On Longitudinal Emittance

W.D. Klotz, wdklotz@alecli.com

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1 Emittance Conversions

The standard formula for an upright ellipse in phase-space $\Delta\phi \otimes w$ is:

$$\frac{\Delta\phi^2}{\Delta\phi_0^2} + \frac{w^2}{w_0^2} = 1 \tag{1}$$

with $\Delta\phi = \phi - \phi_s$ and $w \equiv \delta\gamma = \Delta W/mc^2$.

ϕ_s being the synchronous phase, mc^2 the rest energy, W the total energy and γ the relativistic factor. It has the emittance

$$\epsilon_w = \Delta\phi_0 w_0 \tag{2}$$

and units [rad].

The ellipse intersetcs the $\Delta\phi$-axis at $\Delta\phi_0$ and the w-axis at w_0.

Let's change to new coordinates, for instance the pair of canonical variables $\Delta z \otimes \Delta p/p$, as it is used internally in Trace 3D. The transformation from old to new coordinates is: $|\Delta z| = \kappa|\Delta\phi| = \frac{\beta\lambda}{2\pi}|\Delta\phi|$ and $\Delta p/p = \tau w = \gamma/(\gamma^2-1)w = (\gamma\beta^2)^{-1}w$. This gives the modified ellipse equation:

$$\frac{\Delta z^2}{(\kappa\Delta\phi_0)^2} + \frac{(\Delta p/p)^2}{(\tau w_0)^2} = 1 \tag{3}$$

which has the transformed emittance

$$\epsilon_z = \kappa\Delta\phi_0\tau w_0 = \kappa\tau\epsilon_w = \frac{\beta\lambda}{2\pi}\gamma/(\gamma^2-1)\epsilon_w = \frac{\lambda}{2\pi\gamma\beta}\epsilon_w, \tag{4}$$

with units $[m]$.

For the $\Delta\phi \otimes \Delta W$ phase space, because $\Delta W = mc^2 w$, we have $\kappa = 1$ and $\tau = mc^2$. So that

$$\epsilon_W = mc^2\epsilon_w \tag{5}$$

with units $[rad * eV]$.

Finally for the $\Delta z \otimes \Delta W$ phase space we get the emittance

$$\epsilon_{zW} = \frac{\beta\lambda}{2\pi} mc^2 \epsilon_w \tag{6}$$

with units $[m * eV]$.

2 Twiss Parameter Values

To simplify we assume the twiss parameter $\alpha = 0$. The twiss parameter γ then reduces to $1/\beta$ and only two free parameters ϵ and β remain to describe the ellipse in phase space completely.

For the longitudinal dynamics in the passage of an rf-gap the intersection w_0 on the w-axis is given by

$$w_0 = \frac{\Delta W}{mc^2} = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s) \Delta\phi_0^2 / 2\pi mc^2} \tag{7}$$

$$= \Delta\phi_0 \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s) / 2\pi mc^2} \tag{8}$$

If w_0 is given $\Delta\phi_0$ follows from (8) *and vice versa.* Putting $w_0 = \epsilon_w / \Delta\phi_0$ into (8) we get

$$\Delta\phi_0 = \sqrt{\epsilon_w / \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s) / 2\pi mc^2}} \tag{9}$$

and from (9) we get finally

$$\gamma_0 = \epsilon_w / \Delta\phi_0^2 = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda sin(-\phi_s) / 2\pi mc^2} \tag{10}$$

and

$$\beta_0 = 1/\gamma_0 \tag{11}$$

NOTE: the two twiss parameters γ_0 and β_0 are independent from the emittance ϵ_w and completely defined by the cavity field E_0, rf-phase ϕ_s, rf-wavelength λ and particle impuls $\sim \gamma\beta$.

3 Appendix

Table 1: SIMULAC variable names

ϵ_w = emitw	$\Delta\phi$ = Dphi	$\Delta\phi_0$ = Dphi0	w = w	w_0 = w0
ϵ_W = emitW	Δz = z	ΔW = DW	$\Delta p/p$ = Dp2p	$\Delta p/p_0$ = Dp2p0
ϵ_z = emitz	β_z = betaz	γ_z = gammaz	α_z = alphaz	λ = lamb
Ez_{avg} = EzAvg	Ez_{peak} = EzPeak	ϕ_+ = `phi_1`	ϕ_- = `phi_2`	ψ = psi
γ = gamma	$\gamma\beta$ = gb	β = beta	E_0T = E0T	mc^3 = m0c3
mc^2 = m0c2	ϵ_{xi} = `emitx_i`	ϵ_{yi} = `emity_i`	ϵ_{zi} = `emitz_i`	β_{xi} = `betax_i`
β_{yi} = `betay_i`	α_{xi} = `alfax_i`	α_{yi} = `alfay_i`	γ_{xi} = `gamax_i`	γ_{yi} = `gamay_i`
ω = omg	ϕ = phi	ϕ_s = phis		