Appendix

A. The iteration processes of BSP

Using the operations defined in IV-A, we could deduce the BSP iteration process as following:

$$
\begin{aligned}
X_0 &= (x_0, x_1, \ldots, x_n)\\
X_1 &= X_0 \otimes F\\
&= (x_0, x_1, x_2, \ldots, x_n) \otimes \begin{pmatrix} F_{0,0} & F_{0,1} & \ldots & F_{0,m}\\ F_{1,0} & F_{1,1} & \ldots & F_{1,m}\\ & & \ldots & \\ F_{n,0} & F_{n,1} & \ldots & F_{n,m}\end{pmatrix}\\
&= (\biguplus_{i=0}^{n} F_{i,0}(x_i),\ \biguplus_{i=0}^{n} F_{i,1}(x_i),\ \ldots,\ \biguplus_{i=0}^{n} F_{i,m}(x_i))
\end{aligned}
$$

Let $h(X) = \biguplus_{j=0}^{n} F_{j,i}(x_j)$, which means one round of vector multiplication and one aggregatioin operation then,

$$
\begin{aligned}
&= (h(X_0), h(X_0),\ \ldots, h(X_0))\\
X_2 &= X_1 \otimes F\\
&= (h(X_1), h(X_1),\ \ldots, h(X_1))\\
&= (h(h(X_0)), h(h((X_1)),\ \ldots, h(h(X_1)))\\
&= (h^2(X_0), h^2(X_0),\ \ldots, h^2(X_0))\\
X_3 &= (h(X_2), h(X_2),\ \ldots, h(X_2))\\
&= (h(h(X_1)), h(h(X_1)),\ \ldots, h(h(X_1)))\\
&= (h(h(h(X_0))), h(h((h(X_1))),\ \ldots, h(h(h(X_1))))\\
&= (h^3(X_0), h^3(X_0),\ \ldots, h^3(X_0))\\
&\vdots\\
X_k &= (h^k(X_0), h^k(X_0),\ \ldots, h^k(X_0)) \qquad (4.1)
\end{aligned}
$$

B. The iteration processes of DSP

Using the same deducing method as BSP, we get the formalized representation of DSP iteration processes shown in the following steps:

(1) In each iteration, every processor conducts Δ steps of local computation and local data update. To ensure its conciseness and generality, we just show the transformation on the processor whose segment is from x_p to x_q, and the derivation is shown in Table IV.

(2) After Δ steps of local computation and local data update, it follows a global data synchronization. After that, another superstep restarts again. The process could be described as following:

$$
\begin{aligned}
X_\Delta &= (x_{\Delta,0},\ x_{\Delta,1},\ x_{\Delta,2},\ \ldots,\ x_{\Delta,n})\\
X_{\Delta+1}^{(p,q)} &= X_\Delta \otimes F^{(p,q)}\\
&= (\ldots,\ x_{\Delta,p-1},\ \biguplus_{i=0}^{n} F_{i,p}(x_{\Delta,p}),\ \ldots,\\
&\qquad \biguplus_{i=0}^{n} F_{i,q}(x_{\Delta,q}),\ x_{\Delta,q+1},\ \ldots,\ x_{\Delta,n})\\
&\qquad\vdots
\end{aligned}
$$

$$
\begin{aligned}
X_{2\Delta}^{(p,q)} = (\ldots,\ x_{\Delta,p-1},\ g^{\Delta-1}(&\underbrace{\alpha_{2\Delta,p}}_{\boxed{\biguplus_{i=0}^{n} F_{i,p}(x_{\Delta,p})}},\ \underbrace{\beta_{2\Delta,p}}_{\boxed{\biguplus_{i\notin(p,q)} F_{i,p}(x_{\Delta,p})}}),\ \ldots,\\
&g^{\Delta-1}(\alpha_{2\Delta,q},\ \beta_{2\Delta,q}),\ x_{\Delta,q+1},\ \ldots,\ x_{\Delta,n}),
\end{aligned}
$$

in which each item can be derived as follows :

$$
x_{2\Delta,p} = g^{\Delta-1}(\biguplus_{i=0}^{n} F_{i,p}(\underbrace{x_{\Delta,p}}),\ \biguplus_{i\notin(p,q)} F_{i,p}(\underbrace{x_{\Delta,p}}))
$$

$$
\boxed{x_{\Delta,p} = g^{\Delta-1}(\alpha_p, \beta_p)}
$$

$$
= g^{\Delta-1}(\biguplus_{i=0}^{n} F_{i,p}\left(g^{\Delta-1}(\alpha,\ \beta)\right),\ \biguplus_{i\notin(p,q)} F_{i,p}\left(g^{\Delta-1}(\alpha,\ \beta)\right))
$$

(3) Through the deduction of (1) and (2), we find that the output of current bulk round will be used as the input of next bulk round. Thus one round of iteration process can be expresssed as:

$$
h(g(x, y)),
$$

in which $g(x, y) = \biguplus_{i=0}^{n}(\biguplus_{i\in(p,q)} F_{i,j}(x),\ y)$.

(4) Through l rounds bulk synchronization, we could get $X_{l\Delta}$:

$$
X_{l\Delta} = (x_{l\Delta,0},\ x_{l\Delta,1},\ \ldots,\ x_{l\Delta,m}),
$$

in which each item can be presented as

$$
x_{l\Delta,p} = h^l(x_{\Delta,p}), p = 0, 1, \ldots, m
$$

in which

$$
x_{\Delta,p} = g^{\Delta-1}(\alpha_p,\ \beta_p)
$$

in which $\alpha_p = \biguplus_{i=0}^{n} F_{i,p}(x_{t0,i})$ and $\beta_p = \biguplus_{i\notin(p,q)} F_{i,p}(x_{t0,i})$, finally, we can get

$$
x_{l\Delta,p} = h^l(g^{\Delta-1}(\alpha_p,\ \beta_p)). \qquad (4.2)
$$

$$
\begin{aligned}
X_{t0} &= (x_{t0,0}, x_{t0,1}, \ldots, x_{t0,n}) \\
X_{t1}^{(p,q)} &= X_{t0} \otimes F^{(p,q)} = (x_{t0,0}, x_{t0,1}, \ldots, x_{t0,n}) \otimes
\begin{pmatrix}
1 & 0 & \ldots & 0 & F_{0,p} & \ldots & F_{0,q} & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & F_{1,p} & \ldots & F_{1,q} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & F_{p-1,p} & \ldots & F_{p-1,q} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & F_{p,p} & \ldots & F_{p,q} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & F_{q,p} & \ldots & F_{q,q} & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & F_{q+1,p} & \ldots & F_{q+1,q} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & F_{n,p} & \ldots & F_{n,q} & 0 & \ldots & 1
\end{pmatrix} \\
& \quad Using\ the\ ''matrix\ multiplication'' - like\ transformation\ \otimes,\ we\ get\ the\ follows,\ only\ x_p\ to\ x_q\ are\ updated. \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ \biguplus_{i=0}^{n} F_{i,p}(x_{t0,i}),\ \biguplus_{i=0}^{n} F_{i,p+1}(x_{t0,i}),\ \ldots,\ \biguplus_{i=0}^{n} F_{i,q}(x_{t0,i}),\ x_{t0,q+1},\ \ldots,\ x_{t0,n}) \\
X_{t2}^{(p,q)} &= X_{t1}^{(p,q)} \otimes F^{(p,q)} \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ \biguplus_{i=0}^{n} F_{i,p}(x_{t1,i}),\ \biguplus_{i=0}^{n} F_{i,p+1}(x_{t1,i}),\ \ldots,\ \biguplus_{i=0}^{n} F_{i,q}(x_{t1,i}),\ x_{t0,q+1},\ \ldots,\ x_{t0,n}) \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ \biguplus(\biguplus_{i\in(p,q)} F_{j,p}(\biguplus_{j=0}^{n} F_{j,p}(x_{t0,j})),\ \biguplus_{j\notin(p,q)} F_{j,p}(x_{t0,j})), \\
&\qquad \ldots,\ \biguplus(\biguplus_{i\in(p,q)} F_{i,q}(\biguplus_{j=0}^{n} F_{j,q}(x_{t0,j})),\ \biguplus_{j\notin(p,q)} F_{j,q}(x_{t0,j})),\ x_{t0,q+1},\ \ldots, x_{t0,n}) \\
& \quad Let\ \alpha_p = \biguplus_{j=0}^{n} F_{j,p}(x_{t0,j}),\ \beta_p = \biguplus_{j\notin(p,q)} F_{j,p}(x_{t0,j}) \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ \biguplus(\biguplus_{i\in(p,q)} F_{i,p}(\alpha_p), \beta_p),\ \ldots,\ \biguplus(\biguplus_{i\in(p,q)} F_{i,q}(\alpha_q), \beta_q),\ x_{t0,q+1},\ \ldots, x_{t0,n}) \\
& \quad Let\ g(x,y) = \biguplus(\biguplus_{i\in(p,q)} F_{i,p}(x), y) \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ g(\alpha_p, \beta_p),\ \ldots,\ g(\alpha_q, \beta_q),\ x_{t0,q+1},\ \ldots, x_{t0,n}) \\
X_{t3}^{(p,q)} &= X_{t2}^{(p,q)} \otimes F^{(p,q)} \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ \biguplus(\biguplus_{i\in(p,q)} F_{i,p}(X_{t2}),\ \beta_p),\ \ldots,\ \biguplus(\biguplus_{i\in(p,q)} F_{i,q}(X_{t2}),\ \beta_q),\ x_{t0,q+1},\ \ldots, x_{t0,n}) \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ \biguplus(\biguplus_{i\in(p,q)} F_{i,p}(g(\alpha_p, \beta_p)),\ \beta_p),\ \ldots,\ \biguplus(\biguplus_{i\in(p,q)} F_{i,q}(g(\alpha_q, \beta_q)),\ \beta_q),\ x_{t0,q+1},\ x_{t0,q+1},\ \ldots, x_{t0,n}) \\
&\ \vdots \\
X_{\Delta}^{(p,q)} &= X_{\Delta-1}^{(p,q)} \otimes F^{(p,q)} \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1},\ g(g(\ldots g(\alpha_p, \beta_p), \ldots, \beta_p), \beta_p), \ldots,\ g(g(\ldots g(\alpha_q, \beta_q), \ldots, \beta_q), \beta_q),\ x_{t0,q+1}, \ldots, x_{t0,n}) \\
&= (x_{t0,0}, \ldots,\ x_{t0,p-1}, g^{\Delta-1}(\alpha_p, \beta_p), \ldots,\ g^{\Delta-1}(\alpha_q, \beta_q),\ x_{t0,q+1}, \ldots, x_{t0,n})
\end{aligned}
$$

TABLE IV: Derivation of First Δ Steps of Local Computation and Data Exchange.