On Longitudinal Emittance

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1 Overview

The standard formula for an upright ellipse in phase-space $\Delta\phi \otimes w$ is:

$$\frac{\Delta\phi^2}{\Delta\phi_0^2} + \frac{w^2}{w_0^2} = 1 \tag{1}$$

with $\Delta\phi = \phi - \phi_s$ and $w \equiv \delta\gamma = \Delta W/mc^2$. ϕ_s being the synchronous phase, mc^2 the rest energy, W (aka T) the kinetic energy and γ the Lorentz factor. It has the emittance

$$\epsilon_w = |\Delta\phi_0 w_0| \tag{2}$$

and units [rad]. The ellipse intersetcs the $\Delta\phi$-axis at $\Delta\phi_0$ and the w-axis at w_0. The intersection with the w-axis determines the β-function by the relation $\beta_w = \epsilon_w/w_0^2$. Its units are $[rad]$.

Let's change to new coordinates, for instance the pair of canonical variables $z \otimes \Delta p/p$, as it is used internally in the *TRACE 3-D* program. The transformation from old to new coordinates is: $z = -\kappa\Delta\phi = -\frac{\beta\lambda}{2\pi}\Delta\phi$ and $\Delta p/p = \tau w = \gamma/(\gamma^2-1)w = (\gamma\beta^2)^{-1}w$. This gives the modified ellipse equation:

$$\frac{z^2}{(\kappa\Delta\phi_0)^2} + \frac{(\Delta p/p)^2}{(\tau w_0)^2} = 1 \tag{3}$$

which has the transformed emittance

$$\epsilon_z = \kappa|\Delta\phi_0| * \tau|w_0| = \kappa\tau\epsilon_w = \frac{\beta\lambda}{2\pi}\gamma/(\gamma^2-1)\epsilon_w = \frac{\lambda}{2\pi\gamma\beta}\epsilon_w, \tag{4}$$

with units $[m \times rad]$. Again the β-function is given by

$$\beta_z = \epsilon_z/(\Delta p/p)_0^2 = \kappa\tau\epsilon_w/(\tau w_0)^2 = \kappa/\tau \times \beta_w = \frac{\beta\lambda}{2\pi}\frac{\gamma^2-1}{\gamma}\beta_w, \tag{5}$$

with units $[m]$.

For the $\Delta\phi \otimes \Delta W$ phase space, because $\Delta W = mc^2 w$, we have $\kappa = 1$ and $\tau = mc^2$. So that

$$\epsilon_W = mc^2\epsilon_w \quad [rad \times eV] \tag{6}$$

$$\beta_W = 1/mc^2\beta_w \quad [rad/eV] \tag{7}$$

Finally for the $\Delta z \otimes \Delta W$ phase space we get the emittance

$$\epsilon_{zW} = \frac{\beta\lambda}{2\pi}mc^2\epsilon_w \quad [m \times eV] \tag{8}$$

$$\beta_{zW} = \frac{\beta\lambda}{2\pi}\frac{1}{mc^2}\beta_w \quad [m/eV] \tag{9}$$

The ESS conceptual design report uses the $z \otimes z'$ phase space, i.e. the emittance $\epsilon_{zz'}$. Since $\delta\gamma = w = \beta^2\gamma^3\delta\beta/\beta = \beta^2\gamma^3 z'$ and $\Delta\phi = \frac{2\pi}{\beta\lambda}z$ we have:

$$\epsilon_{zz'} = \frac{\lambda}{2\pi\beta\gamma^3}\epsilon_w \tag{10}$$

with units $[m \times rad]$.

Instead of longitudinal position some people use arrival time. For the $\Delta t \otimes \Delta W$ phase space we use $\Delta t = -(\beta c)^{-1}z = (\beta c)^{-1}\frac{\beta\lambda}{2\pi}\Delta\phi$ and get

$$\epsilon_{tW} = \frac{\lambda}{2\pi c}mc^2\epsilon_w \quad [sec \times eV] \tag{11}$$

2 Full Treatment

The ellipse in normal form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12}$$

defines the emittance ϵ as:

$$\epsilon = |a * b| \tag{13}$$

Changing scales of x, y coordinates: $x = x'/\kappa$ and $y = y'/\tau$ and inserting into normal form:

$$\frac{x'^2}{(a\kappa)^2} + \frac{y'^2}{(b\tau)^2} = 1 \tag{14}$$

gives scaled emittance ϵ'

$$\epsilon' = |(a\kappa) * (b\tau)| = \kappa\tau\epsilon \tag{15}$$

Let $\beta_x = x_0^2/\epsilon$ then $\beta_x = (x'/\kappa)^2/(\epsilon'/\kappa\tau) = \frac{\tau}{\kappa}\beta'_x$, we get

$$\beta'_x = \frac{\kappa}{\tau}\beta_x \tag{16}$$

In phase space $\Delta\phi$, z and Δt are usually used as abscissa and w, ΔW, $\Delta p/p$ and z' as ordinates. We use κ to connect abscissa and τ to connect different ordinates. Six different combinations of abscissa can be made and 11 combinations for ordinates. Their corresponding κ- and τ-values are assembled in the following tables.

κ-values			
wanted↓ in terms of→	$\Delta\phi$ [rad]	z [m]	Δt [sec]
$\Delta\phi$ [rad]	1	$2\pi/\beta\lambda$	$2\pi\beta c/\beta\lambda$
z [m]	$\beta\lambda/2\pi$	1	βc
Δt [sec]	$\beta\lambda/(2\pi\beta c)$	$1/\beta c$	1

τ-values				
wanted↓ in terms of →	$\delta\gamma = w$	ΔW [eV]	$\Delta p/p$	z' [rad]
$\delta\gamma = w$	1	$1/(mc^2)$	$\gamma\beta^2$	$\gamma(\gamma\beta)^2$
ΔW [eV]	mc^2	1	$mc^2\gamma\beta^2$	$mc^2\gamma^3\beta^2$
$\Delta p/p$	$(\gamma\beta^2)^{-1}$	$(mc^2\gamma\beta^2)^{-1}$	1	γ^2
z' [rad]	$\gamma^{-1}(\gamma\beta)^{-2}$	$(mc^2\gamma^3\beta^2)^{-1}$	γ^{-2}	1
with $W = mc^2(\gamma - 1)$ as kinetic energy.				

Example: Phase space $z \otimes \Delta W$ in terms of $\Delta\phi \otimes \delta\gamma$: $\kappa = \beta\lambda/2\pi$, $\tau = mc^2$.

$$\epsilon_{zW} = \kappa\tau\epsilon_w = (\beta\lambda/2\pi)mc^2\epsilon_w. \tag{17}$$

$$\beta_{zW} = \frac{\kappa}{\tau}\beta_w = \frac{\beta\lambda/2\pi}{mc^2}\beta_w. \tag{18}$$

More interesting details about emittance definitions, normalized and unnormalized, and their units can be found in the UserManual of the *TRACEWIN* program.

3 Twiss Parameter Values

To simplify we assume the twiss parameter $\alpha = 0$. The twiss parameter γ then reduces to $1/\beta$ and only two free parameters ϵ and β remain to describe the ellipse in phase space completely.

For small aplitude longitudinal oscillations the separatrix intersects the w-axis at w_0 and is is given by

$$w_0 = \frac{\Delta W}{mc^2} = \sqrt{2qE_0LT\beta_s^3\gamma_s^3\lambda\phi_s^2\sin(-\phi_s)/\pi mc^2} \tag{19}$$

With w_0 and $\Delta\phi_0$ the maximal emittance on the separatrix is given.

$$\epsilon_w = w_0 * \Delta\phi_0 \tag{20}$$

and from 20 we get finally $\gamma_0 = \epsilon_w/\Delta\phi_0^2$ and $\beta_0 = 1/\gamma_0$.

NOTE: the two twiss parameters γ_0 and β_0 are completely defined by the emittance ϵ_w, the cavity field E_0, rf-phase ϕ_s, rf-wavelength λ and particle impuls $\sim \gamma\beta$.

4 Appendix

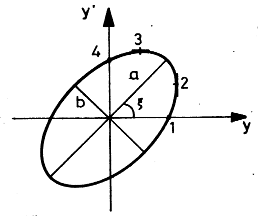
4.1 SIMULAC Variables

Table 1: **Variable Names**

ϵ_w = emitw	$\Delta\phi$ = Dphi	$\Delta\phi_0$ = Dphi0	w = w	w_0 = w0
ϵ_W = emitW	Δz = z	ΔW = DW	$\Delta p/p$ = Dp2p	$\Delta p/p_0$ = Dp2p0
ϵ_z = emitz	β_z = betaz	γ_z = gammaz	α_z = alfaz	λ = lamb
Ez_{avg} = EzAvg	Ez_{peak} = EzPeak	ϕ_+ = phi_1	ϕ_- = phi_2	ψ = psi
γ = gamma	$\gamma\beta$ = gb	β = beta	E_0T = E0T	E_0LT = E0LT
mc^2 = m0c2	mc^3 = m0c3	ϵ_{xi} = emitx_i	ϵ_{yi} = emity_i	ϵ_{zi} = emitz_i
β_{xi} = betax_i	β_{yi} = betay_i	α_{xi} = alfax_i	α_{yi} = alfay_i	γ_{xi} = gamax_i
γ_{yi} = gamay_i	ω = omg	ϕ = phi	ϕ_s = phisoll	W = tkin
$\Delta W/W$ = DW2W	$\Delta T/T$ = DT2T	DW2W = DT2T		

4.2 Relations Between Ellipse and Twiss Parameters

3.4 Geometrical properties of the ellipse



	$\alpha, \beta, \gamma, \epsilon$	$c_1 \quad c_2$ $c_3 \quad c_4$	L, S, ϵ
	$\beta\gamma - \alpha^2 = 1$ $H = \frac{1}{2}(\beta + \gamma)$	$\epsilon = c_1 c_4 - c_2 c_3$ $H = \frac{1}{2}\,(c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$	$H = \frac{1}{2L}\,(L^2 + S^2 + 1)$
y_1	$\sqrt{\epsilon/\gamma}$	$\epsilon/\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon L}$
y_2	$\sqrt{\epsilon\beta}$	$\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L}\,\sqrt{S^2 + L^2}$
y_2'	$-\alpha\sqrt{\epsilon/\beta}$	$(c_1 c_3 + c_2 c_4)/\sqrt{c_1^2 + c_2^2}$	$S\sqrt{\epsilon/L}/\sqrt{S^2 + L^2}$
y_3	$-\alpha\sqrt{\epsilon/\gamma}$	$(c_1 c_3 + c_2 c_4)/\sqrt{c_3^2 + c_4^2}$	$S\sqrt{\epsilon/L}$
y_3'	$\sqrt{\epsilon\gamma}$	$\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
y_4'	$\sqrt{\epsilon/\beta}$	$\epsilon/\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon L}/\sqrt{S^2 + L^2}$
a	$\sqrt{\epsilon/2}\,(\sqrt{H + 1} + \sqrt{H - 1})$		
b	$\sqrt{2\epsilon}/(\sqrt{H + 1} + \sqrt{H - 1}) = \sqrt{\epsilon/2}\,(\sqrt{H + 1} - \sqrt{H - 1})$		
a/b > 1	$H + \sqrt{H^2 - 1}$		
$\tan\xi$	$[-\alpha(H + \sqrt{H^2 - 1})]/[\beta(H + \sqrt{H^2 - 1}) - 1]$	$[c_2 + c_3(H + \sqrt{H^2 - 1})]/[c_1(H + \sqrt{H^2 - 1}) - c_4]$	$S/[L(H + \sqrt{H^2 - 1}) - 1]$
$\sin 2\xi$	$-\alpha/\sqrt{H^2 - 1}$	$(c_1 c_3 + c_2 c_4)/\epsilon\sqrt{H^2 - 1}$	$S/L\sqrt{H^2 - 1}$
$\cos 2\xi$	$(\beta - \gamma)/2\sqrt{H^2 - 1}$	$(c_1^2 + c_2^2 - c_3^2 - c_4^2)/2\epsilon\sqrt{H^2 - 1}$	$(L^2 + S^2 - 1)/2L\sqrt{H^2 - 1}$
$\tan 2\xi$	$-2\alpha/(\beta - \gamma)$	$2(c_1 c_3 + c_2 c_4)/(c_1^2 + c_2^2 - c_3^2 - c_4^2)$	$2S/(L^2 + S^2 - 1)$

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