Phys 772: Week 2 Tuesday

* Scalar fields: $\varphi(x) \in \mathbb{C}$ or $\mathbb{R}$

- charged versus neutral fields:

$\pi^+ \to \varphi$ $\qquad\qquad\qquad \pi^0$

$\updownarrow$ complex conjugate $\qquad \updownarrow$ unchanged: $\varphi^+ = \varphi$

$\pi^- \to \varphi^+$ $\qquad\qquad\qquad \pi^0$

- creation and annihilation operators (scalars = bosons)

$a^+(\bar{p})$ $\qquad\qquad a(p)$

$\hookrightarrow$ operate on vacuum: $|\bar{p}\rangle = a^+(\bar{p})|0\rangle$

$|0\rangle = a(\bar{p})|\bar{p}\rangle$

with normalization given by commutator

$$[a(\bar{p}), a^+(\bar{p}')] = (2\pi)^3\, 2E_p\, \delta^3(\bar{p} - \bar{p}')$$

$\hookrightarrow E_p^2 = \bar{p}^2 + m^2$

particles created or annihilated with specific momentum (quantized quantity in case of harmonic oscillator, particle in a box)

but E-, B- fields more usually thought of as at a fixed position, or π^+, π^0 particle created at fixed position

$\downarrow$

Fourier Transform

- canonical quantization to define field operator φ as Fourier transform of annihilation operator

$$\varphi(x) = \sum_i e^{i\bar{p}_i \cdot \bar{x}} a(\bar{p}_i) \rightarrow \text{field operator}$$

$i \rightarrow$ sum over all particles in state

- complex scalar field $\varphi(x) \neq \varphi^\dagger(x)$

e.g. $\varphi(x)$ annihilates π^+ $\rightarrow$ $\varphi^\dagger(x)$ annihilates π^-
creates π^- $\rightarrow$ creates π^+

$$\mathcal{L}(\varphi, \partial_\mu \varphi, \varphi^\dagger, \partial_\mu \varphi^\dagger)$$
$$= (\partial_\mu \varphi)^\dagger (\partial^\mu \varphi) - m^2 \varphi^\dagger \varphi - V(\varphi, \varphi^\dagger)$$

* real scalar field, $\varphi = \varphi^\dagger$

$$V(\varphi) = \frac{k}{3!}\varphi^3 + \frac{\lambda}{4!}\varphi^4$$ (+ other terms non-renormalizable or can be re-defined away)

* complex scalar field, $\varphi \neq \varphi^\dagger$

$$V(\varphi, \varphi^\dagger) = \frac{\lambda}{4}(\varphi^\dagger \varphi)^2$$ (+ other dimension ≤ 4 terms violate charge conservation)

$\varphi \rightarrow \varphi^\dagger =$ anti-particle

$\downarrow$ see later: U(1) symmetry

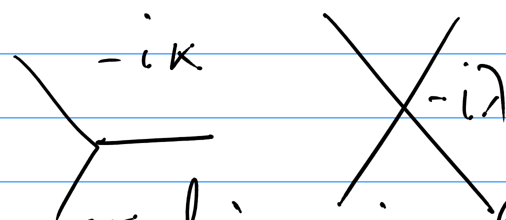
$\mathcal{L} \to$ equation $\dfrac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} = 0 \, , \, \forall \varphi$

* complex scalar field $\to$ 2 equations

$$\begin{cases} (\Box + m^2)\varphi^+ + \dfrac{\lambda}{2}\varphi^+(\varphi^+\varphi) = 0 \\ (\Box + m^2)\varphi + \dfrac{\lambda}{2}\varphi(\varphi^+\varphi) = 0 \end{cases}$$

* real scalar field $\to$ 1 equation

$$\underbrace{(\Box + m^2)\varphi}_{\text{homogenous free field}} + \underbrace{\frac{\kappa}{2!}\varphi^2 + \frac{\lambda}{3!}\varphi^3}_{\text{potential energy term, self-interaction}} = 0$$



coupling given by $i\mathcal{L}$ coefficients

$(\Box + m^2)\varphi = 0$ real Klein-Gordon equ

sol$^{\underline{n}}$ $\hookrightarrow$ $\varphi(x) = \displaystyle\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_p}\left[a(\bar{p})e^{-ip\cdot x} + a^+(\bar{p})e^{+ip\cdot x}\right]$

$\qquad \varphi(x) = \varphi^+(x)$

$\hookrightarrow$ both directions $(\bar{p}, -\bar{p})$ are in $\varphi(x)$

complex scalar: $\varphi(x) \neq \varphi^\dagger(x)$

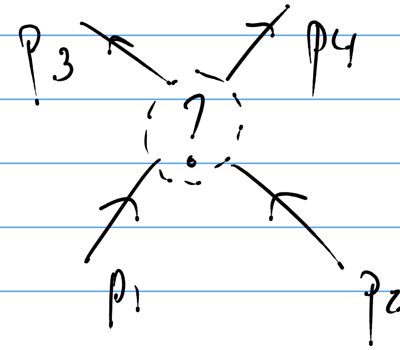
sol^{n}: $\varphi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_p}\left[a(\bar{p})e^{-ip\cdot x} + b^\dagger(\bar{p})e^{+ip\cdot x}\right]$

$\varphi^\dagger(x) = \ldots \quad b(\bar{p}) \;\_\_\; a^\dagger(p) \;\_\_$

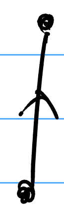
↳ directions different → arrows

- Scattering process requires:

  * free field solutions



  * propagator



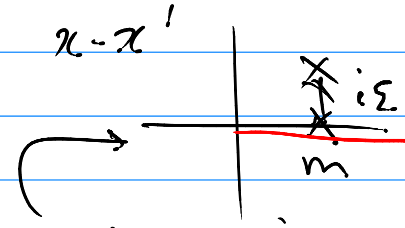
Green's function: general solution to impulse, to be convoluted with non-homogenous source → e.g. E&M, QM

$$(\Box + m^2)\Delta(x) = -\delta^4(x-x')$$

↳ only depends on $x-x'$

$\Delta(x-x')$

Fourier Transform



$$(-k^2+m^2)D(k) = -1 \rightarrow iD(k) = \frac{i}{k^2-m^2}$$

- Underlying symmetry

real scalar field $\rightarrow \mathcal{L} = \frac{1}{2}\left[(\partial_\mu \varphi)^2 - m^2 \varphi^2\right] - \frac{\kappa}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4$

complex scalar field: $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$

$\varphi^+ = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$

with both φ_1 and φ_2 real

with $\mathcal{L} = (\partial_\mu \varphi)^+(\partial_\mu \varphi) + m^2 \varphi^+ \varphi$

$= \frac{1}{2}\left[(\partial_\mu \varphi_1)^2 + m^2 \varphi_1^2\right] + \frac{1}{2}\left[(\partial_\mu \varphi_2)^2 + m^2 \varphi_2^2\right]$

$\rightarrow$ already requirement that φ_1 and φ_2 have same mass, not just any two real scalar fields form a complex scalar field

Also with $V(\varphi, \varphi^+) = \frac{\lambda}{4}(\varphi^+\varphi)^2 = \frac{\lambda}{16}(\varphi_1^2 + \varphi_2^2)^2$

$\rightarrow$ requirement of $\lambda_1 = \lambda_2$ and $\kappa_1 = \kappa_2 = 0$

Third order terms $(\varphi^+\varphi)\varphi$ and $(\varphi^+\varphi)\varphi^+$
→ these violate other symmetry...

- Unitary symmetry ← requires transformation & invariance

$$\varphi \rightarrow \varphi' = e^{i\beta}\varphi \quad : \text{phase transformation}$$

$$\mathcal{L}(\varphi', \partial_\mu\varphi', \varphi'^+, \partial_\mu\varphi'^+) = \mathcal{L}(\varphi, \partial_\mu\varphi, \varphi^+, \partial_\mu\varphi^+)$$

$e^{i\beta} \in U(1)$ = unitary groups of dimension 1
= complex scalars of modulus 1

$e^{i\beta}$ ← global symmetry because invariance does not hold for $\beta(x)$ arbitrary

Back to 3rd order terms:

real scalar field: no $U(1)$ symmetry
but simpler structure

complex scalar field: $(\varphi'^+\varphi')\varphi' \neq (\varphi^+\varphi)\varphi$
→ these terms forbidden in complex scalar field which satisfies $U(1)$ symmetry

- Noether theorem: symmetry → conserved quantity

$$J^\mu = -i\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi)}\varphi - \frac{\delta\mathcal{L}}{\delta(\partial_\mu\varphi^+)}\varphi^+\right]$$

$$J^\mu = -i\left[(\partial^\mu\varphi)^+\varphi - \varphi^+(\partial^\mu\varphi)\right] \rightarrow \delta\mathcal{L} \propto \partial_\mu J^\mu$$

$$J^\mu(x) \longrightarrow \quad Q = \int d^3\bar{x} \; J^0(x)$$

$$\frac{\partial Q}{\partial t} = \int d^3\bar{x} \; \frac{\partial J^0}{\partial t}(x)$$

$$= \int d^3\bar{x}\left(\frac{\partial J^0(x)}{\partial t} + \bar{\nabla}\cdot\bar{J}(x)\right)$$

$$= \int d^3\bar{x} \quad \partial_\mu J^\mu(x) = 0$$

$$\varphi(x) = \int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_p}\left[a(\bar{p})e^{-ip\cdot x} + b^+(\bar{p})e^{+ip\cdot x}\right]$$

$$J^\mu(x) = -i\left[\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E_p}\left[+ip^\mu a^+(\bar{p})e^{+ip\cdot x} - ip^\mu b(\bar{p})e^{-ip\cdot x}\right] \cdot \int\frac{d^3\bar{p}'}{(2\pi)^3}\frac{1}{2E_p'}\left[a(\bar{p}')e^{-ip'\cdot x} + b^+(\bar{p}')e^{+ip'\cdot x}\right] + \ldots\right]$$

$$\downarrow$$

$$Q = N_{\pi^+} - N_{\pi^-}$$

$\downarrow$

3rd order terms do not satisfy conservation

* Vector fields : $A^\mu(x) \in \mathbb{R}^4$

$$\hookrightarrow F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$ (free fields, no interaction)

$$\hookrightarrow \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$$

sol^n $\hookrightarrow$ $$A^\mu(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \left[\sim a(\bar{p}) e^{-ip\cdot x} + \sim a^\dagger(\bar{p}) e^{ip\cdot x} \right]$$

However, there is redundancy in $A^\mu(x)$:

for $A^\mu \to A'^\mu = A^\mu - \frac{1}{e}\partial^\mu \beta(x)$

$F^{\mu\nu} \to F'^{\mu\nu} = F^{\mu\nu}$

$\mathcal{L} \to \mathcal{L}' = \mathcal{L}$

convention

mass term $+\frac{1}{2}m^2 A^\mu A_\mu$ not invariant

U(1) invariance allows us to impose an additional constraint on A^μ, to get a uniquely determined solution it even requires an additional constraint :

* $\partial_\nu A^\nu = 0$ Lorentz gauge

$\hookrightarrow$ covariant, no specification of individual components that depend on frame we work in in Lorentz gauge, can still make transformation with $\Box \beta = 0$

$+ \quad A^0 = 0 \,, \quad \bar{\nabla} \cdot \bar{A} = 0$ → 2 d.o.f. left

Coulomb gauge

↳ not Lorentz invariant but determine unique solution: after Lorentz transformation, require additional gauge transformation to return to Coulomb gauge!

$$\begin{cases} \bar{A}(x) = \displaystyle\int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_\lambda \Big[\bar{\varepsilon}(\bar{p},\lambda)\, a(\bar{p},\lambda)\, e^{-ip\cdot x} + \bar{\varepsilon}^*(\bar{p},\lambda)\, a^\dagger(\bar{p},\lambda)\, e^{ip\cdot x} \Big] \\ A^0(x) = 0 \end{cases}$$

$\bar{\varepsilon}(\bar{p},\lambda)$ = polarization vector for polarization state λ

$\bar{\nabla} \cdot \bar{A} = 0 \longrightarrow \bar{p} \cdot \bar{\varepsilon} = 0$: transverse polarization only

pick polarization basis $\bar{\varepsilon}(\bar{p},1) \perp \bar{\varepsilon}(\bar{p},2)$ | massive field → three states!

→ circular polarization components

$$\begin{cases} \bar{\varepsilon}(\bar{p},L) = \dfrac{1}{\sqrt{2}} \left(\bar{\varepsilon}(\bar{p},1) - i\,\bar{\varepsilon}(\bar{p},2) \right) \\ \bar{\varepsilon}(\bar{p},R) = \dfrac{1}{\sqrt{2}} \left(\bar{\varepsilon}(\bar{p},1) + i\,\bar{\varepsilon}(\bar{p},2) \right) \end{cases}$$

With this basis, and with $\varepsilon(\bar{p},\lambda) = (0, \bar{\varepsilon}(\bar{p},\lambda))$

$$A^\mu(x) = \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_\lambda \left[\varepsilon(\bar{p},\lambda)\, a(\bar{p})\, e^{-ip\cdot x} + \varepsilon^*(\bar{p},\lambda)\, a^+(\bar{p})\, e^{ip\cdot x} \right]$$

satisfies Coulomb gauge, $\lambda = 1,2$ transverse
$\lambda = L, R$ circular

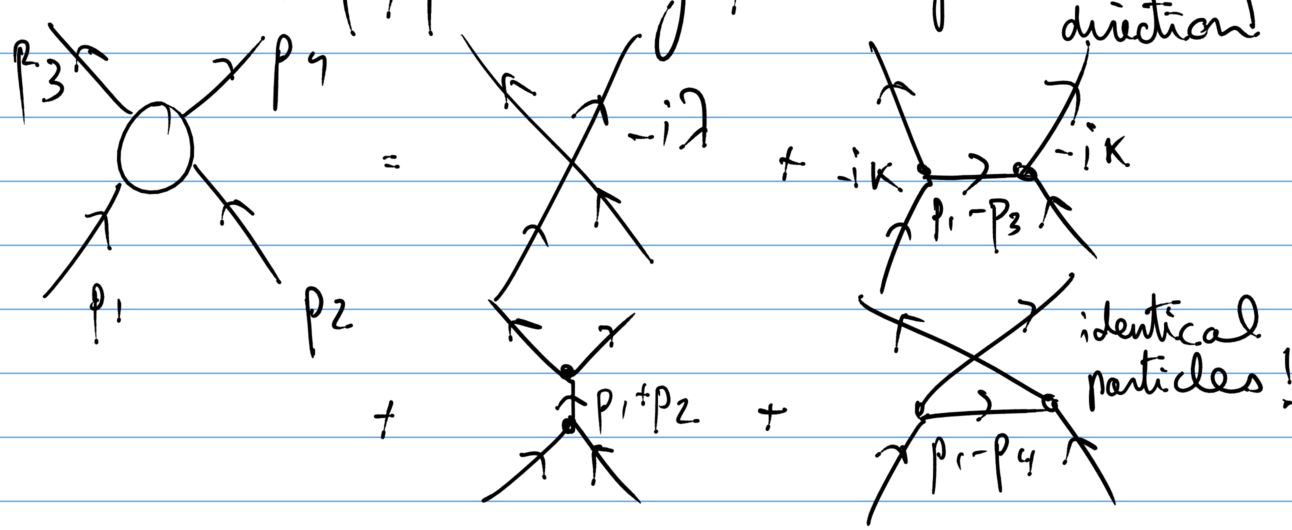
Polarization state:
- initial: averaging
- final: summing

$$\rightarrow \sum_\lambda \varepsilon^\mu(\bar{p},\lambda)\,\varepsilon^{*\nu}(\bar{p},\lambda) = -g^{\mu\nu}$$

Propagator:

$$i D_{\mu\nu}(k) = \frac{-i g_{\mu\nu}}{k^2 + i\varepsilon} \propto \text{Green's function}$$

* Scattering and Feynman diagrams:
- Real scalar φ^3, φ^4 theory, arrows just indicate p direction!



$$\rightarrow M = -i\lambda + (-ik)^2 \left[\frac{i}{(p_1+p_2)^2 - m^2} + \frac{i}{(p_1-p_3)^2 - m^2} + \frac{i}{(p_1-p_4)^2 - m^2} \right]$$

$$= -i\lambda + (-ik)^2 \left[\frac{i}{s^2 - m^2} + \frac{i}{t^2 - m^2} + \frac{i}{u^2 - m^2} \right]$$

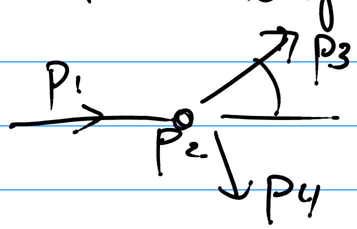
- Depending on kinematics of experiment, detection: s, u, t dominates

e.g. if $s^2 \approx m^2 \rightarrow$ s-channel dominates

$$\begin{cases} t = -4p^2 \sin^2 \frac{\theta_{cm}}{2} \rightarrow -Q^2 \text{ in lab terms} \\ u = -4p^2 \cos^2 \frac{\theta_{cm}}{2} \end{cases}$$

$t \rightarrow 0$ for forward scattering
$u \rightarrow 0$ for backward scattering
when propagator has $m \approx 0$
$\longrightarrow \frac{1}{t^2}$ or $\frac{1}{u^2}$ are dominant

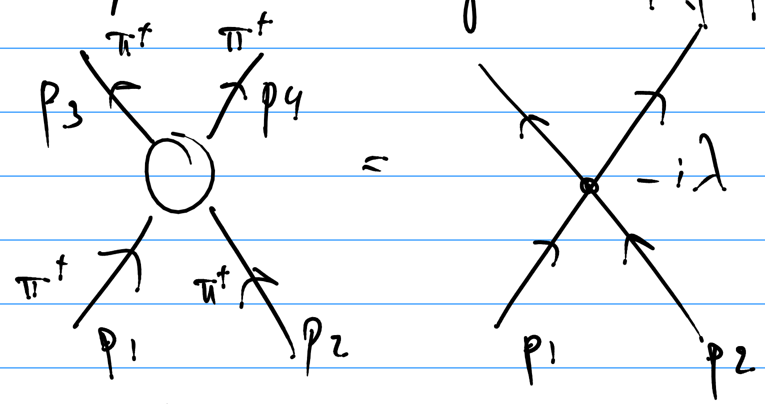
Fixed target: $s = m_1^2 + m_2^2 + 2E_1 m_2$ ← beam: E_1; ← target: m_2

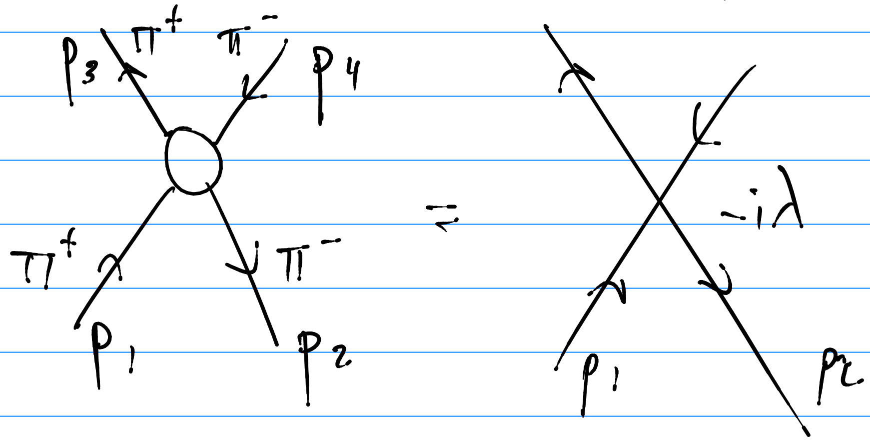


↳ $t = -2E_1E_3(1 - \cos\theta_3)$

- Complex scalar fields, $(\varphi^\dagger\varphi)^2$ interaction only

arrows now





* Scalar/vector interactions

$\mathcal{L}_{\varphi A}(\varphi, \varphi^\dagger, A^\mu)$ must satisfy same gauge invariance under $A^\mu \to A^\mu - \frac{1}{e}\partial^\mu\beta$

while $\varphi \to e^{iq\frac{1}{e}\beta(x)}\varphi$, now a local phase transformation

Minimal substitution : $p^\mu \to p^\mu - qA^\mu$

$i\partial^\mu \to i\partial^\mu - qA^\mu$

or $\partial^\mu \to \partial^\mu + iqA^\mu = D^\mu$

D^μ = covariant derivative

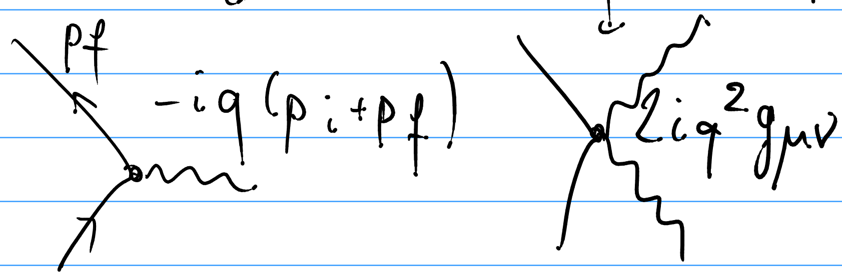
$$D^\mu \varphi \to D'^\mu \varphi' = \left[\partial^\mu + iq\left(A^\mu - \frac{1}{e}\partial^\mu \beta\right)\right] e^{i\frac{q}{e}\beta}\varphi$$

$$= e^{i\frac{q}{e}\beta}\left[\partial^\mu \cancel{+ i\frac{q}{e}\beta\,\partial^\mu} + iqA^\mu \cancel{- i\frac{q}{e}\partial^\mu \beta}\right]\varphi$$

$$= e^{i\frac{q}{e}\beta} D^\mu \varphi$$

→ interaction term from $(D^\mu\varphi)^\dagger (D_\mu \varphi)$:

$$i\mathcal{L}_{\varphi A} = -q(\partial^\mu\varphi)^\dagger \varphi A_\mu + q\varphi^\dagger(\partial^\mu\varphi)A_\mu + iq^2 A_\mu A^\mu \varphi^\dagger \varphi$$



p_f

$-iq(p_i+p_f)$

p_i

$2iq^2 g_{\mu\nu}$

Note : U(1) symmetry of vector fields :

$\varphi \to e^{i\beta\cdot L}\varphi$ with L^i the representation matrix of group

→ that matrix is identity for U(1)

$\varphi \to e^{i\beta}\varphi$ is an U(1) transformation

Examples of $\pi^- K^+$, $\pi^+\pi^-$, $\pi^+\pi^+$, $\gamma\pi^+$ scattering: see e.g. Langacker