Phys 772: Week 3 Thursday

* Gauge invariance : a recap

Requirements on Hamiltonian H

1) time-evolution operator $U = e^{-iHt}$ should be unitary $\rightarrow$ H should be hermitian since if U unitary, then it can be written as e^{iH} with H hermitian.

$H = H^\dagger$

$U^\dagger U = 1$

2) invariance under Lorentz transformations $\Lambda \in SO(3,1)$ with $x'^\mu = \Lambda^\mu{}_\nu x^\nu$
[and translations with $x'^\mu = x^\mu + \xi^\mu$ which we ignore here]

For infinitesimal Lorentz transformation:

$\Lambda^\mu{}_\nu = \underbrace{\delta^\mu_\nu}_{\text{identity } 1} + \omega^\mu{}_\nu + O(\omega^2)$ with ω matrix of boosts, rotations

For finite Lorentz transformations this is:

$\Lambda^\mu{}_\nu = (e^\omega)^\mu{}_\nu$ which

If we have a state in a Hilbert space, transf. is given by unitary operator $U(\Lambda)$ which operates on the Hilbert space.

$U(\Lambda)^\dagger = U(\Lambda)$, $\quad U(\Lambda_1) U(\Lambda_2) = U(\Lambda_1 \Lambda_2)$

For infinitesimal transformations, close to identity

$$U(\omega) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu} + O(\omega^2) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}$$

$$\left[\, U(\xi) = \mathbb{1} - i\xi_\mu \hat{P}^\mu + O(\xi^2) = e^{-i\xi_\mu \hat{P}^\mu} \,\right]$$

$\hat{J}^{\mu\nu}$ = angular momentum operator

$\hat{P}^\mu$ = four-momentum operator

$$U(\omega_1)\,U(\omega_2) = U(\omega_1\omega_2)$$

$$= \left(\mathbb{1} + \frac{i}{2}\omega_{1\mu\nu}\hat{J}^{\mu\nu}\right)\left(\mathbb{1} + \frac{i}{2}\omega_{2\rho\sigma}\hat{J}^{\rho\sigma}\right)$$

$$= \left(\mathbb{1} + \frac{i}{2}\omega_{1\mu\nu}\hat{J}^{\mu\nu} + \frac{i}{2}\omega_{2\rho\sigma}\hat{J}^{\rho\sigma}\right)$$

$$\downarrow$$

$$U(\omega_1\omega_2) - U(\omega_2\omega_1) = -\frac{i^2}{2}\omega_{1\mu\nu}\omega_{2\rho\sigma}\left[\hat{J}^{\mu\nu}, \hat{J}^{\rho\sigma}\right]$$

with $\left[\hat{J}^{\mu\nu}, \hat{J}^{\rho\sigma}\right] = i\left(g^{\nu\sigma}\hat{J}^{\mu\rho} + g^{\mu\rho}\hat{J}^{\nu\sigma} - g^{\mu\sigma}\hat{J}^{\nu\rho} + g^{\nu\rho}\hat{J}^{\mu\sigma}\right)$

By now defining
$$\begin{cases} \hat{J}_i = \frac{1}{2}\varepsilon_{ijk}\hat{J}_{jk} & (i=1,2,3) \\ \hat{K}_i = J^0{}_i & (i=1,2,3) \end{cases}$$

we get
$$\begin{cases} [\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk}\hat{J}_k \\ [\hat{K}_i, \hat{K}_j] = -i\varepsilon_{ijk}\hat{J}_k \\ [\hat{J}_i, \hat{K}_j] = i\varepsilon_{ijk}\hat{K}_k \end{cases}$$

Special combinations : $\hat{L}_i = \frac{\hat{J}_i + i\hat{K}_i}{2}$

$\hat{R}_i = \frac{\hat{J}_i - i\hat{K}_i}{2}$

$$\rightarrow \begin{cases} [\hat{L}_i, \hat{L}_j] = i\varepsilon_{ijk}\hat{L}_k \\ [\hat{R}_i, \hat{R}_j] = i\varepsilon_{ijk}\hat{R}_k \\ [\hat{L}_i, \hat{R}_j] = 0 \end{cases}$$

↳ two subgroups, each satisfying the commutation relations of SU(2)

$\rightarrow SU(2)_L \times SU(2)_R$

↓

will give rise to L and R spinor representations

All previous is operator algebra for Lorentz group SO(3,1) with generators $\hat{J}_k$ for rotations and $\hat{K}_k$ for boosts.

$$e^{\text{parameter} \cdot \underset{\text{matrix}}{\text{generator}}} = \text{operator}$$

Now transition to representations of states, which will require a set of $\hat{J}_k$ and $\hat{K}_k$ that are represented as matrix operators

- general case: $\bar{V}'(x') = \bar{V}(x)$ with $\bar{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_a \end{bmatrix}$

$$V'_a(x) = \underset{\substack{\uparrow \\ \text{transform back}}}{U(\omega)}\, \underset{\substack{\uparrow \\ \text{value}}}{V_a(x)}\, \underset{\substack{\uparrow \\ \text{transform to}}}{U(\omega)^{+}} = \underset{\substack{\uparrow \\ \text{transform back}}}{D^{-1}_{ab}(\omega)}\, \underset{\substack{\uparrow \\ \text{at transf. point}}}{V_b(\Lambda x)}$$

" $D_{ab}(-\omega)$

$\bar{V} = \begin{bmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \\ \psi_u \\ \vdots \\ \varphi_H \end{bmatrix}$ in Standard Model,

4 comp. spinor

but $D^{-1}_{ab}(\omega)$ takes block diagonal form → can be reduced into irreducible representations (irreps) that transform non-independently.

→ find irreps of Lorentz group transformation
↳ $\varphi(x)$, $\psi(x)$, $A^\mu(x)$
with corresponding matrix representations for $D_{ab}(-\omega)$

Irreducible representations:

- scalar representation: $\bar{V} = V_1 = \varphi(x)$
↳ singlet represent.

$(0,0)$ $\quad \varphi'(x') = \varphi(x)$ if

$$U(\omega)\,\varphi(x)\,U^{+}(\omega) = D^{-1}(\omega)\,\varphi(\Lambda x)$$

with $D^{-1}(\omega)$ a "matrix" operating on scalars, i.e. a scalar itself

$J^{\mu\nu} = 0$, $D(-\omega) = \mathbb{1}$ (identity for scalars)

- vector representation: $\bar{V} = A^{\mu}(x)$

$(\frac{1}{2},\frac{1}{2})$ $\quad U(\omega)\,A^{\mu}(x)\,U^{+}(\omega) = (\Lambda^{-1})^{\mu}{}_{\nu}(\omega)\,A^{\nu}(\Lambda x)$

with $D^{-1}(\omega)$ represented by $\Lambda(\omega)^{-1}$

- spinor representations: $\bar{V} = \psi(x)$

simplest non-trivial (non-scalar) matrices that satisfy $[\hat{L}_i, \hat{L}_k] = i\varepsilon_{ijk}\hat{L}_k$ are Pauli matrices:

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma_k}{2}$$

Along with $\hat{K}_i = 0$: → <u>left-handed spinor</u>

$(\frac{1}{2},0)$ $\quad \hat{J}_i = \frac{\sigma_i}{2}$ and $\hat{K}_i = -i\frac{\sigma_i}{2}$

$(0, \frac{1}{2})$ Alternatively: $\hat{L}_i = 0$, $\hat{R}_i = \frac{\sigma_i}{2}$

$\hat{J}_i = \frac{\sigma_i}{2}$ and $\hat{K}_i = i\frac{\sigma_i}{2}$

$\hookrightarrow$ right-handed spinor

Transformation of $\left\{\begin{matrix}\text{Weyl} \\ \text{two-component}\end{matrix}\right.$ spinor:

$$U(\omega)\, \psi_a(x)\, U(\omega)^\dagger = D^{-1}_{ab}\, \psi_b(\Lambda x)$$

with $D(\omega) = e^{-i(\theta_i + i b_i)\frac{\sigma_i}{2}}$

$\rightarrow$ four-component spinors:

$$U(\omega)\, \psi_a(x)\, U(\omega)^\dagger = D^{-1}_{ab}\, \psi_b(\Lambda x)$$

with $D(\omega) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$

$$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$\star$ Gauge boson vector fields, $m = 0$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$A^\mu = \sum_{\lambda = \pm 1} \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \Big[\varepsilon^\mu(\bar{p}, \lambda)\, a(\bar{p}, \lambda)\, e^{ip\cdot x} + \varepsilon^{\mu *}(\bar{p}, \lambda)\, a^*(\bar{p}, \lambda)\, e^{-ip\cdot x} \Big]$$

$$\rightarrow F^{\mu\nu} = \sum_{\lambda=\pm 1} \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{2E_p} \left[\left(ip^\mu \varepsilon^\nu(\bar{p},\lambda) - ip^\nu \varepsilon^\mu(\bar{p},\lambda) \right) \cdot a(\bar{p},\lambda) e^{ip\cdot x} + h.c. \right]$$

with $\varepsilon_\pm \perp \bar{p}$

Consider $\varepsilon^\mu(\bar{p},\lambda) \to \varepsilon^\mu(\bar{p},\lambda) + p^\mu$. No change in $F^{\mu\nu}$. No change in dynamics

$\rightarrow$ can require $p_\mu \varepsilon^\mu(\bar{p},\lambda) = 0$

Gauge field A^μ only defined up to transformation

$$A^\mu \to A^\mu + \partial^\mu \beta(x)$$

$F^{\mu\nu}$ is Lorentz tensor: transforms as

$$U(-\omega) F^{\mu\nu} U(-\omega)^\dagger = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda x)$$

but A^μ is <u>not</u> Lorentz vector:

$$U(-\omega) A^\mu U(-\omega)^\dagger = \Lambda^\mu{}_\nu A^\nu(\Lambda x) + \partial^\mu \beta$$

$\Rightarrow$ Free Lagrangian $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is Lorentz-invariant, but interactions with explicit A^μ dependence have to be made Lorentz-invariant explicitly $\rightarrow$ requirement on any A^μ interaction term!

Invariance maintained by adding more terms or by changing transformation rules.
No additional terms because too high dims.

$\curvearrowright$ some vector functional

Example : $\mathcal{L}_{int} = A_\mu(x)\, J^\mu[\varphi, \partial_\mu\varphi]$

Gauge transformation : $\begin{cases} \delta A_\mu = \partial_\mu\beta(x) \\ \delta\varphi_i = 0 \end{cases}$

$$\delta\mathcal{L}_{int} = \underbrace{\partial_\mu\beta}_{\substack{\text{not zero} \\ \text{because } \delta A_\mu \neq 0}}\, J^\mu[\varphi, \partial_\mu\varphi]$$

To obtain $\delta\mathcal{L} = 0$ we must change the transformation of φ_i : $\begin{cases} \delta A_\mu = \partial_\mu\beta(x) \\ \delta\varphi_i = \varepsilon_a F_i^a[\varphi, \partial_\mu\varphi] \end{cases}$

$\downarrow$

change in $\mathcal{L}_0$ is now (Noether's procedure)

ε_a const : $\delta\mathcal{L}_0 = \dfrac{\partial\mathcal{L}}{\partial\varphi_i}\varepsilon_a F_i^a + \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\varepsilon_a\partial_\mu F_i^a$

$$= \cancel{\left[\frac{\partial\mathcal{L}}{\partial\varphi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\right]}\varepsilon_a F_i^a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}F_i^a\right]\varepsilon_a$$

$$= -\left(\partial_\mu j^\mu_a\right)\varepsilon_a = 0 \quad \text{if} \quad \partial_\mu j^\mu_a = 0$$

$\varepsilon_a(x)$: $\delta\mathcal{L}_0 = -\, j^\mu_a\, \partial_\mu\varepsilon^a(x)$

$\delta\mathcal{L} = 0$ only when $\begin{cases} J^\mu_a = j^\mu_a \\ \varepsilon^a(x) = \beta(x) \end{cases}$

$$\Rightarrow \delta\varphi_i = \beta(x)\, F_i[\varphi, \partial_\mu\varphi]$$

The symmetry that generated j^μ (conserved current) for constant ε or β is now promoted to a gauge symmetry with $\beta(x)$ space-time dependent (local)

$\Downarrow$

General procedure for φ_i fields with symmetry under generators T_a:

1) Start from set of fields $\{\varphi^i\}$ with global symm.

$$\delta\varphi^i = i\,\beta^a (T_a)^i{}_j\, \varphi^j(x) \rightarrow \delta\mathcal{L} = 0$$

__global__ coefficients

generator

sum over all fields

$$\delta\varphi^i \text{ from } \varphi \rightarrow \varphi' = \varphi + i\beta^a T_a \varphi$$

$$= \underbrace{e^{i\vec{\beta}\cdot\vec{T}}}_{U}\varphi$$

U unitary

T hermitian

In general $[T_a, T_b] = i f_{ab}^c T_c$ form a Lie algebra with structure coeff. f_{ab}^c

2) Associate spin-one vector field with each generator T_a

3) Replace covariant derivatives:

$$D_\mu \varphi^i = \partial_\mu \varphi^i - i A^a_\mu (T_a)^i_j \varphi^j$$

("minimal substitution": $p \to p + qA$)

4) Add free Lagrangian for the vector fields:

$$\mathcal{L}_g = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu$$

$\Rightarrow$ Invariant under <u>local</u> gauge transform:

$$\begin{cases} A_\mu \to A_\mu(x) + \partial_\mu \beta^a(x) - f^a_{bc} \beta^b(x) A^c_\mu(x) \\ \varphi^i \to \varphi^i(x) + i \beta^a(x) (T_a)^i_j \varphi^j(x) \end{cases}$$

Example: QED: invariant under global symm with $\varphi \to \varphi' = e^{i\beta} \varphi = 1 + i\beta \varphi$

$$\delta\varphi = i\beta \varphi, \quad T^a = T = 1$$

symmetry group $U(1)$ with irrep

of dimension 1

$f^{a}_{bc} = 0$ because $[T, T] = 0$

Example: QCD; invariant under global symmetry $SU(3)$

$U = e^{i\bar{\beta}\cdot\bar{T}}$ transforms $u_a \to U_{ab} u_b$

but $\mathcal{L} = -\bar{u}(\not{\partial} + m)u$ remains invariant

There are now $8 = N^2 - 1$ generators for $SU(N) \to 8$ gauge fields A^{μ}_{a} and because $f^{c}_{ab} \neq 0$ the transformations change