Phys 772: Week 3 Tuesday

* QED: $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ (massless vector field)

$+ \bar{\psi}(i\not{D} - m_e)\psi$ (fermion field)

$+ (D_\mu\varphi)^\dagger(D^\mu\varphi) - m_\pi^2\varphi^\dagger\varphi$ (scalar field)

with $D_\mu = \partial_\mu + iqA_\mu$ with $\begin{cases} q_\psi = -e \\ q_\varphi = e \end{cases}$

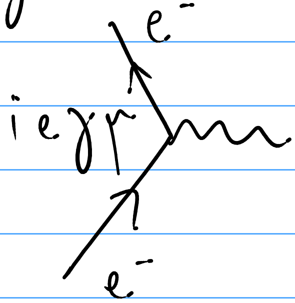
$$\Rightarrow \mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{\partial} - m_e)\psi$$

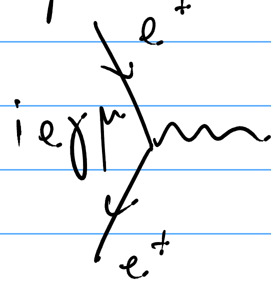
$$+ (\partial_\mu\varphi)^\dagger(\partial^\mu\varphi) - m_\pi^2\varphi^\dagger\varphi$$

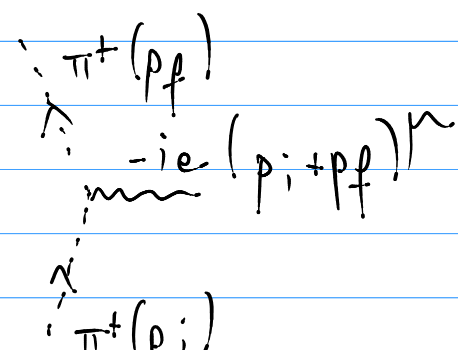
$$+ eA_\mu\bar{\psi}\gamma^\mu\psi \qquad (j^\mu_\psi)$$

$$-ie\left[\varphi^\dagger\partial^\mu\varphi - (\partial^\mu\varphi)^\dagger\varphi\right] \qquad (j^\mu_\varphi)$$

Feynman rules from $i\mathcal{L}$:



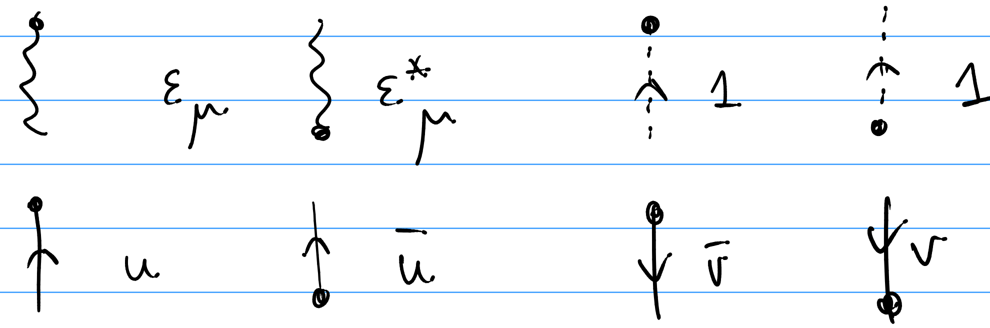




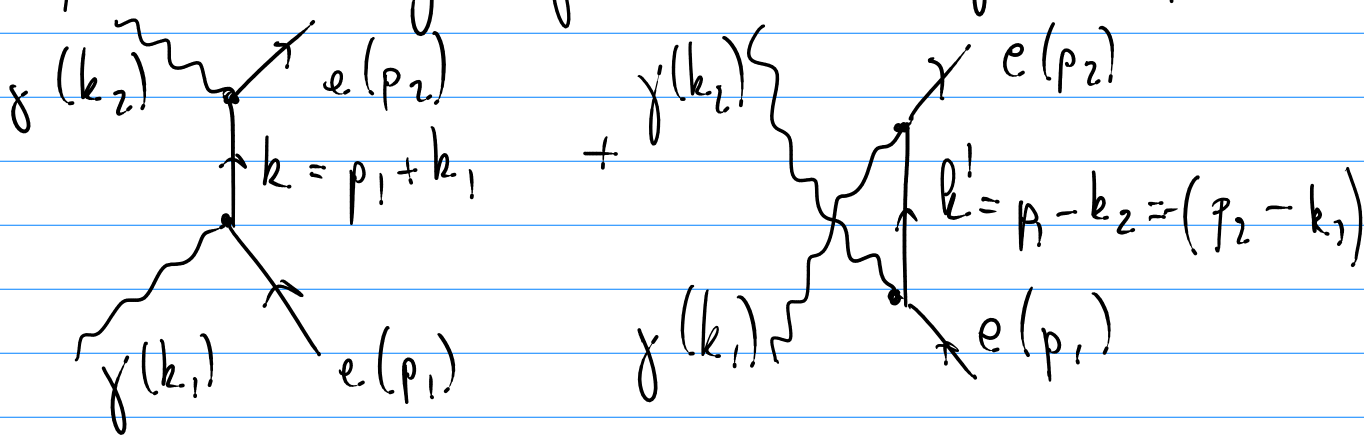
$$\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$$

$$\frac{i}{k^2-m^2+i\varepsilon}$$

$$\frac{-i g_{\mu\nu}}{k^2}$$



$\star$ Compton scattering : $\gamma(k_1)\, e(p_1) \to \gamma(k_2)\, e(p_2)$



$$M = \bar{u}_2\, \varepsilon^*_{2\mu} (ie\gamma^\mu) \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} (ie\gamma^\nu)\, u_1\, \varepsilon_{1\nu}$$

$$+ \bar{u}_2\, \varepsilon^*_{2\mu} (ie\gamma^\nu) \frac{i(\not{k}'+m)}{k'^2-m^2+i\varepsilon} (ie\gamma^\mu)\, u_1\, \varepsilon_{1\nu}$$

$$= \varepsilon^*_{2\mu}\, \varepsilon_{1\nu}\, \overbrace{\bar{u}_2 (ie)^2 i \left[\frac{\gamma^\mu(\not{k}+m)\gamma^\nu}{k^2-m^2+i\varepsilon} + \frac{\gamma^\nu(\not{k}'+m)\gamma^\mu}{k'^2-m^2+i\varepsilon} \right] u_1}^{M^{\mu\nu}}$$

$$\Rightarrow \bar{M} = \frac{1}{2}\sum_{\lambda_1\lambda_2} \frac{1}{2}\sum_{s_1 s_2} |M|^2$$ averaged over initial, summed over final.

Polarization directions:

$$\sum_{\lambda=1}^{2} \varepsilon^\mu(\bar{p},\lambda)\, \varepsilon^{\nu *}(\bar{p},\lambda) = -g^{\mu\nu} + \frac{p^\mu p_\lambda^\nu + p^\nu p_\lambda^\mu}{p\cdot p_\lambda}$$

with $p_\lambda = (E_p, -\bar{p})$
$p = (E_p, \bar{p})$

$\downarrow$

$p\cdot p_\lambda = 2E_p^2$

Because of gauge invariance! second term does not contribute: $\quad p^\mu M_\mu = 0$

$\downarrow$

can just use $-g^{\mu\nu}$ for any external photon lines

[not true for $m \neq 0$ vector fields:

$$\sum_{\lambda=1}^{3} \varepsilon^\mu(\bar{p},\lambda)\, \varepsilon^{\nu *}(\bar{p},\lambda) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$]

$$\Rightarrow \bar{M}^2 = \frac{1}{2}\sum_{\lambda_2} \varepsilon^*_{2\mu}\, \varepsilon_{2\sigma} \sum_{\lambda_1} \varepsilon_{1\nu}\, \varepsilon^*_{1\rho}$$

$$\cdot \frac{1}{2}\sum_{s_2}\sum_{s_1} M^{\mu\nu} M^{*\sigma\rho}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\sum_{s_2}\sum_{s_1} M^{\mu\nu} M^*_{\nu\mu}$$

$$= \frac{e^4}{4}\sum_{s_1 s_2} \bar{u}_2 \left[\frac{\gamma^\mu(\not{p}_1 + \not{k}_1 + m)\gamma^\nu}{(p_1+k_1)^2 - m^2} + \frac{\gamma^\nu(\not{p}_1 - \not{k}_2 + m)\gamma^\mu}{(p_1-k_2)^2 - m^2}\right] u_1$$

$$\bar{u}_1 \left[\frac{\gamma_\mu(\not{p}_1 + \not{k}_1 + m)\gamma_\nu}{(p_1+k_1)^2 - m^2} + \frac{\gamma_\nu(\not{p}_1 - \not{k}_2 + m)\gamma_\mu}{(p_1-k_2)^2 - m^2}\right] u_2$$

$$\sum_{s_1} u_1 \bar{u}_1 \rightarrow \mathrm{Tr}\left[\cdots (\not{p}_1 + m) \cdots \right]$$

$$\sum_{s_2} \bar{u}_2 u_2 \rightarrow \mathrm{Tr}\left[\cdots (\not{p}_2 + m) \cdots \right]$$

$$\Rightarrow \overline{M}^2 = \frac{e^4}{4} \mathrm{Tr}\Big[(\not{p}_1 + m) \times$$

$$\left(\gamma^\mu \frac{(\not{p}_1 + \not{k}_1 + m)}{(p_1 + k_1)^2 - m^2} \gamma^\nu + \gamma^\nu \frac{(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2} \gamma^\mu \right) \times$$

$$(\not{p}_2 + m) \times$$

$$\left(\gamma_\nu \frac{(\not{p}_1 + \not{k}_1 + m)}{\underbrace{(p_1 + k_1)}_{s}{}^2 - m^2} \gamma^\mu + \gamma_\mu \frac{(\not{p}_1 - \not{k}_2 + m)}{\underbrace{(p_1 - k_2)}_{u}{}^2 - m^2} \gamma^\nu \right) \Big]$$

With $m = 0$ (relativistic limit for $E \gg m_e$):

$$\overline{M}^2 = \frac{e^4}{4} \left[\mathrm{Tr}\left[\not{p}_1 \left(\gamma^\mu \frac{\not{p}_1 + \not{k}_1}{s} \gamma^\nu \right) \not{p}_2 \left(\gamma_\nu \frac{\not{p}_1 + \not{k}_1}{s} \gamma_\mu \right) + 3 \text{ terms} \right.\right.$$

$$= \frac{e^4}{4} \left[\frac{1}{s^2} \mathrm{Tr}\Big[\underbrace{\not{p}_1 \gamma^\mu (\not{p}_1 + \not{k}_1) \underbrace{\gamma^\nu \not{p}_2 \gamma_\nu}_{-2\not{p}_2} (\not{p}_1 + \not{k}_1) \gamma_\mu}_{4(\not{p}_1 + \not{k}_1)\not{p}_2(\not{p}_1 + \not{k}_1)} \Big] + \ldots \right.$$

$$= \frac{e^4}{s^2} \mathrm{Tr}\left[\not{p}_1 (\not{p}_1 + \not{k}_1) \not{p}_2 (\not{p}_1 + \not{k}_1) \right] + \cdots$$

$$\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}] = 4(a\cdot b\; c\cdot d + a\cdot d\; b\cdot c - a\cdot c\; b\cdot d)$$

and $p_1^2 = 0,\ p_2^2 = 0$
$k_1^2 = 0,\ k_2^2 = 0$

$$\bar{M}^2 = \frac{8e^4}{s^2}\, p_1\cdot k_1\; k_1\cdot p_2 + \cdots$$

Only terms in $\frac{1}{s^2}$ and $\frac{1}{u^2}$ are non-zero.

Since $m=0$: $2p_1\cdot k_1 + \underset{\overset{\parallel}{0}}{p_1^2} + \underset{\overset{\parallel}{0}}{k_1^2} = s$

$$-2p_2\cdot k_1 + p_2^2 + k_1^2 = u$$

$$\Rightarrow \bar{M}^2 = -2e^4\left(\frac{u}{s} + \frac{s}{u}\right) + \text{terms in } O(m^2)$$

which are zero here

Finally : $$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s}\bar{M}^2 = \frac{e^4}{16\pi s}\left(1+\cos\theta + \frac{2}{1+\cos\theta}\right)$$

$$\begin{cases} s = 4p^2 \\ u = -2p^2(1+\cos\theta)\end{cases} = \frac{e^4}{16\pi s}\left(\frac{4+\cos^2\theta + 9\cos\theta}{1+\cos\theta}\right)$$

Do we get the right result? I.e. does $O(m^2)$ matter compared to $\frac{u}{s} + \frac{s}{u}$?

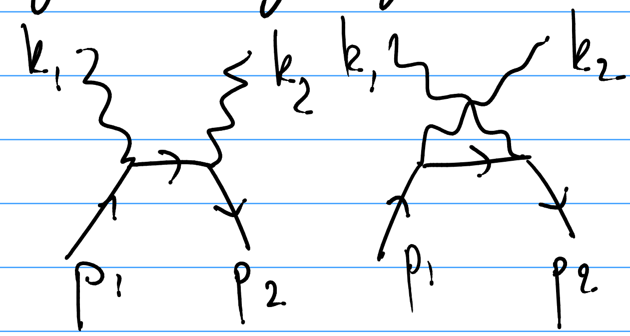
Yes $\rightarrow$ full relativistic calculation (even in the limit $m\rightarrow 0$) requires inclusion of $m \neq 0$ due to mass in s and u chan.

Mechanics is the same, but correct result:

$$\frac{d\sigma}{d\cos\theta} \propto \frac{\alpha^2}{m_e^2} \quad \text{valid for } k_1, k_2 < m_e$$

$\ast$ e^+e^- annihilation to photons

Even if Compton scattering in $m \to 0$ limit is not accurate description, can be used by crossing symmetry to describe $e^+e^- \to \gamma\gamma$



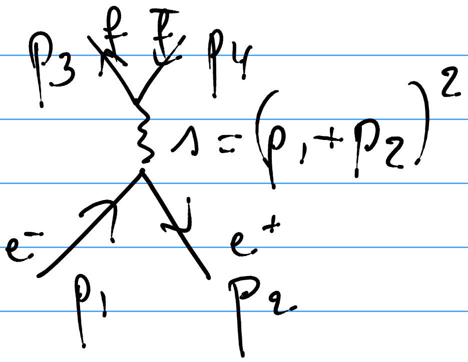
$$\longrightarrow \overline{M}^2 = 2e^4\left(\frac{t}{u} + \frac{u}{t}\right)$$

$$\begin{cases} s = 2p_1\cdot k_1 \to 2p_1\cdot p_2 = t \\ u = 2p_1\cdot k_2 \to 2p_1\cdot k_2 = u \end{cases}$$

Compton $\longrightarrow$ e^+e^-

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)$$

$\ast$ e^+e^- annihilation to fermions



$$\overline{M}^2 = \frac{1}{4}\sum_{s_1,\dots s_4} |M|^2 \quad m_e \simeq 0$$

$$= \frac{Q_f^2 e^4}{4 s^2} \mathrm{Tr}\left[\gamma_\nu(\not{p}_4+m)\gamma_\mu(\not{p}_3+m)\right] \times \mathrm{Tr}\left[\gamma^\nu \not{p}_1 \gamma^\mu \not{p}_2\right]$$

$$= \frac{8Q_f^2 e^4}{s^2}\left(p_1\cdot p_4 \; p_2\cdot p_3 + p_1\cdot p_3 \; p_2\cdot p_4 + m^2 \, p_1\cdot p_2\right)$$

Collider kinematics: $e^- \rightarrow \quad \leftarrow e^+$

$p_1(E,0,0,E)$ $\quad p_2(E,0,0,-E)$

$p_{3,4}\left(E, \pm p\sin\theta, 0, \pm p\cos\theta\right)$

If $m=0$ in the final state ($E \gg m_e$ and m_f):

$$\overline{M}^2 = Q_f^2 e^4 (1+\cos^2\theta)$$

$$\Rightarrow \frac{d\bar{\sigma}}{d\cos\theta} = \frac{\pi Q_f^2 \alpha^2}{2s}(1+\cos^2\theta)$$

$$\Rightarrow \bar{\sigma} = \frac{4\pi Q_f^2 \alpha^2}{3s} \propto Q_f^2$$

→ incoherent sum of all fermions with $m_f < s$

$$R = \frac{\bar{\sigma}(e^+e^- \to \text{strong}) \propto Q_u^2 + Q_d^2 + Q_s^2 + Q_c^2 + Q_b^2}{\bar{\sigma}(e^+e^- \to \mu^+\mu^-) \propto (1)^2}$$

for $s < 175$ GeV

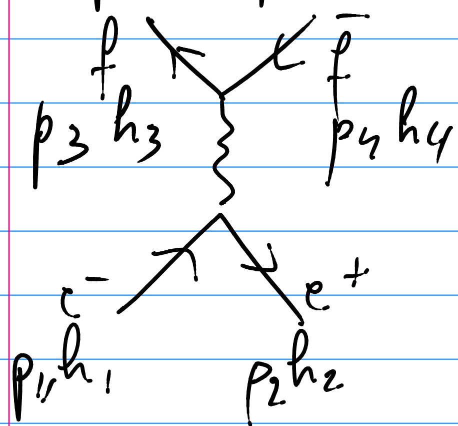
Expt: $R(10\text{ GeV} < s < 40\text{ GeV}) \approx 4$.

Theory: $\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{11}{9}$

→ too small

But quarks have 3 colors → 3 distinguishable final states → $R = 3\sum Q_f^2 = \frac{11}{3} \approx 4$

* Spin dependence in $e^+e^- \to f\bar{f}$ $(f \neq e)$



Since $m_e = m_f = 0 \to$ helicity h_i

$$\longrightarrow M(-+,-+) = M(+-,+-)$$

$$M(-+,+-) = M(+-,-+)$$

$$\to |M|^2 = \frac{Q_f^2 e^4}{s^2} \mathrm{Tr}\left[\gamma_\nu \not{p}_4 \left(\frac{1+\gamma^5 \not{s}_4}{2}\right) \gamma_\mu \not{p}_3 \left(\frac{1+\gamma^5 \not{s}_3}{2}\right)\right]$$

$$\times \mathrm{Tr}\left[\gamma^\nu \not{p}_1 \left(\frac{1+\gamma^5 \not{s}_1}{2}\right) \gamma^\mu \not{p}_2 \left(\frac{1+\gamma^5 \not{s}_2}{2}\right)\right]$$

using $\quad u(\bar{p}, s)\, \bar{u}(p,s) = (\not{p} + m)\left(\dfrac{1+\gamma^5 \not{s}}{2}\right)$

$$\xrightarrow{m=0} \left(\frac{1 \pm \gamma^5}{2}\right) \not{p} = P_{R,L} \not{p}$$

$$v(\bar{p}, s)\, \bar{v}(p,s) = (\not{p} - m)\left(\frac{1+\gamma^5 \not{s}}{2}\right)$$

$$\xrightarrow{m=0} \left(\frac{1 \mp \gamma^5}{2}\right) \not{p} = P_{L,R} \not{p}$$

$$\to |M(+-,+-)|^2 = \frac{Q_f^2 e^4}{s^2} \mathrm{Tr}\left[\gamma_\nu \not{p}_4 \gamma_\mu \not{p}_3 P_L\right] \times$$

$$\mathrm{Tr}\left[\gamma^\nu \not{p}_1 \gamma^\mu \not{p}_2 P_L\right]$$

$$|M(+-,-+)|^2 = \frac{Q_f^2 e^4}{s^2} \mathrm{Tr}\left[\gamma_\nu \not{p}_4 \gamma_\mu \not{p}_3 P_R\right] \times$$

$$\mathrm{Tr}\left[\gamma^\nu \not{p}_1 \gamma^\mu \not{p}_2 P_R\right]$$

$$\rightarrow |M(+-,+-)|^2 = |M(-+,-+)|^2 = Q_f^2 e^4 (1+\cos\theta)^2$$

$$|M(+-,-+)|^2 = |M(-+,+-)|^2 = Q_f^2 e^4 (1-\cos\theta)^2$$

$$\frac{1}{4}\sum |M(\lambda)|^2 = Q_f^2 e^4 (1+\cos^2\theta) \quad \text{indeed}$$

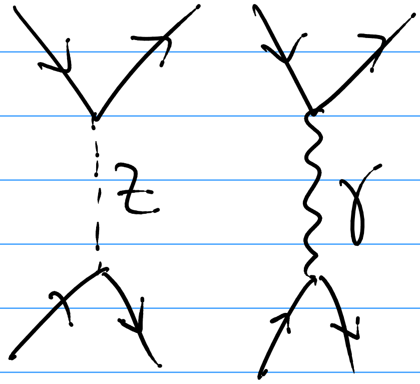
Experimental observables often require polarization asymmetries, e.g. for fixed incoming spins and if we can somehow measure scattered fermion spin:

$$A = \frac{\sigma(+-,+-) - \sigma(+-,-+)}{\sigma(+-,+-) + \sigma(+-,-+)} \quad \left(\text{or } \int_{\text{acceptance}} \frac{d\sigma}{d\cos\theta} d\Omega\right)$$

$$= \frac{Q_f^2 e^4 (1+\cos\theta)^2 - Q_f^2 e^4 (1-\cos\theta)^2}{Q_f^2 e^4 (1+\cos\theta)^2 + Q_f^2 e^4 (1-\cos\theta)^2}$$

$$= \frac{\cos\theta}{1+\cos^2\theta}$$ $\longrightarrow$ independent of Q_f, e; only depends on θ!

$\rightarrow$ see later for A_{FB}, $A_{L,R}$ determination of weak neutral current



$\rightarrow$ identical final state

$\longrightarrow |M_1 + M_2|^2$ with

$$M_1^* M_2 (+s_2, s_3 s_4) \neq M_1^* M_2 (-s_2, s_3 s_4)$$