partial L}{\partial q_i}$

Define Hamiltonian $H = \sum_i p_i \dot{q}_i - L$

$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \Rightarrow$ Hamiltonian is conserved

If only time-independent potentials and constraints:
$H = E = T + V$ = total energy

Lagrangian dynamics:

q_i, $\dot{q}_i$ as independent variables

$L(q_i, \dot{q}_i, t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Hamiltonian dynamics:

q_i and p_i as independent variables

$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

Canonical transformations :

$$(p_i, q_i) \rightsquigarrow (P_i, Q_i) \text{ with } \begin{cases} p_i = p_i(P_1, \dots, P_n, Q_1, \dots, Q_n, t) \\ q_i = q_i(P_1, \dots, P_n, Q_1, \dots, Q_n, t) \end{cases}$$

with $F(q, Q, t)$ the generating function :

$$\begin{cases} p_i = \frac{\partial}{\partial q_i} F(q, Q, t) \\ P_i = -\frac{\partial}{\partial Q_i} F(q, Q, t) \end{cases}$$

$$\Rightarrow \tilde{H}(P, Q, t) = H(p, q, t) + \frac{\partial}{\partial t} F(q, Q, t)$$

Alternative generating functions by Legendre transformation :

$$S(q, P, t) = \sum_i P_i Q_i + F(q, Q, t)$$

$$\begin{cases} p_i = \frac{\partial}{\partial q_i} S(q, P, t) \\ Q_i = \frac{\partial}{\partial P_i} S(q, P, t) \end{cases}$$

$$\Rightarrow \tilde{H}(P, Q, t) = H(p, q, t) + \frac{\partial}{\partial t} S(q, P, t)$$

Hamilton-Jacobi theory :

Find $S(q, P, t)$ such that $\tilde{H}(P, Q, t) \equiv 0 \Rightarrow \begin{cases} P_i = \alpha_i \\ \text{constant} \\ Q_i = \beta_i \end{cases}$

$$\Rightarrow H\left(\frac{\partial S}{\partial q}, q, t\right) + \frac{\partial S}{\partial t} = 0$$

Trajectory is now $q = q(P,Q,t) = q(\alpha,\beta,t)$

$$\Rightarrow S(q,P,t) = S(q(\alpha,\beta,t),\alpha,t) = S(t)$$

and $\frac{dS}{dt} = L \Rightarrow$ S is the action along the trajectory

Conservative, seperable systems : $\frac{dH}{dt} = 0 \Rightarrow H = E$

$$S(q,\alpha,t) = W(q,\alpha) - E(\alpha)\,t$$

$$\Rightarrow H\left(\frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_m}, q_1, \dots, q_m\right) = E(\alpha)$$

$$W(q,\alpha,t) = W_1(q_1,\alpha) + \dots + W_m(q_m,\alpha)$$

$$\Rightarrow H\left(\frac{\partial W_1}{\partial q_1}, \dots, \frac{\partial W_n}{\partial q_n}, q_1, \dots, q_m\right) = E(\alpha)$$

Action angle variables for periodic systems $\nearrow$ libration $\searrow$ rotation

action $J_i = \oint_{\text{cycle}} p_i \, dq_i = \oint \frac{\partial W_i}{\partial q_i} dq_i = J_i(\alpha)$

$$\Rightarrow S(q,P,t) = S(q,\alpha,t) = S(q,\alpha(J),t) = \bar{S}(q,J,t)$$

$$\begin{cases} p_i = \dfrac{\partial \bar{S}}{\partial q_i} \\ \bar{Q}_i = \dfrac{\partial \bar{S}}{\partial J_i} \end{cases} \quad \text{and} \quad \begin{cases} \bar{P}_i = \text{constant} = J_i \\ \bar{Q}_i = \text{constant} = \bar{\beta}_i \end{cases}$$

$$\bar{\beta}_i = \frac{\partial \bar{S}}{\partial J_i} = \frac{\partial \bar{W}}{\partial J_i} - \frac{\partial H}{\partial J_i} t$$

angle $\varphi_i = \nu_i t + \bar{\beta}_i$ with frequency $\nu_i = \frac{\partial H}{\partial J_i}(J)$

(J, φ) are a pair of canonical variables

with $H(J, \varphi) = E(J)$

Poisson brackets:

$$[F, G] = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)$$

$$\frac{dF}{dt} = -[H, F] + \frac{\partial F}{\partial t}$$

$$\hookrightarrow \begin{cases} \dot{q} = -[H, q] = \frac{\partial H}{\partial p} \\ \dot{p} = -[H, p] = -\frac{\partial H}{\partial q} \end{cases}$$

$$\begin{cases} [p_i, p_j] = 0 = [q_i, q_j] \\ [p_i, q_j] = -\delta_{ij} \end{cases}$$

If transformation preserves Poisson brackets

$\longrightarrow$ canonical

Small Oscillations

At equilibrium generalized force Q_i disappears

$$Q_i = \left(-\frac{\partial V}{\partial q_i}\right)_{q^0} = 0$$

Small displacement η around q^0 : $q = q^0 + \eta$

* $$T = \sum_i \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} \sum_k \sum_\ell m_{k\ell} \dot{\eta}_k \dot{\eta}_\ell$$

with $m_{k\ell} = \sum_i m_i \frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_\ell}$ = symmetric

* $$V = V(q^0) + \sum_i \eta_i \left(\frac{\partial V}{\partial q_i}\right)_{q^0} + \frac{1}{2} \sum_k \sum_\ell v_{k\ell} \eta_k \eta_\ell + \mathcal{O}(\eta^3)$$

with $v_{k\ell} = \left(\frac{\partial^2 V}{\partial q_k \partial q_\ell}\right)_{q^0}$ = symmetric

$$\Rightarrow \quad L = T - V = \frac{1}{2} \sum_k \sum_\ell \left(m_{k\ell} \dot{\eta}_k \dot{\eta}_\ell - v_{k\ell} \eta_k \eta_\ell \right)$$

and equations of motion $\sum_\ell \left(m_{k\ell} \ddot{\eta}_\ell - v_{k\ell} \eta_\ell \right) = 0$

$$\Rightarrow \quad M \ddot{\eta} = - V \eta$$

Assumption $\eta = \mathrm{Re}\left(z e^{i\omega t}\right)$

$$\Rightarrow \quad \det\left(- M\omega^2 + V \right) = 0$$

Because M and V symmetric :

$$\exists\, U,\ U^T U = \mathbb{1} = U U^T : \ U^T M U = \Omega = \text{diagonal}$$

$$\Rightarrow \begin{cases} \Omega = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix} & \text{with } z_i^T M z_j = \delta_{ij} \\ U = [z_1 \ \dots \ z_n] & \text{(orthonormality)} \end{cases}$$

possibly after Gram-Schmidt

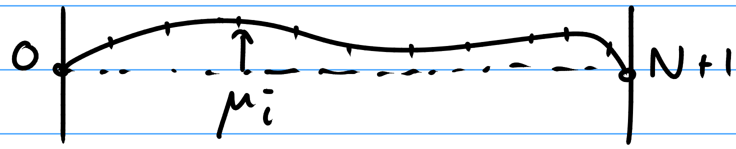
Normal coordinates $\rightarrow$ decouple dynamics

$$\eta = U\xi \quad \Rightarrow \quad \xi = U^T M \eta$$

$$\Rightarrow L = \frac{1}{2}\dot{\xi}^T U^T M U \dot{\xi} - \frac{1}{2}\xi^T U^T V U \xi$$

$$= \frac{1}{2}\dot{\xi}^T \dot{\xi} - \frac{1}{2}\xi^T \Omega \xi$$

Many degrees of freedom :



$$L = \frac{1}{2} m \sum_i \dot{\mu}_i^2 - \frac{1}{2} k \sum_i (\mu_{i+1} - \mu_i)^2$$

$\downarrow$

eigenvalue problem of order N

$$\omega_n^2 = 4 \frac{k}{m} \sin^2 \frac{n\pi}{2(N+1)}$$

Dispersion relation :

Assume solution $\mu(x_i, t) = A e^{i(kx_i - \omega t)}$

$\hookrightarrow$ in Euler - Lagrange equation

$\hookrightarrow$ $\omega^2 = 2 \frac{k}{m} \sin^2 \frac{ka}{2}$ (a = lattice spacing)

Boundary conditions :

* periodic : $\mu(x_i) = \mu(x_{i+N}) = \mu(x_i + Na)$

$\Rightarrow e^{ikNa} = 1 \Rightarrow k_n = \frac{2\pi}{Na} n$

* fixed ends : $\mu(x_0) = \mu(x_{N+1}) = 0$

$\Rightarrow \sin k(N+1)a = 0 \Rightarrow k_n = \frac{n\pi}{a(N+1)}$

Continuous systems:

$$\mathcal{L}(\varphi, \partial_\mu \varphi, x, t)$$

$\hookrightarrow \quad \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \dfrac{\partial \mathcal{L}}{\partial \varphi} = 0$

Example: Klein-Gordon $\mathcal{L} = c^2 \partial_\mu \varphi \, \partial^\mu \varphi^* - m_0^2 c^2 \varphi \varphi^*$

Noether currents:

$$j^\mu{}_s = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \varphi}{\partial s} \right)_{s=0}$$

Rigid Body Dynamics :

Rotations: $U(\theta) = e^{\theta J}$ with J = generator of the rotation

$$\Rightarrow U(\varepsilon) = 1 + \varepsilon J$$

$$\left(\frac{d\bar{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{r}}{dt}\right)_{\text{body}} + \underset{\uparrow}{\bar{\omega}} \times \bar{r}$$

instantaneous angular velocity

$$T = \frac{1}{2}\omega^T I \omega \quad \text{with} \quad I_{ij} = \sum_k^N m_k \left(r_k^2 \delta_{ij} - x_{ki} x_{kj}\right)$$

$$\hookrightarrow \bar{L} = I\bar{\omega} \qquad I_{ij} = \int d^3r\, \rho(\bar{r}) \left(r^2 \delta_{ij} - x_i x_j\right)$$

Parallel axis theorem:

$$I'_{ij} = I_{ij} + M\left(\delta_{ij}\bar{a}^2 - a_i a_j\right)$$ for displacement $\bar{a}$ between origins

Principal axes :

$$T = \frac{1}{2}\omega^T I \omega = \frac{1}{2}\xi^T \underbrace{U^T I U}_{\text{diagonal}} \xi$$

$$U = [\hat{e}_1 \;\; \hat{e}_2 \;\; \hat{e}_3] \,, \quad U^T I U = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix}$$

Euler equations :

$$\bar{\Gamma}^{(e)} = \frac{d}{dt}\bar{L} \Rightarrow \bar{\Gamma}^{(e)} = I\dot{\bar{\omega}} + \bar{\omega} \times (I\bar{\omega})$$

$$\begin{cases} \Gamma_1^{(e)} = I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) \\ \Gamma_2^{(e)} = I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) \\ \Gamma_3^{(e)} = I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) \end{cases}$$

Rotation around principal axis :

↳ $\bar{\omega} = \omega_i \hat{e}_i \Rightarrow \bar{\omega}$ will remain constant

but only two principal axes are stable equilibrium

$I_1 < I_2 < I_3$

$\uparrow$ unstable

Symmetric top : $I_1 = I_2 \neq I_3$

↳ precession of $\bar{\omega}$ around $\hat{e}_3$
at rate $\Omega = \omega_3 \frac{I_3 - I_1}{I_1}$

Euler angles $\alpha, \beta, \gamma \rightarrow$ Lagrangian