Generating Functions

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Uptake

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What is a generating function?

Let $a = a_1, a_2, \ldots$ be a sequence.

Definition

The generating function for a sequence a is

$$G_a(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots$$

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I will cover the "calculus of generating functions" and how basic operations on generating functions correspond to operations on the sequence itself.

The Fibonacci numbers

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The generating function for F is

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Let's multiply it by x

$$\begin{aligned} x \cdot G_F(x) &= F_0 \cdot x + F_1 \cdot x^2 + F_2 \cdot x^3 + \cdots \\ &= F_1 \cdot x^2 + F_2 \cdot x^3 + \cdots \end{aligned}$$

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$$x^2 \cdot G_F(x) = F_0 \cdot x^2 + F_1 \cdot x^3 + F_2 \cdot x^4 + \cdots$$

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The Fibonacci numbers

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$$
\begin{aligned}
x \cdot G_F(x) + x^2 \cdot G_F(x) = &\quad F_1 \cdot x^2 + F_2 \cdot x^3 + \cdots \\
&+ F_0 \cdot x^2 + F_1 \cdot x^3 + F_2 \cdot x^4 + \cdots \\
&= (F_1 + F_0) \cdot x^2 + (F_2 + F_1) \cdot x^3 + \cdots \\
&= F_2 \cdot x^2 + F_3 \cdot x^3 + \cdots \\
&= G_F(x) - x
\end{aligned}
$$

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$$\begin{aligned} x \cdot G_F(x) + x^2 \cdot G_F(x) = &\quad F_1 \cdot x^2 + F_2 \cdot x^3 + \cdots \\ &+ F_0 \cdot x^2 + F_1 \cdot x^3 + F_2 \cdot x^4 + \cdots \\ &= (F_1 + F_0) \cdot x^2 + (F_2 + F_1) \cdot x^3 + \cdots \\ &= F_2 \cdot x^2 + F_3 \cdot x^3 + \cdots \\ &= G_F(x) - x \end{aligned}$$

Solve that equation for G_F and we get

$$G_F(x) = \frac{x}{1 - x - x^2}$$

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We want to simplify

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$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$$

In other words

$$\frac{1}{1-x}$$

is the generating function for $a = 1, 1, 1, \ldots$.

The Fibonacci numbers

So we want the partial fraction decomposition for

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We can use the quadratic formula to factor $1 - x - x^2$.

$$1 - x - x^2 = (1 - \phi_+ x)(1 - \phi_- x)$$

$$\phi_\pm = \frac{1 \pm \sqrt{5}}{2} \approx 1.618, -0.618$$

The Fibonacci numbers

If we do a little bit of algebra we then get

$$1 - x - x^2 = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi_+ x} - \frac{1}{1 - \phi_- x} \right)$$

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If we do a little bit of algebra we then get

$$1 - x - x^2 = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi_+ x} - \frac{1}{1-\phi_- x}\right)$$

But we can use the fact that $\frac{1}{1-y} = 1 + y + y^2 \cdots$ to expand out each of those fractions

$$\frac{1}{\sqrt{5}}\left(\left(1 + \phi_+ \cdot x + \phi_+^2 \cdot x^2 + \cdots\right) - \left(1 + \phi_- \cdot x + \phi_-^2 \cdot x^2 + \cdots\right)\right)$$

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$$F_n = \frac{1}{\sqrt{5}} \left(\phi_+^n - \phi_-^n \right)$$

Since ϕ_-^n gets vanishingly small we can also say

$$F_n \approx 0.447 \cdot 1.618^n$$

We used a lot of little tricks there. I'm going to list out a bunch of tricks we can use to work with generating functions.

The Calculus of Generating Functions

Generating functions preserve linearity

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Multiplying by x shifts the sequence

$x \cdot G_{[A_{n+1}]} = G_A - A_0$

$$\begin{aligned} x \cdot G_{[A_{n+1}]} &= x \cdot (A_1 + A_2 \cdot x + A_3 \cdot x^2 + \cdots) \\ &= A_1 \cdot x + A_2 \cdot x^2 + A_3 \cdot x^3 + \cdots \\ &= G_A - A_0 \end{aligned}$$

Taking the derivative multiplies by n

$x \cdot G'_A(x) = G_{[n*A_n]}$

$$\begin{aligned} x \cdot G'_A(x) &= x \cdot \frac{d(A_0 + A_1 \cdot x + A_2 \cdot x^2 + A_3 \cdot x^3 + \cdots)}{dx} \\ &= x \cdot (A_1 + 2A_2 \cdot x + 3A_3 \cdot x^2 + \cdots) \\ &= A_1 \cdot x + 2A_2 \cdot x^2 + 3A_3 \cdot x^3 + \cdots \\ &= G_{[n*A_n]} \end{aligned}$$

The Calculus of Generating Functions

Let's define the convolution of sequence. For two sequence A and B, $A * B$ is defined as

$$(A * B)_n = \sum_{i+j=n} A_i \cdot B_j$$

Then we get another rule.

Multiplying generating functions takes a convolution

$G_A \cdot G_B = G_{A*B}$

$$\begin{aligned}
G_A \cdot G_B &= (A_0 + A_1 \cdot x + B_2 \cdot x^2 + \cdots) \\
&\quad \cdot (B_0 + B_1 \cdot x + B_2 \cdot x^2 + \cdots) \\
&= (A_0 + B_0) + (A_1 \cdot B_0 + A_0 \cdot B_1) \cdot x \\
&\quad + (A_2 \cdot B_0 + A_1 \cdot B_1 + A_0 \cdot B_2) \cdot x^2 \\
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$$\begin{aligned} G_A \cdot G_B &= (A_0 + A_1 \cdot x + B_2 \cdot x^2 + \cdots) \\ &\quad \cdot (B_0 + B_1 \cdot x + B_2 \cdot x^2 + \cdots) \\ &= (A_0 + B_0) + (A_1 \cdot B_0 + A_0 \cdot B_1) \cdot x \\ &\quad + (A_2 \cdot B_0 + A_1 \cdot B_1 + A_0 \cdot B_2) \cdot x^2 \\ &= G_{A*B} \end{aligned}$$

Let $B = 1, 1, 1, \ldots$. One cool thing to note is that implies $\frac{G_A}{1-x}$ is the generating function for the partial sums of A.

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In other words $\frac{1}{(1-x)^k}$ is the generating function for the number of ways to have an ordered sequence of length k of nonnegative integers add up to n.

Another example

Let's look at all the ways to have 3 nonnegative integers add up to 2. We have

$$\begin{aligned} 2 &= 2+0+0 \\ &= 1+1+0 \\ &= 1+0+0 \\ &= 0+2+0 \\ &= 0+1+1 \\ &= 0+0+2 \end{aligned}$$

Another example

It turns out this is equivalent to allocating $k-1$ plus signs among n ones. Let the $\star$ stand for a 1.

$$\begin{aligned}
2 &= 2+0+0 \Leftrightarrow \star\star++ \\
&= 1+1+0 \Leftrightarrow \star+\star+ \\
&= 1+0+1 \Leftrightarrow \star++\star \\
&= 0+2+0 \Leftrightarrow +\star\star+ \\
&= 0+1+1 \Leftrightarrow +\star+\star \\
&= 0+0+2 \Leftrightarrow ++\star\star
\end{aligned}$$

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So we have $n + k - 1$ slots and we need to choose where to put $k - 1$ plus signs. The number of ways to do this is $\binom{n+k-1}{k-1}$.

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And just to get this to something simpler it turns out the generating function for $\binom{n}{k}$ is $\frac{x^k}{(1-x)^{k+1}}$.

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Generating function of $[n^2]$ is $\frac{x(1+x)}{(1-x)^3}$ by taking the derivative of $\frac{1}{1-x}$ twice.

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Divide by $1 - x$ to get the generating function of $\sum_{k=1}^{n} k^2$ is

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Since dividing by x is equivalent to shifting by n and we have the combinatorial generating function from before, we get formula

$$\sum_{k=1}^{n} k^2 = \binom{n+1}{3} + \binom{n+2}{3} = \frac{n(n+1)(2n+1)}{6}$$

.

Coin counting

Your turn: let's say we have pennies, nickels, dimes, and quarters. What's the generating function for the number of ways to make change for n cents?