Phys 772: Week 2 Thursday

* Dirac spin $\frac{1}{2}$ fermion field:

$\psi_\alpha(x)$, $\alpha = 1 \dots 4$, is the 4-component spinor field

Again, in terms of annihilation, creation operators:

$$\varphi(x) \propto a(\bar{p}) + b^+(\bar{p})$$

annihilation of particle, creation of anti-particle

$\psi_\alpha(x) \propto$ annihilation of 2 particle spin states + creation of 2 anti-particle states

$$\mathcal{L}(\psi) = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

with $\bar{\psi} = \psi^+\gamma^0$, $\partial\!\!\!/ = \gamma^\mu\partial_\mu$; $\gamma^0, \dots \gamma^3, \gamma^5$: Dirac matrices

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} 1_{4\times4}$$

Any 4×4 matrix can be written as lin. comb. of $1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

Bilinear form: $\bar{\psi}\, m 1_{4\times4}\, \psi$

$\overline{(\bar{\psi} 1 \psi)} = \bar{\psi} 1 \psi$, but -1 for γ^5

From $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$

ELE $\downarrow$: $(i\partial\!\!\!/ - m)\psi = 0$ (Dirac equation)

sol$^{\underline{n}}$ $\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \Big[u(\bar{p},s)\, a(\bar{p},s)\, e^{-ip\cdot x}$

$+ v(\bar{p},s)\, b^+(\bar{p},s)\, e^{ip\cdot x} \Big]$

$a^+(\bar{p},s)$ creates particle
$b^+(\bar{p},s)$ creates anti-particle

$s =$ spin state , 1 or 2

$u(\bar{p},s)$ and $v(\bar{p},s)$ are Dirac spinors that are solutions of the Dirac equation's Fourier transform in momentum space :

$$\begin{cases} (p\!\!\!/ + m)\, u(\bar{p},s) = 0 \rightarrow \bar{u}(\bar{p},s)(p\!\!\!/ - m) = 0 \\ (-p\!\!\!/ + m)\, v(\bar{p},s) = 0 \rightarrow \bar{v}(\bar{p},s)(p\!\!\!/ + m) = 0 \end{cases}$$

Normalization: $\bar{u}(\bar{p},s)\, u(\bar{p},s') = 2m\,\delta_{ss'}$

$u^+(\bar{p},s)\, u(\bar{p},s') = 2E_p\,\delta_{ss'}$

$$\sum_s u(\bar{p},s)\,\bar{u}(\bar{p},s) = (p\!\!\!/ + m)$$

$$\begin{cases} u(\bar{p},s)\,\bar{u}(\bar{p},s) = (\not{p}+m)\left(\dfrac{1+\gamma^5\not{s}}{2}\right) \\ v(\bar{p},s)\,\bar{v}(\bar{p},s) = (\not{p}-m)\left(\dfrac{1+\gamma^5\not{s}}{2}\right) \end{cases}$$

with s^μ = spin vector

in rest frame $p = (m, \bar{0})$ and $s = (0, \hat{s})$
with $\hat{s}$ = unit vector in direction of spin

$\Big\downarrow$ boost $\bar{\beta}$ $\rightarrow s^2 = -1$, $p \cdot s = 0$

$\bar{p} = \gamma \bar{\beta} m$ and $s = (\gamma \bar{\beta} \cdot \hat{s}, \gamma \hat{s}_{\parallel} + \hat{s}_{\perp})$

with $\hat{s}_{\parallel} = \hat{s} \cdot \hat{\beta}$, $\hat{s}_{\perp} = \hat{s} - \hat{s}_{\parallel}$
parallel perpendicular
to boost vector

$\rightarrow$ if $\hat{s} \perp \bar{\beta} \rightarrow s = (0, \hat{s})$ helicity state $\downarrow$

if $\hat{s} \parallel \hat{\beta} \rightarrow \hat{s} \cdot \hat{\beta} = \pm 1 \rightarrow s = \pm \gamma (\beta, \hat{\beta})$

$$\Rightarrow \begin{cases} u(\bar{p},s)\,\bar{u}(\bar{p},s) \xrightarrow{m=0} \left(\dfrac{1 \pm \gamma^5}{2}\right)\not{p} \\ v(\bar{p},s)\,\bar{v}(\bar{p},s) \xrightarrow{m=0} \left(\dfrac{1 \mp \gamma^5}{2}\right)\not{p} \end{cases}$$

with $P_{L,R} = \dfrac{1 \mp \gamma^5}{2}$ the helicity projection operator

- Explicit forms for the spinors u and v.

Pauli-Dirac representation:

$$u(\bar{p}, s) = \sqrt{E_p + m} \begin{pmatrix} \varphi_s \\ \frac{\bar{\sigma}\cdot\bar{p}}{E_p + m} \varphi_s \end{pmatrix}$$

with $\begin{cases} \varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow +\hat{z} \\ \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow -\hat{z} \end{cases}$ two different spin state

$$v(\bar{p}, s) = \sqrt{E_p + m} \begin{pmatrix} \frac{\bar{\sigma}\cdot\bar{p}}{E_p + m} \chi_s \\ \chi_s \end{pmatrix}$$

with $\begin{cases} \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \chi_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{cases}$

φ_s and χ_s are Pauli two-component spinors which can also be written in terms of helicity basis with helicity operator $h = \frac{1}{2}\bar{\sigma}\cdot\bar{p}$

$h\varphi_+ = +\frac{1}{2}|\bar{p}|\varphi_+$ & $h\chi_+ = -\frac{1}{2}|\bar{p}|\chi_+$

$h\varphi_- = -\frac{1}{2}|\bar{p}|\varphi_-$ $h\chi_- = +\frac{1}{2}|\bar{p}|\chi_-$

$\longrightarrow$ more complicated explicit forms

Chiral representation:

$$u(\bar{p}, s) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} \left(I - \frac{\bar{\sigma} \cdot \bar{p}}{E+m}\right) \varphi_s \\ \left(I + \frac{\bar{\sigma} \cdot \bar{p}}{E+m}\right) \varphi_s \end{pmatrix}$$

$$v(\bar{p}, s) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} \left(I - \frac{\bar{\sigma} \cdot \bar{p}}{E+m}\right) \chi_s \\ -\left(I + \frac{\bar{\sigma} \cdot \bar{p}}{E+m}\right) \chi_s \end{pmatrix}$$

or

$$u(\bar{p}, s) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \varphi_s \\ \sqrt{p \cdot \bar{\sigma}} & \varphi_s \end{pmatrix}$$

with $\sigma^\mu = (I, \sigma^i)$ and $\bar{\sigma}^\mu = (I, -\sigma^i)$

$$v(\bar{p}, s) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \chi_s \\ -\sqrt{p \cdot \bar{\sigma}} & \chi_s \end{pmatrix}$$

which can be written in helicity basis $\varphi_{\pm}, \chi_{\pm}$

$$u(\bar{p}, \pm) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} \lambda_{\pm} \varphi_{\pm} \\ \lambda_{\mp} \varphi_{\pm} \end{pmatrix} \longrightarrow$$

$$v(\bar{p}, \pm) = \sqrt{\frac{E+m}{2}} \begin{pmatrix} \lambda_{\pm} \chi_{\pm} \\ -\lambda_{\mp} \chi_{\pm} \end{pmatrix}$$

with $\lambda_{\pm} = 1 \mp \frac{p}{E+m}$

For $m \to 0$: $\lambda_- \to 0$ and $\lambda_+ = 1$

and $\begin{cases} u(\bar{p},+) = \sqrt{2E}\begin{pmatrix} 0 \\ \varphi_+ \end{pmatrix} \\ u(\bar{p},-) = \sqrt{2E}\begin{pmatrix} \varphi_- \\ 0 \end{pmatrix} \end{cases}$

$\begin{cases} v(\bar{p},+) = \sqrt{2E}\begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} \\ v(\bar{p},-) = \sqrt{2E}\begin{pmatrix} 0 \\ -\chi_- \end{pmatrix} \end{cases}$

- Weyl fields in chiral representation

$P_L = \frac{1-\gamma^5}{2}$ $\qquad\qquad$ $P_R = \frac{1+\gamma^5}{2}$

$\psi_L = P_L\psi_L$ $\qquad\qquad$ $\psi_R = P_R\psi_R$

ψ_L, ψ_R are independent degrees of freedom when $m=0$. If $m \neq 0 \rightarrow$ mixtures:

$$\mathcal{L} = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$$

Because $P_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

and $P_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\rightarrow \psi_L = \begin{pmatrix} \Psi_L \\ 0 \end{pmatrix}$ and $\psi_R = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix}$

with Ψ_L and Ψ_R two-component Weyl fields such that:

$$\mathcal{L} = \Psi_L^\dagger i\bar{\sigma}^\mu \partial_\mu \Psi_L + \Psi_R^\dagger i\sigma^\mu\partial_\mu \Psi_R - m\left(\Psi_L^\dagger \Psi_R + \Psi_R^\dagger \Psi_L\right)$$

↪ for massless, left-handed only fermions:

$$\mathcal{L} = \Psi_L^\dagger i\bar{\sigma}^\mu \partial_\mu \Psi_L \quad \text{without } \Psi_R$$

Can also apply P_L and P_R not to fermion field ψ but to spinors $u(\bar{p}, s)$ and $v(\bar{p}, s)$

$$u(\bar{p}, \pm) = \begin{pmatrix} u_L \\ u_R \end{pmatrix} \qquad v(\bar{p}, \pm) = \begin{pmatrix} v_L \\ v_R \end{pmatrix}$$

↓

$$\Psi_{L,R}(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \sqrt{2E_p}\left[\varphi_\mp(\bar{p})\, a(\bar{p}, \pm)\, e^{-ip\cdot x} + \chi_\pm(\bar{p})\, b^\dagger(\bar{p}, \pm)\, e^{ip\cdot x}\right]$$

- Propagator for fermion fields:

$$iS(k) = \frac{i}{\not{k} - m + i\varepsilon} = i\,\frac{\not{k} + m}{k^2 - m^2 + i\varepsilon}$$

* QED with fermion fields and massless vector fields :

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \not{D} = \gamma^\mu D_\mu$$

with $D_\mu = \partial_\mu - ieA_\mu$ by minimal substitution

Gauge transformation: $A^\mu \to A^\mu - \frac{1}{e}\partial^\mu \beta(x)$

$$\psi \to e^{-i\frac{q}{e}\beta(x)}\psi$$

$$\downarrow$$

$$\mathcal{L} = \psi(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{eA_\mu\bar{\psi}\gamma^\mu\psi}_{\text{interaction term}}$$

Feynman rule

$ie\gamma^\mu$