On Space Charge and Self Fields

W.D. Klotz, wdklotz@alecli.com

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This document is a draft and will be completed in the future.

1 Debye Shielding (is space charge important?)

In this note I talk about the effects of space charge in the transverse beam optics by using a uniform particle distribution in which both the charge and current density, ρ and $J_z = J$, are independent of the transverse coordinates. This uniform beam model assumes that the beam is continuous in the direction of propagation and has a sharp boundary with $\rho = const$, $J = const$ inside and $\rho = 0$, $J = 0$ outside the boundary. The uniformity of charge and current density assures that the transverse electric and magnetic self fields and the associated forces are linear functions of the transverse coordinates.

The mutual interaction of the charged particles in a beam can be represented by the sum of a *collisional* force and a *smooth* force. The collisional part of the total interaction force arises when a particle sees its immediate neighbors and is therefore affected by their individual positions. This force will cause small random displacements of the particles trajectory and statistical fluctuations in the particle distribution as a whole. In most practical beams, however, this is a relatively small effect, and the mutual interaction between particles can be described largely by a smoothed force in which the graininess of the distribution of discrete particles is washed out. The space-charge potential function in this case obeys Poissons equation, and the resulting force can be treated in the same way as the applied focusing or acceleration forces acting on the beam.

A measure for the relative importance of collisional versus smoothed interaction, or single-particle versus collective effects, is the *Debye length*, λ_D. A fundamental parameter in plasma physics that can also be applied to charged particle beams. If a test charge is placed into a neutral plasma having a temperature T and equal positive ion and electron densities n, the excess electric potential set up by this charge is effectively screened off in a distance λ_D by charge redistribution in the plasma. This effect is known as *Debye shielding*.

$$\lambda_D = \left(\frac{\epsilon_0 k_B T}{q^2 n}\right)^{1/2} \tag{1}$$

$\epsilon_n = \gamma\beta\epsilon$ is the normalized emittance and k_B is Boltzman's constant. For a spherical bunch with RMS projections a, $k_B T = \gamma m_0 c^2 \epsilon_n^2 / a^2$. For accelerator beams this criterion is usually satisfied and the *smoothed* space-charge field describes the main effect of the Coulomb interactions. Thus

$$\lambda_D = \left(\frac{\gamma m_0 c^2 \gamma\beta\epsilon^2}{q^2 n a^2}\right)^{1/2} \tag{2}$$

The current I of a beam of particles with charge q and mean velocity βc is

$$I = q * A * \beta * c * n \tag{3}$$

with $A = \pi a^2$ the beam's cross-section and n the particle density. Inserting (3) into (2) we rewrite the equation for λ_D^2:

$$\lambda_D^2 = \frac{\epsilon_0 m_0 c^2 \epsilon^2 \pi c}{I e_0} (\gamma\beta)^3 \tag{4}$$

with $\epsilon_0 = 8.85 * 10^{12} F/m$ the permettivity in vacuum, $e_0 = 1.6 * 10^{-19} C$ the electron charge, I the current in A and $\gamma\beta = p/m_0 c$ for the average particle impulse. Note that λ_D scales with $p^{3/2}$ and $I^{-1/2}$.

For protons ($m_0 c^2 = 938.3 MeV$) and an emittance $\epsilon = 10^{-6} m * rad$ we get: $\lambda_D^2 = 7.83 * 10^{-6} (\gamma\beta)^3 / I$ in $[A * m^2]$.

W [MeV] kin	$\gamma\beta$	λ_D [m] (I=1.e-9 A)	λ_D [m] (I= 1.e-11 A)
5	0.103	2.94	29.4
10	0.146	4.94	49.4
20	0.210	8.51	85.1
50	0.331	10.8	108
200	0.687	50.4	504

If the Debye length is large compared with the beam radius ($\lambda_D \gg a$), the screening will be ineffective and single-particle behavior will dominate. On the other hand, if the Debye length is small compared to the beam radius ($\lambda_D \ll a$), collective effects due to the self fields of the beam will play an important role.

If collisions can be neglected, the single-particle Hamiltonian, the particle distribution, and Liouvilles theorem can be defined in six-dimensional phase space (r, P). This is possible because the smoothed space-charge forces acting on a particle can be treated like the applied forces. Thus, the six-dimensional phase-space volume occupied by a charged-particle distribution remains constant during propagation or acceleration. If, furthermore, all forces are linear functions of the particle displacement from the beam center, the normalized emittance associated with each direction remains a constant of the motion. For a matched beam in a linear focusing channel we can express this conservation law in terms of the rms beam width $\overline{x}$, rms tranverse momentum $\overline{P_x}$ and rms velocity $\overline{v_x}$ as

$$\overline{\epsilon_n} = \frac{\overline{x}\overline{P_x}}{m_0 c} = \frac{\overline{x}\gamma\overline{v_x}}{c} = const. \tag{5}$$

For the theoretical modeling of beams we can distinguish three regimes that can be characterized by the ratio of the Debye length to the effective beam radius, λ_D/a. When self-field effects dominate the beam physics (i.e., when $\lambda_D \ll a$), it is convenient for the mathematical analysis to neglect the thermal velocity spread altogether and use a laminar-flow model. In laminar flow, all particles at a given point are assumed to have the same velocity, so that particle trajectories do not cross. The particle density for a stationary laminar beam in a linear focusing channel is uniform. Like the external focusing force, the space-charge force is therefore a linear function of position within the beam. As a result, the linear beam optics techniques can be extended in a straightforward manner to include the self-field effects.

When the transverse thermal velocity spread becomes comparable to self-field effects, so that $\lambda_D \approx a$, the density profile of a stationary beam becomes nonuniform, according to a Boltzmann distribution. The forces due to the self fields of the beam are therefore nonlinear, a nonlaminar treatment of the beam is required, and the analysis becomes more complicated. A nonlaminar beam can be represented by the distribution of particles in phase space, f(x, P). The stationary state and the evolution of nonstationary distributions can be analyzed with the aid of the *Vlasov* equation. Analytical techniques are rather limited in usefulness except for the *K–V model* and must be complemented or replaced by particle simulation.

The third regime, the regime of *emittance-dominated* beams, is characterized by $\lambda_D \gg a$, which implies that the self fields of the beam can be ignored. The steady-state density profile is Gaussian. However, the particle motion is entirely governed by the external fields; that is, the standard beam optics techniques with external fields are valid in this regime. This implies that a particle's interaction with its nearest neighbors can be neglected in comparison to the interaction with the average collective field produced by the other particles in the beam. Quantitatively, for this to be the case, the Debye length λ_D, discussed above, must be large compared to the interparticle distance.

Conclusion: The Debye length λ_D for protons at 5 MeV kinetic energy and a beam current of 1nA is about a factor $\approx 10^2 - 10^3$ larger than the typical beam radius a.

2 K-V Envelope Differential Equations

The theory of strong focusing in periodic magnetic channels when space charge effects are negligible is treated in the classic paper by *Courant and Snyder.* The inclusion of the space charge forces usually follows the self-consistent method first developed by *Kapchinsky and Vladimirsky.* In the K-V model, the beam has an elliptic cross section with uniform particle density, and the envelopes X and Y in the two perpendicular transverse directions are determined by the two differential equations

$$\frac{d^2X}{ds^2} + \kappa_x X - \frac{2K}{X+Y} - \frac{\epsilon_x^2}{X^3} = 0, \tag{6}$$

$$\frac{d^2Y}{ds^2} + \kappa_y Y - \frac{2K}{X+Y} - \frac{\epsilon_y^2}{Y^3} = 0, \tag{7}$$

The independent variable s is the path length along the direction of propagation; $\epsilon_{x(y)}\pi$ is the emittance in x(y) direction. The function $\kappa_{x(y)}$ represents the applied forces and is periodic in the variable s with period S, i.e.,

$$\kappa(s+S) = \kappa(s). \tag{8}$$

For a quadrupole channel with pole-tip field B_0 and aperture $2a$, we have

$$\kappa_{x(y)} = \pm\frac{qB'}{m_0 c\beta\gamma} = \frac{qB_0/a}{m_0 c\beta\gamma} \tag{9}$$

where, in the ideal case, the gradient B' is constant inside the magnets and zero between magnets.

The factor K in the envelope equations is the *generalized perveance* and represents the space-charge forces. It is defined in terms of the total beam current as (mks units):

$$K = \frac{qI}{2\pi m_0 c^3\beta^3\gamma^3} = \frac{I}{I_0}\frac{2}{\beta^3\gamma^3}, \tag{10}$$

where

$$I_0 = \frac{4\pi\epsilon_0 m_0 c^3}{q} \tag{11}$$

is the limiting current.

The beam envelopes are related to the amplitude function $\beta(s)$ introduced by Courant and Snyder namely,

$$X(s) = \sqrt{\epsilon_x\beta_x},\ Y(s) = \sqrt{\epsilon_y\beta_y}. \tag{12}$$

For protons I_0 has the value $\cong 3.1\times10^7$ A. I assume again a proton beam with kinetic energy $5MeV$ and a beam current $I \cong 10^{-9}$ A. Then the generalized perveance K is of order $\cong 10^{-14}$. The quadrupole gradient B' for the ALCELI linac is $\cong$ 43 T/m, thus $\kappa_{x(y)} \cong 133$ m^{-2}. Assuming typical beam sizes of $X(Y) \cong 2$ mm and transverse emittances $\epsilon_{x(y)} \cong 1$ mm$\times$ mrad, the second term in (6) and (7) is of order $\cong$ 2.7 10^{-1}, the third term of order $\cong$ 2.9 10^{-11} and the 4th term of order $\cong$ 1.2 10^{-4}.

With these numbers we can expect that self-field effects due to space charge are at least $\approx \sqrt{4 * 10^6} = 2000$ times smaller than those driven by emittance and become less important for increasing energy. Thus can be neglected.

Conclusion: At $5MeV$ injection energy and $1nA$ current, the proton beam will be *emittance-dominated*.

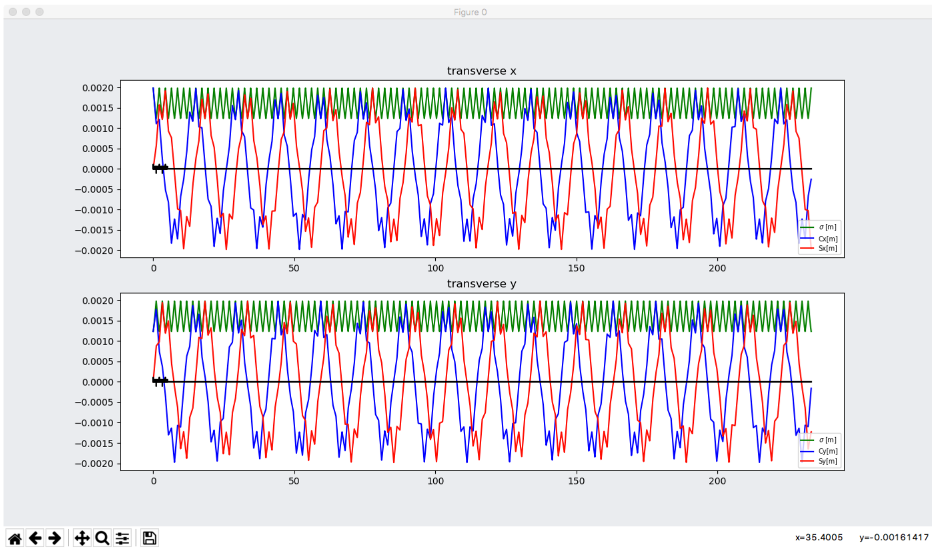
3 The matched FODO Channel
(without space charge and acceleration)

In 1958, Courant and Snyder analyzed the problem of alternating-gradient beam transport and treated a model without focusing gaps or space charge. O.A.Anderson and L.L. LoDestro revisited their work and found the exact solution for matched-beam envelopes in a linear quadrupole lattice [*O.A. Anderson and L.L. LoDestro, Phys. Rev. ST Accel. Beams, 2009*]. They extended that work to include the effect of asymmetric drift spaces. They derive the analytical solution and show exact envelopes for the first two solution bands. From that they get the peak envelope excursion as a function of phase advance σ up to 360^o. They found that solutions for the envelopes $X(Y)(s)$ exist in an infinite number of bands and that these coincide with bands of single-particle stability.

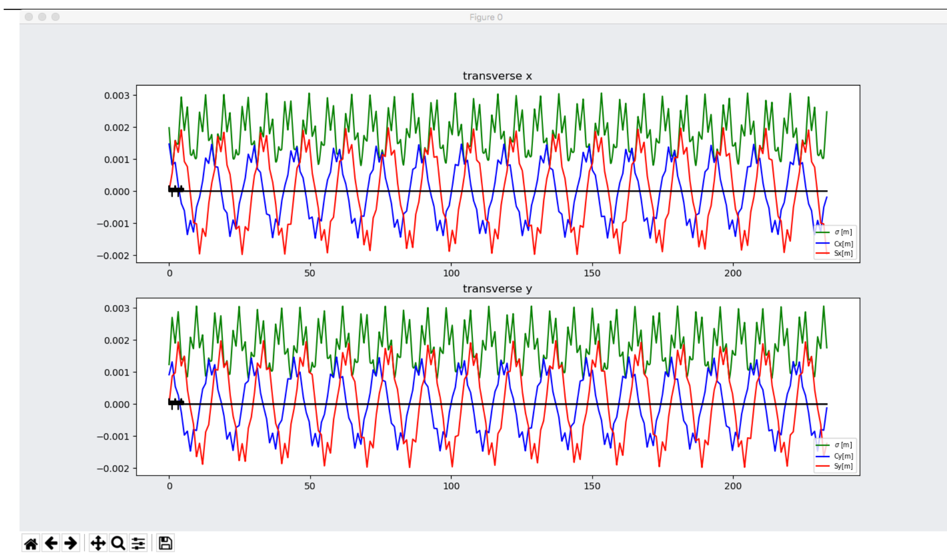
The X envelope has a local maximum where κ_x is focusing and a local minimum where it is defocusing. When the beam is *matched*, the maxima and minima will have the same magnitude along the channel, i.e. $X_{max}(s) = X_{max}(s + S)$ etc. For an *unmatched* beam, there will be a slow variation of the envelope function with a wavelength that is generally large compared to the period L. Superimposed on this slow variation is a *ripple* of short wavelength that exhibits the periodic structure of the channel.

As demonstration the next two figures show SIMULINAC-plots for a proton beam with 5 MeV kin. energy and emittance 1 $mm \times mrad$ in a long FODO-channel. The focussing gradients are set to 6.7 [T/m]. No acceleration applied.

Matched beam

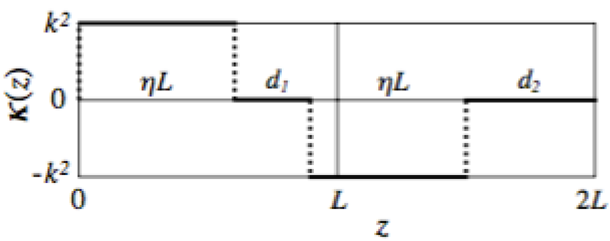


Unmatched beam



In what follows I consider only the first (the largest) solution band of phase advance $\sigma < 180^o$. The FODO model is described for the xz-plane:

$$\kappa_x(s) \equiv \begin{cases} +\kappa_x, & 0 < s < \eta L \\ 0, & \eta L < s < \eta L + d_1 \\ -\kappa_x, & \eta L + d_1 < s < 2\eta L + d_1 \\ 0, & 2L - d_2 < s < 2L \end{cases} \tag{13}$$



Since the FODO lattice cell has equal focus and defocus length, the fields have antisymmetry about each gap center. For a matched beam, this yields a relationship between the envelopes $X(s)$ and $Y(s)$ in the xz- ad yz-planes respectively.

$$Y(s) = X(2s_c - s) \tag{14}$$

wehere s_c is the center of any gap. Therefore it is sufficient to analyze $X(s)$ only in what follows.

For a beam with emittance ϵ, negligible space charge and arbitrary periodic focus function $f(s)$, the xz-plane envelope function $X(s)$ is determined by:

$$X(s)'' + f(s)X - \epsilon^2/X^3 = 0 \tag{15}$$

Without space charge the beam distribution may be KV or a class of physically realistic distributions. After lengthy caclulations - not presented here - the result for an arbitrary point z in the first (focus) segment is the exact focus-segment envelope $X(0 < s < \eta L)$:

$$X^2(\varphi, s) = \epsilon\eta L \frac{F(\varphi, s)}{\varphi\sqrt{1 - \cos(\sigma)^2}} \tag{16}$$

$$\begin{aligned} F(\varphi, s) = (ch + \nu \times sh) \times sn + \mu \times \nu \times sh \times sin[\varphi(1 - 2s/(\eta L)] \\ +B \times cs + (B + sh) \times cos[\varphi(1 - 2s/(etaL)] \end{aligned} \tag{17}$$

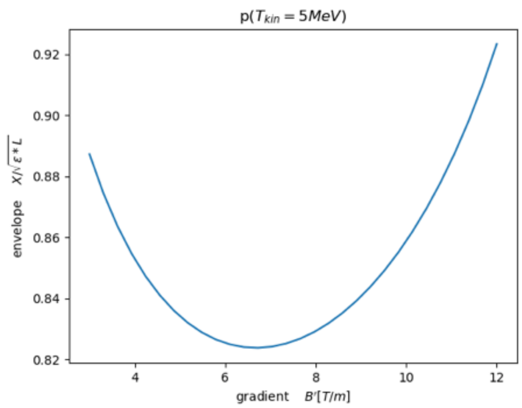
$$B = \nu \times ch + \frac{1 - \mu^2}{2} \times \nu^2 \times sh \tag{18}$$

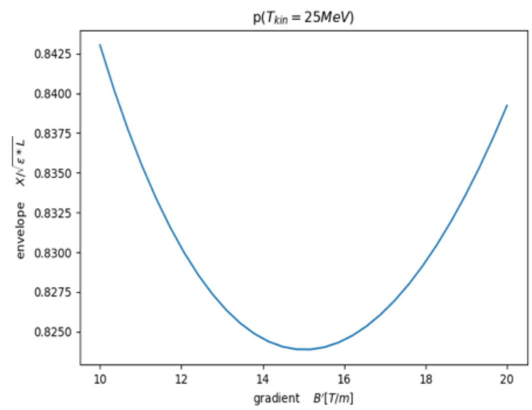
The following definitions have been used:

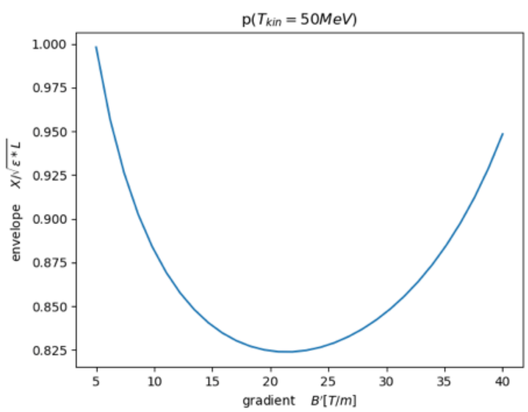
$$\begin{aligned}
&\varphi = \eta L \times k, \quad k = \sqrt{\kappa}, \\
&sn = sin(\varphi), \quad cs = cos(\varphi) \\
&sh = sinh(\varphi), \quad ch = cosh(\varphi), \\
&cos(\sigma) = (ch + \nu \times sh) \times cs - B \times sn, \\
&\nu = k(1-\eta)L, \quad \mu = \frac{d_2 - d_1}{2}, \\
&d = \frac{d_2 + d_1}{2} = (1-\eta)L, \quad d_1 = d(1-\mu), \quad d_2 = d(1+\mu).
\end{aligned} \tag{19}$$

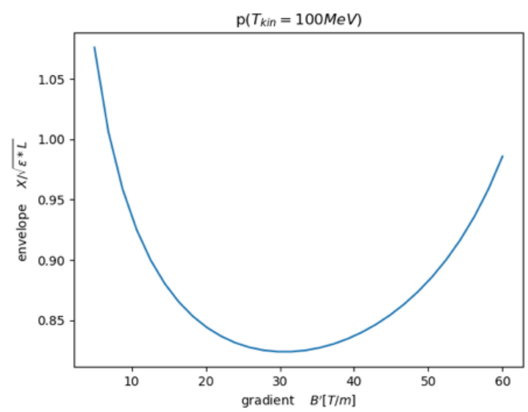
3.1 The Alceli Lattice

The Alceli lattice as it has been studied so far uses a symmetric FODO lattice for tranverse focusing. Evaluation of the above formulas for this case give the results:









The figures show the normalized envelope $X/\sqrt{\epsilon L}$ as function of focusing strength for differen kinetic energies. The envlopes have minima which are all ≈ 0.82 for optimal betatron phase advances $\sigma \approx 48^o$ but at very different focusing gradients.

W [MeV]	σ [deg]	X_n	B' [T/m]
5	49	0,82	6,7
25	49	0,82	15
50	48	0,82	21
100	48	0.82	30

Conclusion: To keep the transverse beam dimension at a minimum, the betatron phase advance per lattice cell should be kept at $\approx 48^o$ and the focusing strength of the quadrupoles have to be increased with increasing energy.