Class Ogev: Computation of Eigenpairs of PDE Problems on Overset Grids using SLEPSc
Documentation and User Guide

William D. Henshaw[a]

[a]*Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180, USA*

Abstract

This document describes some routines that can be used to compute eigenvalues and eigenvectors (eigenpairs) of PDE boundary value problems using the SLEPSc package. Eigenpairs can be computed on two and three dimensional overset grids to second-order and fourth-order accuracy. The driver routine is `genEigs` which makes of functions in the class `Ogev`.

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Email address: `henshw@rpi.edu` (William D. Henshaw)

1. Introduction

This document describes some routines that can be used to compute eigenvalues and eigenvectors (eigenapirs) of PDE boundary value problems using the SLEPSc package. One example is the problem of computing eigenpairs to the negative Laplacian on some domain Ω,

$$-\Delta u = \lambda u, \qquad \mathbf{x} \in \Omega, \tag{1}$$

$$\mathcal{B} u = 0 \qquad \mathbf{x} \in \partial\Omega, \tag{2}$$

where $\mathcal{B}$ denote the boundary condition operator.

The routine `genEigs.bC` and class `Ogev` can be used to compute discrete approximations to eigenfunctions on overset grids using the `SLEPc` solver.

Notes:

1. SLEPc does not generally return orthogonal eigenvectors for the eigenspace of a multiple eigenvalue.
2. The eigenvectors for a multiple eigenvalue can be orthogonalized using the routine genEigs (e.g. using a QR algorithm). However, it appears that sometimes SLEPc returns essentially the same eigenvector more than once for a multiple eigenvalue. In this case the orthogonalization fails – need to fix this. May be able to provide SLEPc with some starting guesses.

2. Computing eigenpairs with genEigs

The routine `eigSolvers/bin/genEigs` can be used to compute eigenpairs of the negative Laplacian on 2D and 3D overset grids.

Usage:

```
genEigs [-noplot] eigs.cmd -problem=<s> -eigCase=<s> -g=<s> -numEigenValues=<i> -tol=<f>
        -bc[123456]=[d|n] -show=<s> -matlab=<s> -table=<s> -orthogonalize=[0|1]
        -discreteEigenValues=[0|1] -go=<s>

 -noplot : run without graphics
 -problem=[laplace|ile] : laplace = (negative) laplacian
                        :ile = incompressible elasticity
 -eigCase=[square|disk|sphere] : compare answers to the known eigenpairs for a square (or box),
       disk (or annulus or cylinder) or sphere.
 -g=gridName : name of overset grid generated by ogen, example -g=square32.order2.hdf
 -numEigenValues=i : number of eigenvalues to compute (and eigenvectors)
 -tol=f : tolerance for the eigenvalues, e.g. -tol=1e-12.
 -bc[1234567]=[d|n] : set boundary condition to Dirichlet or Neumann on a boundary with bc flag 1,2,3,4,6.
             e.g. -bc1=d sets boundaries with bc flag=1 to Dirichlet.
 -orthogonalize=[0|1] : 1=orthogonalize and eigenvectors corresponding to multiple eigenvalues.
 -discreteEigenValues=[0|1] : 1=compare to true discrete eigenvalues (square or box only)
 -matlab=matlabFileName : name of Matlab output file holding eigenvalues and errors,
                          e.g. -matlab=diskG4Order2
 -table=nameOfTableFile : name of file holding a LaTeX table of results,
                          e.g. -table=square32Order4Table.
 -show=showFileName : save results to a show file with this name. e.g. -show=myShowFile.show
                      (use plotStuff to display results from the show file).
 -go=[go|og] : -go=go : run and exit. -go=og (open graphics) : when running with -noplot,
       open graphics windows after commands have been read.
```

Here we are using the `eig.cmd` command file in `eigSolvers/cmd/eig.cmd`. Command line arguments after the command file name are optional and can be in any order.

Command line arguments are also passed to SLEPc. For example, SLEPSc needs to invert a matrix and to solve the linear system. By default a direct sparse solver used. For big problems an iterative method may be necessary since the direct solver requires too much memory. To solve the linear system with GMRES and an ILU(2) preconditioner to relative tolerance of $1e-2$ add the command line options

```
genEigs ... -st_ksp_type gmres -st_pc_type ilu -st_pc_factor_levels 2 -st_ksp_rtol 1e-12
```

3. Class Ogev: Overset Grid EigenValue and EigenVector solver

The overset grid eigenvalue class Ogev contains the functions used to compute eigenpairs using SLEPSc.

todo: Make some doxygen or other documentation for the Ogev functions.

Here is the header file as of Sept. 7, 2024, which lists some of the functions.

```
class Ogev
{

public:
  Ogev();
  ~Ogev();

  aString bcName( int bc );

  int checkResidualsInPsi( int eigc, RealArray & eig, realMappedGridFunction & u, CompositeGrid & cg,
                           CompositeGridOperators & cgop, IntegerArray & bc,
                           int numberOfComponents, int orderOfAccuracy, int useWideStencils, Real mu, int includePressure );

  int computeEigenvalues( const aString & problem, const int numberOfComponents,
                          int orderOfAccuracy, int & numEigenValues, int & numEigenVectors,
                          RealArray & eig, CompositeGrid & cg, realCompositeGridFunction & ucg,
                          CompositeGridOperators & cgop,
                          Real tol, int eigOption, int maxIterations,
                          const int setTargetEigenvalue, const Real targetEigenvalue,
                          IntegerArray & bc, int numGhost, int saveMatlab, int useWideStencils,
                          int maximumProjectedDimension=-1  );

  int
  fillInterpolationCoefficients( Mat & A, realCompositeGridFunction & uu );

  int
  fillMatrixIncompressibleElasticity( const int numberOfComponents, int orderOfAccuracy, MappedGrid & mg, Mat & A, Mat & B,
                                      int numGhost, bool useNew, Real tol, int eigOption, IntegerArray & bc,
                                      int saveMatlab, int useWideStencils );

  int
  fillMatrixLaplacian( int orderOfAccuracy, realCompositeGridFunction & ucg, CompositeGridOperators & cgop,
                       Mat & A, Mat & B, int numGhost, bool useNew,
                       Real tol, int eigOption, IntegerArray & bc, int saveMatlab, Real lambdaShift );

  Real getEigenPairResidual( Real lambda, realCompositeGridFunction & v,
                             realCompositeGridFunction & res,
                             CompositeGridOperators & operators,
                             int component =0  );
  int
  getEigenvaluesBox( int numEigs, RealArray & eigs, CompositeGrid & cg,
                     Real lx =1.0 , Real ly =1.0, Real lz =1.0,
                     RealCompositeGridFunction *eigenvector = NULL,
                     const bool discreteEigenvalues =false  );

  int
  getEigenvaluesCylinder( int numEigs, RealArray & eigs, CompositeGrid & cg,
                          Real ra =0.5, Real rb =1.0, Real za = 0.0, Real zb = 1.0,
                          RealCompositeGridFunction *eigenvector = NULL  );
  int
  getEigenvaluesSphere( int numEigs, RealArray & eigs, CompositeGrid & cg,
                        Real ra =0.0, Real rb =1.0,
                        RealCompositeGridFunction *eigenvector = NULL  );


  int
  getPressureFromDisplacement( realCompositeGridFunction & uv, realCompositeGridFunction & p,
                               IntegerArray & bc, int orderOfAccuracy, Real mu  );

  // count multiplicities, orthogonalize and normalize eigenvectors
  int orthogonalizeEigenvectors( const aString & problem, const int numberOfComponents,
                                 int orderOfAccuracy, int & numEigenValues, int & numEigenVectors,
                                 RealArray & eig, realCompositeGridFunction & ucg,
                                 IntegerArray & eigMultiplicity, IntegerArray & eigStartIndex  );
protected:

  Real getDiscreteSymbol( const Real modeNumber, const Real dx ) const;

  int
  buildGlobalIndexing(CompositeGrid & cg );

  int
  getGlobalIndex( int n, int *iv, int grid, int p );

  int
```

```
  getGlobalIndex( int n, int i1, int i2, int i3, int grid, int p );

  int debug;

  int numberOfComponents;
  int numberOfProcessors;
  int numberOfGridPoints;
  int numberOfGridPointsThisProcessor;

  // --- arrays for global indexing -----
  int *pnab, *pnoffset;

   // Here is the place to store parameters:
  DataBase dbase;

};
```

4. Numerical Examples

The accuracy of the eigen-pairs computed using the EigenWave algorithm is measured in three ways, the relative error in the eigenvalue, the relative error in the eigenvector, and the relative residual defined, respectively, by

$$\lambda_j\text{–err} = \frac{|\lambda_{h,j} - \lambda_{h,j}^{\text{true}}|}{\lambda_{h,j}^{\text{true}}}, \quad \phi_j\text{–err} = \frac{\|V_{\mathbf{i},j} - V_{\mathbf{i},j}^{\text{true}}\|_\infty}{\|V_{\mathbf{i},j}^{\text{true}}\|_\infty}, \quad \texttt{res} = \frac{\|L_{h,p}V_{\mathbf{i},j} + \lambda_{h,j}V_{\mathbf{i},j}\|_\infty}{\lambda_{h,j}}, \tag{3}$$

Some tables also hold columns headed with `multe` and `multc` – these are the multiplicities of the eigenvalues, e=exact, c=computed eigenvalues. The routine to orthogonalize eigenvectors corresponding to multiple eigenvalues attempts to find the multiplicity.

4.1. Square

Table 1 shows results for order=2 comparing to the known discrete eigenvalues. Here is the command used to generate the results.

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=square -g=square64.order2.hdf -eigOption=1
-eps_s=largest_magnitude -numEigenValues=16 -tol=1.0e-14 -bc1=d -show=square64O2Ev16.show
-table=square64O2Ev16DiscreteEigsTable -orthogonalize=1 -discreteEigenvalues=1 -go=go
```

square64.order2.hdf, order=2						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\|A\phi - \lambda\phi\|/\lambda$
0	19.735 + (-0.0e+00) i	1.08e-15	5.11e-15	1	1	1.98e-12
1	49.314 + (-0.0e+00) i	7.20e-16	1.16e-14	2	2	1.84e-12
2	49.314 + (-0.0e+00) i	0.00e+00	7.66e-15	2	2	1.19e-12
3	78.893 + (-0.0e+00) i	4.32e-15	7.88e-15	1	1	8.82e-13
4	98.534 + (-0.0e+00) i	5.34e-15	8.88e-15	2	2	5.27e-13
5	98.534 + (-0.0e+00) i	3.17e-15	5.30e-15	2	2	6.82e-13
6	128.113 + (-0.0e+00) i	3.55e-15	5.45e-15	2	2	5.73e-13
7	128.113 + (-0.0e+00) i	6.66e-16	6.15e-15	2	2	4.66e-13
8	167.275 + (-0.0e+00) i	3.40e-16	9.19e-15	2	2	3.60e-13
9	167.275 + (-0.0e+00) i	5.10e-16	4.41e-14	2	2	3.73e-13
10	177.332 + (-0.0e+00) i	1.76e-15	5.46e-14	1	1	3.86e-13
11	196.854 + (-0.0e+00) i	2.17e-15	1.36e-14	2	2	3.74e-13
12	196.854 + (-0.0e+00) i	4.33e-16	9.79e-15	2	2	2.34e-13
13	246.073 + (-0.0e+00) i	1.96e-15	4.26e-14	2	2	2.27e-13
14	246.073 + (-0.0e+00) i	1.96e-15	2.77e-14	2	2	5.78e-13
15	255.372 + (-0.0e+00) i	4.90e-15	2.33e-14	2	1	5.36e-13

Table 1: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=5.34e-15, max-evect-err=5.46e-14, max-residual=1.98e-12

Table 2 shows results for a square, fourth-order accurate, comparing to the true continuous eigenvalues.

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=square -g=square64.order4.hdf -eigOption=1
-eps_s=largest_magnitude -numEigenValues=16 -tol=1.0e-14 -bc1=d -show=square64O4Ev16.show
-table=square64O4Ev16Table -orthogonalize=1 -go=go
```

square64.order4.hdf, order=4						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\|A\phi - \lambda\phi\|/\lambda$
0	19.739 + (-0.0e+00) i	2.68e-08	2.56e-08	1	1	4.82e-12
1	49.348 + (-0.0e+00) i	3.53e-07	6.44e-07	2	2	1.80e-12
2	49.348 + (-0.0e+00) i	3.53e-07	6.44e-07	2	2	1.98e-12
3	78.957 + (-0.0e+00) i	4.35e-07	8.08e-07	1	1	1.16e-12
4	98.696 + (-0.0e+00) i	2.02e-06	3.64e-06	2	2	1.63e-12
5	98.696 + (-0.0e+00) i	2.02e-06	4.49e-06	2	2	1.03e-12
6	128.305 + (-0.0e+00) i	1.69e-06	3.61e-06	2	2	9.49e-13
7	128.305 + (-0.0e+00) i	1.69e-06	3.61e-06	2	2	7.01e-13
8	167.782 + (-0.0e+00) i	6.88e-06	1.71e-05	2	2	7.62e-13
9	167.782 + (-0.0e+00) i	6.88e-06	1.71e-05	2	2	4.71e-13
10	177.652 + (-0.0e+00) i	2.25e-06	6.03e-06	1	1	5.56e-13
11	197.391 + (-0.0e+00) i	5.93e-06	2.40e-05	2	2	3.84e-13
12	197.391 + (-0.0e+00) i	5.93e-06	2.33e-05	2	2	5.00e-13
13	246.739 + (-0.0e+00) i	5.48e-06	2.32e-05	2	2	5.07e-13
14	246.739 + (-0.0e+00) i	5.48e-06	2.32e-05	2	2	4.07e-13
15	256.605 + (-0.0e+00) i	1.78e-05	5.74e-05	2	2	3.75e-13
16	256.605 + (-0.0e+00) i	1.78e-05	5.44e-05	2	2	5.33e-13
17	286.214 + (-0.0e+00) i	1.60e-05	6.66e-05	2	1	3.22e-13

Table 2: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=1.78e-05, max-evect-err=6.66e-05, max-residual=4.82e-12

4.2. Disk

Here are results for a disk.

- The eigenvectors seem to be converging at a faster rate than expected, see Figure ??.

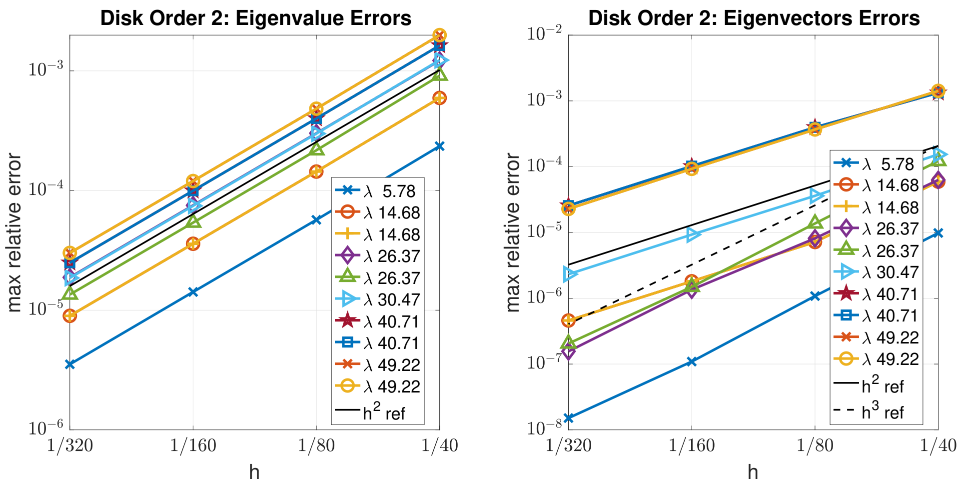

Figure 1: Accuracy of some computed eigenvalues and eigenvalues on a disk compared to the true continuous values.

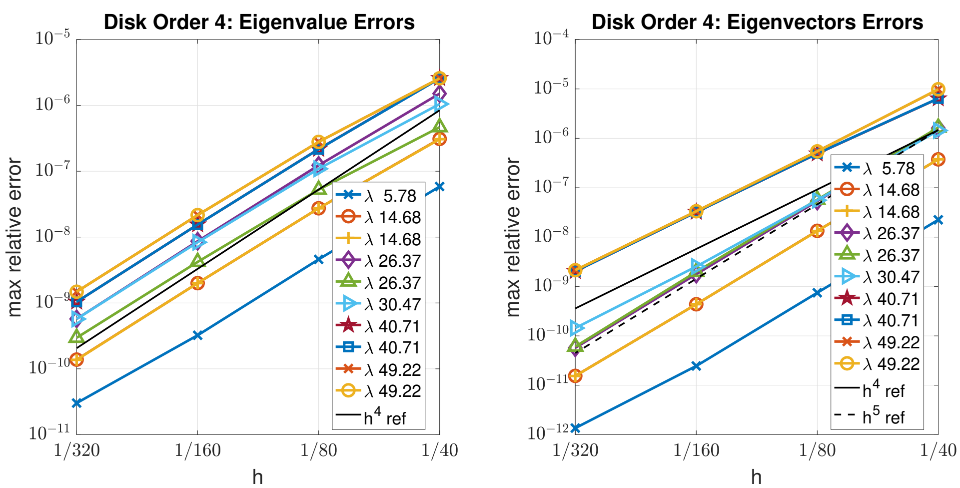

Figure 2: Accuracy of some computed eigenvalues and eigenvalues on a disk compared to the true continuous values.

Table 3 shows fourth order accurate results using:

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=disk -g=sice4.order4.ng3 -numEigenValues=32
     -tol=1.0e-14 -bc1=d -show=sice8O2EigsEv32.show -table=diskG8O4Ev32Table -orthogonalize=1 -go=go
```

sice4.order4.ng3.hdf, order=4						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\Vert A\phi - \lambda\phi\Vert/\lambda$
0	5.783 + (-0.0e+00) i	5.86e-08	2.24e-08	1	1	3.16e-12
1	14.682 + (-0.0e+00) i	3.10e-07	3.75e-07	2	2	2.06e-12
2	14.682 + (-0.0e+00) i	3.07e-07	3.76e-07	2	2	1.60e-12
3	26.375 + (-0.0e+00) i	1.51e-06	1.61e-06	2	2	5.92e-13
4	26.375 + (-0.0e+00) i	4.67e-07	1.65e-06	2	2	3.57e-11
5	30.471 + (-0.0e+00) i	1.05e-06	1.44e-06	1	1	9.61e-13
6	40.706 + (-0.0e+00) i	2.58e-06	6.38e-06	2	2	5.23e-13
7	40.706 + (-0.0e+00) i	2.56e-06	6.43e-06	2	2	2.59e-10
8	49.218 + (-0.0e+00) i	2.58e-06	9.80e-06	2	2	4.51e-13
9	49.218 + (-0.0e+00) i	2.58e-06	9.84e-06	2	2	1.97e-12
10	57.583 + (-0.0e+00) i	5.51e-06	9.25e-06	2	2	4.55e-13
11	57.583 + (-0.0e+00) i	5.50e-06	1.14e-05	2	2	1.03e-09
12	70.849 + (-0.0e+00) i	9.25e-06	1.87e-05	2	2	4.10e-13
13	70.850 + (-0.0e+00) i	2.20e-06	1.77e-05	2	2	3.15e-11
14	74.887 + (-0.0e+00) i	5.41e-06	2.16e-05	1	1	5.94e-13
15	76.938 + (-0.0e+00) i	1.04e-05	1.99e-05	2	2	3.88e-13
16	76.938 + (-0.0e+00) i	1.02e-05	2.05e-05	2	2	2.98e-09
17	95.277 + (-0.0e+00) i	1.12e-05	8.65e-05	2	2	3.65e-13
18	95.277 + (-0.0e+00) i	1.12e-05	8.56e-05	2	2	1.85e-10
19	98.725 + (-0.0e+00) i	1.75e-05	3.85e-05	2	2	2.45e-13
20	98.725 + (-0.0e+00) i	1.73e-05	3.13e-05	2	2	6.98e-09
21	103.498 + (-0.0e+00) i	1.08e-05	7.82e-05	2	2	2.31e-13
22	103.498 + (-0.0e+00) i	1.08e-05	7.79e-05	2	2	2.09e-12
23	122.425 + (-0.0e+00) i	1.97e-05	1.50e-04	2	2	2.18e-13
24	122.425 + (-0.0e+00) i	1.96e-05	7.84e-05	2	2	6.21e-10
25	122.904 + (-0.0e+00) i	2.75e-05	8.46e-05	2	2	2.56e-13
26	122.904 + (-0.0e+00) i	2.73e-05	7.57e-05	2	2	1.41e-08
27	135.016 + (-0.0e+00) i	3.20e-05	8.93e-05	2	1	1.98e-13
28	135.020 + (-0.0e+00) i	7.74e-06	9.27e-05	2	1	2.99e-13
29	139.038 + (-0.0e+00) i	1.99e-05	1.07e-04	1	1	4.39e-13
30	149.447 + (-0.0e+00) i	4.08e-05	1.15e-04	2	2	1.76e-13
31	149.447 + (-0.0e+00) i	4.07e-05	1.04e-04	2	2	2.57e-08
32	152.236 + (-0.0e+00) i	3.19e-05	1.61e-04	2	2	1.45e-13
33	152.236 + (-0.0e+00) i	3.15e-05	1.59e-04	2	2	1.50e-09

Table 3: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=4.08e-05, max-evect-err=1.61e-04, max-residual=2.57e-08

4.3. Ellipse

Here we compute eigenpairs of an ellipse.

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=none -g=ellipseGride4.order2.hdf -numEigenValues=16
    -tol=1.0e-12 -bc1=d -show=junk.show -table=ellipseG4O2Table -orthogonalize=1 -go=go
```

ellipseGride4.order2.hdf, order=2			
j	λ_j	multc	$\Vert A\phi - \lambda\phi\Vert/\lambda$
0	4.894 + (-0.0e+00) i	1	3.36e-12
1	11.296 + (-0.0e+00) i	1	9.90e-13
2	13.535 + (-0.0e+00) i	1	1.05e-12
3	20.562 + (-0.0e+00) i	1	5.43e-13
4	22.272 + (-0.0e+00) i	1	7.56e-13
5	27.483 + (-0.0e+00) i	1	9.85e-13
6	32.422 + (-0.0e+00) i	1	4.70e-13
7	33.462 + (-0.0e+00) i	1	4.84e-13
8	39.752 + (-0.0e+00) i	1	4.87e-13
9	46.202 + (-0.0e+00) i	1	3.56e-13
10	46.483 + (-0.0e+00) i	1	3.14e-13
11	46.978 + (-0.0e+00) i	1	3.13e-13
12	55.220 + (-0.0e+00) i	1	4.66e-13
13	61.324 + (-0.0e+00) i	1	2.83e-11
14	62.481 + (-0.0e+00) i	1	2.92e-13
15	62.674 + (-0.0e+00) i	1	2.41e-13
16	69.829 + (-0.0e+00) i	1	3.19e-13
17	73.996 + (-0.0e+00) i	1	1.33e-11

Table 4: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1

Order=4: check me: residuals?

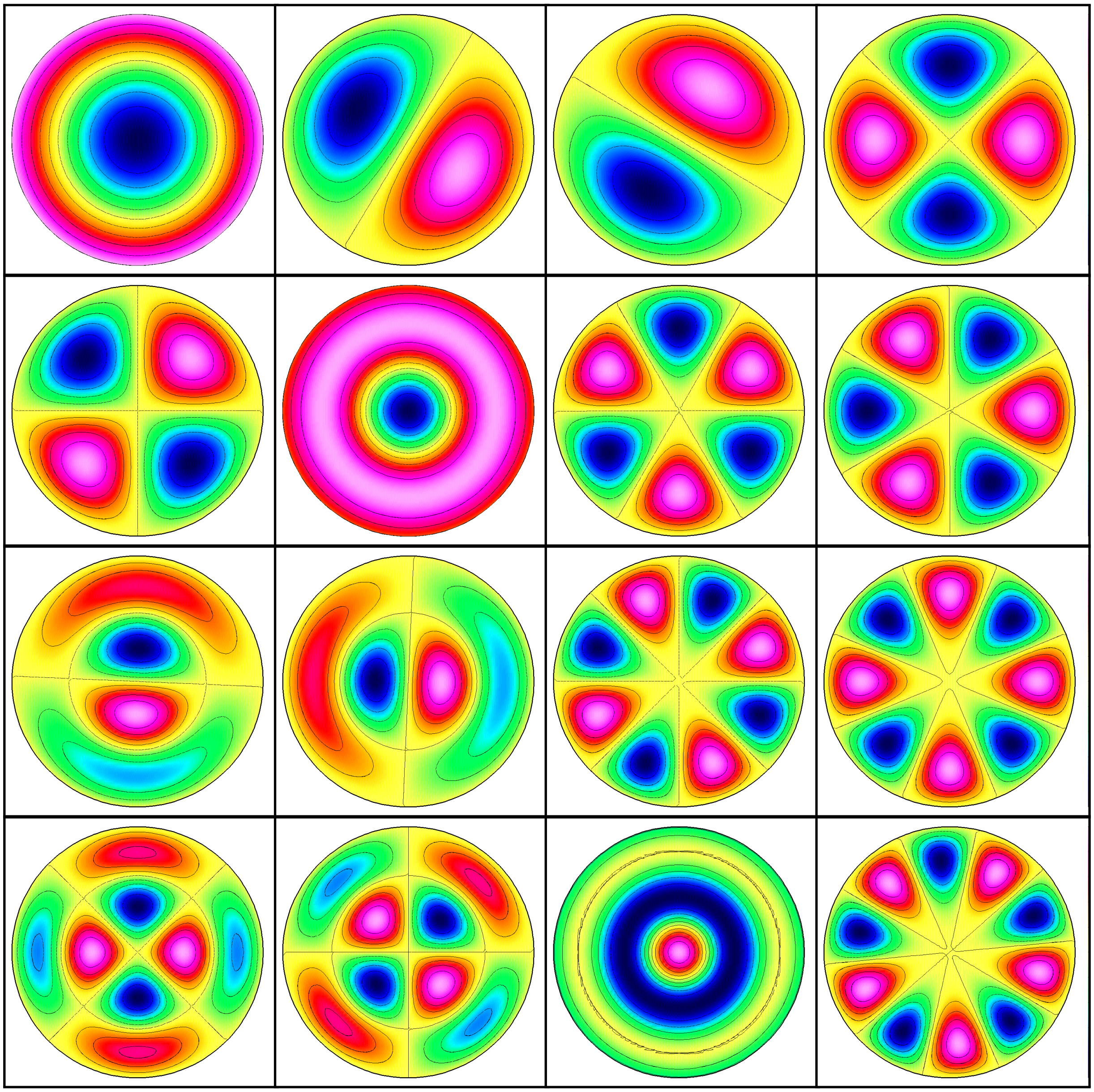

Figure 3: Computed eigenfunctions of a disk, arranged by magnitude of the eigenvalues.

ellipseGride4.order4.ng3.hdf, order=4			
j	λ_j	multc	$\Vert A\phi - \lambda\phi\Vert/\lambda$
0	4.895 + (-0.0e+00) i	1	1.20e-05
1	11.301 + (-0.0e+00) i	1	1.19e-05
2	13.543 + (-0.0e+00) i	1	7.08e-12
3	20.581 + (-0.0e+00) i	1	1.09e-05
4	22.289 + (-0.0e+00) i	1	2.61e-12
5	27.516 + (-0.0e+00) i	1	1.15e-06
6	32.469 + (-0.0e+00) i	1	9.80e-06
7	33.501 + (-0.0e+00) i	1	4.82e-11
8	39.809 + (-0.0e+00) i	1	2.58e-06
9	46.296 + (-0.0e+00) i	1	1.18e-12
10	46.574 + (-0.0e+00) i	1	8.53e-06
11	47.059 + (-0.0e+00) i	1	2.04e-10
12	55.333 + (-0.0e+00) i	1	3.98e-06
13	61.458 + (-0.0e+00) i	1	1.52e-12
14	62.633 + (-0.0e+00) i	1	7.32e-06
15	62.819 + (-0.0e+00) i	1	7.37e-10
16	70.046 + (-0.0e+00) i	1	1.02e-07

Table 5: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1

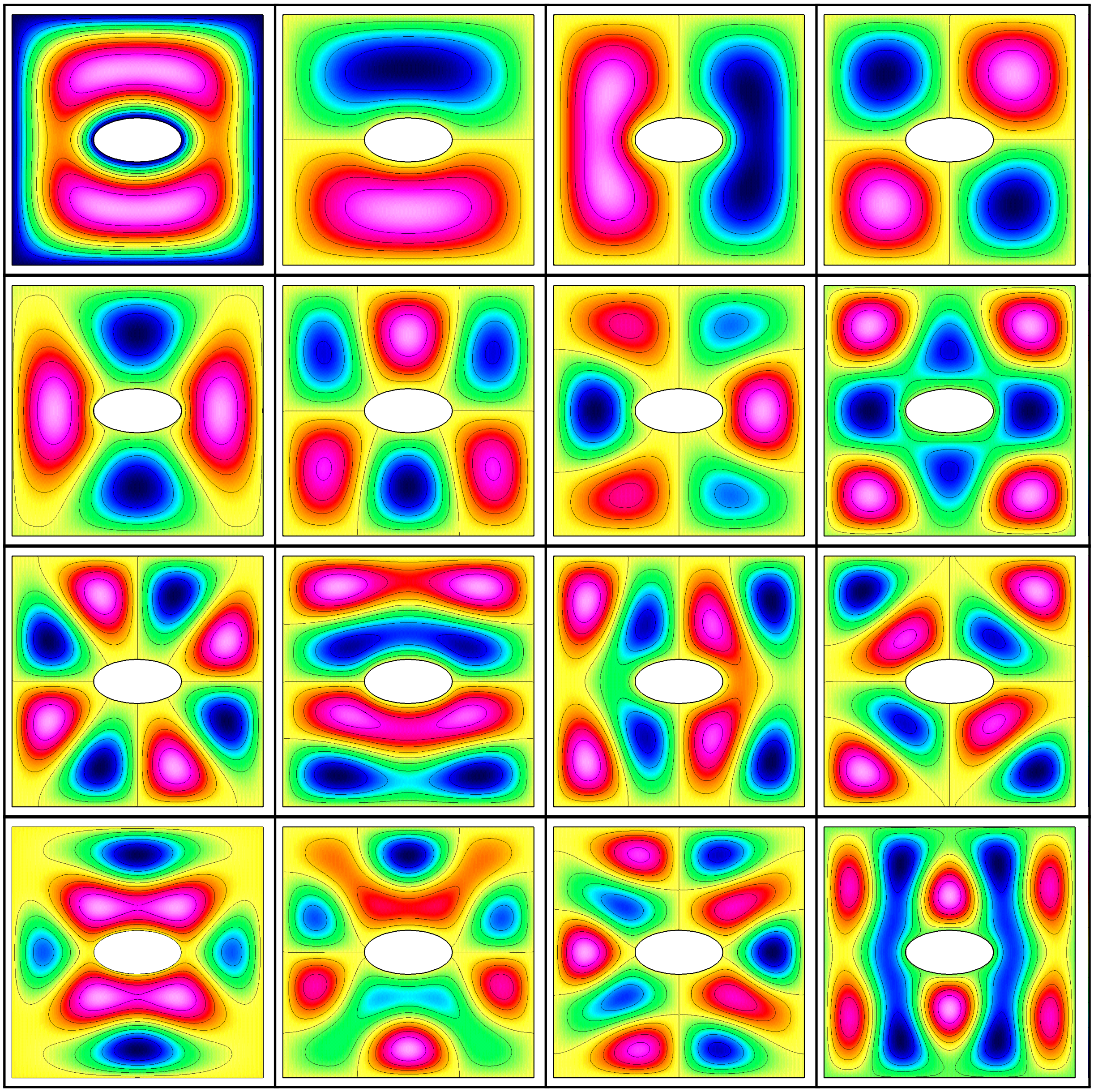

Figure 4: Computed eigenfunctions of an ellipse in a box, arranged by magnitude of the eigenvalues.

4.4. Shapes in two dimensions

Eigenpairs for some shapes.

Command:

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=none -g=shapese4.order2.hdf -numEigenValues=16
  -tol=1.0e-12 -show=junk.show -table=shapesG4O2Table -orthogonalize=1 -discreteEigenvalues=1 -go=go
```

rpiGride4.order2.hdf, order=2			
j	λ_j	multc	$\Vert A\phi-\lambda\phi\Vert/\lambda$
0	5.627 + (-0.0e+00) i	1	2.20e-12
1	5.959 + (-0.0e+00) i	1	2.47e-12
2	6.033 + (-0.0e+00) i	1	3.30e-12
3	7.051 + (-0.0e+00) i	1	1.87e-12
4	8.774 + (-0.0e+00) i	1	1.08e-12
5	9.658 + (-0.0e+00) i	1	1.10e-12
6	10.128 + (-0.0e+00) i	1	8.81e-11
7	10.276 + (-0.0e+00) i	1	1.88e-12
8	10.889 + (-0.0e+00) i	1	1.09e-12
9	11.385 + (-0.0e+00) i	1	1.00e-12
10	11.607 + (-0.0e+00) i	1	1.80e-12
11	12.747 + (-0.0e+00) i	1	1.47e-12
12	13.420 + (-0.0e+00) i	1	8.60e-13
13	13.793 + (-0.0e+00) i	1	7.95e-13
14	14.209 + (-0.0e+00) i	1	3.63e-12
15	14.835 + (-0.0e+00) i	1	7.24e-13
16	15.264 + (-0.0e+00) i	1	3.27e-11

Table 6: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-residual=8.81e-11

shapese4.order4.ng3.hdf, order=4			
j	λ_j	multc	$\Vert A\phi-\lambda\phi\Vert/\lambda$
0	21.525 + (-0.0e+00) i	1	9.29e-12
1	22.319 + (-0.0e+00) i	1	9.43e-12
2	36.366 + (-0.0e+00) i	1	1.77e-11
3	36.984 + (-0.0e+00) i	1	4.79e-11
4	37.762 + (-0.0e+00) i	1	8.74e-12
5	47.658 + (-0.0e+00) i	1	4.83e-12
6	49.037 + (-0.0e+00) i	1	7.16e-12
7	52.132 + (-0.0e+00) i	1	8.40e-12
8	55.546 + (-0.0e+00) i	1	6.09e-12
9	60.082 + (-0.0e+00) i	1	1.08e-11
10	67.634 + (-0.0e+00) i	1	4.73e-12
11	72.607 + (-0.0e+00) i	1	5.31e-12
12	73.063 + (-0.0e+00) i	1	4.07e-12
13	77.470 + (-0.0e+00) i	1	3.51e-12
14	82.325 + (-0.0e+00) i	1	3.72e-12
15	88.056 + (-0.0e+00) i	1	3.42e-12

Table 7: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-residual=4.79e-11

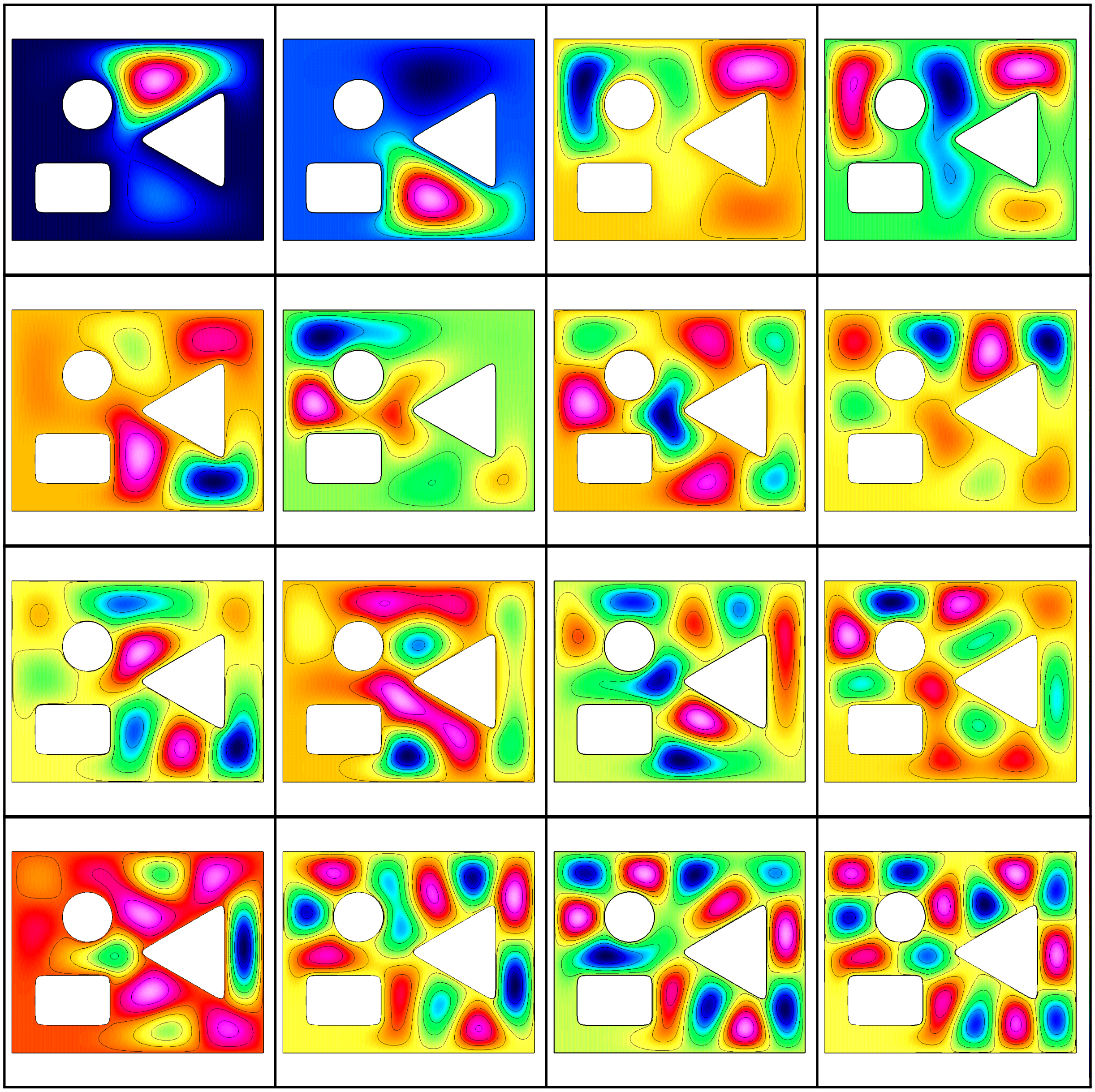

Figure 5: Computed eigenfunctions of some shapes in a box, arranged by magnitude of the eigenvalues.

4.5. Two-dimensional cylinder in a channel

Cylinder in a channel, periodic in y.

cice8.order2.hdf, order=2			
j	λ_j	multc	$\|A\phi - \lambda\phi\|/\lambda$
0	3.320 + (-0.0e+00) i	1	2.44e-11
1	3.936 + (-0.0e+00) i	2	2.19e-11
2	3.936 + (-0.0e+00) i	2	3.43e-09
3	5.146 + (-0.0e+00) i	1	1.63e-11
4	6.566 + (-0.0e+00) i	1	1.05e-11
5	8.484 + (-0.0e+00) i	2	6.71e-12
6	8.484 + (-0.0e+00) i	2	1.19e-08
7	10.217 + (-0.0e+00) i	1	6.67e-12
8	12.354 + (-0.0e+00) i	1	9.23e-12
9	12.793 + (-0.0e+00) i	2	6.70e-12
10	12.793 + (-0.0e+00) i	2	1.70e-08
11	13.400 + (-0.0e+00) i	1	6.75e-12
12	15.081 + (-0.0e+00) i	1	6.99e-12
13	17.292 + (-0.0e+00) i	2	4.14e-11
14	17.292 + (-0.0e+00) i	2	4.01e-09
15	17.926 + (-0.0e+00) i	1	4.16e-12

Table 8: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-residual=1.70e-08

Figure 6 shows some eigenfunctions.

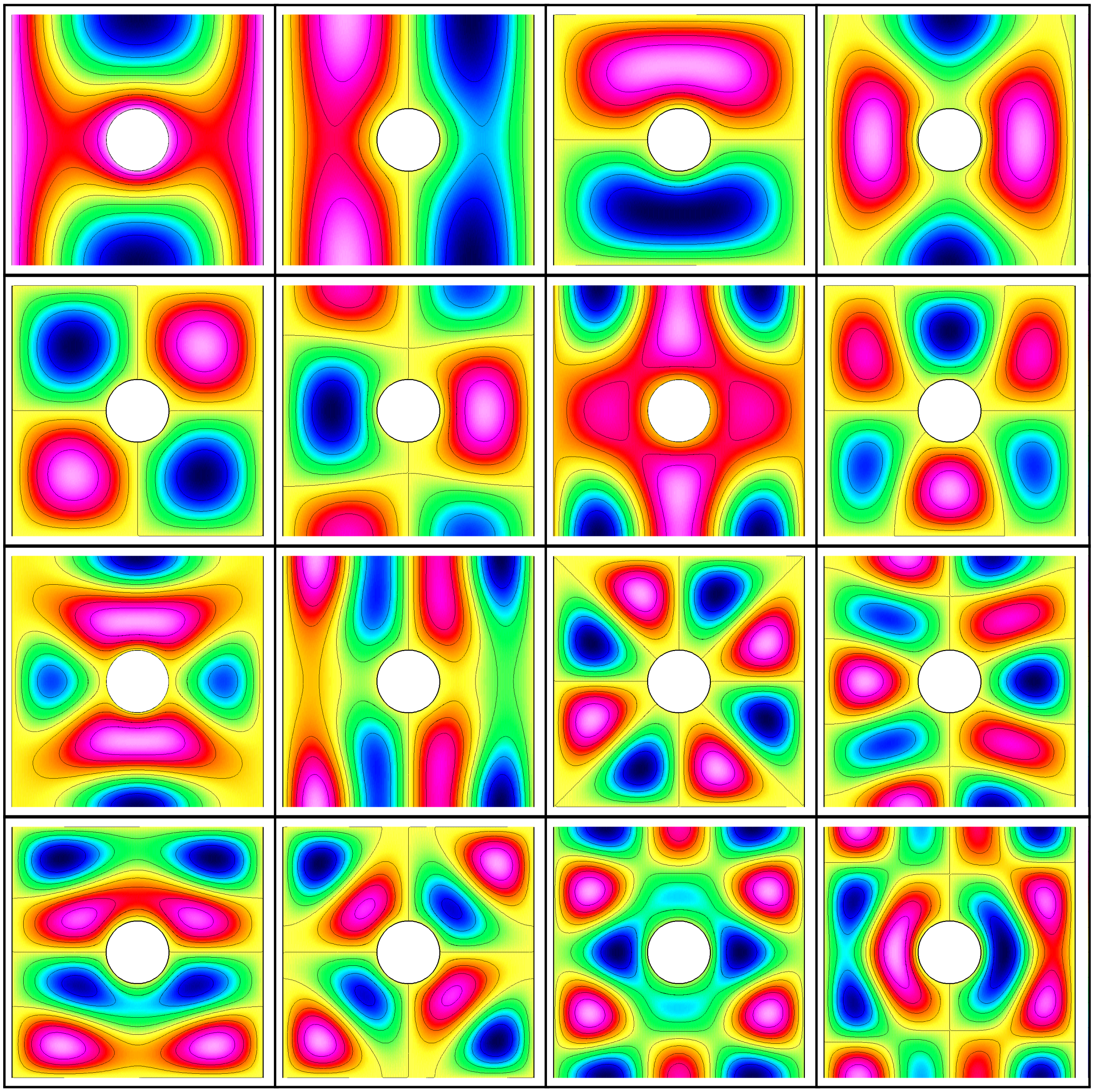

Figure 6: Computed eigenfunctions of a cylinder in a channel (periodic in y), arranged by magnitude of the eigenvalues.

4.6. RPI letters

Command:

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=none -g=rpiGride4.order2.hdf -numEigenValues=16
  -tol=1.0e-12 -show=junk.show -table=shapesG4O2Table -orthogonalize=1 -discreteEigenvalues=1 -go=go
```

rpiGride4.order2.hdf, order=2			
j	λ_j	multc	$\Vert A\phi - \lambda\phi\Vert/\lambda$
0	5.627 + (-0.0e+00) i	1	2.20e-12
1	5.959 + (-0.0e+00) i	1	2.47e-12
2	6.033 + (-0.0e+00) i	1	3.30e-12
3	7.051 + (-0.0e+00) i	1	1.87e-12
4	8.774 + (-0.0e+00) i	1	1.08e-12
5	9.658 + (-0.0e+00) i	1	1.10e-12
6	10.128 + (-0.0e+00) i	1	8.81e-11
7	10.276 + (-0.0e+00) i	1	1.88e-12
8	10.889 + (-0.0e+00) i	1	1.09e-12
9	11.385 + (-0.0e+00) i	1	1.00e-12
10	11.607 + (-0.0e+00) i	1	1.80e-12
11	12.747 + (-0.0e+00) i	1	1.47e-12
12	13.420 + (-0.0e+00) i	1	8.60e-13
13	13.793 + (-0.0e+00) i	1	7.95e-13
14	14.209 + (-0.0e+00) i	1	3.63e-12
15	14.835 + (-0.0e+00) i	1	7.24e-13
16	15.264 + (-0.0e+00) i	1	3.27e-11

Table 9: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-residual=8.81e-11

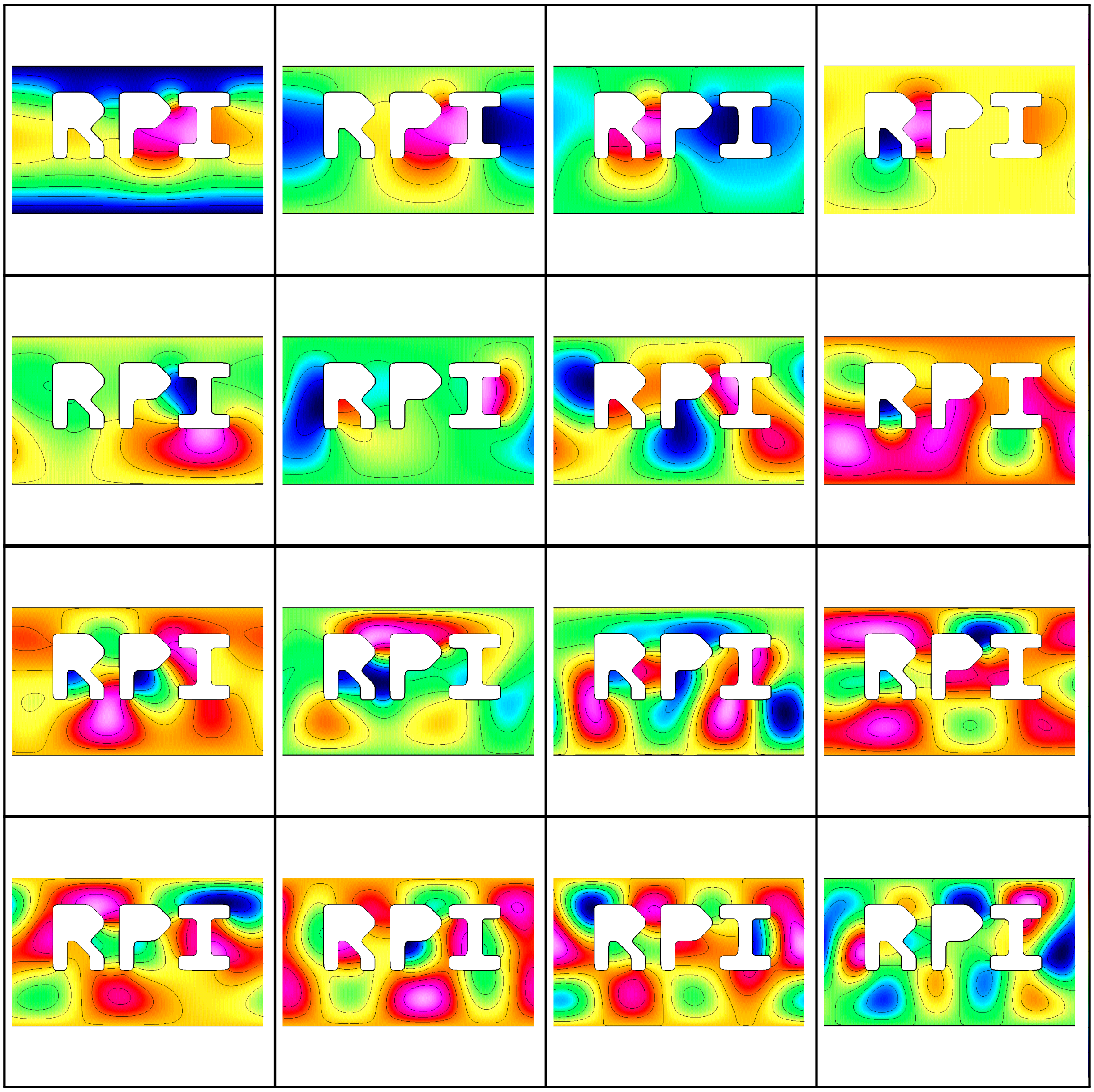

Figure 7: Computed eigenfunctions for the letters RPI, arranged by magnitude of the eigenvalues. Outer boundary is periodic in x and Dirichlet on top/bottom, Neumann BCs on the letters.

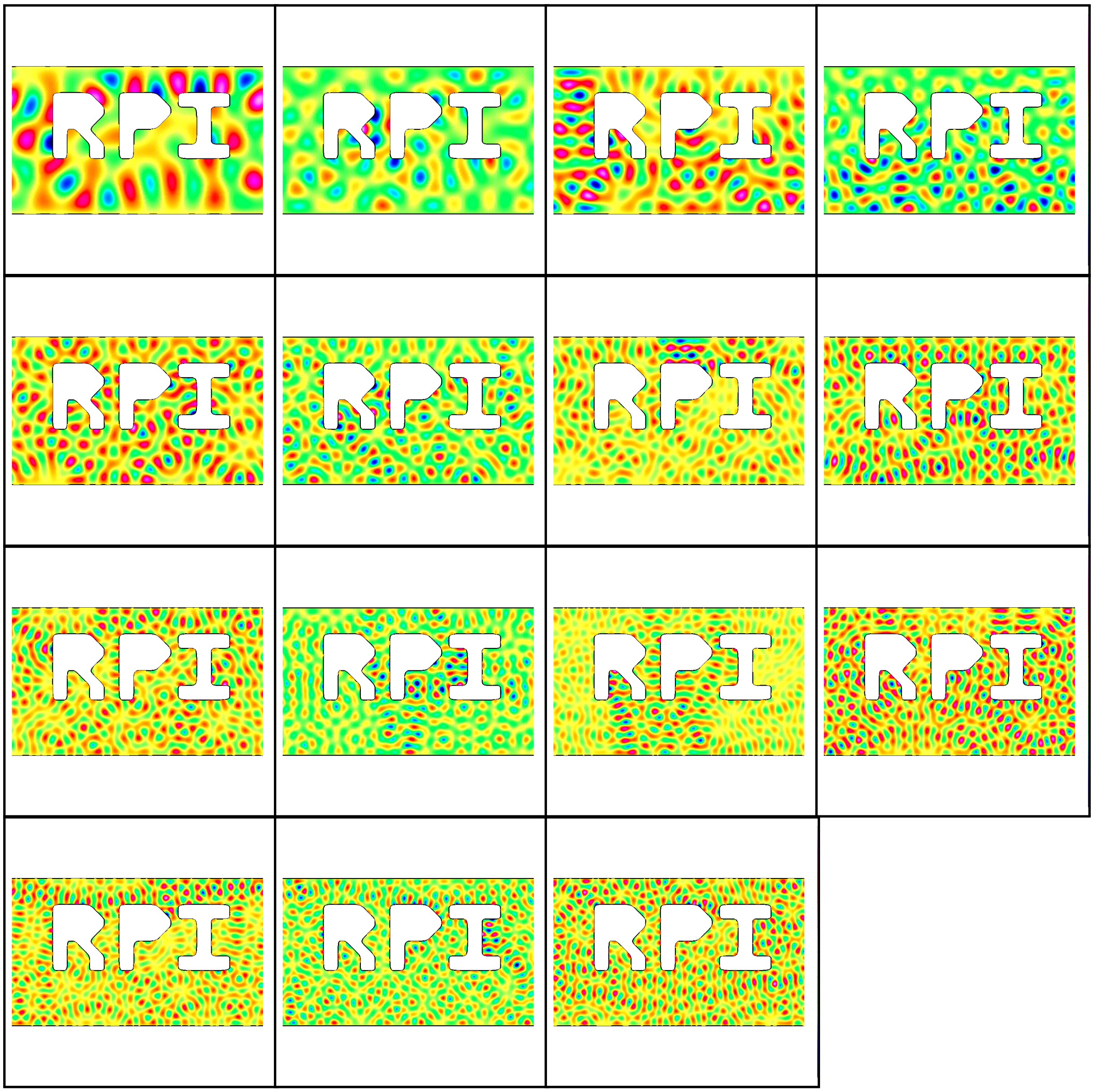

Figure 8: Selected computed eigenfunctions for the letters RPI, arranged by magnitude of the eigenvalues. Outer boundary is periodic in x and Dirichlet on top/bottom, Neumann BCs on the letters. Modes 64, 128, ...

4.7. Box

Here we compute eigenpairs of the unit box.

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=square -g=box2.order2.hdf -numEigenValues=16
   -tol=1.0e-12 -bc1=d -show=junk.show -table=boxG2O2Table -orthogonalize=1  -discreteEigenvalues=1 -go=go
```

box2.order2.hdf, order=2						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\Vert A\phi-\lambda\phi\Vert/\lambda$
0	29.548 + (-0.0e+00) i	3.85e-15	1.79e-14	1	1	5.99e-13
1	58.853 + (-0.0e+00) i	2.29e-15	8.56e-14	3	3	4.51e-12
2	58.853 + (-0.0e+00) i	1.81e-15	1.22e-14	3	3	2.67e-13
3	58.853 + (-0.0e+00) i	3.02e-15	1.65e-14	3	3	7.52e-13
4	88.159 + (-0.0e+00) i	1.45e-15	3.05e-14	3	3	8.93e-13
5	88.159 + (-0.0e+00) i	1.13e-15	2.87e-14	3	3	9.62e-13
6	88.159 + (-0.0e+00) i	1.61e-16	1.50e-14	3	3	1.90e-13
7	106.893 + (-0.0e+00) i	6.65e-16	7.96e-15	3	3	2.44e-13
8	106.893 + (-0.0e+00) i	1.33e-15	1.17e-14	3	3	1.91e-13
9	106.893 + (-0.0e+00) i	1.99e-15	1.09e-14	3	3	1.81e-13
10	117.464 + (-0.0e+00) i	1.33e-15	3.56e-14	1	1	1.67e-13
11	136.199 + (-0.0e+00) i	2.71e-15	2.61e-14	6	3	2.72e-13
12	136.199 + (-0.0e+00) i	2.50e-15	1.35e-14	6	3	1.64e-13
13	136.199 + (-0.0e+00) i	2.09e-16	1.41e-13	6	3	2.29e-12
14	165.504 + (-0.0e+00) i	4.46e-15	4.72e-14	3	2	1.06e-13
15	165.504 + (-0.0e+00) i	3.95e-15	3.96e-14	3	2	1.26e-13
16	172.485 + (-0.0e+00) i	3.46e-15	8.77e-13	3	2	3.27e-12
17	172.485 + (-0.0e+00) i	4.94e-15	7.68e-12	3	2	2.28e-11

Table 10: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=4.94e-15, max-evect-err=7.68e-12, max-residual=2.28e-11

```
genEigs -noplot eigs.cmd -problem=laplace -eigCase=square -g=box2.order4.hdf -numEigenValues=16
   -tol=1.0e-12 -bc1=d -show=junk.show -table=boxG2O4Table -orthogonalize=1 -go=go
```

box2.order4.hdf, order=4						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\Vert A\phi-\lambda\phi\Vert/\lambda$
0	29.609 + (-0.0e+00) i	5.49e-06	9.03e-06	1	1	2.09e-12
1	59.220 + (-0.0e+00) i	4.73e-05	1.20e-04	3	3	7.39e-13
2	59.220 + (-0.0e+00) i	4.73e-05	1.60e-04	3	3	9.07e-13
3	59.220 + (-0.0e+00) i	4.73e-05	1.41e-04	3	3	1.18e-12
4	88.832 + (-0.0e+00) i	6.13e-05	1.61e-04	3	3	8.31e-13
5	88.832 + (-0.0e+00) i	6.13e-05	1.79e-04	3	3	6.04e-13
6	88.832 + (-0.0e+00) i	6.13e-05	1.62e-04	3	3	8.41e-13
7	108.583 + (-1.9e-13) i	1.60e-04	4.89e-04	3	3	8.50e-13
8	108.583 + (+1.9e-13) i	1.60e-04	6.62e-04	3	3	2.96e-12
9	108.583 + (-0.0e+00) i	1.60e-04	7.49e-04	3	3	9.13e-12
10	118.443 + (-0.0e+00) i	6.82e-05	1.98e-04	1	1	3.20e-13
11	138.194 + (-0.0e+00) i	1.45e-04	6.66e-04	6	3	6.80e-13
12	138.194 + (-0.0e+00) i	1.45e-04	7.61e-04	6	3	4.10e-13
13	138.194 + (-0.0e+00) i	1.45e-04	9.79e-04	6	3	3.99e-13
14	167.806 + (-0.0e+00) i	1.35e-04	8.73e-04	3	2	2.02e-13
15	167.806 + (-0.0e+00) i	1.35e-04	8.91e-04	3	2	2.75e-13

Table 11: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=1.60e-04, max-evect-err=9.79e-04, max-residual=9.13e-12

4.8. Cylindrical pipe

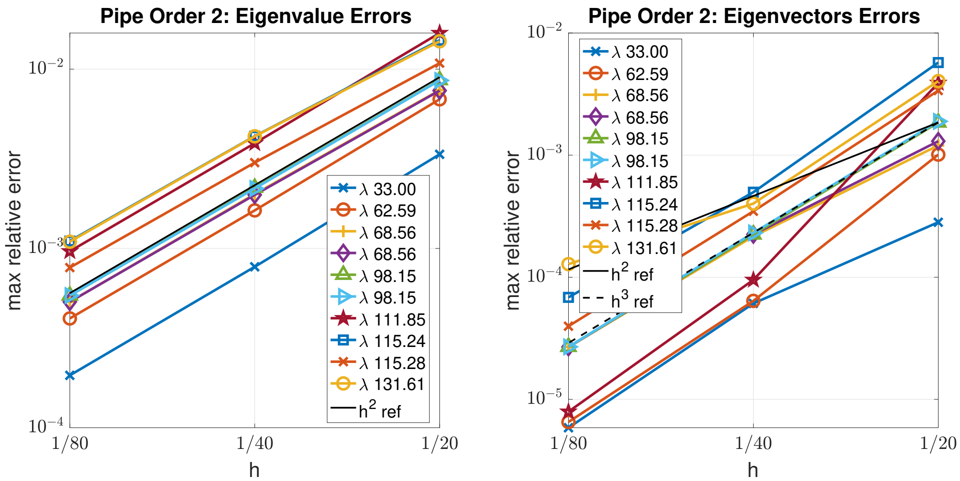

Figure 9: Accuracy of some computed eigenvalues and eigenvalues on a cylindrical pipe (order 2) compared to the true continuous values.

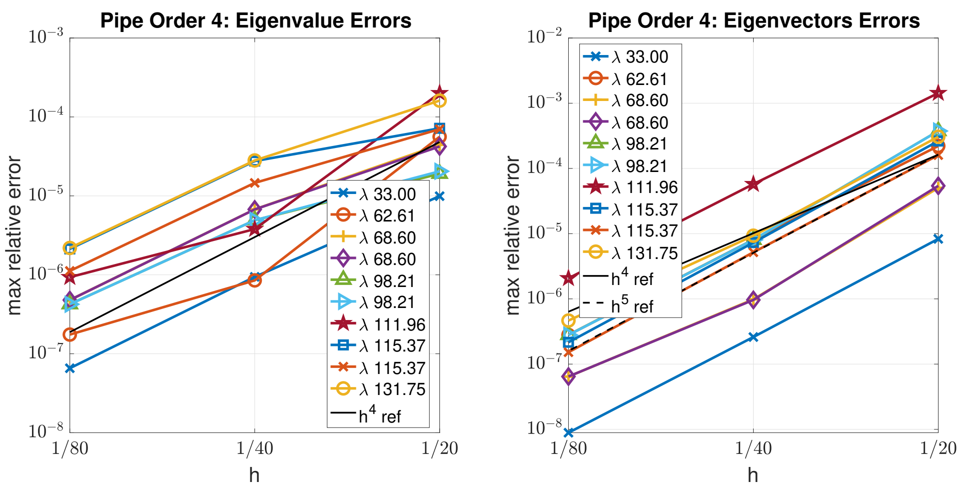

Figure 10: Accuracy of some computed eigenvalues and eigenvalues on a cylindrical pipe (order 4) compared to the true continuous values.

Here are results for a solid cylinder in 3D.

- At order 2, the eigenvectors seem to be converging at a faster rate than expected (?).
- At order 4, the eigenvectors seem to be converging at a faster rate than expected (?).

Command:

```
genEigs -noplot eigs.cmd -problem=laplace -g=pipeze4.order2.hdf -eigCase=disk -numEigenValues=32 -tol=1.0e-12 -disc
```

check this case – orthogonalize may be wrong.

pipeze2.order2.hdf, order=2						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\|A\phi - \lambda\phi\|/\lambda$
0	32.892 + (-0.0e+00) i	3.35e-03	2.82e-04	1	1	2.88e-11
1	62.186 + (-0.0e+00) i	6.79e-03	1.01e-03	1	1	1.58e-11
2	68.074 + (-0.0e+00) i	7.64e-03	1.20e-03	2	2	1.73e-11
3	68.077 + (-0.0e+00) i	7.59e-03	1.30e-03	2	2	5.06e-06
4	97.353 + (-0.0e+00) i	8.69e-03	1.85e-03	2	2	1.28e-11
5	97.357 + (-0.0e+00) i	8.65e-03	1.89e-03	2	2	3.54e-06
6	110.180 + (-0.0e+00) i	1.59e-02	3.90e-03	1	1	1.30e-11
7	113.692 + (-0.0e+00) i	1.45e-02	5.73e-03	2	1	8.82e-12
8	114.126 + (-0.0e+00) i	1.08e-02	3.38e-03	2	1	6.20e-12
9	129.866 + (-0.0e+00) i	1.43e-02	4.04e-03	1	1	1.31e-11
10	142.956 + (-0.0e+00) i	1.39e-02	5.47e-03	2	1	2.58e-11
11	143.388 + (-0.0e+00) i	1.10e-02	3.56e-03	2	1	5.43e-11
12	145.283 + (-0.0e+00) i	2.11e-03	1.00e+00	2	2	1.66e-11
13	145.286 + (-0.0e+00) i	2.13e-03	1.00e+00	2	2	2.38e-06
14	159.125 + (-0.0e+00) i	1.39e-02	3.72e-03	1	1	1.06e-11
15	169.537 + (-0.0e+00) i	1.83e-02	1.86e-02	2	2	8.85e-12
16	169.619 + (-0.0e+00) i	1.78e-02	1.83e-02	2	2	1.03e-04
17	175.588 + (-0.0e+00) i	1.68e-02	1.00e+00	2	1	5.47e-12
18	190.819 + (-0.0e+00) i	1.80e-02	8.17e-03	2	2	1.57e-11
19	191.238 + (-0.0e+00) i	1.59e-02	8.20e-03	2	2	2.10e-11
20	198.783 + (-0.0e+00) i	1.74e-02	1.84e-02	2	2	8.90e-12
21	198.865 + (-0.0e+00) i	1.70e-02	1.81e-02	2	2	8.81e-05
22	202.221 + (-0.0e+00) i	4.12e-04	1.00e+00	2	2	9.08e-12
23	202.305 + (-0.0e+00) i	2.38e-06	1.00e+00	2	2	9.86e-12
24	206.964 + (-0.0e+00) i	1.07e-03	1.00e+00	2	1	1.35e-11
25	210.464 + (-0.0e+00) i	1.18e-03	1.00e+00	1	2	7.30e-12
26	210.467 + (-0.0e+00) i	1.16e-03	1.00e+00	1	2	1.58e-06
27	231.461 + (-0.0e+00) i	2.07e-02	2.44e-02	2	2	1.17e-11
28	231.545 + (-0.0e+00) i	2.03e-02	2.15e-02	2	2	9.40e-12
29	234.309 + (-0.0e+00) i	8.65e-03	1.00e+00	2	2	3.96e-12
30	234.568 + (-0.0e+00) i	7.55e-03	1.00e+00	2	2	1.98e-04
31	246.566 + (-0.0e+00) i	2.02e-02	1.81e-02	2	2	9.71e-12
32	246.648 + (-0.0e+00) i	1.99e-02	1.77e-02	2	2	7.09e-05
33	255.789 + (-0.0e+00) i	1.64e-02	1.00e+00	2	1	1.95e-11

Table 12: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=2.07e-02, max-evect-err=1.00e+00, max-residual=1.98e-04

pipeze4.order2.hdf, order=2						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\|A\phi - \lambda\phi\|/\lambda$
0	32.976 + (-0.0e+00) i	7.90e-04	6.10e-05	1	1	2.01e-11
1	62.509 + (-0.0e+00) i	1.63e-03	6.40e-05	1	1	2.40e-11
2	68.460 + (-0.0e+00) i	2.01e-03	2.16e-04	2	2	1.06e-11
3	68.461 + (-0.0e+00) i	1.99e-03	2.24e-04	2	2	1.19e-11
4	97.993 + (-0.0e+00) i	2.18e-03	2.24e-04	2	2	2.07e-11
5	97.994 + (-0.0e+00) i	2.16e-03	2.31e-04	2	2	2.16e-11
6	111.527 + (-0.0e+00) i	3.86e-03	9.51e-05	1	1	1.71e-11
7	114.880 + (-0.0e+00) i	4.23e-03	4.97e-04	2	1	1.10e-11
8	115.020 + (-0.0e+00) i	3.01e-03	3.47e-04	2	1	1.34e-11
9	131.198 + (-0.0e+00) i	4.22e-03	4.01e-04	1	1	1.81e-11
10	144.413 + (-0.0e+00) i	3.89e-03	5.10e-04	2	1	2.45e-11
11	144.553 + (-0.0e+00) i	2.92e-03	3.67e-04	2	1	1.69e-11
12	147.009 + (-0.0e+00) i	3.69e-03	1.93e-04	2	2	1.12e-11
13	147.011 + (-0.0e+00) i	3.68e-03	2.60e-04	2	2	9.45e-12
14	160.731 + (-0.0e+00) i	3.92e-03	4.12e-04	1	1	1.59e-11
15	171.736 + (-0.0e+00) i	5.56e-03	5.19e-03	2	2	1.69e-11
16	171.738 + (-0.0e+00) i	5.55e-03	5.50e-03	2	2	2.14e-11
17	179.729 + (-0.0e+00) i	7.28e-03	5.25e-04	1	1	1.64e-11
18	193.428 + (-0.0e+00) i	4.61e-03	5.71e-04	2	1	1.66e-11
19	193.569 + (-0.0e+00) i	3.89e-03	4.50e-04	2	1	1.57e-11
20	201.269 + (-0.0e+00) i	5.12e-03	5.21e-03	2	2	1.92e-11
21	201.271 + (-0.0e+00) i	5.11e-03	5.54e-03	2	2	1.42e-11
22	205.356 + (-0.0e+00) i	6.71e-03	5.07e-03	2	2	1.62e-11
23	205.364 + (-0.0e+00) i	6.67e-03	5.00e-03	2	2	8.93e-12
24	209.746 + (-0.0e+00) i	4.58e-03	4.10e-04	1	1	1.94e-11
25	215.209 + (-0.0e+00) i	6.61e-03	8.66e-04	2	2	1.09e-11
26	215.210 + (-0.0e+00) i	6.61e-03	8.64e-04	2	2	1.27e-11
27	234.889 + (-0.0e+00) i	6.19e-03	5.10e-03	2	2	2.15e-11
28	234.897 + (-0.0e+00) i	6.16e-03	5.03e-03	2	2	1.96e-11
29	238.326 + (-0.0e+00) i	7.81e-03	1.25e-03	2	2	1.59e-11
30	238.358 + (-0.0e+00) i	7.67e-03	5.64e-03	2	2	1.77e-11
31	250.282 + (-0.0e+00) i	5.44e-03	5.20e-03	2	2	1.27e-11
32	250.285 + (-0.0e+00) i	5.44e-03	5.52e-03	2	2	9.13e-12

Table 13: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=7.81e-03, max-evect-err=5.64e-03, max-residual=2.45e-11

pipeze8.order2.hdf, order=2						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\|A\phi - \lambda\phi\|/\lambda$
0	32.996 + (-0.0e+00) i	1.96e-04	5.85e-06	1	1	1.66e-11
1	62.586 + (-0.0e+00) i	4.07e-04	6.52e-06	1	1	1.16e-11
2	68.563 + (-4.6e-09) i	5.04e-04	2.68e-05	2	2	1.70e-11
3	68.563 + (+4.6e-09) i	5.04e-04	2.68e-05	2	2	3.21e-07
4	98.153 + (-6.3e-09) i	5.46e-04	2.69e-05	2	2	2.14e-11
5	98.153 + (+6.3e-09) i	5.46e-04	2.69e-05	2	2	2.24e-07
6	111.851 + (-0.0e+00) i	9.64e-04	7.92e-06	1	1	1.18e-11
7	115.241 + (-0.0e+00) i	1.10e-03	6.84e-05	2	2	2.36e-11
8	115.278 + (-0.0e+00) i	7.82e-04	3.97e-05	2	2	7.18e-07
9	131.610 + (-0.0e+00) i	1.09e-03	1.28e-04	1	1	1.41e-11
10	144.831 + (-0.0e+00) i	1.00e-03	6.91e-05	2	2	4.80e-10
11	144.868 + (-0.0e+00) i	7.53e-04	4.00e-05	2	2	5.72e-07
12	147.418 + (-0.0e+00) i	9.22e-04	3.31e-05	2	2	2.13e-11
13	147.418 + (-0.0e+00) i	9.22e-04	3.31e-05	2	2	1.49e-07
14	161.200 + (-0.0e+00) i	1.01e-03	1.28e-04	1	1	1.42e-11
15	172.440 + (-0.0e+00) i	1.48e-03	1.61e-03	2	2	1.16e-11
16	172.440 + (-0.0e+00) i	1.48e-03	1.60e-03	2	2	1.26e-06
17	180.717 + (-0.0e+00) i	1.82e-03	2.68e-05	1	1	1.39e-11
18	194.097 + (-0.0e+00) i	1.17e-03	6.41e-05	2	2	1.93e-10
19	194.133 + (-0.0e+00) i	9.86e-04	4.74e-05	2	2	4.27e-07
20	202.030 + (-1.1e-08) i	1.36e-03	1.60e-03	2	2	1.08e-06
21	202.030 + (+1.1e-08) i	1.36e-03	1.61e-03	2	2	1.43e-11
22	206.371 + (-2.8e-09) i	1.80e-03	1.47e-03	2	2	1.34e-11
23	206.371 + (+2.8e-09) i	1.80e-03	1.47e-03	2	2	3.96e-07
24	210.466 + (-0.0e+00) i	1.17e-03	1.28e-04	1	1	1.17e-11
25	216.284 + (-0.0e+00) i	1.65e-03	6.75e-05	2	2	1.83e-11
26	216.284 + (-0.0e+00) i	1.65e-03	6.75e-05	2	2	1.02e-07
27	235.961 + (-7.8e-09) i	1.66e-03	1.47e-03	2	2	1.60e-11
28	235.961 + (+7.8e-09) i	1.66e-03	1.47e-03	2	2	3.46e-07
29	239.692 + (-0.0e+00) i	2.12e-03	2.04e-03	2	2	8.99e-12
30	239.694 + (-0.0e+00) i	2.11e-03	2.14e-04	2	2	2.02e-06
31	251.295 + (-7.7e-09) i	1.42e-03	1.60e-03	2	2	8.67e-07
32	251.295 + (+7.7e-09) i	1.42e-03	1.61e-03	2	2	6.93e-11
33	262.963 + (-0.0e+00) i	1.71e-03	9.37e-05	2	2	1.19e-10
34	262.999 + (-0.0e+00) i	1.57e-03	8.45e-05	2	2	3.15e-07
35	269.076 + (-0.0e+00) i	2.72e-03	1.00e+00	2	1	3.73e-11

Table 14: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=2.72e-03, max-evect-err=1.00e+00, max-residual=2.02e-06

pipeze4.order4.ng3.hdf, order=4						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\\|A\phi - \lambda\phi\\|/\lambda$
0	33.002 + (-0.0e+00) i	9.42e-07	2.60e-07	1	1	1.91e-11
1	62.611 + (-0.0e+00) i	8.50e-07	8.09e-06	1	1	2.65e-11
2	68.597 + (-0.0e+00) i	6.75e-06	9.84e-07	2	2	2.79e-11
3	68.597 + (-0.0e+00) i	6.74e-06	9.64e-07	2	2	8.69e-08
4	98.206 + (-0.0e+00) i	4.91e-06	8.05e-06	2	2	1.75e-11
5	98.206 + (-0.0e+00) i	4.91e-06	8.04e-06	2	2	6.07e-08
6	111.959 + (-0.0e+00) i	3.80e-06	5.75e-05	1	1	1.10e-11
7	115.365 + (-0.0e+00) i	2.76e-05	7.29e-06	2	2	1.73e-11
8	115.366 + (-0.0e+00) i	1.46e-05	5.18e-06	2	2	2.19e-07
9	131.751 + (-0.0e+00) i	2.81e-05	9.31e-06	1	1	1.84e-11
10	144.974 + (-0.0e+00) i	2.20e-05	9.47e-06	2	2	1.51e-11
11	144.975 + (-0.0e+00) i	1.18e-05	8.71e-06	2	2	1.74e-07
12	147.553 + (-0.0e+00) i	5.80e-06	5.76e-05	2	2	1.23e-11
13	147.553 + (-0.0e+00) i	5.80e-06	5.76e-05	2	2	4.04e-08
14	161.360 + (-0.0e+00) i	2.30e-05	1.01e-05	1	1	1.41e-11
15	172.687 + (-0.0e+00) i	4.73e-05	8.52e-05	2	2	1.60e-11
16	172.687 + (-0.0e+00) i	4.71e-05	8.36e-05	2	2	5.24e-07
17	181.043 + (-0.0e+00) i	1.70e-05	2.28e-04	1	1	3.23e-11
18	194.321 + (-0.0e+00) i	1.84e-05	6.12e-05	2	2	1.99e-11
19	194.323 + (-0.0e+00) i	1.07e-05	6.00e-05	2	2	1.30e-07
20	202.296 + (-0.0e+00) i	4.04e-05	8.54e-05	2	2	2.94e-11
21	202.296 + (-0.0e+00) i	4.03e-05	8.37e-05	2	2	4.47e-07
22	206.730 + (-7.2e-07) i	6.73e-05	1.08e-04	2	2	4.43e-07
23	206.730 + (+7.2e-07) i	6.73e-05	1.09e-04	2	2	5.18e-10
24	210.707 + (-0.0e+00) i	1.94e-05	5.58e-05	1	1	1.42e-11
25	216.638 + (-0.0e+00) i	1.60e-05	2.21e-04	2	2	1.27e-11
26	216.638 + (-0.0e+00) i	1.60e-05	2.21e-04	2	2	2.75e-08
27	236.338 + (-7.2e-07) i	5.89e-05	1.08e-04	2	2	4.54e-10
28	236.338 + (+7.2e-07) i	5.89e-05	1.08e-04	2	2	3.88e-07
29	240.179 + (-0.0e+00) i	9.32e-05	5.55e-05	2	2	1.13e-11
30	240.181 + (-0.0e+00) i	8.45e-05	1.36e-04	2	2	9.52e-07
31	251.644 + (-0.0e+00) i	3.40e-05	8.70e-05	2	2	1.74e-11
32	251.644 + (-0.0e+00) i	3.39e-05	8.55e-05	2	2	3.59e-07

Table 15: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=9.32e-05, max-evect-err=2.28e-04, max-residual=9.52e-07

pipeze8.order4.ng3.hdf, order=4						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\\|A\phi - \lambda\phi\\|/\lambda$
0	33.002 + (-0.0e+00) i	6.56e-08	8.87e-09	1	1	2.98e-11
1	62.611 + (-0.0e+00) i	1.75e-07	2.80e-07	1	1	2.86e-11
2	68.597 + (-0.0e+00) i	4.80e-07	6.43e-08	2	2	3.20e-11
3	68.597 + (-0.0e+00) i	4.79e-07	6.45e-08	2	2	6.14e-10
4	98.206 + (-0.0e+00) i	4.25e-07	2.80e-07	2	2	1.06e-11
5	98.206 + (-0.0e+00) i	4.24e-07	2.79e-07	2	2	4.35e-10
6	111.959 + (-0.0e+00) i	9.34e-07	2.07e-06	1	1	3.35e-11
7	115.368 + (-0.0e+00) i	2.13e-06	2.16e-07	2	2	3.69e-11
8	115.368 + (-0.0e+00) i	1.12e-06	1.52e-07	2	2	1.93e-09
9	131.754 + (-0.0e+00) i	2.19e-06	4.65e-07	1	1	3.68e-11
10	144.977 + (-0.0e+00) i	1.75e-06	2.71e-07	2	2	2.35e-11
11	144.977 + (-0.0e+00) i	9.51e-07	2.75e-07	2	2	1.53e-09
12	147.554 + (-0.0e+00) i	9.18e-07	2.07e-06	2	2	4.76e-11
13	147.554 + (-0.0e+00) i	9.18e-07	2.07e-06	2	2	2.87e-10
14	161.363 + (-0.0e+00) i	1.84e-06	4.87e-07	1	1	1.83e-11
15	172.695 + (-0.0e+00) i	3.93e-06	7.56e-06	2	2	2.62e-11
16	172.695 + (-0.0e+00) i	3.92e-06	7.57e-06	2	2	5.27e-09
17	181.046 + (-0.0e+00) i	3.23e-06	8.46e-06	1	1	3.38e-11
18	194.325 + (-0.0e+00) i	1.79e-06	2.06e-06	2	2	1.75e-11
19	194.325 + (-0.0e+00) i	1.19e-06	2.06e-06	2	2	1.14e-09
20	202.304 + (-0.0e+00) i	3.40e-06	7.55e-06	2	2	2.43e-11
21	202.304 + (-0.0e+00) i	3.39e-06	7.57e-06	2	2	4.50e-09
22	206.742 + (-0.0e+00) i	5.77e-06	8.47e-06	2	2	1.83e-11
23	206.742 + (-0.0e+00) i	5.76e-06	8.47e-06	2	2	8.54e-10
24	210.711 + (-0.0e+00) i	1.86e-06	2.05e-06	1	1	2.13e-11
25	216.641 + (-0.0e+00) i	2.84e-06	8.45e-06	2	2	1.84e-11
26	216.641 + (-0.0e+00) i	2.84e-06	8.45e-06	2	2	1.98e-10
27	236.351 + (-0.0e+00) i	5.08e-06	8.44e-06	2	2	3.11e-11
28	236.351 + (-0.0e+00) i	5.08e-06	8.45e-06	2	2	7.44e-10
29	240.199 + (-0.0e+00) i	7.88e-06	2.75e-06	2	2	2.74e-11
30	240.199 + (-0.0e+00) i	7.87e-06	1.26e-05	2	2	1.23e-08
31	251.652 + (-0.0e+00) i	3.11e-06	7.53e-06	2	2	2.01e-11
32	251.652 + (-0.0e+00) i	3.10e-06	7.54e-06	2	2	3.62e-09

Table 16: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=7.88e-06, max-evect-err=1.26e-05, max-residual=1.23e-08

4.9. Sphere

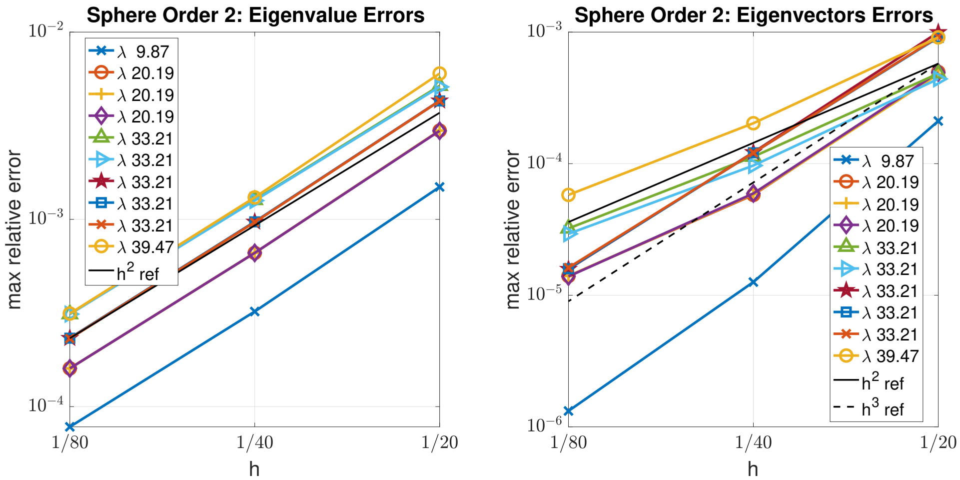

Figure 11: Accuracy of some computed eigenvalues and eigenvalues on a solid sphere (order 2) compared to the true continuous values.

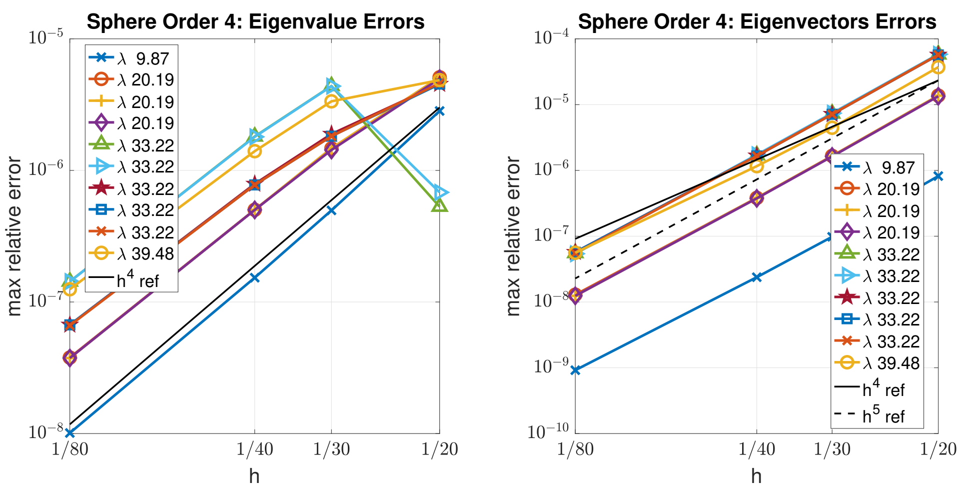

Figure 12: Accuracy of some computed eigenvalues and eigenvalues on a solid sphere (order 4) compared to the true continuous values.

In this section eigenpairs of a solid sphere are computed. The eigenfunctions of a sphere of radius a take the form

$$\phi_{m_\phi,m_r,m_\theta}(r,\phi,\theta) = r^{-1/2} J_{m_\phi+1/2}(\lambda_{m_\phi,m_r} r)\, P_{m_\phi}^{m_\theta}(\cos(\phi)) \begin{cases} \cos(m_\theta \theta), & m_\theta = 0,1,\ldots,m_\phi, \\ \sin(m_\theta \theta), & m_\theta = 1,2,\ldots,m_\phi, \end{cases} \tag{4a}$$

where $P_{m_\phi}^{m_\theta}$ are the associated Legendre polynomials. The eigenvalues are

$$\lambda_{m_\phi,m_r} = \zeta_{m_\phi,m_r}/a, \qquad m_\phi = 0,1,2,\ldots, \quad m_r = 1,2,3,\ldots, \tag{4b}$$

where ζ_{m_ϕ,m_r} are the zeros of $J_{m_\phi+1/2}(\zeta)$, for $m_r = 1,2,3,\ldots$. Note that eigenvalue λ_{m_ϕ,m_r} has multiplicity $2m_\phi+1$ and thus there are eigenvalues with high multiplicity which could cause difficulties for numerical algorithms.

The composite grids for the solid sphere, denoted by $\mathcal{G}^{(j)}_{\text{sphere}}$, consist of four component grids, each with grid spacing approximately equal to $\Delta s^{(j)} = 1/(10j)$. The sphere, of radius $a = 1$, is covered with three boundary-fitted patches near the surface as shown on the left in Figure **??**. There is one patch specified using spherical polar coordinates that covers much of the sphere except near the poles. To remove the polar singularities there are two patches that cover the north and south poles, defined by orthographic mappings. A background Cartesian grid covers the interior of the sphere.

spheree4.order2.hdf, order=2						
j	λ_j	λ_j-err	ϕ_j-err	multe	multc	$\Vert A\phi - \lambda\phi\Vert/\lambda$
0	9.866 + (-0.0e+00) i	3.22e-04	1.26e-05	1	1	1.44e-11
1	20.177 + (-2.1e-07) i	6.62e-04	5.79e-05	3	3	2.35e-10
2	20.177 + (+2.1e-07) i	6.62e-04	5.87e-05	3	3	8.54e-07
3	20.177 + (-0.0e+00) i	6.62e-04	5.94e-05	3	3	2.01e-07
4	33.175 + (-0.0e+00) i	1.27e-03	1.13e-04	5	5	9.17e-12
5	33.176 + (-0.0e+00) i	1.26e-03	9.68e-05	5	5	1.30e-11
6	33.185 + (-7.7e-07) i	9.70e-04	1.22e-04	5	5	4.72e-08
7	33.185 + (+7.7e-07) i	9.70e-04	1.21e-04	5	5	1.11e-06
8	33.185 + (-0.0e+00) i	9.65e-04	1.21e-04	5	5	4.51e-07
9	39.427 + (-0.0e+00) i	1.31e-03	2.03e-04	1	1	5.36e-12
10	48.746 + (-0.0e+00) i	1.75e-03	1.28e-03	7	7	1.01e-11
11	48.746 + (-0.0e+00) i	1.75e-03	1.27e-03	7	7	2.30e-06
12	48.746 + (-0.0e+00) i	1.75e-03	1.22e-03	7	7	7.25e-08
13	48.756 + (-0.0e+00) i	1.55e-03	2.24e-04	7	7	1.45e-11
14	48.756 + (-5.5e-07) i	1.54e-03	2.33e-04	7	7	1.63e-07
15	48.756 + (+5.5e-07) i	1.54e-03	2.34e-04	7	7	1.58e-09
16	48.770 + (-0.0e+00) i	1.25e-03	2.17e-04	7	7	5.46e-12
17	59.561 + (-3.0e-07) i	1.98e-03	1.89e-03	3	3	6.69e-07
18	59.561 + (+3.0e-07) i	1.98e-03	1.90e-03	3	3	2.02e-10
19	59.561 + (-0.0e+00) i	1.98e-03	1.84e-03	3	3	1.38e-08
20	66.791 + (-0.0e+00) i	2.44e-03	1.38e-03	9	9	8.95e-12
21	66.798 + (-0.0e+00) i	2.34e-03	2.59e-04	9	9	4.30e-06
22	66.798 + (-2.7e-06) i	2.33e-03	2.44e-04	9	9	8.56e-08
23	66.798 + (+2.7e-06) i	2.33e-03	2.47e-04	9	9	9.83e-07
24	66.807 + (-0.0e+00) i	2.20e-03	1.61e-03	9	9	2.17e-11
25	66.807 + (-0.0e+00) i	2.20e-03	1.85e-03	9	9	9.31e-12
26	66.829 + (-0.0e+00) i	1.88e-03	9.83e-04	9	9	2.90e-08
27	66.829 + (-0.0e+00) i	1.88e-03	9.74e-04	9	9	2.52e-06
28	66.831 + (-0.0e+00) i	1.85e-03	8.19e-04	9	9	1.39e-11
29	82.457 + (-0.0e+00) i	3.17e-03	2.22e-03	5	2	1.27e-11
30	82.458 + (-0.0e+00) i	3.16e-03	2.22e-03	5	2	1.11e-11
31	82.516 + (-7.3e-07) i	2.45e-03	1.22e-03	5	3	6.18e-07
32	82.516 + (+7.3e-07) i	2.45e-03	1.23e-03	5	3	6.20e-10
33	82.516 + (-0.0e+00) i	2.45e-03	1.21e-03	5	3	1.28e-11

Table 17: Computed eigenvalues, relative-error in the eigenvalues, and relative error in the eigenvectors. orthogonalize=1, max-rel-err=3.17e-03, max-evect-err=2.22e-03, max-residual=4.30e-06