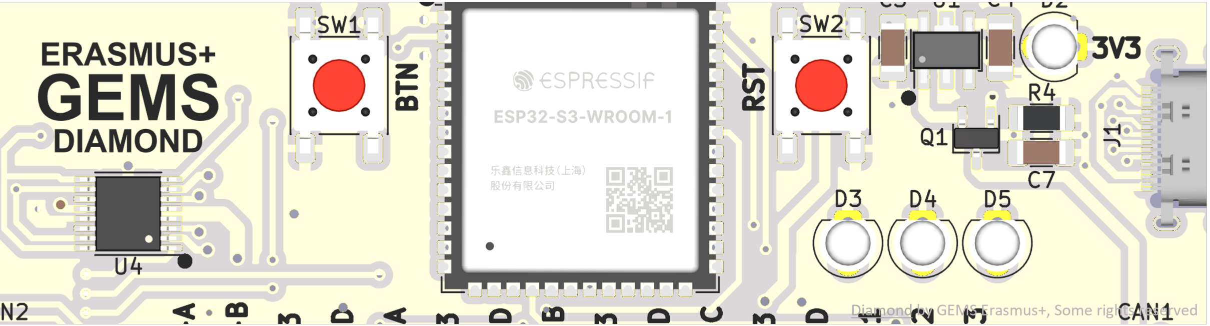


Tutorial 2

Laplace Transform and its Application

Dr. ir. Sachin Yadav
DCE&S, ESE, TU Delft

1. What is the Laplace transform?
2. Advantages of the Laplace Transform
3. Worked example

Main Idea

- Many real-life problems change over time: motion, voltage, temperature, etc.
- These are described by functions of time, $f(t)$.
- The Laplace Transform changes $f(t)$ into a new function $F(s)$.
- In this new world (the s-domain), math often becomes easier!

Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{1}$$

- $f(t)$: original function (time-domain)
- $F(s)$: transformed function (Laplace or s-domain)
- s: a complex variable that helps describe how things grow or decay

$$s = \sigma + j\omega$$

Simpler Math

- Differentiation and integration become simple multiplication or division by s.
- Example:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) \tag{2}$$

- So a hard differential equation becomes a simple algebraic equation.

Built-In Initial Conditions

- The Laplace transform includes initial values directly:

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) \tag{3}$$

- No need to find constants of integration later.
- Example:

$$y'(t) + y(t) = 0,\ y(0) = 5 \Rightarrow Y(s) = \frac{5}{s+1} \Rightarrow y(t) = 5e^{-t} \tag{4}$$

System Behavior

- Once you have $F(s)$, you can study:
    - Stability (poles in the left/right half-plane)
    - Oscillations and damping
    - System response to inputs (e.g., step or impulse)
- Used heavily in control systems and circuit analysis.

Real-World Connection

- Foundation for engineering tools:
    - Control theory ($H(s)$, transfer functions)
    - Signal processing and filter design
    - Circuit analysis (RLC circuits)
- The Laplace domain connects time, frequency, and system design in one framework.

Motor model

Motor circuit model:

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + K\Omega \tag{5}$$

$$i_a = \frac{T}{K} \tag{6}$$

$$T = J\frac{d\Omega}{dt} + b\Omega \tag{7}$$

Motor model

Substituting (7) in (6) and the result to (5), we get

$$V_a = \frac{L_a J}{K} \frac{d^2\Omega}{dt^2} + \left(\frac{R_a J + L_a b}{K} \right) \frac{d\Omega}{dt} + \left(\frac{R_a b + K^2}{K} \right) \Omega \tag{8}$$

Solving this equation would require the solver to know the solutions of second order differential equations -> Non-trivial!

Differential equation to Laplace equation

Taking the Laplace transform of (8),

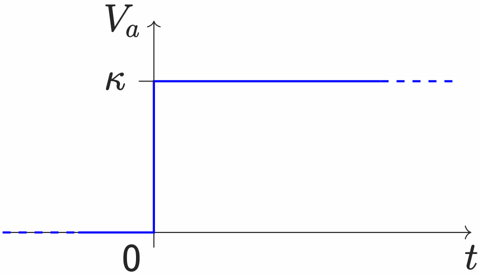
$$V_a(s) = \frac{L_a J}{K} s^2 \Omega(s) + \left(\frac{R_a J + L_a b}{K} \right) s\Omega(s) + \left(\frac{R_a b + K^2}{K} \right) \Omega(s) \quad (9)$$

Rearranging, we get,

$$\Omega(s) = \frac{K V_a(s)}{L_a J s^2 + (R_a J + L_a b)\, s + (R_a b + K^2)} \quad (10)$$

$$= \frac{K}{L_a J} \frac{V_a(s)}{s^2 + \frac{R_a J + L_a b}{L_a J} s + \frac{R_a b + K^2}{L_a J}} \quad (11)$$

Step response



$$\mathcal{L}\{\kappa u(t)\} = \frac{\kappa}{s}$$

Partial fraction

$$\Omega(s) = \frac{KV_a}{L_aJ}\frac{1}{s\left(s^2 + \frac{R_aJ+L_ab}{L_aJ}s + \frac{R_ab+K^2}{L_aJ}\right)} = \frac{\alpha}{s(s^2+\beta s+\gamma)} \tag{12}$$

$$\alpha = \frac{KV_a}{L_aJ}, \qquad \beta = \frac{R_aJ+L_ab}{L_aJ}, \qquad \gamma = \frac{R_ab+K^2}{L_aJ}$$

Using partial fraction:

$$\frac{\alpha}{s(s^2+\beta s+\gamma)} = \frac{A}{s} + \frac{Bs+C}{s^2+\beta s+\gamma} \tag{13}$$

$$A = \frac{\alpha}{\gamma}, \qquad B = \frac{-\alpha}{\gamma}, \qquad C = \frac{-\alpha\beta}{\gamma}$$

Some standard solutions

$$\mathcal{L}\{\kappa\} = \frac{\kappa}{s} \qquad \mathcal{L}\{sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\{cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$

Shifting property:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Solution

Continuing from (13):

$$\Omega(s) = \frac{\alpha}{\gamma s} - \frac{\alpha}{\gamma}\left[\frac{s+\beta}{s^2+\beta s+\gamma}\right] \tag{14}$$

Rearranging:

$$\Omega(s) = \frac{\alpha}{\gamma s} - \frac{\alpha}{\gamma}\left[\frac{s+\frac{\beta}{2}+\frac{\beta}{2}}{\left(s+\frac{\beta}{2}\right)^2+\lambda^2}\right] \tag{15}$$

$$\lambda = \sqrt{-\left(\frac{\beta}{2}\right)^2+\gamma}$$

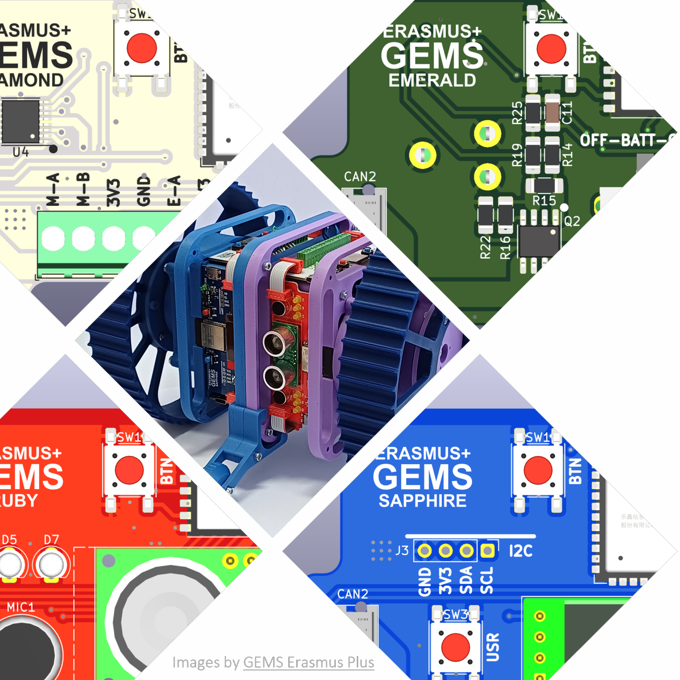
Solution

Using standard solution and shifting property:

$$\Omega(t) = \frac{\alpha}{\gamma} - \frac{\alpha e^{-\frac{\beta}{2}t}}{\gamma}\left[cos(\lambda t) + \frac{\beta}{2\lambda}sin(\lambda t)\right] \tag{16}$$

Conclusion

- Review of Laplace transform.
- Solution of speed of motor.



Images by GEMS Erasmus Plus

Thank you for watching!

This video was created by Delft University of Technology for the GEMS Erasmus+ project (a collaboration between University of Ljubljana, University of Alcala, Teaching Factory Competence Center, and Delft University of Technology).

Co-funded by the European Union

Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or CMEPIUS. Neither the European Union nor the granting authority can be held responsible for them.