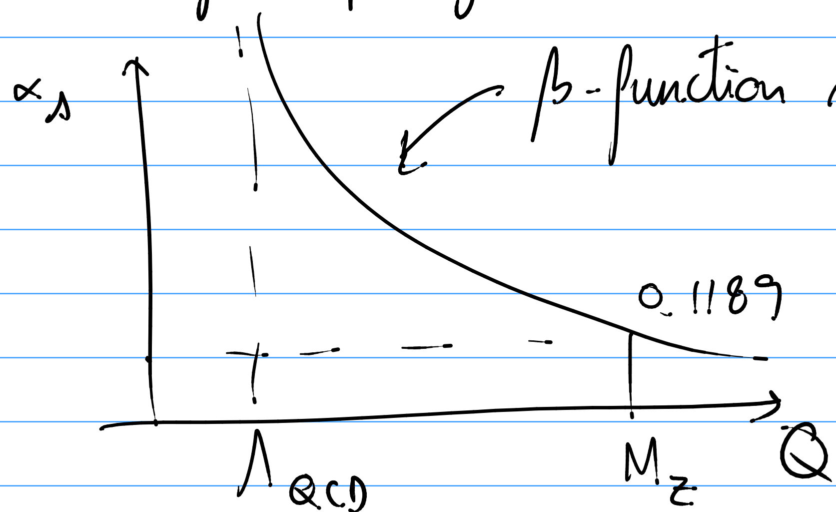
Phys 772 : Week 6 Thursday

* Running coupling in QCD



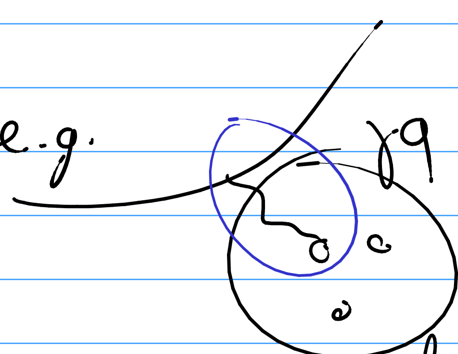
β-function negative, $b' < 0$

$$\frac{d\alpha_s}{d\ln Q^2} = \beta = b'\alpha_s^2 + O(\alpha_s^3)$$

Λ_{QCD}	M_Z
$\sim 200-300\,\text{MeV}$	$\sim 90\,\text{GeV}$
long distances	short distances
non-perturbative	perturbative
	weakly coupled q and g
$\downarrow$	$\downarrow$
hadron spectroscopy	deep inelastic scattering

Factorization : most processes can be "factorized" in long distance and short distance contributions

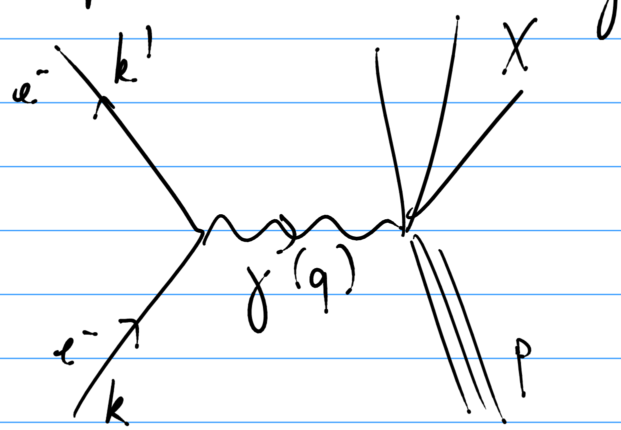
e.g.



qg scattering,
short distance
weakly coupled ("free") quarks or gluons

p, hadron,
long distance
non-perturbative

- Deep inelastic scattering : $e^- p \to e^- X$



$Q^2 = -q^2 \gg 1\,\mathrm{GeV}^2$

↳ t-channel

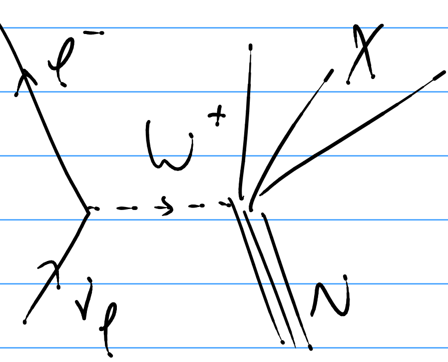
↓

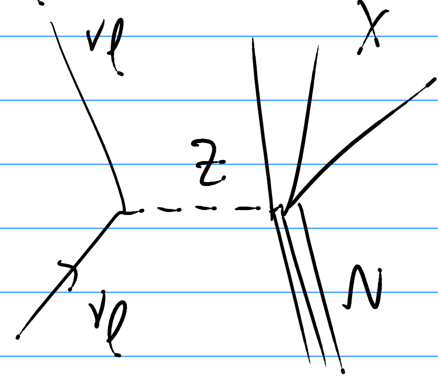
large momentum transfer

→ small distance scales

→ free quark/gluons

Generally : $e^\pm N$, $\mu^\pm N$, $\nu_\ell N$, $\bar{\nu}_\ell N$





Kinematics : $q = k - k'$

typically $k \approx E$, $k' \approx E'$ as $k, k' \gg m_e$, $E, E' \gg m_e$

momentum transfer $Q^2 = -q^2 = -(k-k')^2$

$= 2kk'(1-\cos\theta)$ in lab frame

$\approx 2EE'(1-\cos\theta)$ for $E, E' \gg m_e$

energy transfer $\nu = E - E' = \frac{p \cdot q}{M}$ $\quad (p = (M, \vec{0}))$

invariant mass $W^2 = p_X^2 = (p+q)^2 = M^2 + 2M\nu - Q^2$

dimensionless $x = \frac{Q^2}{2M\nu}$ and $y = \frac{\nu}{E}$, both $[0, 1]$

Only Lorentz scalars are ν, Q^2, W^2, but W^2 depends on ν, Q^2

$\to$ hadronic parts will only depend on ν, Q^2

1) Elastic scattering : $X = p \to W^2 = M^2$

$\to 2M\nu = Q^2$

$\to x = 1$

2) Single pion inelastic scattering : $X = p + \pi^0, n + \pi^+$

$$W^2 = (p_p + p_\pi)^2 = (M_p + M_\pi)^2 + \dots \geq (M_p + M_\pi)^2$$

3) Complicated final states : W^2 continuous

* Elastic scattering $e^- p \to e^- p \to k, k', \theta$: pick 2

$$d\sigma = \frac{(2\pi)^4 \delta^4(\Sigma p)}{4Mk} \frac{d^3k'}{(2\pi)^3 2E_k'} \frac{d^3p'}{(2\pi)^3 2E_p'} \frac{e^4}{q^4} L_e^{\mu\nu} L_{p\,\mu\nu}$$

with leptonic tensor $L_e^{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\gamma^\mu (\not{k} + m) \gamma^\nu (\not{k}' + m)\right]$

and hadronic tensor $L_{p\,\mu\nu} = \frac{1}{2} \mathrm{Tr}\left[\gamma_\mu (\not{p} + M) \gamma_\nu (\not{p}' + M)\right]$

$$L_e^{\mu\nu} = 2\left[k^\mu k'^\nu + k'^\mu k^\nu + g^{\mu\nu}\frac{q^2}{2}\right]$$

$$L_{p\mu\nu} = 2\left[p_\mu p'_\nu + p'_\mu p_\nu + g_{\mu\nu}\frac{q^2}{2}\right]$$

$$\longrightarrow \frac{d\sigma}{dk'\,d\Omega} = \frac{\alpha^2}{q^4}\frac{k'}{k} L_e^{\mu\nu} W_{\mu\nu}$$ for specific $W_{\mu\nu}$ tensor which contains all info about proton

δ function requiring elasticity

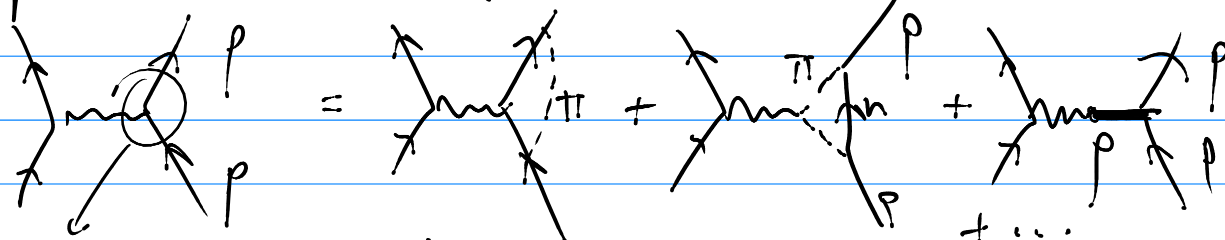
If proton is elementary in pure QED:

$$M = (ie)\bar{u}(k')\gamma^\mu u(k)\left(\frac{-i g_{\mu\nu}}{q^2}\right)(-ie)\bar{u}(p')\gamma^\nu u(p)$$

and $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{q^4}\frac{k'}{k}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right)$$

actually $4k^2\sin^4\frac{\theta}{2}$

But proton contains QCD structure:



$$-ie\gamma^\nu \longrightarrow -ie\Gamma^\nu$$ with

$$\Gamma^\mu = \gamma^\mu F_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M} q_\nu F_2^p(q^2) + q^\mu F_3^p(q^2) + \gamma^\mu\gamma^5 g_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M}\gamma^5 q_\nu g_2^p(q^2) + q^\mu\gamma^5 g_3^p(q^2)$$

g_1^p, g_2^p, g_3^p require parity violation → not present in QED + QCD but allowed by weak interaction in proton

Because $q_\mu J^\mu = q_\mu \bar{u}(p') \Gamma^\mu u(p) = 0$

$$\rightarrow \begin{cases} F_3^p(q^2) = 0 \\ \text{and} \\ 2M g_1^p(q^2) + q^2 g_3^p(q^2) = 0 \end{cases}$$

⇒ QED + QCD elastic scattering :

$$\Gamma^\mu = \gamma^\mu F_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M} q_\nu F_2^p(q^2)$$

with electromagnetic form factors

$$\begin{cases} G_E = F_1 - \tau F_2 \\ G_M = F_1 + F_2 \end{cases}$$ are Sachs $\begin{matrix}\text{electric}\\ \text{form factors}\\ \text{magnetic}\end{matrix}$

with $\tau = \frac{Q^2}{4M}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha}{q^4} \frac{k'}{k} \left[\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right]$$

↳ Rosenbluth formula

with $G_E^p(0) = 1$, $G_M^p(0) = 1 + \kappa$

Also $F_1^n, F_2^n, G_E^n, G_M^n$ for neutron scattering

For proton: $F_1^p(0) = 1$, $F_2^p(0) = \kappa_p$ = anomalous magnetic moment ≈ 1.79

$$\downarrow$$

$$\Gamma^\mu(0) = \gamma^\mu + \frac{i\sigma^{\mu\nu}}{2M} q_\nu \kappa_p \approx \gamma^\mu \text{ for } Q^2 < M^2$$

with κ_p level corrections $\gg \frac{\alpha}{2\pi}$ QED NLO

For neutron: $F_1^n(0) = 0$, $F_2^n(0) = \kappa_n \approx -1.91$

$F^V = \frac{1}{2}(F^p - F^n)$ isovector $\leftrightarrow$ $F^S = \frac{1}{2}(F^p + F^n)$ isoscalar

Dipole form factor

$$G_E^p(Q^2) \approx G_D(Q^2) = \frac{1}{\left(1 + Q^2/Q_0^2\right)^2} \quad \text{with } Q_0^2 = 0.71\ \text{GeV}^2$$

$$G_M^p(Q^2) = (1+\kappa_p)\, G_E^p(Q^2)$$

Proton charge radius:

$$G_E^p(Q^2) \approx 1 - \frac{1}{6} R_p^2 Q^2$$

$$\text{with } R_p^2 = \langle r^2 \rangle = -6 \left.\frac{d G_E^p(Q^2)}{dQ^2}\right|_{Q^2=0}$$

$$= \int d^3r\, |\vec{r}|^2 \, |\psi(\vec{r})|^2 = \text{RMS charge radius}$$

* Inelastic scattering : $e^- p \to e^- X$

Same form for $\frac{d\sigma}{dk' d\Omega}$ can be used, with $W_{\mu\nu}$

$W_{\mu\nu}$ can only depend on p and q ($p' = p + q$)

- symmetric tensors : $p^\mu p^\nu$, $q^\mu q^\nu$, $p^\mu q^\nu + p^\nu q^\mu$, $g^{\mu\nu}$

- anti-symmetric tensors : $p^\mu q^\nu - p^\nu q^\mu$, $\varepsilon^{\mu\nu\rho\sigma} g_{\rho\sigma}$

But $L_e^{\mu\nu}$ is symmetric $\to$ $W_{\mu\nu}$ must be too

Most general form (ignoring spin-dependence)

$$W_{\mu\nu} = \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] W_1(Q^2, \nu)$$
$$+ \frac{1}{M^2}\left[p_\mu - \frac{p\cdot q}{q^2} q_\mu\right]\left[p_\nu - \frac{p\cdot q}{q^2} q_\nu\right] W_2(Q^2, \nu)$$

W_1, W_2 are structure functions

$$\frac{d\sigma}{dk' d\Omega} = \frac{\alpha^2}{q^4}\left[W_2(Q^2, \nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2, \nu)\sin^2\frac{\theta}{2}\right]$$

$\to$ W_1, W_2 generalize F_1, F_2 for elastic scatt. but have dependence on ν as well

$$\begin{cases} F_1(x, Q^2) = M W_1(Q^2, \nu) \\ F_2(x, Q^2) = \nu W_2(Q^2, \nu) \end{cases}$$ notations

Bjorken scaling: as $Q^2 \to \infty$

$F_1(x, Q^2) \to F_1(x)$ (constant charge)

instead of $F_1(x, Q^2) \to 0$ (negligible charge)

$\Rightarrow$ scattering independent of small scale $\sim \frac{1}{Q^2}$

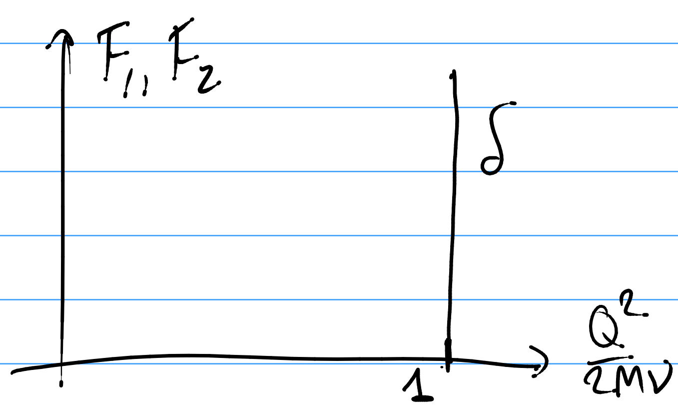
$\rightarrow$ hard partons inside proton off which the electrons, virtual photons scatter

* Quark parton model (as the precursor to a full understanding of QCD)

– elastic scattering:

$$\begin{cases} W_1(Q^2, \nu) = \dfrac{Q^2}{4M^2}\,\delta\left(\nu - \dfrac{Q^2}{2M}\right) = \dfrac{Q^2}{4M^2\nu}\,\delta\left(1 - \dfrac{Q^2}{2M\nu}\right) \\ W_2(Q^2, \nu) = \delta\left(\nu - \dfrac{Q^2}{2M}\right) = \dfrac{1}{\nu}\,\delta\left(1 - \dfrac{Q^2}{2M\nu}\right) \end{cases}$$

$$F_1 = MW_1 = \frac{1}{2}\,\delta\left(1 - \frac{Q^2}{2M\nu}\right) \;\&\; F_2 = \delta\left(1 - \frac{Q^2}{2M\nu}\right)$$



- scattering off point-like partons:

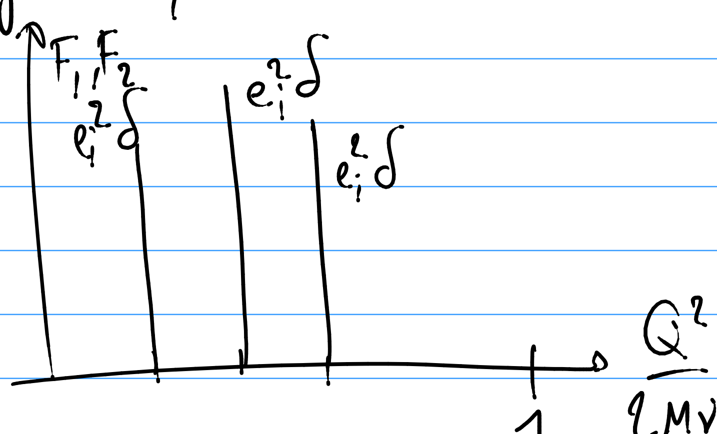
$M \to x_i M$ (momentum fraction), charge e_i^2

$$\begin{cases} W_1^i(Q^2,\nu) = e_i^2 \dfrac{Q^2}{4x_i M^2 \nu}\, \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \\ W_2^i(Q^2,\nu) = e_i^2 \dfrac{x_i}{\nu}\, \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \end{cases}$$

x_i here moved outside of δ

$$\to \begin{cases} F_1^i = M W_1^i = \dfrac{1}{2} e_i^2\, \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \\ F_2 = \nu W_2^i = e_i^2 x_i\, \delta\left(x_i - \dfrac{Q^2}{2M\nu}\right) \end{cases}$$

e.g. 3 quarks



- distribution of valence quarks inside proton:

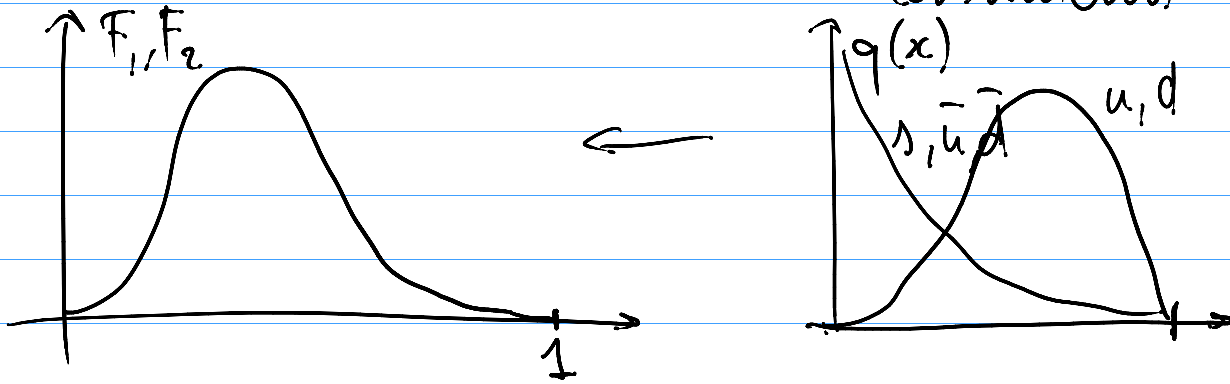
$q_i(x_i)$ probability density = parton distribution function PDF

$$F_2(x,Q^2) = \sum_i \int_0^1 dx_i\, q_i(x_i)\, e_i^2 x_i\, \delta\left(x_i - \frac{Q^2}{2M\nu}\right)$$

$$= 2x F_1(x,Q^2) = \sum_i e_i^2\, x\, q_i(x), \quad x = \frac{Q^2}{2M\nu}$$

Callan-Gross relation: $F_2(x, Q^2) = 2x F_1(x, Q^2)$

↳ indicates scattering off point-like constituents



⇒ started with x = convenient dimensionless kinematic variable

↓

now x has interpretation as momentum fraction carried by struck quark

$q(x)$ = non-perturbative quantity that is messy to calculate

– distribution of sea quarks inside the proton

$$F_2(x, Q^2) = \sum_i e_i^2 \left(q_i(x) + \bar{q}_i(x) \right)$$

valence + sea   sea

$\int_0^1 u_v(x)\, dx = 2$, $\int_0^1 d_v(x)\, dx = 1$ for proton

$$\sum_i \int_0^1 x\left(q_i(x)+\bar{q}_i(x)\right)dx + \int_0^1 x\,G(x)\,dx = 1$$

$\rightarrow$ entire momentum fraction explained by all partons

* Evolution equations:

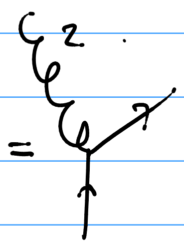
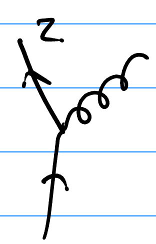
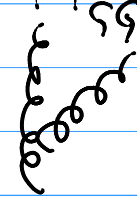
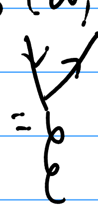
All previous is assuming α_s is constant, but we know that α_s runs with $\ln Q^2$

$\rightarrow$ modifications to $q(x)$, $G(x)$ depending on scale at which processes are considered

$\rightarrow$ effective Q^2 dependence is (re) introduced by QCD because partons are NOT independent

DGLAP equations:

$$\frac{dG(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dw}{w}\left[\sum_i P_{Gq}\Big(\underbrace{\frac{x}{w}}_{z}\Big)\, q_j(w,Q^2) + P_{GG}\left(\frac{x}{w}\right) G(w,Q^2)\right]$$

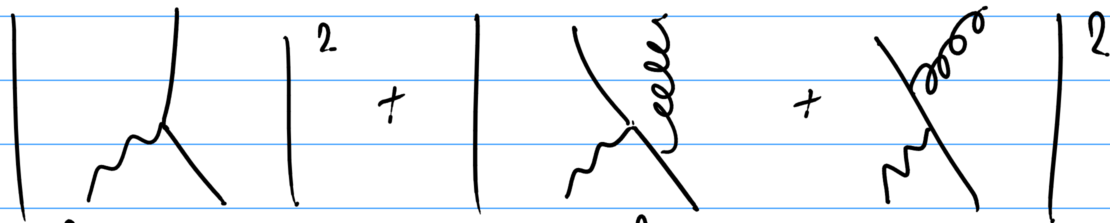
$P_{Gq} =$  $P_{qq} =$  $P_{GG} =$  $P_{qG} =$ 

$$\frac{dq_i(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dw}{w}\left[P_{qq}\left(\frac{x}{w}\right) q_i(w,Q^2) + P_{qG}\left(\frac{x}{w}\right) G(w,Q^2)\right]$$

e.g. $P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right)$ in LO

as $z \to 1$: gluon very soft $\to$ infrared divergence

if $z<1$: hard gluon emission
$\to$ original quark receives transverse momentum p_T



0 gluon emission

$\underbrace{\text{1 gluon emission}}_{P_{qq}(z)}$

Why is DGLAP relevant to Standard Model?

LHC : $p\bar{p} \to F + X$

F: desired final state

X: other stuff

p and $\bar{p}$ are really $q, \bar{q}$ or g

$$\Rightarrow \sigma(p\bar{p} \to F+X) = \sum_i \sum_j \int_0^1 dx_i \int_0^1 dx_j \underbrace{f_i^p(x_i)\, f_j^{\bar{p}}(x_j)}_{\text{at } Q^2} \hat{\sigma}(ij \to F+X)(x_i, x_j)$$