ECE 595: Introduction to Quantum Computing
Final Project: Review of Quantum Phase Estimation

Warren D. (Hoss) Craft

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Note: References throughout the homework to "Mermin" refer to Mermin's (2007) *Quantum Computer Science: An Introduction* [1], and references to "Nielsen & Chuang" or just "N&C" refer to Nielsen & Chuang's (2010) *Quantum Computation and Quantum Information (10th Anniversary Edition)* [2], and further bracketed citations are omitted.

(1) Nielsen & Chuang's Exercise 5.7 (pg 222):

> **Exercise 5.7:** Additional insight into the circuit in Figure 5.2 may be obtained by showing, as you should now do, that the effect of the sequence of controlled-U operations like that in Figure 5.2 is to take the state $|j\rangle |u\rangle$ to $|j\rangle U^j |u\rangle$. (Note that this does not depend on $|u\rangle$ being an eigenstate of U.)

Solution. We take $|j\rangle = |j\rangle_t$ — *i.e.*, j is an integer value expressible in t bits, so that

$$j = \sum_{i=0}^{t-1} \alpha_i 2^i, \text{ for } \alpha_i \in \{0, 1\} \tag{1}$$

and thus

$$|j\rangle = |\alpha_{t-1}\rangle |\alpha_{t-1}\rangle \dots |\alpha_1\rangle |\alpha_0\rangle \tag{2}$$

Then the effect of the sequence of controlled-U operations like that in Figure 5.2 is to effect the transformation:

$$|j\rangle |u\rangle \mapsto |j\rangle \prod_{i=0}^{t-1} e^{2\pi i(\alpha_i 2^i \varphi)} |u\rangle \tag{3}$$

because for each non-zero α_i in the expansion of j, we will obtain a (non-unitary) factor of $e^{2\pi i(2^i \varphi)}$. The product of the exponential functions can be re-expressed as sum in the exponent, where the sum is exactly our formula for j:

$$\begin{align}
|j\rangle \prod_{i=0}^{t-1} e^{2\pi i(\alpha_i 2^i \varphi)} |u\rangle &= |j\rangle e^{2\pi i(\sum_{i=0}^{t-1}(\alpha_i 2^i)\varphi)} |u\rangle \tag{4}\\
&= |j\rangle e^{2\pi i(j\varphi)} |u\rangle \tag{5}\\
&= |j\rangle \left(e^{2\pi i\varphi}\right)^j |u\rangle \tag{6}\\
&= |j\rangle U^j |u\rangle \tag{7}\\
& \tag{8}
\end{align}$$

where in the end we used the fact that $U|u\rangle = e^{2\pi i\varphi}|u\rangle$ and thus $U^j|u\rangle = (e^{2\pi i\varphi})^j |u\rangle$.

Some simple concrete examples of this result are shown worked out from scratch in Table 1 for j defined on $t = 3$ Qbits.

$\vert j\rangle\,\vert u\rangle$ input	output	simplified output
$\vert 001\rangle\,\vert u\rangle$	$\vert 001\rangle\, U^{2^0}\,\vert u\rangle$	$\vert 001\rangle\, U^1\,\vert u\rangle = \vert 1\rangle_3\, U^1\,\vert u\rangle$
$\vert 010\rangle\,\vert u\rangle$	$\vert 010\rangle\, U^{2^1}\,\vert u\rangle$	$\vert 010\rangle\, U^2\,\vert u\rangle = \vert 2\rangle_3\, U^2\,\vert u\rangle$
$\vert 011\rangle\,\vert u\rangle$	$\vert 011\rangle\, U^{2^1}U^{2^0}\,\vert u\rangle$	$\vert 011\rangle\, U^2U^1\,\vert u\rangle = \vert 3\rangle_3\, U^3\,\vert u\rangle$
$\vert 100\rangle\,\vert u\rangle$	$\vert 100\rangle\, U^{2^2}\,\vert u\rangle$	$\vert 100\rangle\, U^4\,\vert u\rangle = \vert 4\rangle_3\, U^4\,\vert u\rangle$
$\vert 101\rangle\,\vert u\rangle$	$\vert 101\rangle\, U^{2^2}U^{2^0}\,\vert u\rangle$	$\vert 101\rangle\, U^4U^1\,\vert u\rangle = \vert 5\rangle_3\, U^5\,\vert u\rangle$
$\vert 110\rangle\,\vert u\rangle$	$\vert 110\rangle\, U^{2^2}U^{2^1}\,\vert u\rangle$	$\vert 110\rangle\, U^4U^2\,\vert u\rangle = \vert 6\rangle_3\, U^6\,\vert u\rangle$

Table 1: Examples of results for Exercise 5.7, with j defined on 3 Qbits.

References

[1] N. David Mermin. *Quantum Computer Science: An Introduction*. New York, NY: Cambridge University Press, 2007.

[2] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information (10th Anniversary Edition)*. New York, NY: Cambridge University Press, 2010.