

On Longitudinal Emittance

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1 Overview

The standard formula for an upright ellipse in phase-space $\Delta\phi \otimes w$ is:

$$\frac{\Delta\phi^2}{\Delta\phi_0^2} + \frac{w^2}{w_0^2} = 1 \quad (1)$$

with $\Delta\phi = \phi - \phi_s$ and $w \equiv \delta\gamma = \Delta W/mc^2$. ϕ_s being the synchronous phase, mc^2 the rest energy, W the total energy and γ the Lorentz factor. It has the emittance

$$\epsilon_w = \Delta\phi_0 w_0 \quad (2)$$

and units [rad]. The ellipse intersects the $\Delta\phi$ -axis at $\Delta\phi_0$ and the w -axis at w_0 . The intersection with the w -axis determines the β -function by the relation $\beta_w = \epsilon_w/w_0^2$. Its units are [rad].

Let's change to new coordinates, for instance the pair of canonical variables $z \otimes \Delta p/p$, as it is used internally in Trace 3D. The transformation from old to new coordinates is: $z = \kappa|\Delta\phi| = \frac{\beta\lambda}{2\pi}|\Delta\phi|$ and $\Delta p/p = \tau w = \gamma/(\gamma^2 - 1)w = (\gamma\beta^2)^{-1}w$. This gives the modified ellipse equation:

$$\frac{z^2}{(\kappa\Delta\phi_0)^2} + \frac{(\Delta p/p)^2}{(\tau w_0)^2} = 1 \quad (3)$$

which has the transformed emittance

$$\epsilon_z = \kappa|\Delta\phi_0|\tau w_0 = \kappa\tau\epsilon_w = \frac{\beta\lambda}{2\pi}\gamma/(\gamma^2 - 1)\epsilon_w = \frac{\lambda}{2\pi\gamma\beta}\epsilon_w, \quad (4)$$

with units [$m \times rad$]. Again the β -function is given by

$$\beta_z = \epsilon_z/(\Delta p/p)_0^2 = \kappa\tau\epsilon_w/(\tau w_0)^2 = \kappa/\tau \times \beta_w = \frac{\beta\lambda}{2\pi} \frac{\gamma^2 - 1}{\gamma} \beta_w, \quad (5)$$

with units [m].

For the $\Delta\phi \otimes \Delta W$ phase space, because $\Delta W = mc^2 w$, we have $\kappa = 1$ and $\tau = mc^2$. So that

$$\epsilon_W = mc^2 \epsilon_w \quad [rad \times eV] \quad (6)$$

$$\beta_W = 1/mc^2 \beta_w \quad [rad/eV] \quad (7)$$

Finally for the $\Delta z \otimes \Delta W$ phase space we get the emittance

$$\epsilon_{zW} = \frac{\beta\lambda}{2\pi} mc^2 \epsilon_w \quad [m \times eV] \quad (8)$$

$$\beta_{zW} = \frac{\beta\lambda}{2\pi} \frac{1}{mc^2} \beta_w \quad [m/eV] \quad (9)$$

The ESS conceptual design report uses the $z \otimes z'$ phase space, i.e. the emittance $\epsilon_{zz'}$. Since $\delta\gamma = w = \beta^2 \gamma^3 \delta\beta/\beta = \beta^2 \gamma^3 z'$ and $\Delta\phi = \frac{2\pi}{\beta\lambda} z$ we have:

$$\epsilon_{zz'} = \frac{\lambda}{2\pi\beta\gamma^3} \epsilon_w \quad (10)$$

with units $[m \times rad]$.

Instead of longitudinal position some people use arrival time. For the $\Delta t \otimes \Delta W$ phase space we use $|\Delta t| = (\beta c)^{-1} |\Delta z| = (\beta c)^{-1} \frac{\beta\lambda}{2\pi} |\Delta\phi|$ and get

$$\epsilon_{tW} = \frac{\lambda}{2\pi c} mc^2 \epsilon_w \quad [sec \times eV] \quad (11)$$

2 Full Treatment

The ellipse in normal form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (12)$$

defines the emittance ϵ as:

$$\epsilon = a * b \quad (13)$$

Changing scales of x, y coordinates: $x = x'/\kappa$ and $y = y'/\tau$ and inserting into normal form:

$$\frac{x'^2}{(a\kappa)^2} + \frac{y'^2}{(b\tau)^2} = 1 \quad (14)$$

gives scaled emittance ϵ'

$$\epsilon' = (a\kappa) * (b\tau) = \kappa\tau\epsilon \quad (15)$$

Let $\beta_x = x_0^2/\epsilon$ then $\beta_x = (x'/\kappa)^2/(\epsilon'/\kappa\tau) = \frac{\tau}{\kappa} \beta'_x$, we get

$$\beta'_x = \frac{\kappa}{\tau} \beta_x \quad (16)$$

In phase space $\Delta\phi$, z and Δt are usually used as abscissa and w , ΔW , $\Delta p/p$ and z' as ordinates. We use κ to connect abscissa and τ to connect different ordinates. Six different combinations of abscissa can be made and 11 combinations for ordinates. Their corresponding κ - and τ -values are assembled in the following tables.

| κ-values | | | |
|-----------------------------------|------------------------------|----------------|----------------------------|
| wanted↓ in terms of→ | $\Delta\phi$ [rad] | z [m] | Δt [sec] |
| $\Delta\phi$ [rad] | 1 | $2\pi/\lambda$ | $2\pi\beta c/\beta\lambda$ |
| z [m] | $\beta\lambda/2\pi$ | 1 | βc |
| Δt [sec] | $\beta\lambda/(2\pi\beta c)$ | $1/\beta c$ | 1 |

| τ-values | | | | |
|--|---------------------------------|------------------------------|---------------------|-------------------------|
| wanted↓ in terms of → | $\delta\gamma = w$ | ΔW [eV] | $\Delta p/p$ | z' [rad] |
| $\delta\gamma = w$ | 1 | $1/(mc^2)$ | $\gamma\beta^2$ | $\gamma(\gamma\beta)^2$ |
| ΔW [eV] | mc^2 | 1 | $mc^2\gamma\beta^2$ | $mc^2\gamma^3\beta^2$ |
| $\Delta p/p$ | $(\gamma\beta^2)^{-1}$ | $(mc^2\gamma\beta^2)^{-1}$ | 1 | γ^2 |
| z' [rad] | $\gamma^{-1}(\gamma\beta)^{-2}$ | $(mc^2\gamma^3\beta^2)^{-1}$ | γ^{-2} | 1 |
| with $W = mc^2(\gamma - 1)$ as kinetic energy. | | | | |

Example: Phase space $z \otimes \Delta W$ in terms of $\Delta\phi \otimes \delta\gamma$: $\kappa = \beta\lambda/2\pi$, $\tau = mc^2$.

$$\epsilon_{zW} = \kappa\tau\epsilon_w = (\beta\lambda/2\pi)mc^2\epsilon_w. \quad (17)$$

$$\beta_{zW} = \frac{\kappa}{\tau}\beta_w = \frac{\beta\lambda/2\pi}{mc^2}\beta_w. \quad (18)$$

More interesting details about emittance definitions, normalized and unnormalized, and their units can be found in the UserManual of the *TraceWin* program.

3 Twiss Parameter Values

To simplify we assume the twiss parameter $\alpha = 0$. The twiss parameter γ then reduces to $1/\beta$ and only two free parameters ϵ and β remain to describe the ellipse in phase space completely.

For the longitudinal dynamics in the passage of an rf-gap the intersection w_0 on the w -axis is given by

$$w_0 = \frac{\Delta W}{mc^2} = \Delta\phi_0 \sqrt{qE_0 L T \beta_s^3 \gamma_s^3 \lambda \sin(-\phi_s) / 2\pi mc^2} \quad (19)$$

If w_0 is given $\Delta\phi_0$ follows from (19) *and vice versa*. Putting $w_0 = \epsilon_w/\Delta\phi_0$ into (19) we get

$$\Delta\phi_0 = \sqrt{\epsilon_w / \sqrt{qE_0LT\beta_s^3\gamma_s^3\lambda\sin(-\phi_s)}/2\pi mc^2} \quad (20)$$

and from (20) we get finally

$$\gamma_0 = \epsilon_w/\Delta\phi_0^2 = \sqrt{qE_0LT\beta_s^3\gamma_s^3\lambda\sin(-\phi_s)}/2\pi mc^2 \quad (21)$$

and

$$\beta_0 = 1/\gamma_0 \quad (22)$$

NOTE: the two twiss parameters γ_0 and β_0 are completely defined by the emittance ϵ_w and the cavity field E_0 , rf-phase ϕ_s , rf-wavelength λ and particle impuls $\sim \gamma\beta$.

4 Appendix

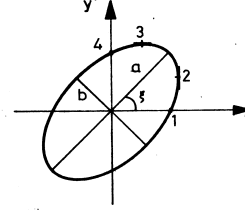
4.1 SIMULAC Variables

Table 1: **Variable Names**

| | | | | |
|--------------------------------|-------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\epsilon_w = \text{emitw}$ | $\Delta\phi = \text{Dphi}$ | $\Delta\phi_0 = \text{Dphi0}$ | $w = \text{w}$ | $w_0 = \text{w0}$ |
| $\epsilon_W = \text{emitW}$ | $\Delta z = \text{z}$ | $\Delta W = \text{DW}$ | $\Delta p/p = \text{Dp2p}$ | $\Delta p/p_0 = \text{Dp2p0}$ |
| $\epsilon_z = \text{emitz}$ | $\beta_z = \text{betaz}$ | $\gamma_z = \text{gammaz}$ | $\alpha_z = \text{alphaz}$ | $\lambda = \text{lamb}$ |
| $Ez_{avg} = \text{EzAvg}$ | $Ez_{peak} = \text{EzPeak}$ | $\phi_+ = \text{phi.1}$ | $\phi_- = \text{phi.2}$ | $\psi = \text{psi}$ |
| $\gamma = \text{gamma}$ | $\gamma\beta = \text{gb}$ | $\beta = \text{beta}$ | $E_0T = \text{E0T}$ | $E_0LT = \text{E0LT}$ |
| $mc^2 = \text{m0c2}$ | $mc^3 = \text{m0c3}$ | $\epsilon_{xi} = \text{emitx.i}$ | $\epsilon_{yi} = \text{emity.i}$ | $\epsilon_{zi} = \text{emitz.i}$ |
| $\beta_{xi} = \text{betax.i}$ | $\beta_{yi} = \text{betay.i}$ | $\alpha_{xi} = \text{alfax.i}$ | $\alpha_{yi} = \text{alfay.i}$ | $\gamma_{xi} = \text{gamax.i}$ |
| $\gamma_{yi} = \text{gamay.i}$ | $\omega = \text{omg}$ | $\phi = \text{phi}$ | $\phi_s = \text{phis}$ | |

4.2 Relations Between Ellipse and Twiss Parameters

5.4 Geometrical properties of the ellipse



| | $\alpha, \beta, \gamma, \epsilon$ | c_1, c_2, c_3, c_4 | L, S, ϵ |
|-------------|--|---|---|
| | $\beta\gamma - \alpha^2 = 1$ $\Pi = 1/2(\beta + \gamma)$ | $\epsilon = c_1c_4 - c_2c_3$ $\Pi = 1/2(c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$ | $H = \frac{1}{2L}(L^2 + S^2 + 1)$ |
| y_1 | $\sqrt{\epsilon/\gamma}$ | $\epsilon/\sqrt{c_3^2 + c_4^2}$ | $\sqrt{\epsilon/L}$ |
| y_2 | $\sqrt{\epsilon\beta}$ | $\sqrt{c_1^2 + c_2^2}$ | $\sqrt{\epsilon/L} \sqrt{S^2 + L^2}$ |
| y_3 | $-\alpha\sqrt{\epsilon/\beta}$ | $(c_1c_3 + c_2c_4)/\sqrt{c_1^2 + c_2^2}$ | $S\sqrt{\epsilon/L} / \sqrt{S^2 + L^2}$ |
| y_4 | $-\alpha\sqrt{\epsilon/\gamma}$ | $(c_1c_3 + c_2c_4)/\sqrt{c_3^2 + c_4^2}$ | $S\sqrt{\epsilon/L}$ |
| y_5 | $\sqrt{\epsilon\gamma}$ | $\sqrt{c_3^2 + c_4^2}$ | $\sqrt{\epsilon/L}$ |
| y_6 | $\sqrt{\epsilon/\beta}$ | $\epsilon/\sqrt{c_1^2 + c_2^2}$ | $\sqrt{\epsilon/L} / \sqrt{S^2 + L^2}$ |
| a | $\sqrt{\epsilon/2} (\sqrt{\Pi+1} + \sqrt{\Pi-1})$ | | |
| b | $\sqrt{2\epsilon}/(\sqrt{\Pi+1} + \sqrt{\Pi-1}) = \sqrt{\epsilon/2} (\sqrt{\Pi+1} - \sqrt{\Pi-1})$ | | |
| $a/b > 1$ | $\Pi + \sqrt{\Pi^2 - 1}$ | | |
| $\tan \xi$ | $[-\alpha(H + \sqrt{\Pi^2 - 1})] / [\beta(H + \sqrt{\Pi^2 - 1}) - 1]$ | $[c_2 + c_3(H + \sqrt{\Pi^2 - 1})] / [c_1(H + \sqrt{\Pi^2 - 1}) - c_4]$ | $S/[L(H + \sqrt{\Pi^2 - 1}) - 1]$ |
| $\sin 2\xi$ | $-\alpha/\sqrt{\Pi^2 - 1}$ | $(c_1c_3 + c_2c_4)/\epsilon\sqrt{\Pi^2 - 1}$ | $S/L\sqrt{\Pi^2 - 1}$ |
| $\cos 2\xi$ | $(\beta - \gamma)/2\sqrt{\Pi^2 - 1}$ | $(c_1^2 + c_2^2 - c_3^2 - c_4^2)/2\epsilon\sqrt{\Pi^2 - 1}$ | $(L^2 + S^2 - 1)/2L\sqrt{\Pi^2 - 1}$ |
| $\tan 2\xi$ | $-2\alpha/(\beta - \gamma)$ | $2(c_1c_3 + c_2c_4)/(c_1^2 + c_2^2 - c_3^2 - c_4^2)$ | $2S/(L^2 + S^2 - 1)$ |