PHYS 107 - Week 07 - Wednesday

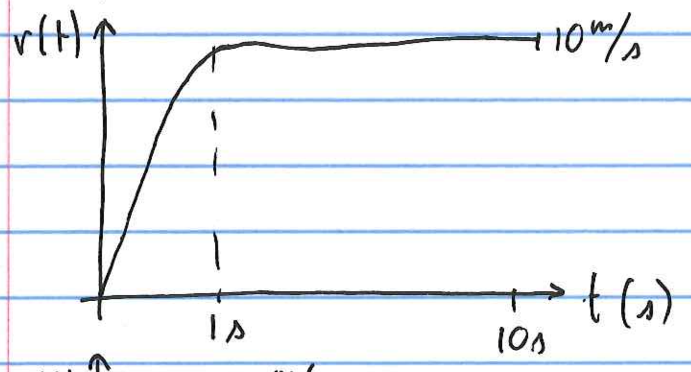
* Energy : $KE = \frac{1}{2}mv^2$ (kinetic), $PE = \frac{1}{2}kx^2$ (springs), $PE = mgh$ (gravity)

  Other energies : $W_{nc} = Fd\cos\theta$

* Power : rate at which energy is used
  ——— work is done

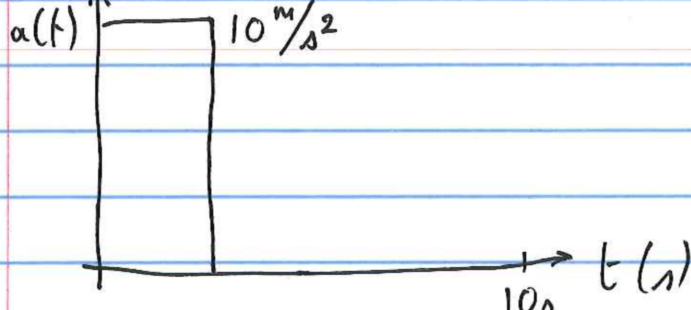
$$P = \frac{W}{\Delta t} , \quad 1 \text{ Watt} = 1W = 1\frac{J}{s}$$

* Sprinter Usain Bolt with $m = 60kg$



$KE = \frac{1}{2}mv^2 = 3000J$

$P = \frac{3000J}{1s} = 3000W$



typical power for strenuous excercise at duration
$\sim 200W$

* Momentum : $\vec{p} = m\vec{v}$ (vector quantity)

units $kg\frac{m}{s}$ or $N \cdot s$

kinematic quantity

why? useful in describing collisions

* Connection with force

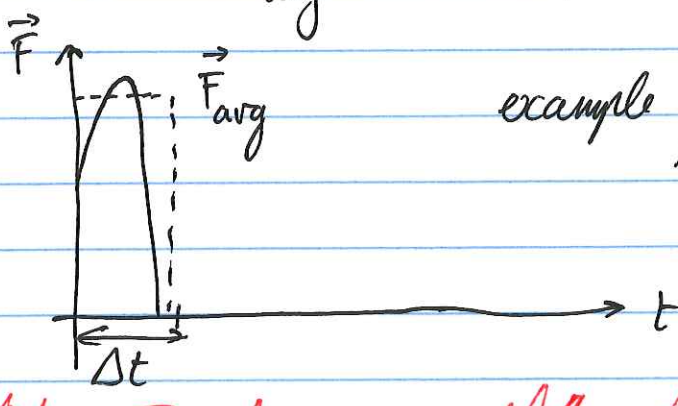
$$\vec{F} = m\vec{a} = m\frac{\vec{v}_f - \vec{v}_i}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t} = \frac{\Delta \vec{p}}{\Delta t}$$

Newton's 2nd law can then be written

$\vec{F} = \frac{\Delta \vec{p}}{\Delta t}$ or force is change of momentum over unit of time

* Impulse :

$\vec{I} = \vec{F}_{avg} \cdot \Delta t$ (in units $N \cdot s$) $= \underline{\Delta \vec{p}}$



example for model rockets (Stine & Stine, p 77)

$\vec{I} = \Delta \vec{p}$

Video Impulse : model rocket launch of perka-potty

* Example: height of 1200 feet → 400 m
mass of porta-potty = 100 kg (200 lbs)
(empty ...)

$$\Delta \vec{p}_h = m\vec{v}_f - m\vec{v}_i = m\vec{v}_f \rightarrow \text{need } \vec{v}_f$$

$$h = \frac{v^2}{2g} \rightarrow v = \sqrt{2gh} = \sqrt{2(10)(400)}\ \text{m/s} = 90\ \text{m/s}$$

$$\vec{I} = \Delta\vec{p} = (100\ \text{kg})(90\ \text{m/s}) = \underline{9000\ \text{N·s}}\ (\text{up})$$

↓

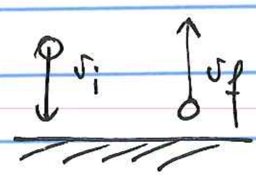
M-type motor
(typical $\Delta\vec{p} = \vec{I} =$ few N·s)

Q 1-a, b Δp

* Example: ball bouncing vertically of the floor, $m = 0.1$ kg

$|\vec{v}_i| = |\vec{v}_f| = 5.5$ m/s

$\hookrightarrow \begin{cases} v_i = -5.5\ \text{m/s} \\ v_f = +5.5\ \text{m/s} \end{cases}$



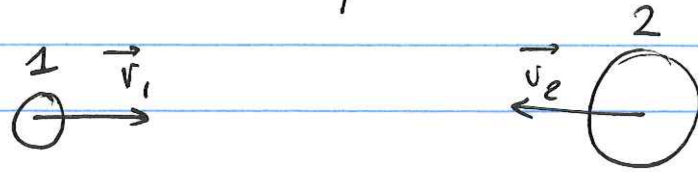
1) What is the impulse?
2) What is the force if the impact lasts for ~~less than~~ 0.02 s?

$$\vec{I} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = (+0.55 - (-0.55))\ \text{N·s} = \underline{1.1\ \text{N·s}}$$

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{1.1\ \text{N·s}}{0.02\ \text{s}} = \underline{550\ \text{N}}$$

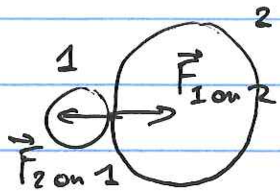
Q 2-a Stopping bowling ball vs. ping pong ball

* Conservation of momentum :



2 colliding masses

At the moment when they collide :



$\vec{F}_{1 \text{ on } 2} = - \vec{F}_{2 \text{ on } 1}$ (Newton's 3rd law)

They collide for a time Δt

↪ $\vec{I}_1 = \vec{F}_{1 \text{ on } 2} \Delta t = - \vec{F}_{2 \text{ on } 1} \Delta t = - \vec{I}_2$

$\Delta \vec{p}_1 = \vec{p}_{1,f} - \vec{p}_{1,i}$ $\qquad$ $\vec{p}_{2,f} - \vec{p}_{2,i} = \Delta \vec{p}_2$

↪ $\vec{p}_{1,f} - \vec{p}_{1,i} = -(\vec{p}_{2,f} - \vec{p}_{2,i})$

$\Rightarrow \vec{p}_{1,f} + \vec{p}_{2,f} = \vec{p}_{1,i} + \vec{p}_{2,i}$

$\vec{p}_1 + \vec{p}_2$ is <u>constant before and after collision</u>

Q Collision 4-a elastic

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On the performance of Usain Bolt in the 100 m sprint

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Abstract

Many university texts on mechanics consider the effect of air drag force, using the slowing down of a parachute as an example. Very few discuss what happens when the drag force is proportional to both u and u^2. In this paper we deal with a real problem to illustrate the effect of both terms on the speed of a runner: a theoretical model of the world-record 100 m sprint of Usain Bolt during the 2009 World Championships in Berlin is developed, assuming a drag force proportional to u and to u^2. The resulting equation of motion is solved and fitted to the experimental data obtained from the International Association of Athletics Federations, which recorded Bolt's position with a laser velocity guard device. It is worth noting that our model works only for short sprints.

(Some figures may appear in colour only in the online journal)

1. Introduction

In Zürich, Switzerland, 21 June 1960, the German Armin Harry astounded the sports world by equalling what was considered an insuperable physiological and psychological barrier for the 100 m sprint: the 10 s race. It was not until 20 June 1968, at Sacramento, USA, that Jim Hines ran 100 m in 9.9 s, breaking through this barrier. Many sprinters have run this distance in under 10 s subsequently, but 31 years were needed to lower Harry's record by 0.14 s (Carl Lewis, 25 August 1991, at Tokyo, Japan). The current world record of 9.58 s was established by Usain Bolt (who also held the 200 m world record of 19.19 s up to 2012) in the 12th International Association of Athletics Federations (IAAF) World Championships in Athletics (WCA) at Berlin, Germany in 2009.

The performance of Usain Bolt in the 100 m sprint is of physical interest because he can achieve speeds and accelerations that no other runner has been able to. Several mathematical models to fit the position and velocity (or both) of a sprinter have been proposed [1–6]. Recently, Helene *et al* [6] fitted Bolt's performance during both the summer Olympics in 2008 at Beijing and the world championships in 2009 at Berlin, using a simple exponential model for the time dependence of the speed of the runner.

2. Theoretical model

The important forces acting during the race are the horizontal force that Bolt exerts and a drag force that depends upon the horizontal velocity (speed). Other factors affecting the mechanics of his motion, such as humidity, altitude above sea level (36 m), oxygen intake and his turning his head to watch other runners, are not taken into account. Based on the fact that Bolt's 200 m time is almost twice that for 100 m, our main assumption is that in the 100 m sprint he is able to develop a constant horizontal force F_0 during the whole race. The drag force, $D(u)$, is a function of Bolt's horizontal speed with respect to the ground $u(t)$, with or without wind. This force causes a reduction of his acceleration, so his speed tends to a constant value (terminal speed). Thus, the equation of motion is

$$m\dot{u} = F_0 - D(u). \tag{1}$$

This equation can readily be cast as a quadrature,

$$t - t_0 = m\int_{u_0}^{u} \frac{\mathrm{d}u'}{F_0 - D(u')}. \tag{2}$$

The integral above does not have an analytical solution for a general drag function; however, the drag force can be expanded in Taylor series,

$$D(u) \simeq D(0) + \left.\frac{\mathrm{d}D(u)}{\mathrm{d}u}\right|_0 u + \frac{1}{2}\left.\frac{\mathrm{d}^2D(u)}{\mathrm{d}u^2}\right|_0 u^2 + O(u^3). \tag{3}$$

The constant term of the expansion is zero, because the runner experiences no drag when at rest. The second and third terms must be retained. While the term proportional to the speed represents the basic effects of resistance, the term proportional to the square of the speed takes into account hydrodynamic drag, obviously present due to the highly non-uniform geometry of the runner. In general, for relatively small speeds it suffices to take only the first three terms of the expansion.

Renaming the u and u^2 coefficients as γ and σ, respectively, the equation of motion (1) takes the form

$$m\dot{u} = F_0 - \gamma u - \sigma u^2, \tag{4}$$

whose solution follows straightforwardly from equation (2),

$$u(t) = \frac{AB\left(1 - \mathrm{e}^{-kt}\right)}{A + B\,\mathrm{e}^{-kt}}, \tag{5}$$

where the coefficients are related by $\sigma = \mathrm{km}/(A + B)$, $F_0 = \mathrm{km}AB/(2A + 2B)$ and $\gamma = \mathrm{km}(A - B)/(A + B)$.

The position can be obtained by integrating equation (5),

$$x(t) = \frac{A}{k}\ln\left(\frac{A + B\,\mathrm{e}^{-kt}}{A + B}\right) + \frac{B}{k}\ln\left(\frac{A\,\mathrm{e}^{kt} + B}{A + B}\right), \tag{6}$$

while the acceleration can also be calculated by deriving equation (5),

$$a(t) = ABk(A + B)\frac{\mathrm{e}^{-kt}}{\left(A + B\,\mathrm{e}^{-kt}\right)^2}. \tag{7}$$

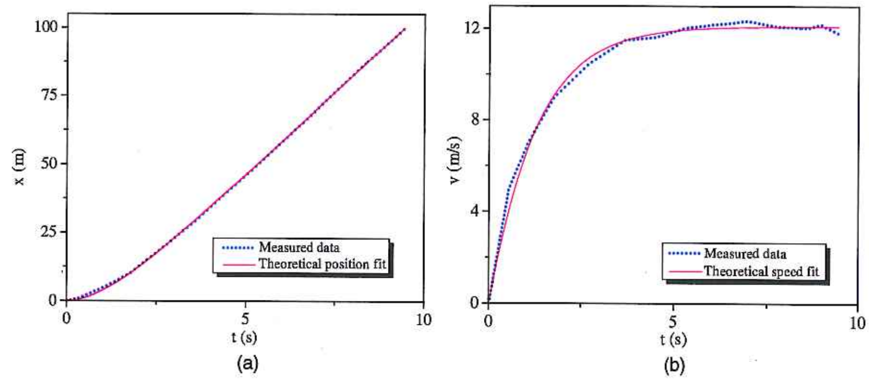

Figure 1. Position (a) and speed (b) of Bolt in the 100 m sprint at the 12th IAAF WCA. The dotted (blue) line corresponds to the experimental data while the solid (red) one corresponds to the theoretical fitting.

Table 1. Fitted values of the parameters A, B and k.

Parameter	Position fitting	Velocity fitting
A (m s^{-1})	110.0	110.0
B (m s^{-1})	12.2	12.1
k (1 s^{-1})	0.9	0.8

Table 2. Values of the physical parameters F_0, γ and σ.

Constant	Value
F_0 (N)	815.8
γ (kg s^{-1})	59.7
σ (kg m^{-1})	0.6

3. Experimental data fitting

The experimental data we used were from the 12th IAAF WCA, which were obtained from [7], and consist of Bolt's position and speed every 1/10 s. To corroborate the accuracy of the data obtained from [7], we reproduced the velocity versus position plot given in [8] with them, obtained by the IAAF using a laser velocity guard device. The parameters A, B and k were fitted using a least-squares analysis via Origin 8.1 (both position and speed data sets) considering a reaction time of 0.142 s [6]. In figures 1(a) and (b) we show such fittings, together with the experimental data.

The parameter values for both fittings are shown in table 1. We do not report errors, because the standard error of the fitting on each parameter lies between the second and the third significant digit, which is finer than the measurement error in the data.

The accuracy of the position and velocity fittings is $R_p^2 = 0.999$ and $R_v^2 = 0.993$, respectively, so we use the results of the parameters A, B and k from the position fitting henceforth. The computed values of the magnitude of the constant force, F_0, and the drag coefficients, γ and σ, are shown in table 2, taking Bolt's mass as 86 kg [9].

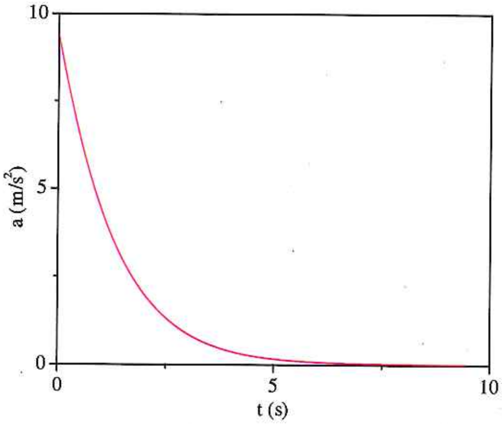

Figure 2. Theoretical acceleration of Bolt in the 100 m sprint at the 12th IAAF WCA.

We also show in figure 2 the plot of the magnitude of the acceleration we obtained; no fitting was made because there are no experimental data available.

4. Results

As for any mechanical system subject to drag, the runner experiments a terminal velocity u_T that is formally obtained when $\dot{u} = 0$ in the equation of motion (1); that is, by solving the equation

$$F_0 = D(u_T) \tag{8}$$

for u_T. Nevertheless, the solution of the equation for the terminal velocity can also be found when $t \to \infty$ in equation (5), and it turns out to be $u_T = B$. Therefore, in this model the runner acquires a terminal speed of $u_T = 12.2$ m s^{-1}, which is physically feasible (see figure 1(b)). According to the data obtained from [7] the average speed in the second half of the sprint (surprisingly equal to 99% of the maximum speed recorded [7]) is 12.15 m s^{-1}. Moreover, the initial acceleration of Bolt is $a(0) = 9.5$ m s^{-1}, which is of the order of the acceleration of gravity, g; this value for the initial acceleration is reasonable, considering that the acceleration a man must exert in order to be able to jump half of his own height should be slightly greater than g. Furthermore, the value of the constant force in table 2, $F_0 = 815.8$ N, is entirely consistent with the fact that one expects that the maximum constant (horizontal) force he could exert should be of the order of his weight, i.e. $w = 842.8$ N.

Now, $\sigma = 0.5\rho C_d A$ represents the hydrodynamic drag, where ρ is the density of air, C_d the drag coefficient of the runner and A his cross section area. The density of air at the time of the spring can be approximated as follows. Berlin has a mean altitude of 34 m above sea level, and an average mean temperature for the month of August[1] [10] of 18.8 °C. Bearing in mind that the race took place at night, we consider an average temperature between the average mean temperature and the mean daily minimum temperature for August in Berlin, which is 14.3 °C. Thus, the density of air is $\rho = 1.215$ kg m^{-3} and the drag coefficient of Bolt

[1] The sprint took place 16 August 2009.

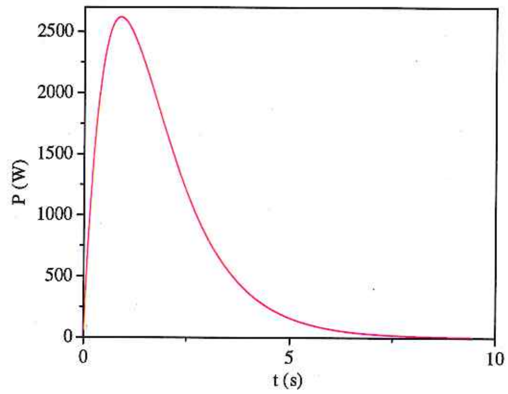

Figure 3. Theoretical power of Bolt in the 100 m sprint at the 12th IAAF WCA.

is $C_d = 2\sigma/\rho A = 1.2$, where the cross section area of Bolt[2] was estimated as $A = 0.8$ m^2. This value of C_d lies in the typical range for human beings reported in the literature (between 1.0 and 1.3) [11–13].

The instantaneous power that Bolt develops, considering the drag effect, is simply

$$P(t) = Fu = m\dot{u}u = mABk(A+B)\frac{(1-\mathrm{e}^{-kt})\,\mathrm{e}^{-kt}}{(A+B\,\mathrm{e}^{-kt})^3}. \tag{9}$$

In figure 3 we plot the power of the sprint for Bolt and the drag. It is remarkable that the maximum power of $P_{\max} = 2619.5$ W (3.5 HP) was at time $t_{P\max} = 0.89$ s, when the speed $u(t_{P_{\max}}) = 6.24$ m s^{-1} was only about half of the maximum speed. The fact that the maximum instantaneous power arises in such a short time indicates the prompt influence of the drag terms in the dynamics of the runner.

The effective work (considering the effect of the drag force) is then

$$W_{\mathrm{Eff}} = \int_0^{\tau} P(t)\,\mathrm{d}t = \int_0^{\tau} \frac{1}{2} m\,\mathrm{d}u^2 = \frac{1}{2} mu^2(\tau), \tag{10}$$

where τ is the running time (the official time of the sprint minus the reaction time of the runner). The effective work is the area under the curve of figure 3, and it is $W_{\mathrm{Eff}} = 6.36$ kJ. On the other hand, as Bolt is assumed to develop an essentially constant force, his mechanical work is simply $W_{\mathrm{B}} = F_0 d = 81.58$ kJ, where d is the length of the sprint (100 m). This means that from the total energy that Bolt develops only 7.79% is used to achieve the motion, while 92.21% is absorbed by the drag; that is, 75.22 kJ are dissipated by the drag, which is an incredible amount of lost energy.

5. Discussion

As mentioned in section 2, a central assumption in our model is that a 100 m sprinter (not only Bolt) is able to develop a constant force during a race (except in the initial few tenths of a second, where he pushes himself against the starting block). To delimit how good this

[2] To calculate such a cross section area, we used a similar procedure to that used in [9], where instead of a circle we estimated the area of the head with an ellipse. We averaged several scaled measures from Bolt pictures taken from [14].

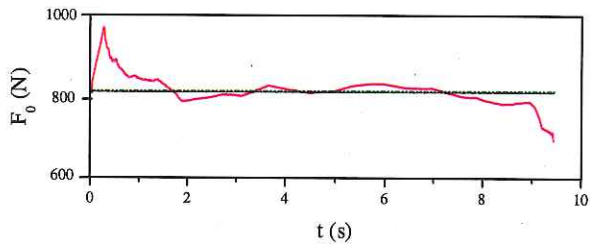

Figure 4. Force exerted by the runner during the race. The red line is calculated with the experimental data, the dash-dot-dot (green) line is the average force of 818.3 N, while the short-dash-dot (black) line is the value of the force F_0 obtained from the adjustment (815.7 N).

assumption is we use the experimental values of u, the calculated acceleration and the fitted values of the constants γ and σ to compute F_0. The result is shown in figure 4. It is interesting to note that the average value of the force obtained from this figure is 818.3 N, which is very close to the value obtained from the fitting of the data, 815.7 N. The high value of the force in the first part of the race is due do the acceleration he obtains when he pushes himself from the starting block.

At first glance, observing the values of the drag coefficients in table 2, one is impelled to argue that because $\sigma \ll \gamma$ the hydrodynamic drag could have been neglected. However, one can calculate the drag terms in the equation of motion at the terminal speed u_T, attaining $\gamma u_T = 725.59$ N and $\sigma {u_T}^2 = 90.18$ N. Thus, from the total drag $\gamma u_T + \sigma u_T^2$, 11.05% corresponds to turbulent drag, which turns to be an important contribution.

If we would like to make predictions considering different wind corrections, this can be done as follows. Once a runner acquires the wind speed (almost instantly), the second term in the right side (γu) of equation (1) behaves as if the sprinter were running in still air, because γ is proportional to the air viscosity, which is independent of air pressure. However, that is not the case for the third term in (1), (σu^2), which arises from the collisions per unit time of the air molecules against the sprinter and it is proportional to the speed of the runner with respect to the ground. In a simple model, the hydrodynamical drag force is $D_H = \sigma(\rho)(v + v_w)^2$, where v is the speed achieved by the runner without wind and v_w is the speed of the wind. The value of σ depends on the number of molecules that impact on the runner per unit time and should be different in still air conditions. Then, the equation of motion (1) can be rewritten as

$$m\dot{u} = m\dot{v} = F_0 - \gamma v - \sigma(v + v_w)^2, \tag{11}$$

and without wind as

$$m\dot{v} = F_0 - \gamma v - \sigma' v^2. \tag{12}$$

Subtracting (11) and (12), we obtain

$$\sigma(v^2 + 2vv_w + v_w^2) = \sigma' v^2, \tag{13}$$

so then

$$\sigma' = \sigma\left(1 + \frac{2v_w}{v} + \frac{v_w^2}{v^2}\right) \sim \sigma\left(1 + \frac{2v_w}{v}\right), \tag{14}$$

where the third term in the second expression has been neglected ($v_w \ll v$). To estimate the value of σ', we consider v as the terminal speed of Bolt, u_T. With these conditions, $\sigma' = 0.69$ with still air ($v_w = 0$ m s^{-1}) and $\sigma' = 0.49$ with a tailwind of $v_w = 2$ m s^{-1}. It should be clear that this calculation is only a crude way to estimate the differences of running time with

Table 3. Predictions of the running time for Bolt without tailwind, and with a tailwind of 2 m s^{-1}.

v_w (m s^{-1})	Estimated running time (s)
0	9.68
0.9	9.58
2	9.46

and without wind. The results, which are close to the values reported in the literature [15], are summarized in table 3.

Although this is a simple way to calculate a correction due to wind, it turns out to be a good proposal for it. A more realistic assumption would be to modify equation (14) to be

$$\sigma' = \sigma\left(1 + \frac{\alpha v_w}{u_T}\right), \tag{15}$$

with the parameter α lying between 1 and 2.

The results obtained, together with the facts indicated in this discussion, show the appropriateness and quality of the model developed in this paper. We look forward to the next IAAF WCA, which will be held in Moscow, Russia, 10–18 August 2013, to test our model with the experimental data obtained from such sprints, as well as waiting to see if the fastest man on earth is able to beat his own world record again.

Acknowledgment

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* Collision could be considered broadly:

an astronaut throws a hammer in outer space

$m_h = 2\,kg \qquad v_h = 4\,m/s$

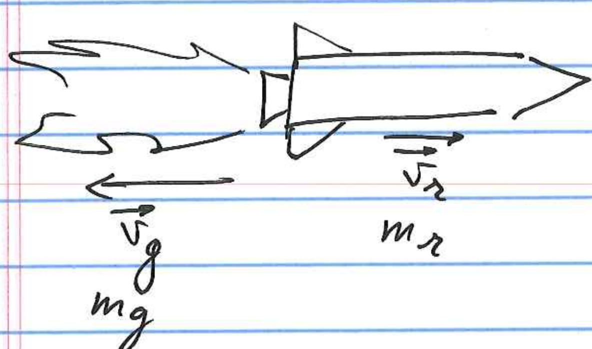
$m_a = 80\,kg \longrightarrow$ how fast does the astronaut recoil?

$$\vec{p}_{initial} = 0 \longrightarrow \vec{p}_{final} = 0 = m_h \vec{v}_h + m_a \vec{v}_a$$

$$\longrightarrow \vec{v}_a = -\frac{m_h}{m_a}\vec{v}_h = -\left(\frac{2}{80}\right) 4\,m/s$$

$$= -0.1\,m/s$$

Similar in rocket propulsion



$$\vec{v}_r = -\frac{m_g}{m_r}\vec{v}_g$$

but $\vec{v}_g$ is large, even though $m_g < m_r$

* Kinetic energy: $KE = \frac{1}{2} m v^2$

Is energy conserved when the astronaut throws the hammer? YES

Is mechanical energy conserved? NO
(KE)

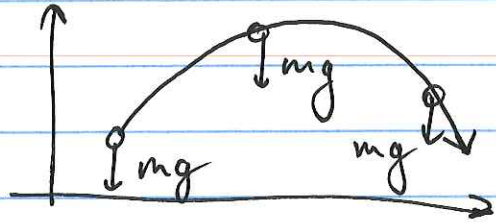
$KE_i = \frac{1}{2} m v_i^2 = 0$ $\quad KE_f = \frac{1}{2} m_h v_h^2 + \frac{1}{2} m_a v_a^2 > 0$

$OE_i + KE_i = OE_f + KE_f$

$OE_f < OE_i$ because astronaut has to spend energy to throw hammer

* Conservation of momentum in general situations (no collisions)

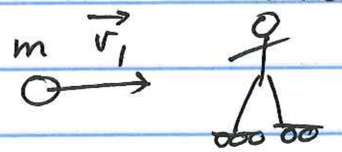
ball thrown, $\vec{F} = \Delta \vec{p}$



p_y is not conserved because $F_y \neq 0$

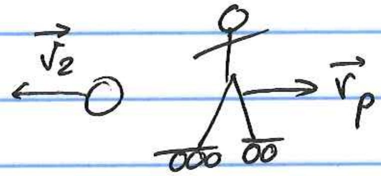
p_x is conserved / constant because $F_x = 0$

* Example: catching and throwing a ball on roller skates (no friction)



$m = 2kg$ $|\vec{v}_1| = 8\,m/s$

$M = 80kg$ $|\vec{v}_2| = 2\,m/s$



1) What is recoil speed v_p?
2) Is energy conserved?

$$\left.\begin{array}{l}\vec{p}_i = m\vec{v}_1 \\ \vec{p}_f = m\vec{v}_2 + M\vec{v}_p\end{array}\right\} \vec{p}_i = \vec{p}_f$$

$$\rightarrow \quad m\vec{v}_1 = m\vec{v}_2 + M\vec{v}_p$$

$$\vec{v}_p = \frac{m}{M}(\vec{v}_1 - \vec{v}_2) = \frac{2kg}{80kg}\left(+8\,m/s - (-2\,m/s)\right)$$

$$= +0.25\,m/s$$

$$KE_i = \frac{1}{2}mv_1^2 = \frac{1}{2}(2kg)(8\,m/s)^2 = 64J$$

$$KE_f = \frac{1}{2}mv_2^2 + \frac{1}{2}Mv_p^2 = \frac{1}{2}(2kg)(2\,m/s)^2 + \frac{1}{2}(80kg)(0.25\,m/s)^2$$

$$= 4J + 2.5J = 6.5J$$

$\rightarrow$ nearly 60J of energy was converted into other energy OE_f

* Classification of collisions : ALWAYS $\vec{p}_i = \vec{p}_f$

$KE_i = KE_f$ → elastic collision
$KE_i \neq KE_f$ → inelastic collision

if KE_f is as low as possible → perfectly inelastic collision