Data Structures and Algorithms

Chapter 11

Learning Objectives

- Define Binary Search Trees
- Be able to implement a binary search tree and use it in practice
- Define an AVL tree
- Understand and implement rotation and restructuring to maintain balance in a binary tree

Binary Search Trees

- Note: A “position” in the slides is a “node” in a tree.
- Each internal position *p* in a binary search tree stores (*k*, *v*) pair.

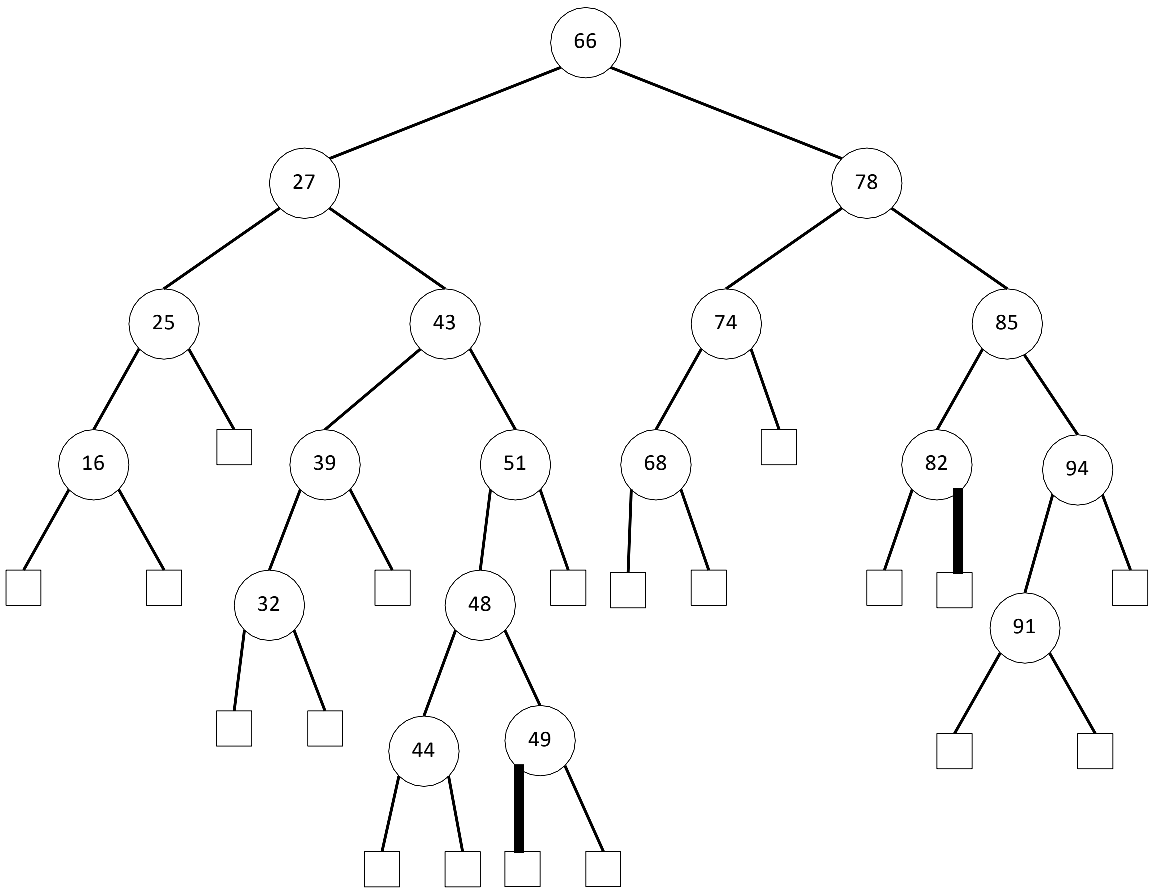
- A Binary search tree is a *proper binary tree* with the following properties:

  For each internal position *p* with entry (*k*, *v*) pair,
  - Keys stored in the left subtree of *p* are less than *k*.
  - Keys stored in the right subtree of *p* are greater than *k*.

- Note: In this definition, external nodes (or leaves) are “placeholders,” which are shown as small squares in the graph.

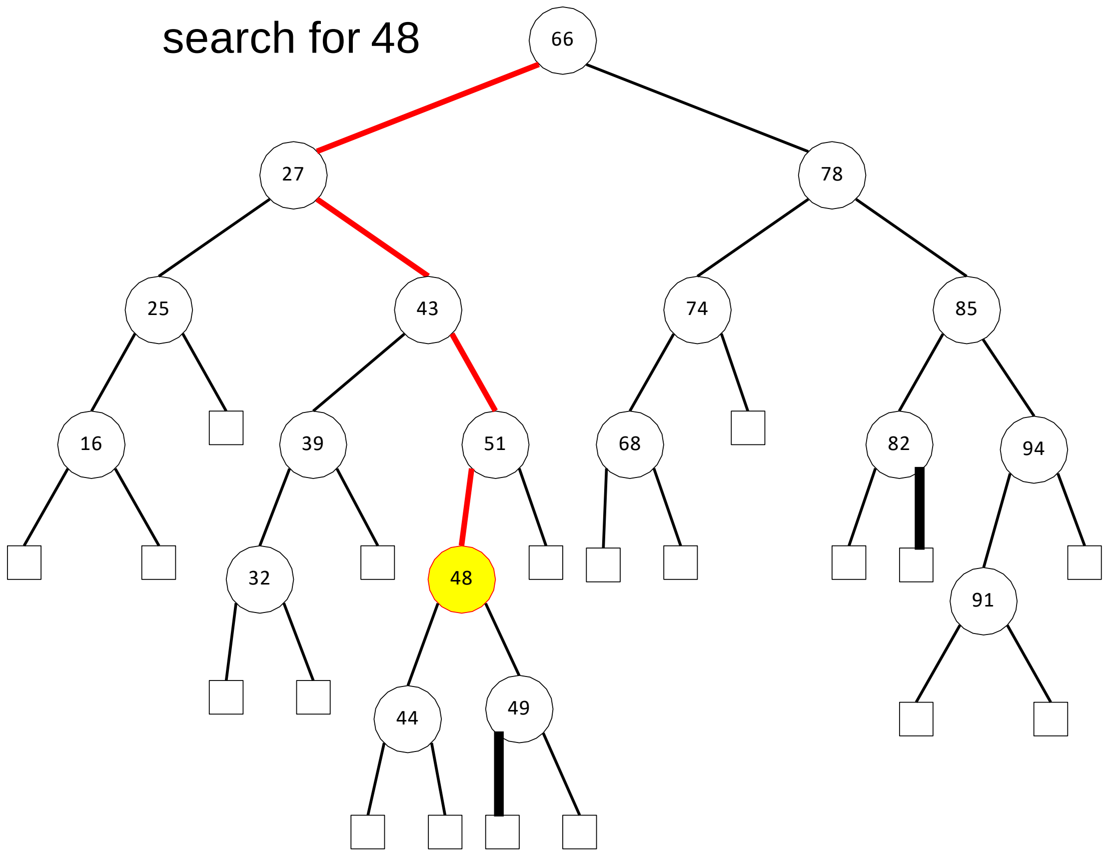
Binary Search Trees

- Example (only keys are shown):



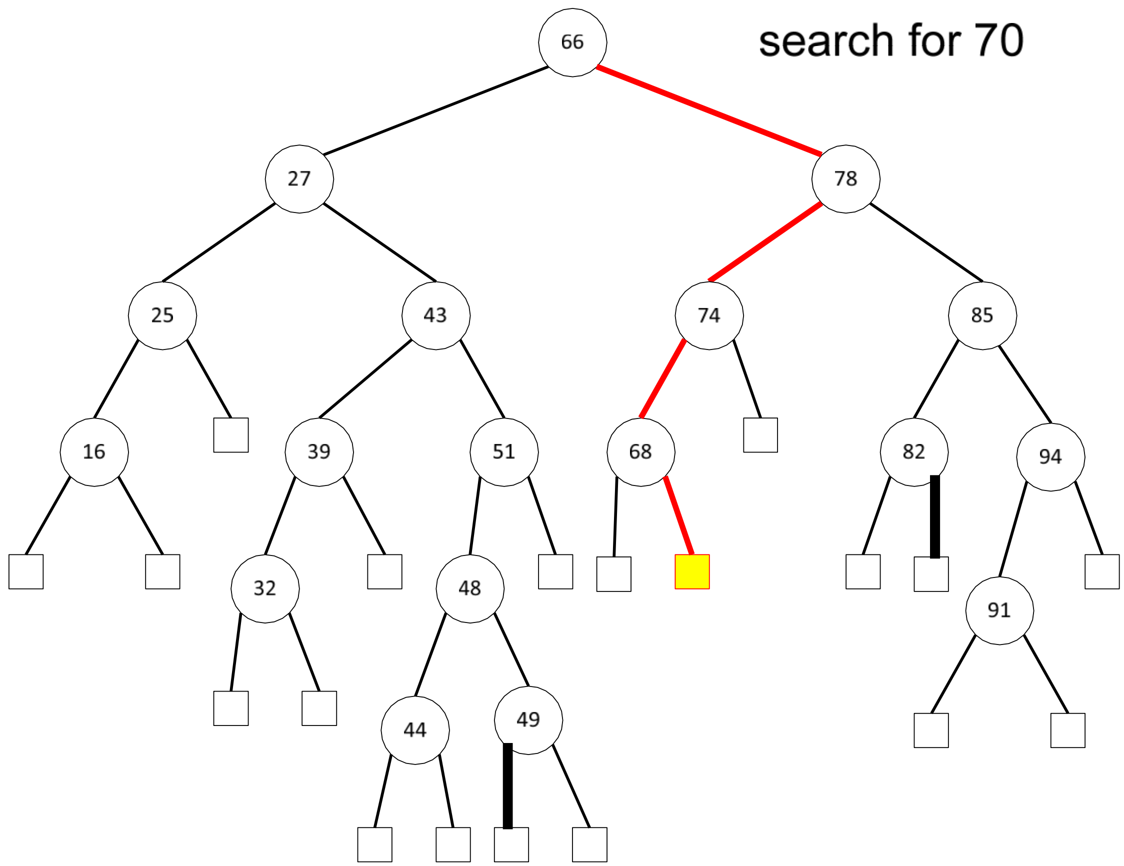
Binary Search Trees

- Search (successful search)



Binary Search Trees

- Search (unsuccessful search)



Binary Search Trees

- Search pseudocode

```
Algorithm TreeSearch(p, k)
if p is external then                      // unsuccessful search
  return p
else if k == key(p)                        // successful search
  return p
else if k < key(p)
  return TreeSearch(left(p), k)            // recurse on left subtree
else
  return TreeSearch(right(p), k)           // recurse on right subtree
```

- Running time: O(h)

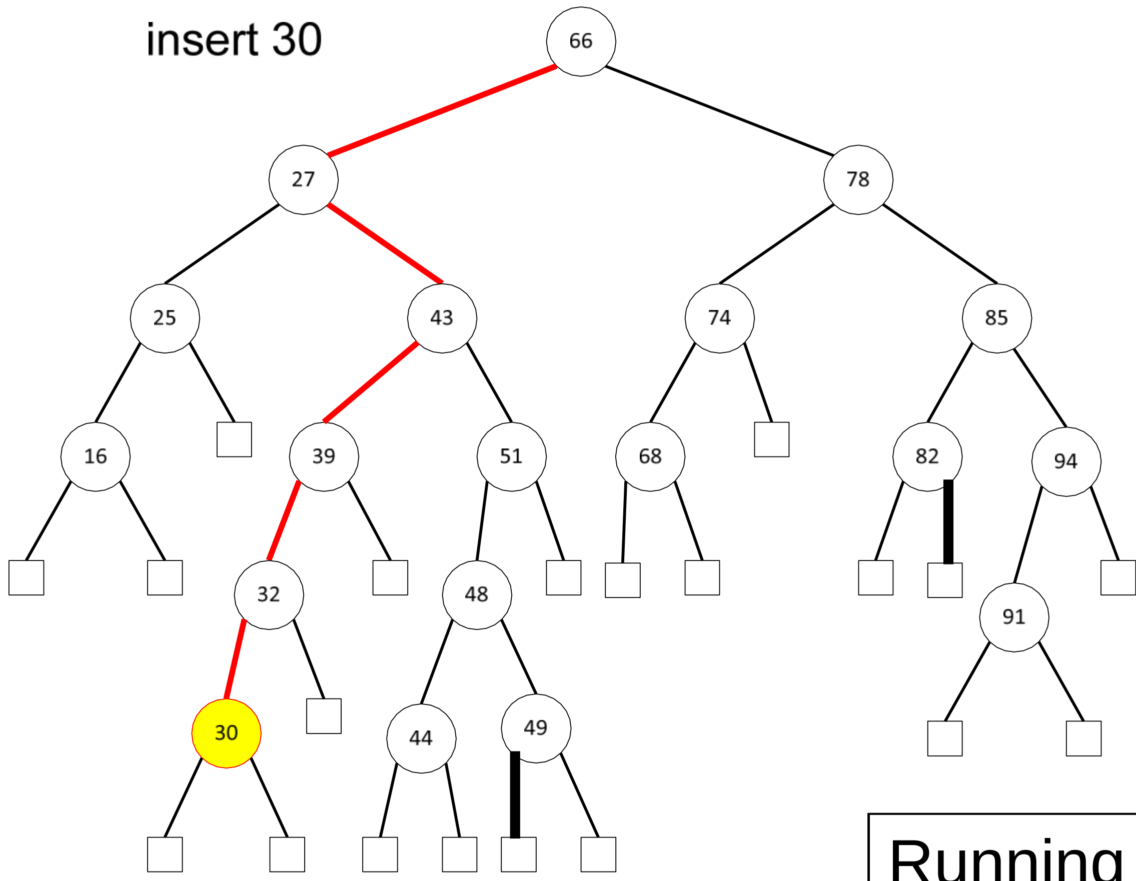
Binary Search Trees

- Inserting an entry with (k, v)

  - Perform a search operation.
  - If an entry with key *k* is found (i.e., successful search), the existing value is replaced with the new value *v*.
  - If there is no entry with key *k*, then we add an entry at the leaf node where the unsuccessful search ended up.

Binary Search Trees

- Insert illustration



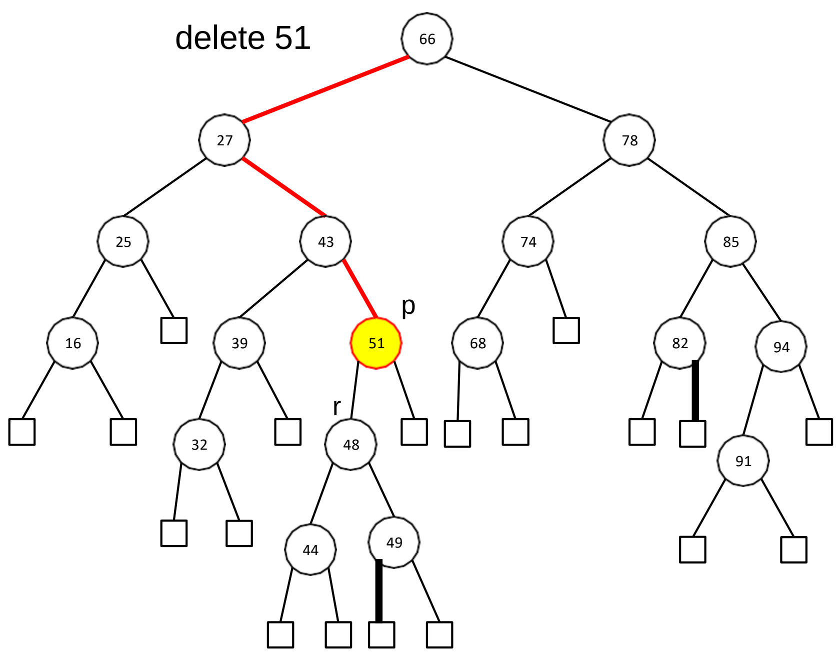
Running time: *O*(h)

Binary Search Trees

- Deleting an entry with (k, v)
  - Slightly more complex
  - Perform search
    - If we reach a leaf node, do nothing
    - If we find the entry at position *p*
      - Case 1: at most one child of *p* is an internal node
      - Case 2: *p* has two children, both of which are internal

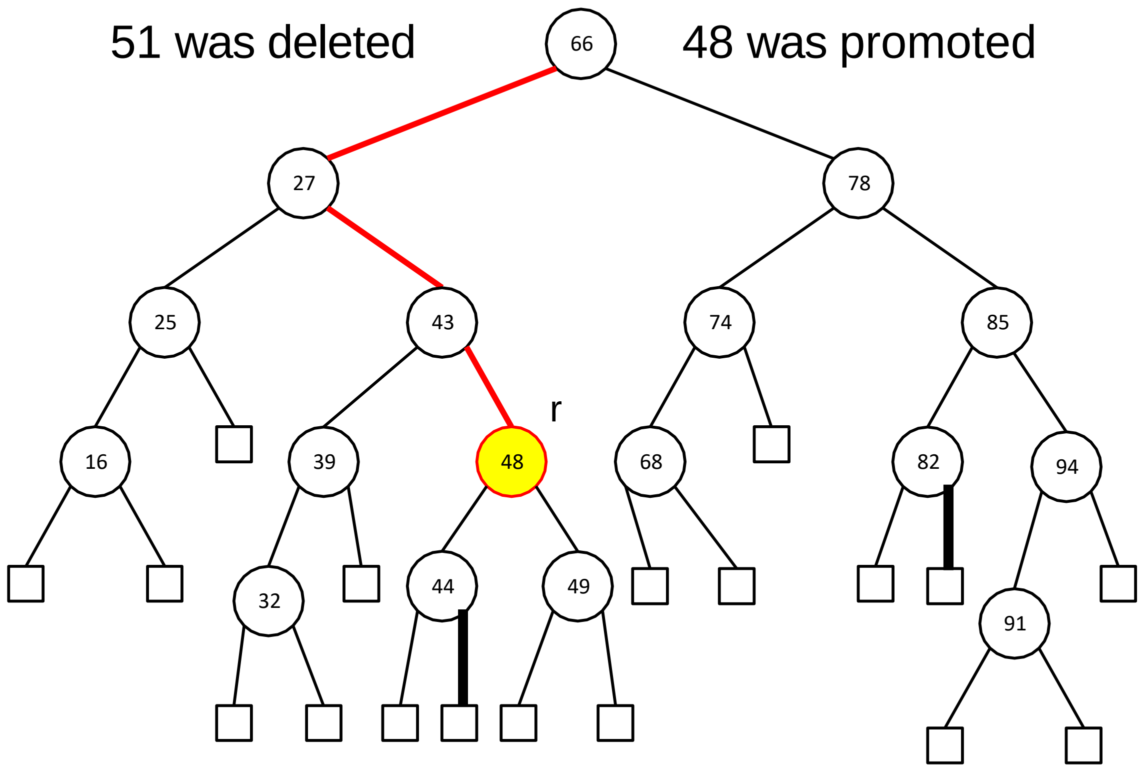
Binary Search Trees

- Deletion Case 1
  - If both children are leaf nodes, then *p* is replaced with a leaf node.
  - If *p* has one internal-node child, then that child node replaces *p*



Binary Search Trees

- Deletion Case 1
  - If *p* has one internal-node child (continued)

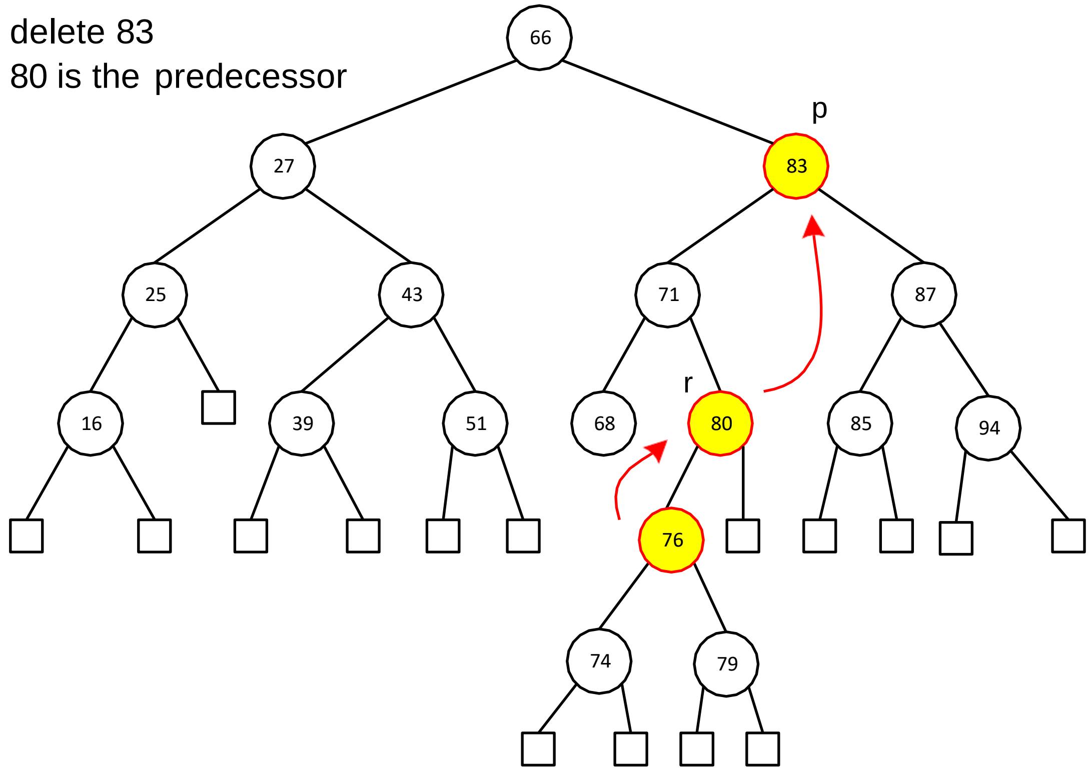


Binary Search Trees

- Deletion Case 2
  - First, we find the node *r* that has the largest key that is strictly less than *p*'s key. This node is called the *predecessor* of *p* in the ordering of keys, which is the rightmost node in *p*'s left subtree.
  - We let *r* replace *p*.
  - Since *r* is the rightmost node in *p*'s left subtree, it does not have a right child. It has only a left child.
  - The node *r* is removed and the subtree rooted at *r*'s left child is promoted to *r*'s position.

Binary Search Trees

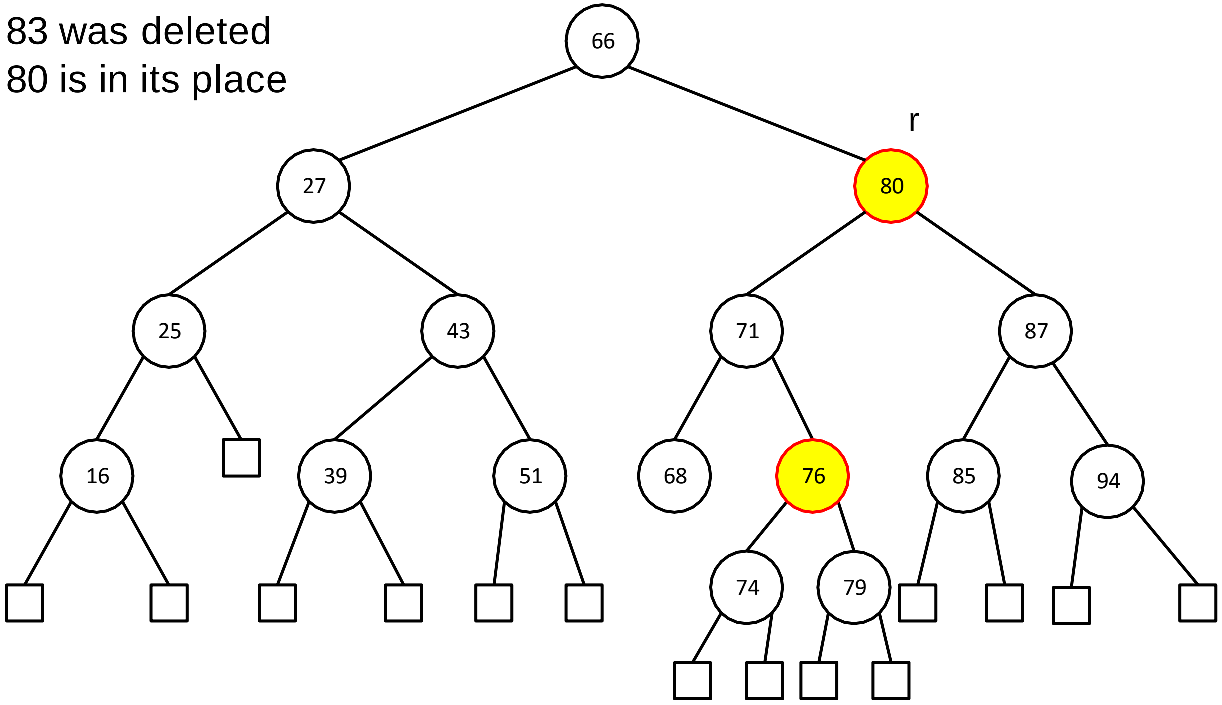
- Deletion Case 2



Binary Search Trees

- Deletion Case 2

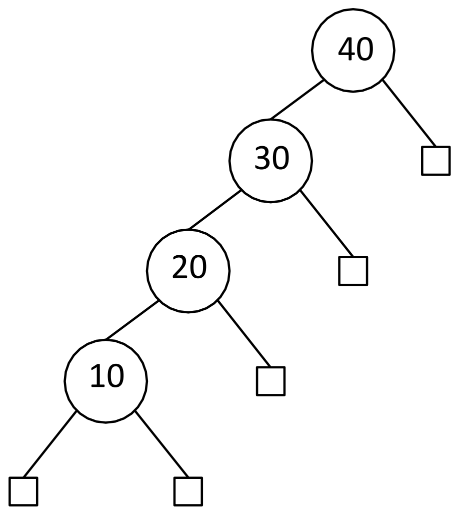
83 was deleted
80 is in its place



- Running time: *O*(h)

Binary Search Trees

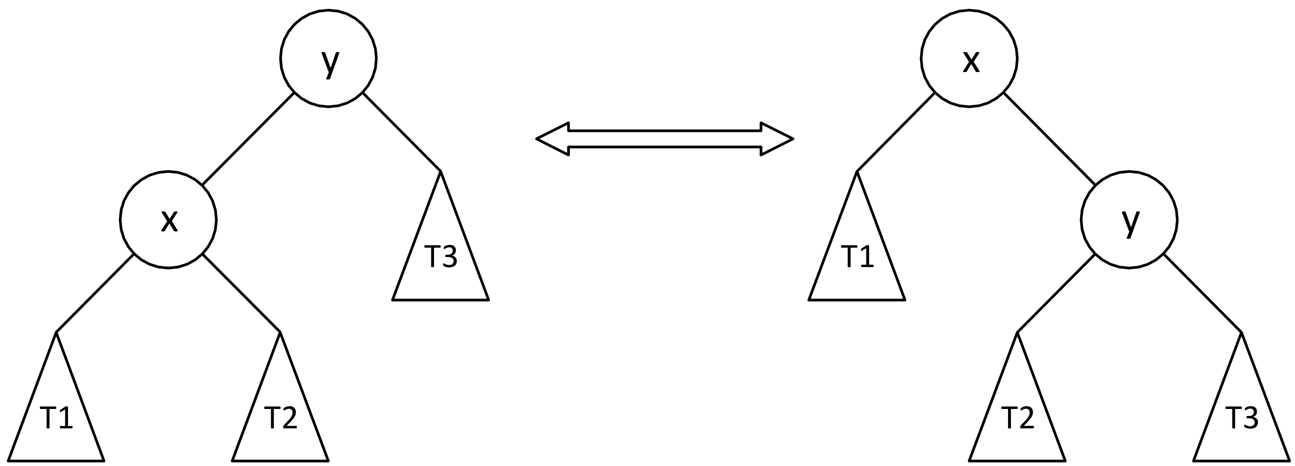
- Most binary search tree operations run in *O*(h).
- In the worst case, a tree is just a linked list. In this case, running times are *O*(n).



- To guarantee *O*(h), a tree needs to be balanced.

Balanced Search Trees

- When a binary search tree is unbalanced, it is necessary to *rebalance* the tree.
- Primary operation for rebalancing a binary search tree is *rotation*.



- Can rotate in either direction.
- Binary search tree property is maintained after rotation.

Balanced Search Trees

- A *trinode restructuring* performs a broader rebalancing.
- It involves three positions: *x*, *y*, and *z*
- *y* is the parent of *x* and *z* is the grandparent of *x*.
- Goal: Restructure the subtree rooted at *z* to reduce the path length from *z* to *x* and its subtrees.

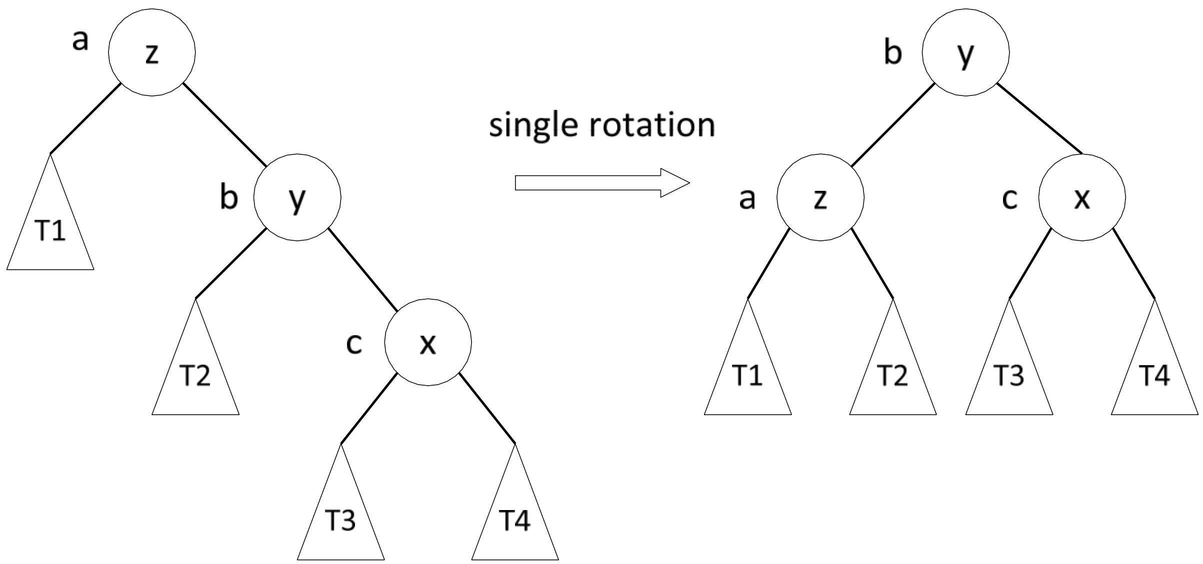
- Use secondary labels, *a*, *b*, and *c*, for the three positions such that *a* comes before *b* and *b* comes before *c* in an inorder tree traversal of the tree.
- There are four different configurations. This secondary labels allow us to describe the trinode restruring operations in a uniform way.

Balanced Search Trees

- Outline of the algorithm:
  - (T_1, T_2, T_3, T_4) are left-to-right listing of subtrees of *x*, *y*, and *z*.
  - The subtree rooted at *z* is replaced with the subtree rooted at *b*.
  - Make *a* the left child of *b*.
  - Make T_1 and T_2 the left and right subtree of *a*, respectively.
  - Make *c* the right child of *b*.
  - Make T_3 and T_4 the left and right subtree of *c*, respectively.

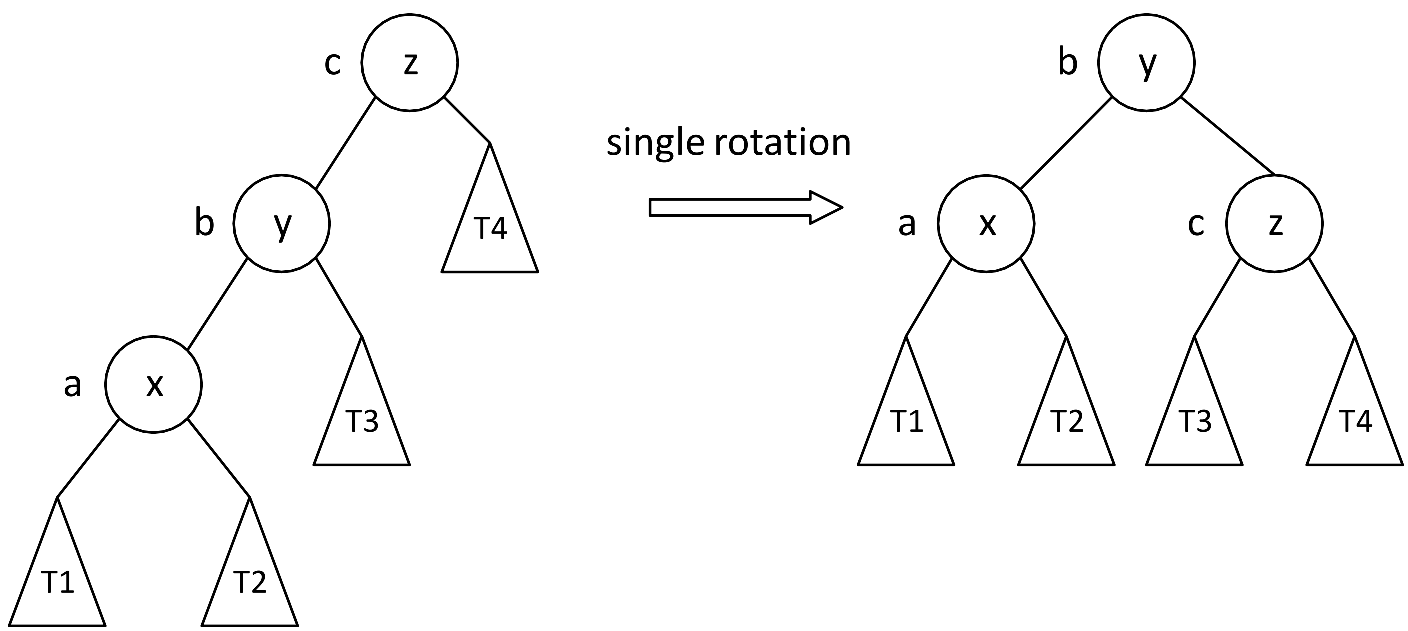
Balanced Search Trees

- Trinode restructuring: single rotation 1



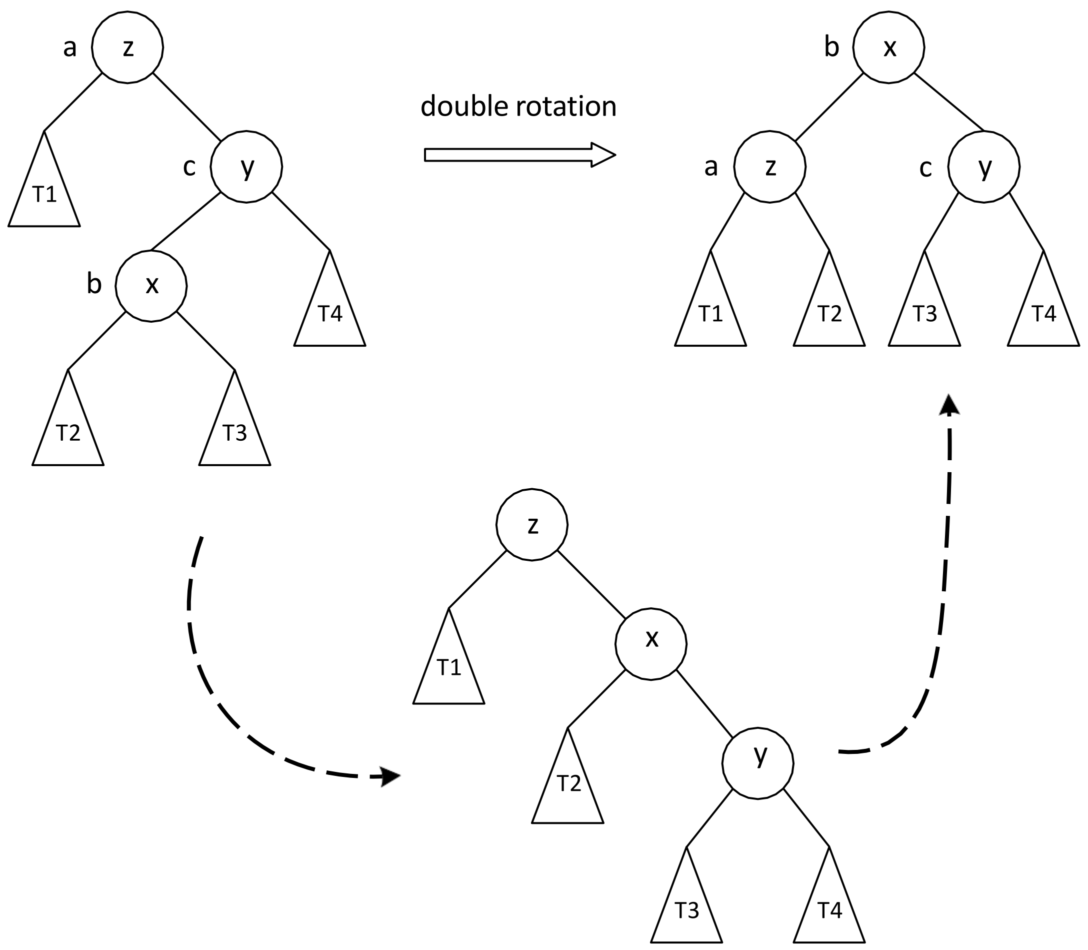
Balanced Search Trees

- Trinode restructuring: single rotation 2



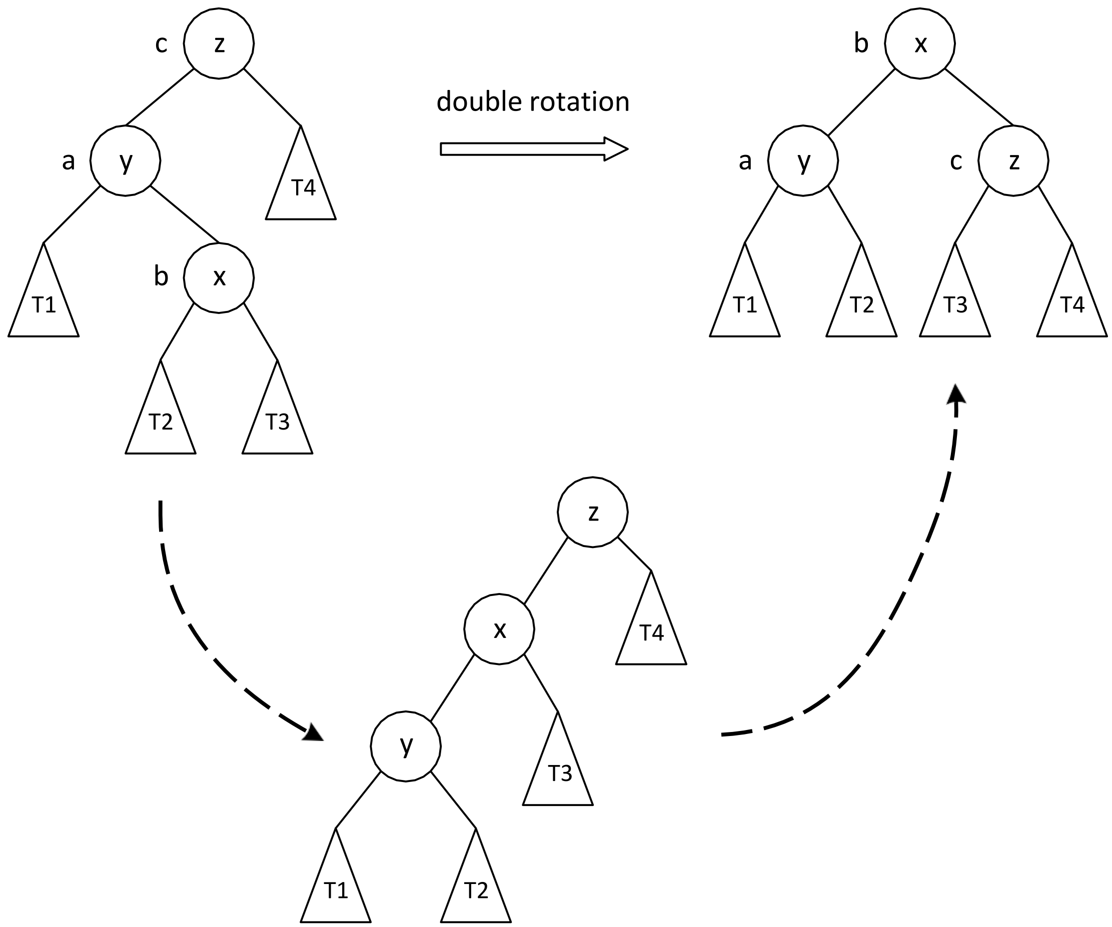
Balanced Search Trees

- Trinode restructuring: double rotation 1



Balanced Search Trees

- Trinode restructuring: double rotation 2



AVL Trees

- Recall
  - The *height of a node* is the number of edges on the longest path from that node to a leaf node.
  - The *height of a tree* (or a subtree) is the height of the root of the tree (or a subtree).
  - The height of a leaf node is zero.

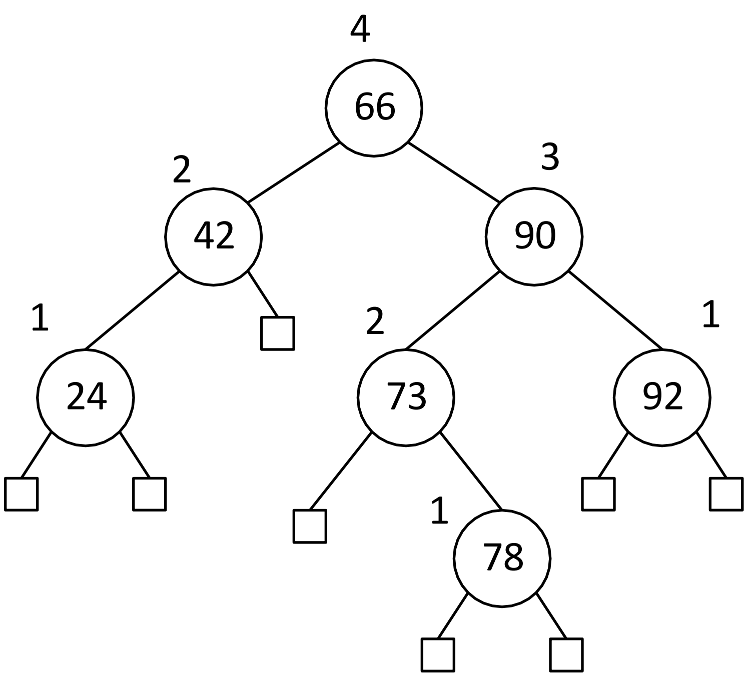
- An AVL tree is a binary search tree that satisfies the following *height-balance property*:

  For every internal node *p* of *T*, the heights of the children of *p* differ by at most one.

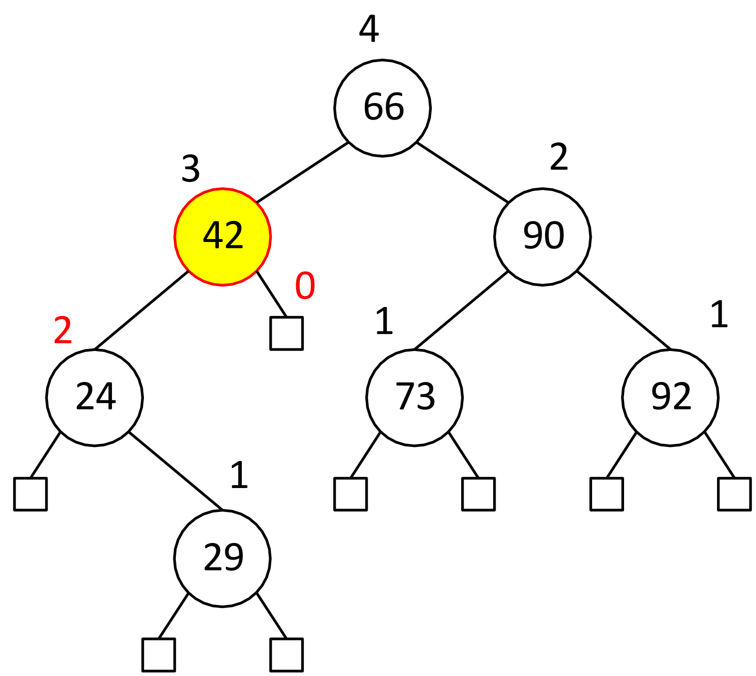
AVL Trees

- AVL tree example:

AVL tree



Not an AVL tree



AVL Trees

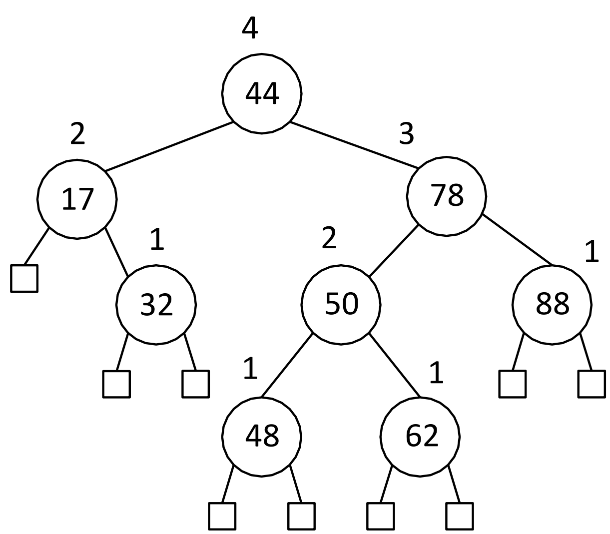
- Updating an AVL tree
  - A node *p* in a binary search tree is said to be *balanced* if the heights of *p*'s children differ by at most one.
  - Otherwise, a node is said to be *unbalanced.*
  - Therefore, every node in an AVL tree is balanced.

  - When we insert a node to an AVL tree or remove a node from an AVL tree, the resulting tree may violate the height-balance property.
  - So, we need to perform *post-processing*.

  - We will discuss only insertion.

AVL Trees

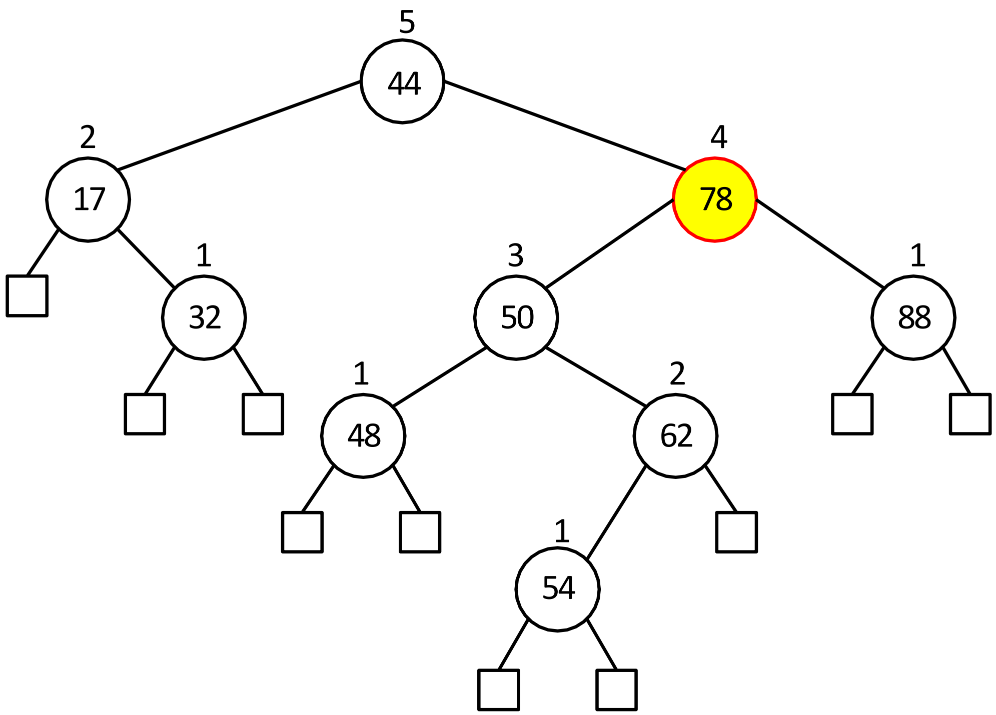
- When a node is inserted, the leaf node *p* where the new node is inserted becomes an internal node (with the entry of the new node).
- So, ancestors of *p* may be unbalanced.
- Restructuring is necessary.
- Consider the following tree:



AVL Trees

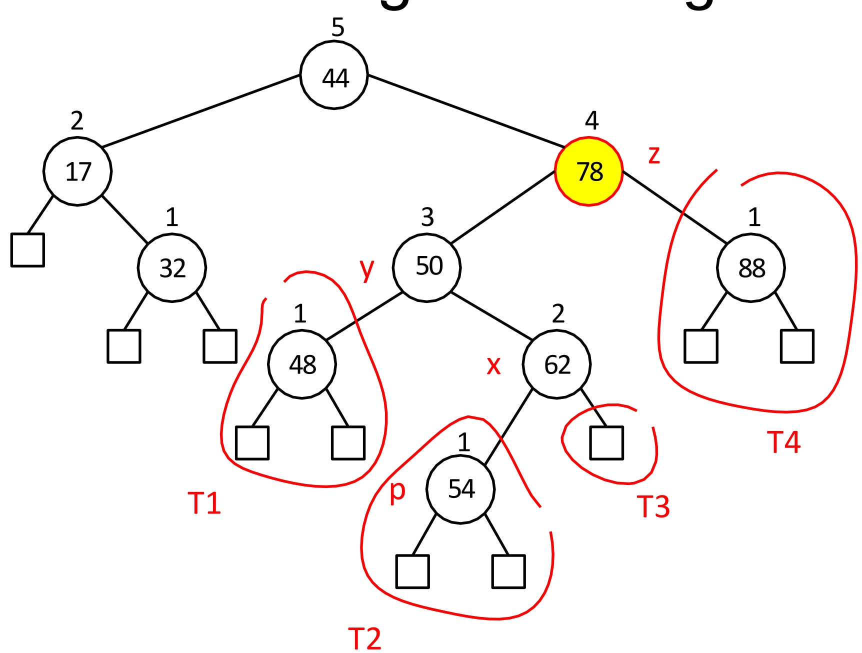
- After inserting 54, the node with 78 is unbalanced

After insertion, before rebalancing



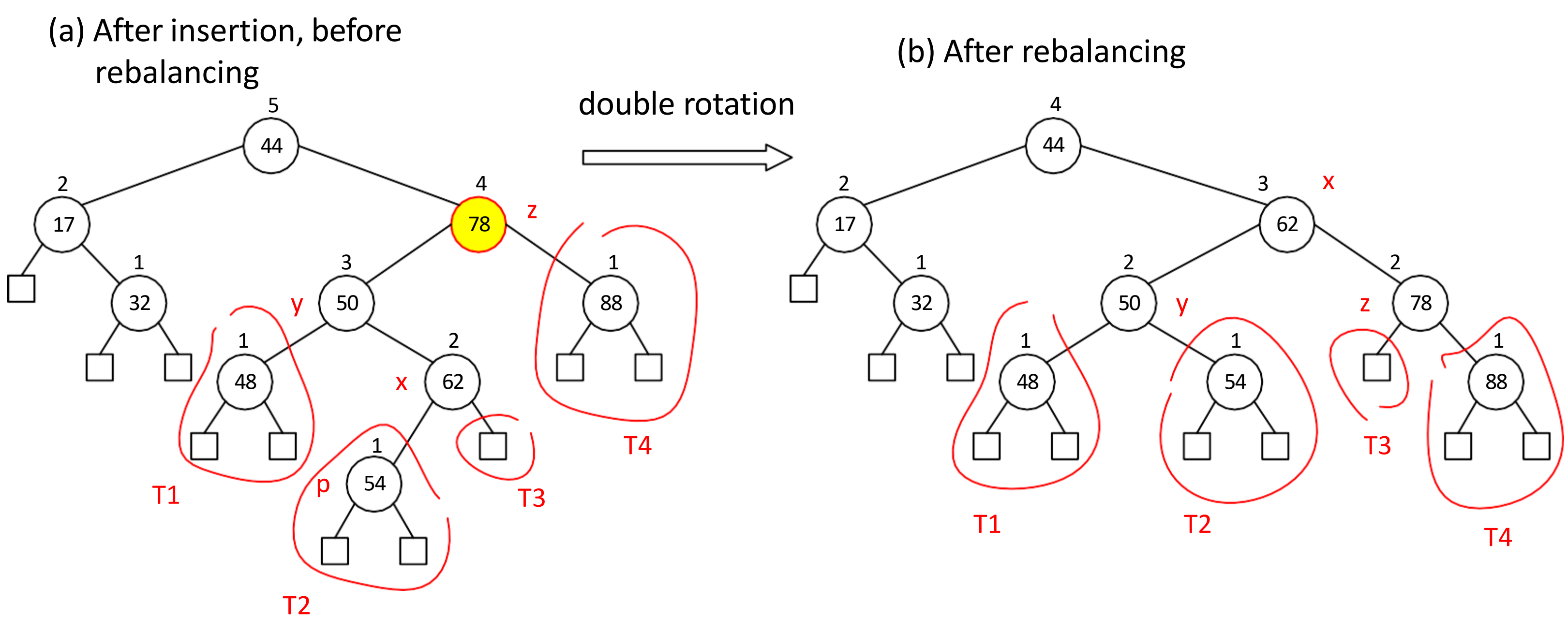
AVL Trees

- Post-processing
  - Search-and-repair strategy
  - Search a node *z* that is the lowest (in height) ancestor of *p* that is unbalanced.
  - *y* is *z*'s child with the greater height
  - *x* is *y*'s child with the greater height



AVL Trees

- Perform double rotation to rebalance the tree



References

- M.T. Goodrich, R. Tamassia, and M.H. Goldwasser, “Data Structures and Algorithms in Java,” Sixth Edition, Wiley, 2014.