On Longitudinal Emittance

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1 Emittance Conversions

The standard formula for an upright ellipse in phase-space $\Delta\phi \otimes w$ is:

$$\frac{\Delta\phi^2}{\Delta\phi_0^2} + \frac{w^2}{w_0^2} = 1 \tag{1}$$

with $\Delta\phi = \phi - \phi_s$ and $w \equiv \delta\gamma = \Delta W/mc^2$. ϕ_s being the synchronous phase, mc^2 the rest energy, W the total energy and γ the Lorentz factor. It has the emittance

$$\epsilon_w = \Delta\phi_0 w_0 \tag{2}$$

and units [rad]. The ellipse intersetcs the $\Delta\phi$-axis at $\Delta\phi_0$ and the w-axis at w_0. The intersection with the w-axis determines the β-function by the relation $\beta_w = \epsilon_w/w_0^2$. Its units are $[rad]$.

Let's change to new coordinates, for instance the pair of canonical variables $\Delta z \otimes \Delta p/p$, as it is used internally in Trace 3D. The transformation from old to new coordinates is: $|\Delta z| = \kappa|\Delta\phi| = \frac{\beta\lambda}{2\pi}|\Delta\phi|$ and $\Delta p/p = \tau w = \gamma/(\gamma^2-1)w = (\gamma\beta^2)^{-1}w$. This gives the modified ellipse equation:

$$\frac{\Delta z^2}{(\kappa\Delta\phi_0)^2} + \frac{(\Delta p/p)^2}{(\tau w_0)^2} = 1 \tag{3}$$

which has the transformed emittance

$$\epsilon_z = \kappa\Delta\phi_0\tau w_0 = \kappa\tau\epsilon_w = \frac{\beta\lambda}{2\pi}\gamma/(\gamma^2-1)\epsilon_w = \frac{\lambda}{2\pi\gamma\beta}\epsilon_w, \tag{4}$$

with units $[m]$. Again the β-function is given by

$$\beta_z = \epsilon_z/(\Delta p/p)_0^2 = \kappa\tau\epsilon_w/(\tau w_0)^2 = \kappa/\tau \times \beta_w = \frac{\beta\lambda}{2\pi}\frac{\gamma^2-1}{\gamma}\beta_w, \tag{5}$$

with units $[m/rad]$.

For the $\Delta\phi \otimes \Delta W$ phase space, because $\Delta W = mc^2 w$, we have $\kappa = 1$ and $\tau = mc^2$. So that

$$\epsilon_W = mc^2 \epsilon_w \quad [rad \times eV] \tag{6}$$

$$\beta_W = 1/mc^2 \beta_w \quad [rad/eV] \tag{7}$$

Finally for the $\Delta z \otimes \Delta W$ phase space we get the emittance

$$\epsilon_{zW} = \frac{\beta\lambda}{2\pi} mc^2 \epsilon_w \quad [m \times eV] \tag{8}$$

$$\beta_{zW} = \frac{\beta\lambda}{2\pi} \frac{1}{mc^2} \beta_w \quad [m/eV] \tag{9}$$

The ESS conceptual design report uses the $z \otimes z'$ phase space, i.e. the emittance $\epsilon_{zz'}$. Since $\delta\gamma = \delta\beta/\beta = z'$ and $\Delta\phi = \frac{2\pi}{\beta\lambda} z$ we have:

$$\epsilon_{zz'} = \frac{\lambda}{2\pi\beta\gamma^3} \epsilon_w \tag{10}$$

with units $[m \times rad]$ like the transverse emittances.

Instead of longitudinal position some people use arrival time. For the $\Delta t \otimes \Delta W$ phase space we use $|\Delta t| = (\beta c)^{-1} |\Delta z| = (\beta c)^{-1} \frac{\beta\lambda}{2\pi} |\Delta\phi|$ and get

$$\epsilon_{tW} = \frac{\lambda}{2\pi c} mc^2 \epsilon_w \quad [sec \times eV] \tag{11}$$

More details about emittance definitions, normalized and unnormalized, and their units can be found in the UserManual of the TraceWin program.

2 Twiss Parameter Values

To simplify we assume the twiss parameter $\alpha = 0$. The twiss parameter γ then reduces to $1/\beta$ and only two free parameters ϵ and β remain to describe the ellipse in phase space completely.

For the longitudinal dynamics in the passage of an rf-gap the intersection w_0 on the w-axis is given by

$$w_0 = \frac{\Delta W}{mc^2} = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s) \Delta\phi_0^2 / 2\pi mc^2} \tag{12}$$

$$= \Delta\phi_0 \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s) / 2\pi mc^2} \tag{13}$$

If w_0 is given $\Delta\phi_0$ follows from (13) *and vice versa*. Putting $w_0 = \epsilon_w / \Delta\phi_0$ into (13) we get

$$\Delta\phi_0 = \sqrt{\epsilon_w / \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s)/2\pi mc^2}} \tag{14}$$

and from (14) we get finally

$$\gamma_0 = \epsilon_w / \Delta\phi_0^2 = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s)/2\pi mc^2} \tag{15}$$

and

$$\beta_0 = 1/\gamma_0 \tag{16}$$

NOTE: the two twiss parameters γ_0 and β_0 are completely defined by the emittance ϵ_w and the cavity field E_0, rf-phase ϕ_s, rf-wavelength λ and particle impuls $\sim \gamma\beta$.

3 Appendix

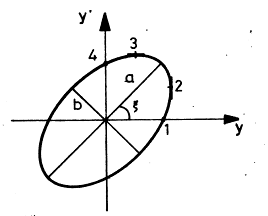
3.1 SIMULAC Variables

Table 1: variable names

ϵ_w = emitw	$\Delta\phi$ = Dphi	$\Delta\phi_0$ = Dphi0	w = w	w_0 = w0
ϵ_W = emitW	Δz = z	ΔW = DW	$\Delta p/p$ = Dp2p	$\Delta p/p_0$ = Dp2p0
ϵ_z = emitz	β_z = betaz	γ_z = gammaz	α_z = alphaz	λ = lamb
Ez_{avg} = EzAvg	Ez_{peak} = EzPeak	ϕ_+ = `phi_1`	ϕ_- = `phi_2`	ψ = psi
γ = gamma	$\gamma\beta$ = gb	β = beta	E_0T = E0T	mc^3 = m0c3
mc^2 = m0c2	ϵ_{xi} = `emitx_i`	ϵ_{yi} = `emity_i`	ϵ_{zi} = `emitz_i`	β_{xi} = `betax_i`
β_{yi} = `betay_i`	α_{xi} = `alfax_i`	α_{yi} = `alfay_i`	γ_{xi} = `gamax_i`	γ_{yi} = `gamay_i`
ω = omg	ϕ = phi	ϕ_s = phis		

3.2 Relations Between Ellipse and Twiss Parameters

3.4 Geometrical properties of the ellipse



	$\alpha, \beta, \gamma, \epsilon$	$c_1 \quad c_2$ $c_3 \quad c_4$	L, S, ϵ
	$\beta\gamma - \alpha^2 = 1$ $H = \frac{1}{2}(\beta + \gamma)$	$\epsilon = c_1c_4 - c_2c_3$ $H = \frac{1}{2}(c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$	$H = \frac{1}{2L}(L^2 + S^2 + 1)$
y_1	$\sqrt{\epsilon/\gamma}$	$\epsilon/\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon L}$
y_2	$\sqrt{\epsilon\beta}$	$\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L}\,\sqrt{S^2 + L^2}$
y_2'	$-\alpha\sqrt{\epsilon/\beta}$	$(c_1c_3 + c_2c_4)/\sqrt{c_1^2 + c_2^2}$	$S\sqrt{\epsilon/L}/\sqrt{S^2 + L^2}$
y_3	$-\alpha\sqrt{\epsilon/\gamma}$	$(c_1c_3 + c_2c_4)/\sqrt{c_3^2 + c_4^2}$	$S\sqrt{\epsilon/L}$
y_3'	$\sqrt{\epsilon\gamma}$	$\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
y_4'	$\sqrt{\epsilon/\beta}$	$\epsilon/\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon L}/\sqrt{S^2 + L^2}$
a	$\sqrt{\epsilon/2}\,(\sqrt{H + 1} + \sqrt{H - 1})$		
b	$\sqrt{2\epsilon}/(\sqrt{H + 1} + \sqrt{H - 1}) = \sqrt{\epsilon/2}\,(\sqrt{H + 1} - \sqrt{H - 1})$		
a/b > 1	$H + \sqrt{H^2 - 1}$		
$\tan \xi$	$[-\alpha(H + \sqrt{H^2 - 1})]/[\beta(H + \sqrt{H^2 - 1}) - 1]$	$[c_2 + c_3(H + \sqrt{H^2 - 1})]/[c_1(H + \sqrt{H^2 - 1}) - c_4]$	$S/[L(H + \sqrt{H^2 - 1}) - 1]$
$\sin 2\xi$	$-\alpha/\sqrt{H^2 - 1}$	$(c_1c_3 + c_2c_4)/\epsilon\sqrt{H^2 - 1}$	$S/L\sqrt{H^2 - 1}$
$\cos 2\xi$	$(\beta - \gamma)/2\sqrt{H^2 - 1}$	$(c_1^2 + c_2^2 - c_3^2 - c_4^2)/2\epsilon\sqrt{H^2 - 1}$	$(L^2 + S^2 - 1)/2L\sqrt{H^2 - 1}$
$\tan 2\xi$	$-2\alpha/(\beta - \gamma)$	$2(c_1c_3 + c_2c_4)/(c_1^2 + c_2^2 - c_3^2 - c_4^2)$	$2S/(L^2 + S^2 - 1)$

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