CURVED BEAM ELEMENTS

KINEMATICS:

$$\varepsilon_0 = u_0' + h' v' + \frac{1}{2}(v')^2 \quad \& \quad \phi = v''$$

↳ $u_0(x)$ IS DISPLACEMENT PARALLEL TO x-AXIS

↳ $v(x)$ IS DEFLECTION ⊥ TO x AXIS

WEAK FORM:

$$\delta\Pi(u_0, v) = G(u_0, v; \delta u_0, \delta v)$$

$$= \int_L \delta\tilde{W}(\varepsilon_0, \phi)\,dx + \int_L w(x)\,\delta v(x)\,dx - \bar{P}\,\delta u_{0L} + \bar{R}\,\delta v_L - \bar{M}\,\delta v'_L$$

$$= \int_L \left(F(\varepsilon_0,\phi)\,\delta\varepsilon_0 + M(\varepsilon_0,\phi)\,\delta\phi\right)dx + \int_L w(x)\,\delta v(x)\,dx - \bar{P}\,\delta u_{0L} + \bar{R}\,\delta v_L - \bar{M}\,\delta v'_L$$

* FOR LINEAR ELASTIC CONSTITUITIVE LAWS

↳ $F(\varepsilon_0, \phi) = EA\varepsilon_0$

↳ $M(\varepsilon_0, \phi) = EI\phi$

* PRIMARY VARIBLES

↳ $u_0(x)$

↳ $v(x)$

① DERIVE THE LINEARIZED WEAK FORM, $d\delta\Pi(u_0, v; du_0, dv; \delta u_0, \delta v)$

$$\delta\Pi = \int_L \left[EA\varepsilon_0\,\delta\varepsilon_0 + EI\phi\,\delta\phi\right]dx + \int_L w(x)\,\delta v(x)\,dx - \bar{P}\,\delta u_{0L} + \bar{R}\,\delta v_L - \bar{M}\,\delta v'_L$$

$$d\delta\Pi = \int_L \left[EA\left(\frac{\partial\varepsilon_0}{\partial u_0}du_0 + \frac{\partial\varepsilon_0}{\partial v}dv\right)\delta\varepsilon_0 + EA\varepsilon_0\,d\delta\varepsilon_0\right]dx$$

$$+ \int_L \left[EI\frac{d\phi}{dv}dv\,\delta\phi + EI\phi\,d\delta\phi\right]dx$$

$$+ \int_L w(x)\,\cancel{d}\,\delta v\,dx - \bar{P}\,\cancel{d}\,\delta u_{0L} + \bar{R}\,\cancel{d}\,\delta v_L - \bar{M}\,\cancel{d}\,\delta v'_L$$

* LOAD TERMS INDEPENDENT OF DISPLACEMENT

$$d\delta\Pi = \int \left[\frac{dF}{d\varepsilon_0}d\varepsilon_0\,\delta\varepsilon_0 + F\,d\delta\varepsilon_0 + \frac{dM}{d\phi}d\phi\,\delta\phi\right]dx$$

$\frac{dF}{d\varepsilon_0} = EA$ $\qquad$ $\frac{dM}{d\phi} = EI$

$d\varepsilon_0 = du_0' + (h' + v')\,dv'$ $\qquad$ $d\phi = dv''$

$\delta\varepsilon_0 = \delta u_0' + (h' + v')\,\delta v'$ $\qquad$ $\delta\phi = \delta v''$

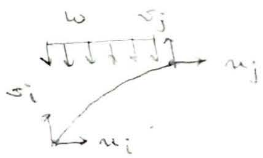
$d\delta\varepsilon_0 = d\left(\delta u_0' + (h' + v')\,\delta v'\right)$

$= dv'\,\delta v'$

① CONT'D

$$d\delta\Pi = \int \Big[EA\left(du_0' + (h' + v')\,dv'\right)\left(\delta u_0' + (h' + v')\,\delta v'\right) + EA\varepsilon_0\, \delta v'\, dv' + EI\, dv''\, \delta v'' \Big] dx$$

② FINITE ELEMENT FOR CURVED BEAM



- SINGLE ELEMENT

↳ SHAPE FUNCTIONS

$$u_0^h = \left(1 - \frac{x}{L_x}\right) u_i + \left(\frac{x}{L_x}\right) u_j$$

$$v^h = N_1(x) v_i + N_2(x)\theta_i + N_3(x) v_j + N_4(x)\theta_j$$

$$N_1(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$N_2(\xi) = L(\xi - 2\xi^2 + \xi^3)$$

$$N_3(\xi) = 3\xi^2 - 2\xi^3$$

$$N_4(\xi) = L(-\xi^2 + \xi^3)$$

$$\xi = \frac{x}{L}$$

↳ ELEVATION FUNCTION

$$h(x) = \left(1 - \frac{x}{L_e}\right) y_i + \left(\frac{x}{L_e}\right) y_j$$

- FINDING RESIDUAL FORCE VECTOR

$$\{q\} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\begin{Bmatrix} u_0^h(x) \\ v_0^h(x) \end{Bmatrix} = \begin{bmatrix} N_{u_1} & 0 & 0 & N_{u_2} & 0 & 0 \\ 0 & N_{v_1} & N_{v_2} & 0 & N_{v_3} & N_{v_4} \end{bmatrix} \{q\}$$

$$u_0^h(x) = \begin{bmatrix} N_{u_1} & N_{u_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\underline{N_u}]\{q_u\}$$

$$v_0^h(x) = \begin{bmatrix} N_{v_1} & N_{v_2} & N_{v_3} & N_{v_4} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = [\underline{N_v}]\{\underline{q}_v\}$$

② CONT'D

FIND STRAINS

$$\left.\begin{aligned} u_o^{h\,\prime} &= [\underline{N}_u]'\{q_u\} \\ v_o^{h\,\prime} &= [\underline{N}_v]'\{q_v\} \end{aligned}\right\} \quad \varepsilon_o = u_o^{h\,\prime} + h'v' + \tfrac{1}{2}(v')^2$$

$$v_o^{h\,\prime\prime} = [\underline{N}_v]''\{q_v\} \quad \Big\} \quad \phi = v_o''$$

TAKE VARIATIONS

$$\delta\varepsilon_o = \delta u_o' + (h' + v')\delta v' = [\underline{N}_u]'\{q_u\} + (h' + v')[\underline{N}_v]'\{q_v\}$$

$$\delta\phi = \delta v'' = [\underline{N}_v]''\{q_v\}$$

REARRANGE TO FIND STRAIN-DISPLACEMENT MATRICES $[B_\varepsilon]$ & $[B_\phi]$

LET $\begin{Bmatrix} \delta\varepsilon_o \\ \delta\phi \end{Bmatrix} = \begin{bmatrix} B_\varepsilon \\ B_\phi \end{bmatrix} \{\delta q\}$

$$[\underline{B}_\varepsilon] = \left[N_{u_1} \quad (h'+v')N_{v_1}' \quad (h'+v')N_{v_2}' \quad N_{u_2}' \quad (h'+v')N_{v_3}' \quad (h'+v')N_{v_4}' \right]$$

$$[\underline{B}_\phi] = \left[0 \quad N_{v_1}'' \quad N_{v_2}'' \quad 0 \quad N_{v_3}'' \quad N_{v_4}'' \right]$$

TAKE VARIATION OF POTENTIAL ENERGY

$$\Pi = \Pi_{INT} - \Pi_{EXT} = \int_L [W_\varepsilon(\varepsilon) + W_\phi(\phi)]\,dx - \int w(x)v(x)\,dx$$

$$\delta\Pi = \int_L \left(\frac{\partial W}{\partial \varepsilon}\delta\varepsilon + \frac{\partial W}{\partial \phi}\delta\phi \right) dx + \int w\,\delta v\,dx$$

$$= \int_L \left(F\,\delta\varepsilon + M\,\delta\phi + w\,\delta v \right) dx$$

$$= \int \Big(F[\underline{B}_\varepsilon]\{\delta q\} + M[\underline{B}_\phi]\{\delta q\} + \underbrace{w[\underline{N}_v]\{\delta q_v\}}\Big)\,dx$$

NEED TO GET IN SAME DIMENSION AS $\{\delta q\}$

DEFINE $[\hat{N}_v]\{\delta q\}$ S.T. THERE IS NO CONTRIBUTION IN THE $\{q_u\}$ DIRECTION

$\downarrow$

$$[\hat{N}_v] = [0 \quad N_{v_1} \quad N_{v_2} \quad 0 \quad N_{v_3} \quad N_{v_4}]$$

② CONT'D

$$\boxed{\begin{aligned}\delta\Pi &= \int\left(F[\underline{B}_\varepsilon]\{\delta\underline{q}\} + M[\underline{B}_\phi]\{\delta\underline{q}\} + w[\hat{N}_v]\{\delta\underline{q}\}\right)dx \\ &= \underbrace{\{\delta\underline{q}\}^T\Big(\underbrace{\int F[B_\varepsilon]^T dx + \int M[B_\phi]^T dx}_{\underline{F}^{INT}} + \underbrace{\int w[\hat{N}_v]^T dx}_{-\underline{P}}\Big)}_{\equiv \underline{R} = 0}\end{aligned}}$$

• FIND TANGENT STIFFNESS MATRIX (PERFORM LINEARIZATION)

$$d(\delta\Pi) = d\int(F\delta\varepsilon_0 + M\delta\phi + w\delta v)\,dx$$

$$= \int\Big(\underbrace{\frac{\partial F}{\partial\varepsilon_0}}_{EA}\underbrace{d\varepsilon_0}_{[B_\varepsilon]\{dq\}}\underbrace{\delta\varepsilon_0}_{[B_\varepsilon]\{\delta q\}} + \underbrace{F\,d\delta\varepsilon_0}_{F\,dv'\delta v'} + \underbrace{\frac{\partial M}{\partial\phi}}_{EI}\underbrace{d\phi}_{[B_\phi]\{dq\}}\underbrace{\delta\phi}_{[B_\phi]\{\delta q\}} + \cancel{w\,d\delta v}\Big)dx$$

$$\left.\begin{aligned} dv' &= [\hat{N}_v']\{d\underline{q}\} \\ \delta v' &= [\hat{N}_v']\{\delta\underline{q}\}\end{aligned}\right\}\quad dv'\delta v' = \{\delta q\}^T[\hat{N}_v']^T[\hat{N}_v']\{dq\}$$

$$d\delta\Pi = \underbrace{\{\delta q\}^T\Big(\underbrace{\int_L [B_\varepsilon]^T EA[B_\varepsilon]\,dx}_{\text{AXIAL STIFFNESS}} + \underbrace{\int_L [B_\phi]^T EI[B_\phi]\,dx}_{\text{BENDING STIFFNESS}} + \underbrace{\int[\hat{N}_v']^T F[\hat{N}_v']\,dx}_{\text{GEOMETRIC STIFFNESS}}\Big)}_{[\underline{K}_T]}$$

OR, WRITE $[\underline{K}_T]$ I.T.O. SHAPE FUNCTIONS BY PARTITIONING

$$[\underline{K}_T] = \left[\begin{array}{c|c} \underset{2\times 2}{\int[N_u']^T EA[N_u]\,dx} & \underset{2\times 4}{\int EA(h'+v')[N_u'][N_v']\,dx} \\ \hline \underset{4\times 2}{\int EA(h'+v')[N_u'][N_v']^T dx} & \underset{4\times 4}{\begin{aligned}&\int EA(h'+v')^2[N_v']^T[N_v']\,dx \\ &+\int EI[N_v'']^T[N_v'']\,dx \\ &+\int F[N_v']^T[N_v']\,dx\end{aligned}}\end{array}\right]$$