CgWaveHoltz: A Composite Grid Helmholtz Solver using the WaveHoltz Algorithm

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Abstract

Here are notes on CgWaveHoltz Helmholtz solver that is based on solving the Helmholtz equation using the wave equation solver CgWave.

Keywords: Maxwell

Contents

1. Introduction

Here are some notes on the CgWaveHoltz solver based on WaveHoltz scheme [**?**] of Appelö et.al. The WaveHoltz algorithm solves the time-harmonic wave equation (Helmholtz equation) using a time-dependent wave equation solver. CgWaveHoltz uses the CgWave wave equation solver.

Things to do:

1. Compatibility boundary conditions for direct Helmholtz solver.
2. Sixth and eight-order accurate direct Helmholtz solver.

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[1]This work was performed under DOE contracts from the ASCR Applied Math Program.
[2]Research supported by the National Science Foundation under grant DMS-1519934.
[3]Research supported by a U.S. Presidential Early Career Award for Scientists and Engineers.

2. WaveHoltz algorithm

The Helmholtz solution $u(\mathbf{x}, t)$ satisfies,

$$-\omega^2 u = c^2 \Delta u + f(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \Omega, \tag{1a}$$

$$Bu = g(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial\Omega, \tag{1b}$$

Here $Bw = g$ denotes some appropriate boundary conditions.

Let $w(\mathbf{x}, t)$ be a solution to the wave equation in second-order form with a time-harmonic forcing,

$$\partial_t^2 w = c^2 \Delta w + f(\mathbf{x})\, e^{i\omega t}, \qquad \text{for } \mathbf{x} \in \Omega,\ t > 0, \tag{2a}$$

$$Bw = g(\mathbf{x})\, e^{i\omega t} \qquad \text{for } \mathbf{x} \in \partial\Omega,\ t > 0, \tag{2b}$$

$$w(x, 0) = v_0(\mathbf{x}), \quad \partial_t w(\mathbf{x}, 0) = v_1(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \partial\Omega. \tag{2c}$$

and initial conditions for u and u_t given by $v_0(\mathbf{x})$ and $v_1(\mathbf{x})$.

The WaveHoltz algorithm [**?**] defines an affine operator Π that acts on the initial conditions $\mathbf{v} = [v_0,\, v_1]^T$ by solving the wave equation (2) with initial conditions $\mathbf{v}$ over a time period $T = 2\pi/\omega$ and then forming the time integral

$$\Pi \mathbf{v} = \frac{2}{T} \int_0^T \left(\cos(\omega t) - \frac{1}{4} \right) \begin{bmatrix} w(\mathbf{x}, t) \\ w_t(\mathbf{x}, t) \end{bmatrix} dt. \tag{3}$$

Note that w implicitly depends on v and f, $w = w(\mathbf{x}, t; v, f)$. Note that Π is an affine operator of the form

$$\Pi v = Lv + b = (I - A)v + b, \tag{4}$$

where L and A are a linear operators. The solution to the Helmholtz problem (1) is found as the solution to the fixed point iteration

$$\mathbf{v}^n = \Pi \mathbf{v}^{n-1}, \tag{5}$$

where $\mathbf{v}^0 = 0$ (or some other initial guess).

3. Using Krylov methods to accelerate the WaveHoltz algorithm

The fixed point iteration (5) can be accelerated with a Krylov method such as GMRES. A Krylov method solves a linear system

$$Ax = b. \tag{6}$$

At the most basic level Krylov methods only require a function to evaluate b and a function to compute A times a vector. For the fixed point iteration (5), b is defined as the application of Π to the vector $\mathbf{v}^0 = \mathbf{0}$ (zero initial conditions),

$$b \stackrel{\text{def}}{=} \Pi\, \mathbf{0}, \tag{7}$$

while the application of A to an iterate $\mathbf{v}^n$ is defined as

$$A\mathbf{v}^n = \mathbf{v}^n - \Pi \mathbf{v}^n + b. \tag{8}$$

Formula (8) looks a bit funny at first glance but this is the correct definition.

We use the Krylov solvers from PETSc [1] to solve the fixed point iteration (5). PETSc has a "matrix-free" option that allows one to provide function to compute a matrix-vector product. PETSc has many Krylov solvers suh as CG, GMRES, bi-CG-stab, etc.

4. Numerical Results

Here we present some numerical results.

4.1. Exact trigonometric Helmholtz solution

An exact solution to the Helmholtz equation on a square domain $[0,1]^2$ is given by the trigonometric function

$$u_e(\mathbf{x},t) \stackrel{\text{def}}{=} \sin(k_x x)\,\sin(k_y y)\,\cos(\omega t),$$

where the forcing function is

$$f(\mathbf{x},t) = \Big(-\omega^2 + c^2(k_x^2+k_y^2)\Big) u_e(\mathbf{x},t)$$

This solution is also used on other domains in which case we allow for in-homogeneous BCs.

FD22s Errors and convergence of WaveHoltz solutions on a square.

grid	N	u	r
square8	1	5.5e-2	
square16	2	1.3e-2	4.04
square32	4	3.4e-3	4.01
square64	8	8.4e-4	3.99
rate		2.00	

Table 1: CgWaveHoltz, boxHelmholtz, max norm, solver=krylov, order=2, $t = 0.5$, cfl=0.9, ad4=1, omega=2, tol=1e-05 Sat May 22 06:17:07 2021

FD44s Errors and convergence of WaveHoltz solutions on a square.

grid	N	u	r
square8	1	3.4e-3	
square16	2	3.4e-4	10.12
square32	4	1.5e-5	22.97
square64	8	5.1e-7	28.96
square128	16	3.8e-8	13.34
square256	32	2.7e-9	14.03
square512	64	1.9e-10	14.52
rate		4.10	

Table 2: CgWaveHoltz, boxHelmholtz, max norm, solver=krylov, order=4, $t = 0.5$, cfl=0.9, ad4=1, omega=2, tol=1e-8 Sat May 22 06:36:05 2021

******** OLD STUFF *********

Here are some initial results for $k_x = 2\pi$, $k_y = 2\pi$ and $\omega = 2$.
Notes on output:

- line with "TIME INTEGRAL v" gives the true solution error.
- lines with "Maximum residual = 9.319e-05" gives the residual in the Helmholtz equation.
- lines with "`it=3:  max(|v-vOld|)=8.47e-06 (tol=0.0001)`" give the residual in the WaveHoltz iteration $\|v^{n+1} - \mathbf{v}^n\|_\infty$

Square : fixed-point iteration. SECOND-ORDER - looks OK

```
RESIDUAL IS SMOOTH

../bin/cgwh -grid=square64.order2 -cmd=boxHelmholtz.cmd -omega=2 -tol=1e-4
../bin/cgwh -grid=square32.order2 -cmd=boxHelmholtz.cmd -omega=2 -tol=1e-3

G64
cgWave: t=5.030e-01 (65 steps) maxErr= 4.55e-04
cgWave: t=1.006e+00 (130 steps) maxErr= 3.53e-04
cgWave: t=1.509e+00 (195 steps) maxErr= 8.45e-04
cgWave: t=2.012e+00 (260 steps) maxErr= 5.42e-04
cgWave: t=2.515e+00 (325 steps) maxErr= 2.72e-04
cgWave: t=3.018e+00 (390 steps) maxErr= 8.21e-04
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 8.45e-04
cgWave: t=3.142e+00 (406 steps) maxErr= 8.54e-04
it=3:  max(|v-vOld|)=8.47e-06 (tol=0.0001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =3 ##########################
Maximum residual = 9.319e-05


G32
advWave: sosupParameter=  0.10E+01
cgWave: t=4.952e-01 (32 steps) maxErr= 2.00e-03
cgWave: t=9.905e-01 (64 steps) maxErr= 9.78e-04
cgWave: t=1.486e+00 (96 steps) maxErr= 3.71e-03
cgWave: t=1.981e+00 (128 steps) maxErr= 2.45e-03
cgWave: t=2.476e+00 (160 steps) maxErr= 1.27e-03
cgWave: t=2.971e+00 (192 steps) maxErr= 3.05e-03
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 3.38e-03
cgWave: t=3.142e+00 (203 steps) maxErr= 3.81e-03
it=2:  max(|v-vOld|)=4.51e-04 (tol=0.001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =2 ##########################
Maximum residual = 1.058e-03
```

Square : fixed-point iteration. FOURTH-ORDER – not bad, errors may be skewed by poor BC's

```
LOOKS BETTER -- residual largest near the boundary -- need to fix ghost
COMPARE ERRORS IN TIME INTEGRAL ratio= 30

../bin/cgwh -grid=square64.order4 -cmd=boxHelmholtz.cmd -omega=2 -tol=1e-4
../bin/cgwh -grid=square32.order4 -cmd=boxHelmholtz.cmd -omega=2 -tol=1e-3

G64
cgWave: t=5.030e-01 (65 steps) maxErr= 2.20e-06
cgWave: t=1.006e+00 (130 steps) maxErr= 7.10e-06
cgWave: t=1.509e+00 (195 steps) maxErr= 5.93e-06
cgWave: t=2.012e+00 (260 steps) maxErr= 4.95e-06
cgWave: t=2.515e+00 (325 steps) maxErr= 7.88e-06
cgWave: t=3.018e+00 (390 steps) maxErr= 1.40e-06
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 5.34e-07
cgWave: t=3.142e+00 (406 steps) maxErr= 8.17e-06
it=3:  max(|v-vOld|)=8.09e-06 (tol=0.0001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =3 ##########################
```

```
Maximum residual = 7.956e-04


G32
cgWave: t=9.905e-01 (64 steps) maxErr= 3.30e-04
cgWave: t=1.486e+00 (96 steps) maxErr= 3.31e-04
cgWave: t=1.981e+00 (128 steps) maxErr= 1.31e-04
cgWave: t=2.476e+00 (160 steps) maxErr= 4.08e-04
cgWave: t=2.971e+00 (192 steps) maxErr= 1.18e-04
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 1.63e-05
cgWave: t=3.142e+00 (203 steps) maxErr= 3.85e-04
it=2:  max(|v-vOld|)=4.00e-04 (tol=0.001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =2 ##########################
Maximum residual = 5.925e-03
```

CIC grid : fixed-point iteration. SECOND-ORDER - looks OK

```
G4  - ORDER=2
cgWave: t=5.045e-01 (44 steps) maxErr= 2.23e-03 (max(|u|)= 5.35e-01)
cgWave: t=1.009e+00 (88 steps) maxErr= 2.95e-03 (max(|u|)= 4.35e-01)
cgWave: t=1.513e+00 (132 steps) maxErr= 4.82e-03 (max(|u|)= 9.98e-01)
cgWave: t=2.018e+00 (176 steps) maxErr= 1.64e-03 (max(|u|)= 6.28e-01)
cgWave: t=2.522e+00 (220 steps) maxErr= 3.43e-03 (max(|u|)= 3.30e-01)
cgWave: t=3.027e+00 (264 steps) maxErr= 4.38e-03 (max(|u|)= 9.78e-01)
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 4.58e-03
cgWave: t=3.142e+00 (274 steps) maxErr= 3.90e-03 (max(|u|)= 1.00e+00)
it=3:  max(|v-vOld|)=4.52e-04 (tol=0.001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =3 ##########################
Maximum residual = 4.049e-03

G2 - ORDER=2
cgWave: t=5.094e-01 (24 steps) maxErr= 9.28e-03 (max(|u|)= 5.33e-01)
cgWave: t=1.019e+00 (48 steps) maxErr= 7.21e-03 (max(|u|)= 4.57e-01)
cgWave: t=1.528e+00 (72 steps) maxErr= 1.72e-02 (max(|u|)= 1.01e+00)
cgWave: t=2.038e+00 (96 steps) maxErr= 1.14e-02 (max(|u|)= 6.05e-01)
cgWave: t=2.547e+00 (120 steps) maxErr= 6.96e-03 (max(|u|)= 3.80e-01)
cgWave: t=3.057e+00 (144 steps) maxErr= 1.74e-02 (max(|u|)= 1.00e+00)
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 1.74e-02
cgWave: t=3.142e+00 (148 steps) maxErr= 1.80e-02 (max(|u|)= 1.02e+00)
it=5:  max(|v-vOld|)=9.19e-04 (tol=0.001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =5 ##########################
Maximum residual = 8.508e-03
```

CIC grid : Krylov iteration. FOURTH-ORDER - v error looks OK, check residual compared to order=2, residual is large near boundaries

```
../bin/cgwh boxHelmholtz.cmd -g=cice4.order4.ng3.hdf -omega=2 -solver=krylov -tol=1e-5 -imode=1
../bin/cgwh boxHelmholtz.cmd -g=cice2.order4.ng3.hdf -omega=2 -solver=krylov -tol=1e-5 -imode=1

G4 : ORDER=4
advWave: ADVANCE dim=2 order=4                grid=curvilinear... t=  0.11E-01
cgWave: t=4.966e-01 (46 steps) maxErr= 1.12e-05, ||u||= 5.46e-01
cgWave: t=9.932e-01 (92 steps) maxErr= 8.34e-06, ||u||= 4.04e-01
cgWave: t=1.490e+00 (138 steps) maxErr= 2.00e-05, ||u||= 9.87e-01
cgWave: t=1.986e+00 (184 steps) maxErr= 1.39e-05, ||u||= 6.74e-01
cgWave: t=2.483e+00 (230 steps) maxErr= 5.05e-06, ||u||= 2.51e-01
cgWave: t=2.980e+00 (276 steps) maxErr= 1.97e-05, ||u||= 9.48e-01
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 2.05e-05
cgWave: t=3.142e+00 (291 steps) maxErr= 2.07e-05, ||u||= 1.00e+00
it=-1:  max(|v-vOld|)=1.30e-06, tol=1e-05

 *************** DONE KYRLOV ITERATIONS -- numberOfIterations=8 *************

Maximum residual = 6.903e-03

G2  order=4
```

```
cgWave: t=4.950e-01 (26 steps) maxErr= 2.82e-04, ||u||= 5.49e-01
cgWave: t=9.901e-01 (52 steps) maxErr= 2.00e-04, ||u||= 3.98e-01
cgWave: t=1.485e+00 (78 steps) maxErr= 4.84e-04, ||u||= 9.86e-01
cgWave: t=1.980e+00 (104 steps) maxErr= 3.73e-04, ||u||= 6.83e-01
cgWave: t=2.475e+00 (130 steps) maxErr= 1.88e-04, ||u||= 2.36e-01
cgWave: t=2.970e+00 (156 steps) maxErr= 4.49e-04, ||u||= 9.42e-01
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 4.98e-04
cgWave: t=3.142e+00 (165 steps) maxErr= 4.97e-04, ||u||= 1.00e+00
it=-1:  max(|v-vOld|)=3.40e-06, tol=1e-05

 ################ DONE KYRLOV ITERATIONS -- numberOfIterations=8 #############
Maximum residual = 3.699e-02
```

CIC grid : Krylov iteration. SECOND-ORDER - looks OK, residual is smooth

```
  ../bin/cgwh boxHelmholtz.cmd -g=cice4.order2.hdf -omega=2 -solver=krylov -tol=1e-4
ORDER=2
advWave: ADVANCE dim=2 order=2                grid=curvilinear... t=  0.11E-01
cgWave: t=5.045e-01 (44 steps) maxErr= 1.46e-03, ||u||= 5.34e-01
cgWave: t=1.009e+00 (88 steps) maxErr= 1.16e-03, ||u||= 4.34e-01
cgWave: t=1.513e+00 (132 steps) maxErr= 2.72e-03, ||u||= 9.96e-01
cgWave: t=2.018e+00 (176 steps) maxErr= 1.78e-03, ||u||= 6.28e-01
cgWave: t=2.522e+00 (220 steps) maxErr= 9.08e-04, ||u||= 3.27e-01
cgWave: t=3.027e+00 (264 steps) maxErr= 2.66e-03, ||u||= 9.76e-01
cgWave: t=3.142e+00 TIME INTEGRAL v:  maxErr= 2.74e-03
cgWave: t=3.142e+00 (274 steps) maxErr= 2.74e-03, ||u||= 1.00e+00
plot: finish chosen...
it=-1:  max(|v-vOld|)=9.43e-05, tol=0.0001

 ********** DONE KYRLOV ITERATIONS -- numberOfIterations=6 ************

Maximum residual = 4.955e-04
```

4.2. Gaussian forcing

We consider solving the Helmholtz problem with a source term consisting of a sum of generalized Gaussians,

$$f(x,y,z,t) = \sum_m a_m \cos(\omega_m(t - t_{0,m}))\, e^{-\beta_m[(x-x_{0,m})^2+(y-y_{0,m})^2+(z-z_{0,m})^2]^{p_m}}. \tag{9}$$

4.2.1. Gaussian source in a square

Figure 1 shows results from CgWaveHoltz for a square. Source term: $\omega = 17.17$, $a = 100$, $(x_0, y_0) = (.7, .7)$, $\beta = 50$, $p = 1$.

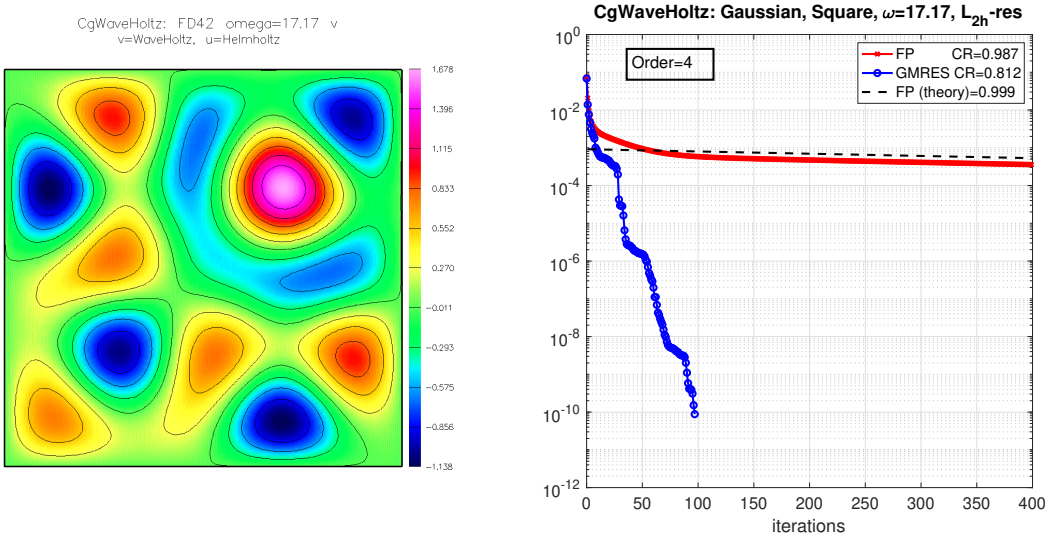

Figure 1: Gaussian source, $\omega = 17.17$, square64, order 4

Figure 2 shows results from CgWaveHoltz for a square. Source term: $\omega = 15$, $a = 100$, $(x_0, y_0) = (.7, .7)$, $\beta = 50$, $p = 1$. The FPI convergence rate matches the theory.

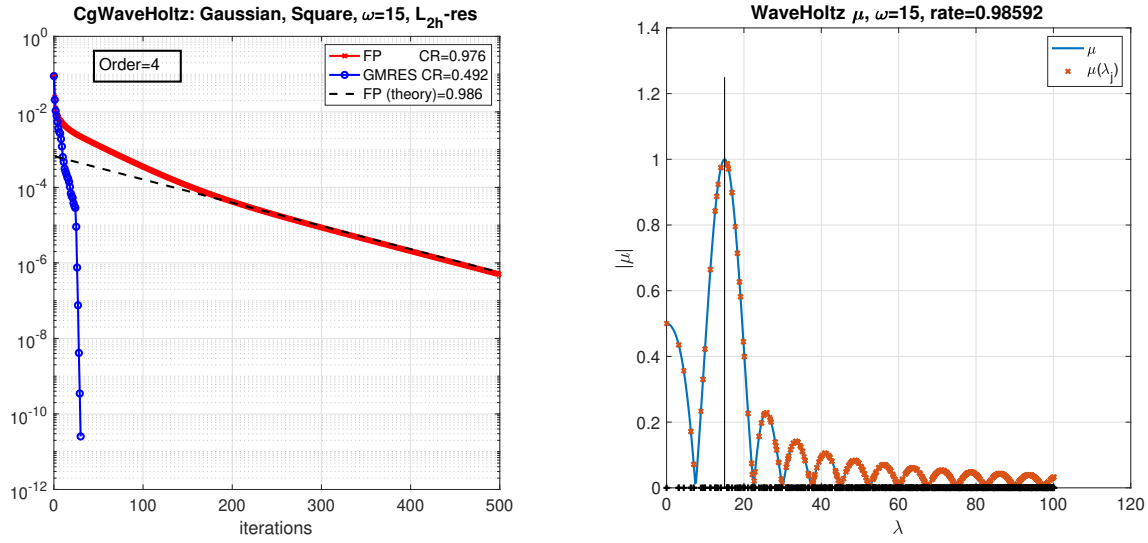

Figure 2: Gaussian source, $\omega = 15$, square128, order 4. Right: β function with distribution of eigenvalues λ_j

```
 ################ DONE KYRLOV ITERATIONS -- KSP residual=2.54e-11 (tol=1.00e-10) numberOfIterations=31 #############
CgWaveHoltz::residual: c=1, omega= 1.50000e+01, omegas= 1.49974e+01 (from symbol of D+D-), dt=4.273544e-03, adjustOmega=1
CgWaveHoltz::residual: max-res=4.010e-04 (using omega), max-res=5.004e-02 (using omega from discrete symbol)
CgWaveHoltz: omega=1.500e+01, max-res=4.010e-04, CR=0.492 CR-perPeriod=0.492, cpu= 1.58e+00(s)
CgWaveHoltz::Results saved to file gaussianWHSquare128O4Omega15Krylov.m
```

4.2.2. Gaussian source in an annulus

Figure 3 shows results from CgWaveHoltz for an Annulus. Eigenvalues for this problem were computed in `cg/ad/codes/annulusEigenvalues.maple`.

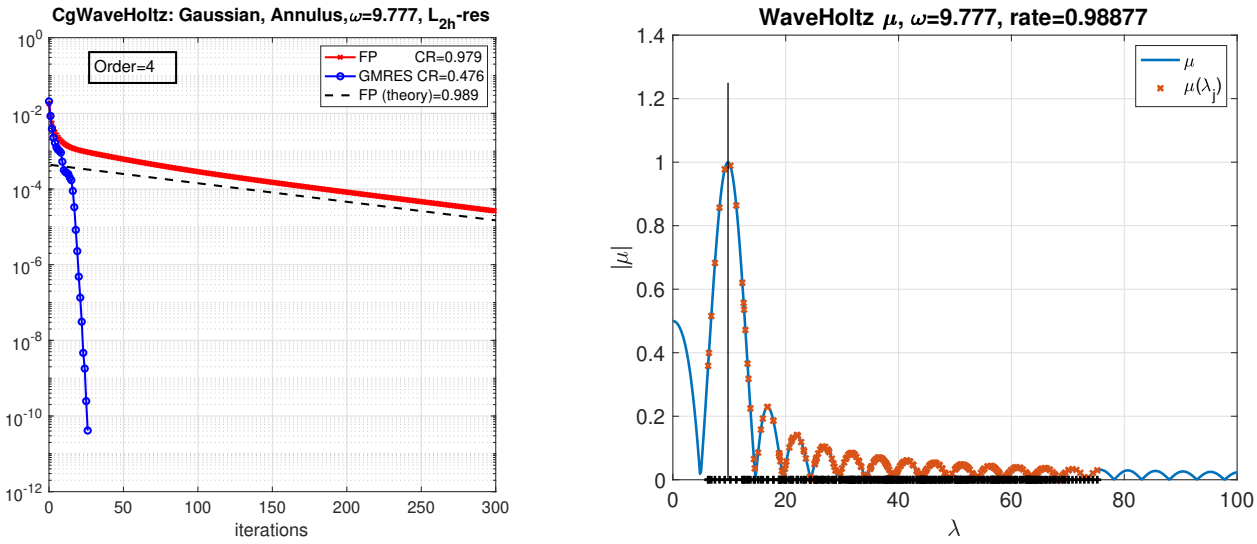

Figure 3: Gaussian source, $\omega = 9.777$, annulus8, order 4. Right: β function with distribution of eigenvalues λ_j

Figure 4 shows results from CgWaveHoltz for an Annulus for $\omega = 18.4$. It takes quite a iterations for the CR of the FPI to approach the theory; likely due to there being a few eigenvalues near the worst case. GMRES is also quite slow.

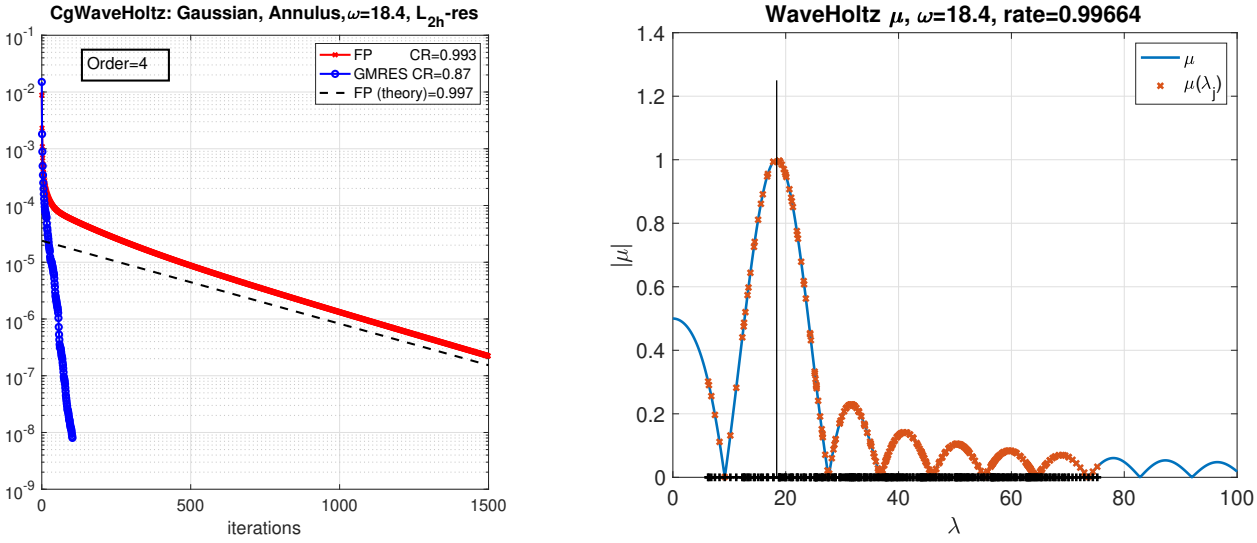

Figure 4: Gaussian source, $\omega = 18.4$, annulus8, order 4. Right: β function with distribution of eigenvalues λ_j

4.2.3. Gaussian source in a disk

Figure 5 shows results from CgWaveHoltz for an disk. Eigenvalues for this problem were computed in `cg/ad/codes/diskEigenvalues.maple`.

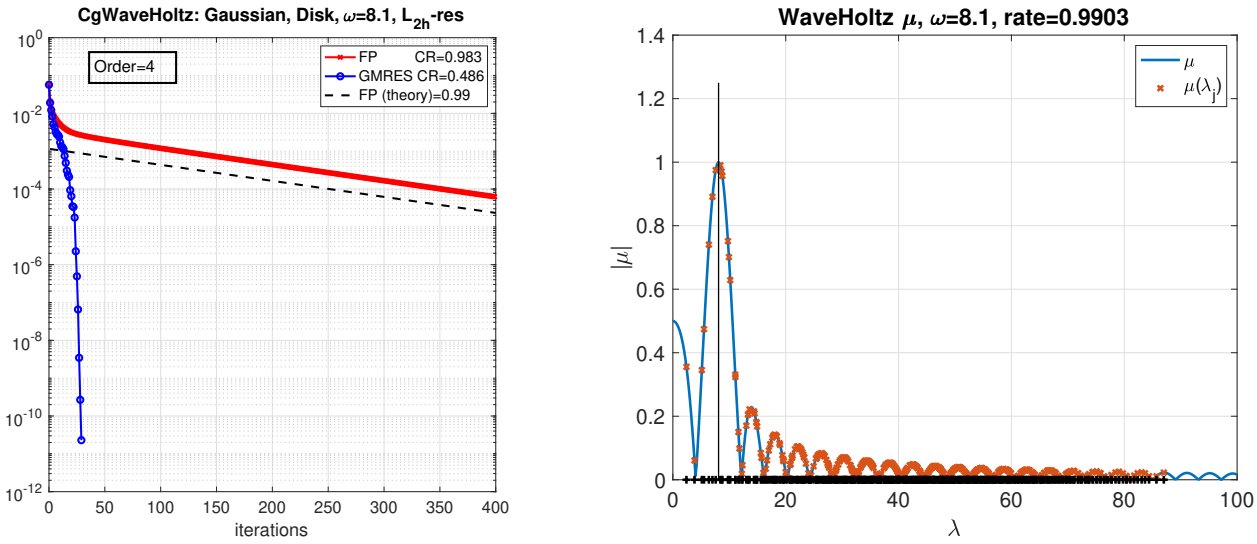

Figure 5: Gaussian source, $\omega = 8.1$, disk8, order 4. Right: β function with distribution of eigenvalues λ_j

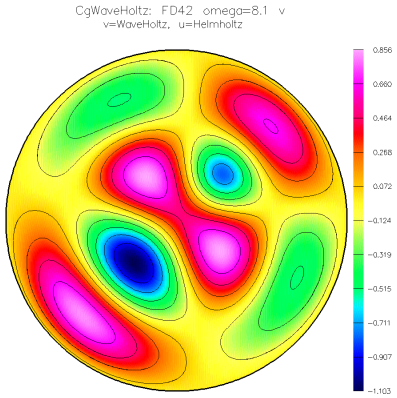

Figure 6: Gaussian source, $\omega = 8.1$, disk8, order 4.

Solution is close to the direct solver.

```
 ################ DONE KYRLOV ITERATIONS -- KSP residual=2.29e-11 (tol=1.00e-10) numberOfIterations=30 #############
CgWaveHoltz::residual: c=1, omega= 8.10000e+00, omegas= 8.09901e+00 (from symbol of D+D-), dt=6.687085e-03, adjustOmega=0
CgWaveHoltz::residual: max-res=1.776e-02 (using omega), max-res=9.750e-05 (using omega from discrete symbol)
CgWaveHoltz: omega=8.100e+00, max-res=1.776e-02, CR=0.486 CR-perPeriod=0.486, cpu= 3.41e+00(s)
CgWaveHoltz::Results saved to file gaussianWHDisk04Omega8p1Krylov.m
CgWaveHoltz: max-diff=3.25e-03 (between WaveHoltz and Direct Helmholtz solution)
```

4.2.4. Gaussian source in a box

Figure 7 shows results from CgWaveHoltz for a 3D box.

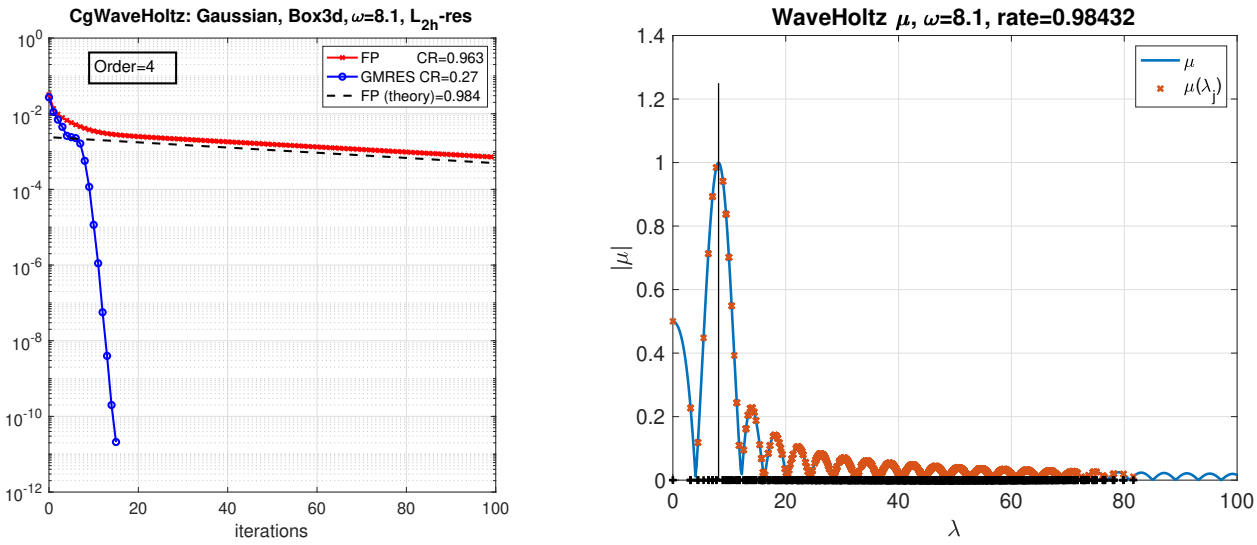

Figure 7: Gaussian source, $\omega = 8.1$, box4, order 4. Right: β function with distribution of eigenvalues λ_j

4.2.5. Gaussian source in a sphere

Figure 8 shows results from CgWaveHoltz for a 3D solid sphere.

FIX ME – EIGENVALUES FOR A SPHERE NEEDED

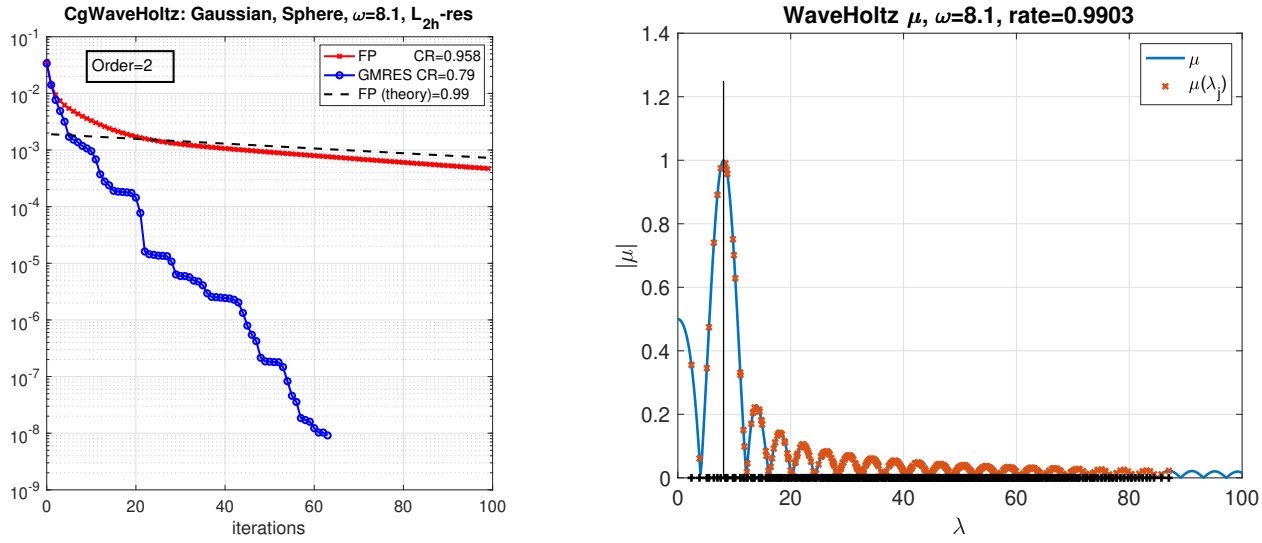

Figure 8: Gaussian source, $\omega = 8.1$, sphere4, order 2. Right: β function with distribution of eigenvalues λ_j

4.2.6. Gaussian source in a domain with three shapes

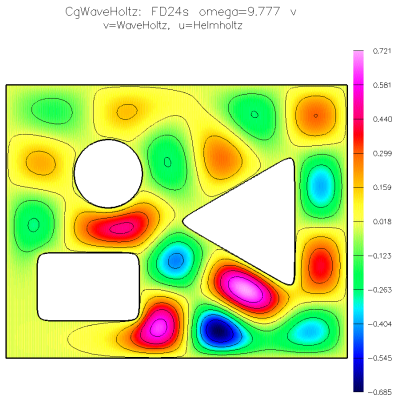

Figure 9: Gaussian pulse for shapes, $\omega = 9.777$, source at $(.3, -.75)$

```
SHAPES: omega=9.777
################ DONE KYRLOV ITERATIONS -- KSP residual=9.71e-09 (tol=1.00e-08) numberOfIterations=93 ##############
CgWaveHoltz::residual: c=1, omega= 9.77700e+00, omegas= 9.77691e+00 (from symbol of D+D-), dt=1.501518e-03, adjustOmega=0
CgWaveHoltz::residual: max-res=3.500e-02 (using omega), max-res=3.382e-02 (using omega from discrete symbol)
CgWaveHoltz: omega=9.777e+00, max-res=3.500e-02, CR=0.872 CR-perPeriod=0.872, cpu= 9.54e+01(s)
CgWaveHoltz::Results saved to file cgWaveHoltz.m
CgWaveHoltz: max-diff=1.16e-02 (between WaveHoltz and Direct Helmholtz solution)
```

********* OLD STUFF *********

- The Krylov method works much better than the fixed-point iteration.

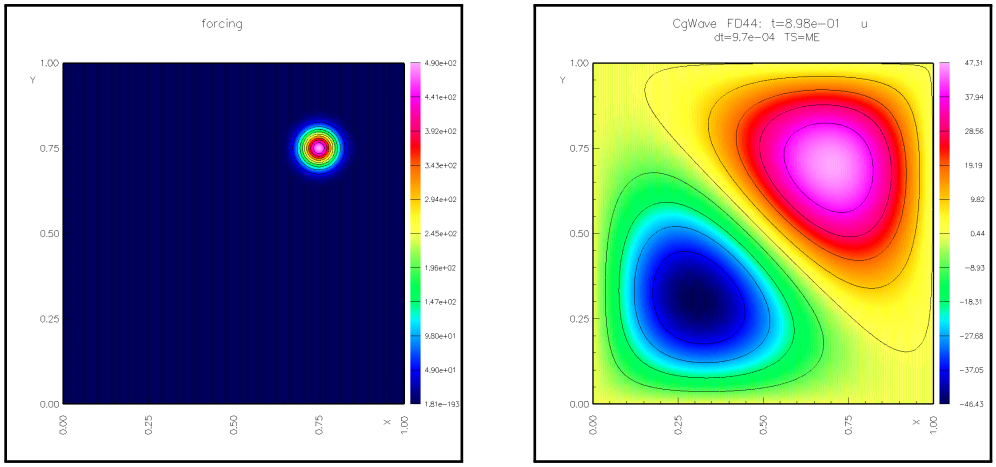

Figure 10: Gaussian forcing on a square. Left forcing. Right solution for $\omega = 7$.

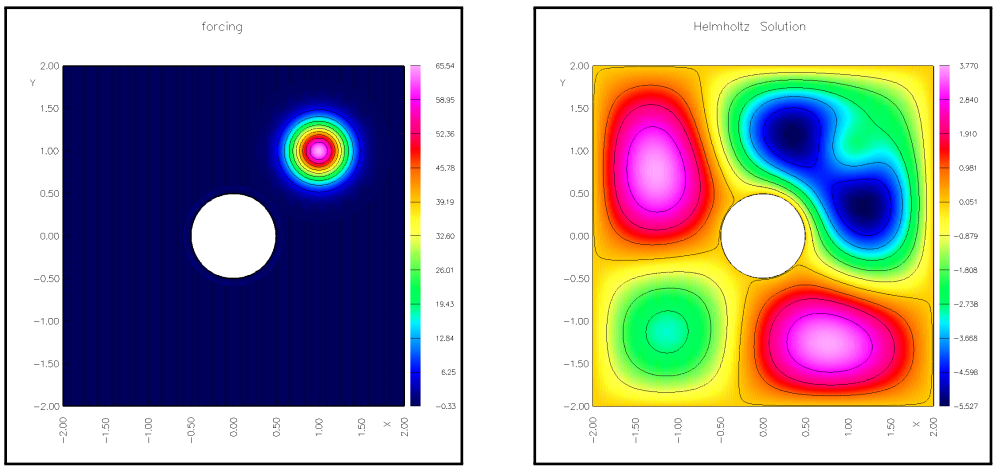

Figure 11: Gaussian forcing on the cic. Left forcing. Right solution for $\omega = 2.56$.

Square : Krylov iteration. SECOND-ORDER - looks OK

```
================  ORDER=2 -- looks OK

cgwh waveHoltz.cmd -g=square512.order2.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-5 -tp=1 -ad4=0 -imode=1
cgwh waveHoltz.cmd -g=square256.order2.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-4 -tp=1 -ad4=0 -imode=1
cgwh waveHoltz.cmd -g=square128.order2.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-3 -tp=1 -ad4=0 -imode=1

 ################ DONE KYRLOV ITERATIONS -- numberOfIterations=11 #############
CgWaveHoltz::residual: c=1, omega=7

G512
 ################ DONE KYRLOV ITERATIONS -- KSP residual=3.35e-04 (tol=1.00e-05) numberOfIterations=12 #############
CgWaveHoltz::residual: c=1, omega=7
Maximum residual = 8.888e-03

G256
 ################ DONE KYRLOV ITERATIONS -- KSP residual=3.17e-03 (tol=1.00e-04) numberOfIterations=11 #############
CgWaveHoltz::residual: c=1, omega=7
```

```
Maximum residual = 3.792e-02

G128
 ################ DONE KYRLOV ITERATIONS -- KSP residual=1.64e-03 (tol=1.00e-03) numberOfIterations=11 #############
CgWaveHoltz::residual: c=1, omega=7
Maximum residual = 1.449e-01

++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
G128 FIXED-POINT ITERATION
FIXED-POINT -omega=7 has trouble, may be close to a resonance!! TRY -omega=6.869 ...converges NOTE numPeriods
cgwh waveHoltz.cmd -g=square128.order2.hdf -x0=.75 -y0=.75 -omega=6.869 -solver=fixedPoint -tol=1e-3 -tp=1 -ad4=0 -imode=1 -maxl

it=75:  max(|v-vOld|)=9.70e-04 (tol=0.001)
#################### DONE CgWaveHoltz: CALL cgWave : number of WaveHoltz iteration =76 ########################
Maximum residual = 4.527e-02
```

Square : Krylov iteration. FOURTH-ORDER - not bad – RESIDUAL LARGE AT BOUNDARY probably degrades convergence

```
cgwh waveHoltz.cmd -g=square512.order4.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-6 -tp=1 -ad4=0 -imode=1
cgwh waveHoltz.cmd -g=square256.order4.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-5 -tp=1 -ad4=0 -imode=1
cgwh waveHoltz.cmd -g=square128.order4.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-3 -tp=1 -ad4=0 -imode=1
cgwh waveHoltz.cmd -g=square64.order4.hdf -x0=.75 -y0=.75 -omega=7 -solver=krylov -tol=1e-3 -tp=1 -ad4=0 -imode=1

G512
################ DONE KYRLOV ITERATIONS -- KSP residual=1.54e-05 (tol=1.00e-06) numberOfIterations=13 #############
Maximum residual = 4.732e-05
################ DONE KYRLOV ITERATIONS -- KSP residual=7.44e-07 (tol=1.00e-07) numberOfIterations=14 #############
Maximum residual = 4.734e-05


G256
 ################ DONE KYRLOV ITERATIONS -- KSP residual=1.67e-04 (tol=1.00e-05) numberOfIterations=12 #############
Maximum residual = 5.578e-04

--- tol=1e-4 not enough
################ DONE KYRLOV ITERATIONS -- KSP residual=3.14e-03 (tol=1.00e-04) numberOfIterations=11 #############
Maximum residual = 4.488e-03


G128
################ DONE KYRLOV ITERATIONS -- KSP residual=1.57e-03 (tol=1.00e-03) numberOfIterations=11 #############
Maximum residual = 1.392e-02


G64
################ DONE KYRLOV ITERATIONS -- KSP residual=7.93e-04 (tol=1.00e-03) numberOfIterations=11 #############
CgWaveHoltz::residual: c=1, omega=7
Maximum residual = 1.531e-01
```

CIC : Krylov iteration. SECOND-ORDER – NOT BAD, some wiggles in RESIDUAL near interpolation points, worse for cice16.

```
cgwh waveHoltz.cmd -g=cice8.order2 -beta=10 -x0=1 -y0=1 -omega=2.56 -solver=krylov -tol=1e-5 -tp=1 -ad4=1 -imode=1
cgwh waveHoltz.cmd -g=cice4.order2 -beta=10 -x0=1 -y0=1 -omega=2.56 -solver=krylov -tol=1e-3 -tp=1 -ad4=1 -imode=1


G8
################ DONE KYRLOV ITERATIONS -- KSP residual=1.31e-03 (tol=1.00e-05) numberOfIterations=14 #############
omega=2.560e+00, Maximum residual = 7.921e-04

G4
```

```
 ################ DONE KYRLOV ITERATIONS -- KSP residual=5.26e-02 (tol=1.00e-03) numberOfIterations=12 ##############
omega=2.560e+00, Maximum residual = 3.118e-03

G2
cgwh waveHoltz.cmd -g=cice2.order2 -beta=10 -x0=1 -y0=1 -omega=2.56 -solver=krylov -tol=1e-3 -tp=1 -ad4=1 -imode=1

################ DONE KYRLOV ITERATIONS -- KSP residual=3.38e-02 (tol=1.00e-03) numberOfIterations=12 ##############
omega=2.560e+00, Maximum residual = 1.640e-02
```

References

[1] S. Balay, W. D. Gropp, L. C. McInnes, B. F. Smith, The portable extensible toolkit for scientific computation, Tech. Rep. http://www.mcs.anl.gov/petsc/petsc.html, Argonne National Laboratory (1999).