# Phys 772: Week 4 Tuesday

* Lie groups:

<u>group</u> $G$: $g_1, g_2, \ldots \in G$ and multiplication

$Z_4 = \{1, 90^\circ, 180^\circ, 270^\circ\} \rightarrow$ 4 elements

↳ subgroup $Z_2 = \{1, 180^\circ\}$

$G$ = rotations angle $\theta \rightarrow$ continuous

properties: $\forall g_1, g_2 \in G : g_1 g_2 \in G$ closure

$g_1(g_2 g_3) = (g_1 g_2) g_3$ associative

$\exists 1 : g1 = 1g = g$ identity

$\forall g \in G, \exists g^{-1} \in G : g g^{-1} = g^{-1} g = 1$ inverse

abelian: $\forall g_1, g_2 \in G : g_1 g_2 = g_2 g_1 \rightarrow [g_1, g_2] = 0$

non-abelian: $\exists g_1, g_2 \in G : g_1 g_2 \neq g_2 g_1 \rightarrow [g_1, g_2] \neq 0$

<u>Lie group</u> $G$: continuous group, usually compact
differentiable multiplication law

↳ allows for differentiation around the identity element $1 = U(0)$

small $\beta$ : $U(\bar{\beta}) \simeq 1 - i\bar{\beta}\cdot\bar{T}$ for small $\bar{\beta}$

general : with $\bar{T} = T_i$, $i = 1 \dots N$, hermitian

$U(\bar{\beta}) = e^{-i\bar{\beta}\cdot\bar{T}}$ for any $\bar{\beta}$ in compact group connected to $\mathbb{1}$

$(U(\bar{\beta}))^{-1} = U(-\bar{\beta}) = e^{i\bar{\beta}\cdot\bar{T}} = (U(\bar{\beta}))^{+}$

$\hookrightarrow$ U is unitary when T is hermitian

Lie algebra: $[T_i, T_j] = i c_{ijk} T^k$

$c_{ijk}$ = structure constants

If $\exists L_i \in \mathbb{C}_{n\times n}$, $i = 1 \dots N$, that satisfy this same set of structure constant commutator relations

$\downarrow$

$L_i$ is a representation of Lie algebra $G$ and $\{L_i\}$ are generators of Lie group

Example: Lie group $U(1)$ = phases of complex numbers, rotations in plane, etc

generator T such that $U(\beta) = e^{-i\beta T}$

representation with $L = 1 \to U(\beta) = e^{-i\beta}$

$\Rightarrow 1\times 1$ unitary matrices

Example: $U(2) = 2\times 2$ unitary matrices

$SU(2) = U(2)$ with determinant 1

$SU(N)$: $U \in SU(N)$ is special, unitary

$U = e^{iH}$ with $H$ hermitian $H^{+} = H$

and

$\det U = e^{i \,\mathrm{Tr}\, H}$ with $\mathrm{Tr}\, H = 0 \rightarrow \det U = 1$

$N\times N$ complex matrix $\rightarrow 2N^2$ d.o.f.

Hermitian: $N^2$ eqn $\rightarrow N^2$ d.o.f.

traceless: 1 eqn $\rightarrow N^2 - 1$ d.o.f.

$\hookrightarrow N^2 - 1$ generators

$SU(2)$: $\Rightarrow N^2 - 1 = 3$ generators $L_i = \frac{\sigma_i}{2}$ for $N = 2$

$\Rightarrow U(\vec{\beta}) = e^{-i\vec{\beta}\cdot\frac{\vec{\sigma}}{2}}$, $\mathrm{Tr}\, \sigma_i = 0$

$\rightarrow \det U(\vec{\beta}) = 1$

$SU(2)$ homomorphic with $SO(3)$

= group of rotation on the unit sphere in 3D space with determinant 1

$O \in SO(3)$: $O^T O = 1 = O O^T$

$\det O = 1$

Euler angles: $\begin{pmatrix} \cos\frac{\theta}{2} e^{i\frac{\varphi+\psi}{2}} & i\sin\frac{\theta}{2} e^{i\frac{\varphi-\psi}{2}} \\ i\sin\frac{\theta}{2} e^{-i\frac{\varphi-\psi}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\varphi+\psi}{2}} \end{pmatrix} \in SU(2)$

SU(3): $N^2-1=8$ generators $T_i$, $i=1,\dots,8$ hermitian

representation in $3\times3$ matrices: $T_i = \frac{\lambda_i}{2}$

$\lambda_i$ = Gellman matrices

$$[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$$

Applications: U(1) theory is symmetric under U(1), i.e. phase transformations

SU(2) theory is invariant under

e.g. isospin exchange $\begin{cases} u \leftrightarrow d \\ p \leftrightarrow n \end{cases}$

rotational invariance

SU(3) theory is invariant under

e.g. color r, g, b exchange

eight lightest hadrons/mesons

* Gauge theories : recipe of last week's lecture

1) Start from set of fields with global symmetry

$$\Phi \to \Phi' = e^{-i\vec{\beta}\cdot\vec{T}}\, \Phi\, e^{i\vec{\beta}\cdot\vec{T}} = U(\beta)\,\varphi\, U(\beta)^{+}$$

matrix representation $n \times n$

$$\varphi^i \to \varphi'^i = \varphi^i + i\vec{\beta}\cdot\vec{L}_{ij}\,\varphi_j = e^{i\vec{\beta}\cdot\vec{L}}\varphi$$

with $[T^i, \Phi_a] = -L^i_{ab}\,\Phi_b$

$\to$ leaves $\mathcal{L}$ unchanged.

examples: $\Phi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$ real fields, SU(2) isospin symmetry:

$\hookrightarrow$ $3\times3$ representation, $\pi^0 \leftrightarrow \pi^0$, $\pi^+ \leftrightarrow \pi^-$

$\Phi = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ $\Phi^+ = \begin{pmatrix} K^- \\ \bar{K}^0 \end{pmatrix}$, complex fields SU(2) isospin

$\hookrightarrow$ $2\times2$ representation

$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$, SU(2) isospin

$\hookrightarrow$ $2\times2$ representation on all of $\psi_n, \psi_p$

$$L^i_\psi = \frac{\sigma^i}{2}$$

$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, SU(3) u, d, s light quarks

$\hookrightarrow$ $3\times3$ representation

$\rightarrow \quad L^i_\psi = \frac{\lambda^i}{2}$ for Gellman matrices

$\psi = \begin{pmatrix} u_r \\ u_g \\ u_b \end{pmatrix}$, SU(3) color symmetry

2) Associate new vector boson with each generator

examples: U(1) QED, $\psi \rightarrow e^{-i\beta(x)}\psi$, $T = L = 1$

$A^\mu \rightarrow A^\mu - \frac{1}{e}\partial^\mu\beta$ for photon

SU(3) QCD, $\psi \rightarrow e^{-i\bar{\beta}\cdot\bar{L}}\psi$

$A^{i\mu}$ for 8 gluons $(N^2-1)$

3) Introduce covariant derivative

$D^\mu = \partial^\mu + ig\,\bar{A^\mu}\cdot\bar{L}$

examples: QED: $D^\mu = \partial^\mu + igA^\mu$ with $g = -e$

QCD: $D^\mu = \partial^\mu + igA^{i\mu}L^i$

With requirement $(D^\mu\psi) \rightarrow e^{-i\bar{\beta}\cdot\bar{L}}(D_\mu\psi)$
such that $\bar{\psi}\, i\not{D}\psi \rightarrow \bar{\psi}\, i\, U^\dagger U \not{D}\psi = \bar{\psi}\, i\not{D}\psi$
leaves the fermion kinetic term invariant
Also $(D^\mu\varphi)^\dagger(D_\mu\varphi)$ invariant.

4) Each vector boson field receives a free term :

$$\mathcal{L} = -\frac{1}{4} F^i_{\mu\nu} F^{\mu\nu}_i \propto \mathrm{Tr}\left[\left(F^i_{\mu\nu} \cdot L_i\right)^2\right]$$

which is invariant when we take

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\, c_{ijk} A^j_\mu A^k_\nu$$

since then $(\bar{F}_{\mu\nu} \cdot \bar{L}) \longrightarrow U (\bar{F}_{\mu\nu} \cdot \bar{L}) U^\dagger$

and $(\bar{A}_\mu \cdot \bar{L}) \longrightarrow U (\bar{A}_\mu \cdot \bar{L}) U^\dagger + \frac{i}{g} (\partial_\mu U) U^\dagger$

$\Rightarrow$ Summary for QCD part of standard model

$q_{r\alpha}$ = quark fermion fields $\quad r = u, d, s, c, b, t$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \alpha = r, g, b$

$\alpha \rightarrow$ gauge symmetry index

$G^i_\mu$ = gauge field for ith boson fields

$$G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_s f_{ijk} G^j_\mu G^k_\nu$$

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^i_{\mu\nu} G^{\mu\nu}_i + \sum_r \bar{q}_r (i\not{D} - m_r) q_r$$

with $(D^\mu)_\alpha{}^\beta = \partial^\mu \delta_\alpha{}^\beta + i \frac{g_s}{\sqrt{2}} G^\mu{}_\alpha{}^\beta$

Comments: - could add CP violating term $\theta_{QCD}$ but small

- $m_q$ not fundamental → Higgs

$SU(3)_{color}$ is gauge symmetry, but there are still others:

$SU(2) \times SU(2)$ : $m_u \approx m_d = 0$ isospin, L, R

↪ $SU(3) \times SU(3)$ : $m_u \approx m_d \approx m_s = 0$, light quarks

$U(1)$ : conserved baryon number B

* Higgs mechanism and spontaneous symmetry breaking: in class exercise (?)

Consider complex scalar:

$$\mathcal{L} = (\partial_\mu \varphi)^\dagger (\partial^\mu \varphi) - V(\varphi) \rightarrow \underline{U(1) \text{ symmetry}}$$

$$V(\varphi) = \mu^2 (\varphi^+ \varphi) + \lambda (\varphi^+ \varphi)^2$$

transform using $\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2)$

with $\varphi_1, \varphi_2$ hermitian

$$V = \frac{\mu^2}{2} (\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4} (\varphi_1^2 + \varphi_2^2)^2$$

Vacuum state $v_1, v_2$ at lowest $V$ requires

$$\left.\frac{\partial V}{\partial \varphi_i}\right|_{v_1, v_2} = 0 \quad , \text{ eigenvalues } m^2_{1,2} \text{ of } \left.\frac{\partial^2 V}{\partial \varphi_i \partial \varphi_j}\right|_{v_1 v_2}$$

$\mu^2 > 0 \rightarrow m^2_{1,2} = \mu^2$

$\mu^2 < 0 \rightarrow$ hat potential, minima away from $v_1 = v_2 = 0$
$\rightarrow$ no more $SO(2)$ $U(1)$ symmetry

Expand around $v_1 = v, v_2 = 0$ (vev) with $\varphi_1', \varphi_2'$

$\rightarrow m_1^2 = -2\mu^2$

Now consider additional fermion field $\psi = \psi_L + \psi_R$ where only

$$\begin{aligned}\mathcal{L} = & \bar{\psi}_L i \not\partial \psi_L + \bar{\psi}_R i \not\partial \psi_R \\ & - h \bar{\psi}_L \psi_R \varphi - h^* \bar{\psi}_R \psi_L \varphi^+ \\ & + (\partial_\mu \varphi)^+ (\partial^\mu \varphi) - V(\varphi)\end{aligned}$$

$U(1)$ symmetry : $\begin{cases} \varphi \rightarrow e^{i\beta} \varphi \\ \psi_R \rightarrow e^{-i\beta} \psi_R \end{cases}$

Rewrite for $\mu^2 < 0$, $\varphi_1', \varphi_2'$

$\downarrow$

$$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$$

$$\bar{\psi}\gamma^5\psi = \bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L$$

$\bar{\psi}\psi$ term with $m_\psi = \frac{h v}{\sqrt{2}}$