

On Longitudinal Emittance

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1 Overview

The standard formula for an upright ellipse in phase-space $\Delta\phi \otimes w$ is:

$$\frac{\Delta\phi^2}{\Delta\phi_0^2} + \frac{w^2}{w_0^2} = 1 \quad (1)$$

with $\Delta\phi = \phi - \phi_s$ and $w \equiv \delta\gamma = \Delta W/mc^2$. ϕ_s being the synchronous phase, mc^2 the rest energy, W the total energy and γ the Lorentz factor. It has the emittance

$$\epsilon_w = \Delta\phi_0 w_0 \quad (2)$$

and units [rad]. The ellipse intersects the $\Delta\phi$ -axis at $\Delta\phi_0$ and the w -axis at w_0 . The intersection with the w -axis determines the β -function by the relation $\beta_w = \epsilon_w/w_0^2$. Its units are [rad].

Let's change to new coordinates, for instance the pair of canonical variables $\Delta z \otimes \Delta p/p$, as it is used internally in Trace 3D. The transformation from old to new coordinates is: $|\Delta z| = \kappa|\Delta\phi| = \frac{\beta\lambda}{2\pi}|\Delta\phi|$ and $\Delta p/p = \tau w = \gamma/(\gamma^2 - 1)w = (\gamma\beta^2)^{-1}w$. This gives the modified ellipse equation:

$$\frac{\Delta z^2}{(\kappa\Delta\phi_0)^2} + \frac{(\Delta p/p)^2}{(\tau w_0)^2} = 1 \quad (3)$$

which has the transformed emittance

$$\epsilon_z = \kappa\Delta\phi_0\tau w_0 = \kappa\tau\epsilon_w = \frac{\beta\lambda}{2\pi}\gamma/(\gamma^2 - 1)\epsilon_w = \frac{\lambda}{2\pi\gamma\beta}\epsilon_w, \quad (4)$$

with units [$m \times rad$]. Again the β -function is given by

$$\beta_z = \epsilon_z/(\Delta p/p)_0^2 = \kappa\tau\epsilon_w/(\tau w_0)^2 = \kappa/\tau \times \beta_w = \frac{\beta\lambda}{2\pi} \frac{\gamma^2 - 1}{\gamma} \beta_w, \quad (5)$$

with units [m].

For the $\Delta\phi \otimes \Delta W$ phase space, because $\Delta W = mc^2 w$, we have $\kappa = 1$ and $\tau = mc^2$. So that

$$\epsilon_W = mc^2 \epsilon_w \quad [rad \times eV] \quad (6)$$

$$\beta_W = 1/mc^2 \beta_w \quad [rad/eV] \quad (7)$$

Finally for the $\Delta z \otimes \Delta W$ phase space we get the emittance

$$\epsilon_{zW} = \frac{\beta\lambda}{2\pi} mc^2 \epsilon_w \quad [m \times eV] \quad (8)$$

$$\beta_{zW} = \frac{\beta\lambda}{2\pi} \frac{1}{mc^2} \beta_w \quad [m/eV] \quad (9)$$

The ESS conceptual design report uses the $z \otimes z'$ phase space, i.e. the emittance $\epsilon_{zz'}$. Since $\delta\gamma = w = \beta^2 \gamma^3 \delta\beta/\beta = \beta^2 \gamma^3 z'$ and $\Delta\phi = \frac{2\pi}{\beta\lambda} z$ we have:

$$\epsilon_{zz'} = \frac{\lambda}{2\pi\beta\gamma^3} \epsilon_w \quad (10)$$

with units $[m \times rad]$.

Instead of longitudinal position some people use arrival time. For the $\Delta t \otimes \Delta W$ phase space we use $|\Delta t| = (\beta c)^{-1} |\Delta z| = (\beta c)^{-1} \frac{\beta\lambda}{2\pi} |\Delta\phi|$ and get

$$\epsilon_{tW} = \frac{\lambda}{2\pi c} mc^2 \epsilon_w \quad [sec \times eV] \quad (11)$$

2 Full Treatment

The ellipse in normal form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (12)$$

defines the emittance ϵ as:

$$\epsilon = a * b \quad (13)$$

Changing scales of x, y coordinates: $x = x'/\kappa$ and $y = y'/\tau$ and inserting into normal form:

$$\frac{x'^2}{(a\kappa)^2} + \frac{y'^2}{(b\tau)^2} = 1 \quad (14)$$

gives scaled emittance ϵ'

$$\epsilon' = (a\kappa) * (b\tau) = \kappa\tau\epsilon \quad (15)$$

Let $\beta_x = x_0^2/\epsilon$ then $\beta_x = (x'/\kappa)^2/(\epsilon'/\kappa\tau) = \frac{\tau}{\kappa} \beta'_x$, we get

$$\beta'_x = \frac{\kappa}{\tau} \beta_x \quad (16)$$

In phase space $\Delta\phi$, z and Δt are usually used as abscissa and w , ΔW , $\Delta p/p$ and z' as ordinates. We use κ to connect abscissa and τ to connect different ordinates. Six different combinations of abscissa can be made and 11 combinations for ordinates. Their corresponding κ - and τ -values are assembled in the following tables.

Table of κ -values:

wanted↓ in terms of→	$\Delta\phi$ [rad]	z [m]	Δt [sec]
$\Delta\phi$ [rad]	1	$2\pi/\lambda$	$2\pi\beta c/\beta\lambda$
z [m]	$\beta\lambda/2\pi$	1	βc
Δt [sec]	$\beta\lambda/(2\pi\beta c)$	$1/\beta c$	1

Table of τ -values:

wanted↓ in terms of→	$\delta\gamma = w$	ΔW [eV]	$\Delta p/p$	z' [rad]
$\delta\gamma = w$	1	$1/(m_0c^2)$	$\gamma\beta^2$	$\gamma(\gamma\beta)^2$
ΔW [eV]	m_0c^2	1	$m_0c^2\gamma\beta^2$	$m_0c^2\gamma^3\beta^2$
$\Delta p/p$	$(\gamma\beta^2)^{-1}$	$(m_0c^2\gamma\beta^2)^{-1}$	1	γ^2
z' [rad]	$\gamma^{-1}(\gamma\beta)^{-2}$	$(m_0c^2\gamma^3\beta^2)^{-1}$	γ^{-2}	1
with $W = m_0c^2(\gamma - 1)$ as kinetic energy.				

Example: Phase space $[z \otimes \Delta W]$ in terms of $[\Delta\phi \otimes \delta\gamma]$. $\kappa = \beta\lambda/2\pi$. $\tau = m_0c^2$.

$$\epsilon_{zW} = \kappa\tau\epsilon_w = (\beta\lambda/2\pi)m_0c^2\epsilon_w. \quad (17)$$

$$\beta_{zW} = \frac{\kappa}{\tau}\beta_w = \frac{\beta\lambda/2\pi}{m_0c^2}\beta_w. \quad (18)$$

More interesting details about emittance definitions, normalized and unnormalized, and their units can be found in the UserManual of the *TraceWin* program.

3 Twiss Parameter Values

To simplify we assume the twiss parameter $\alpha = 0$. The twiss parameter γ then reduces to $1/\beta$ and only two free parameters ϵ and β remain to describe the ellipse in phase space completely.

For the longitudinal dynamics in the passage of an rf-gap the intersection w_0 on the w-axis is given by

$$w_0 = \frac{\Delta W}{mc^2} = \sqrt{qE_0T\beta_s^3\gamma_s^3\lambda\sin(-\phi_s)\Delta\phi_0^2/2\pi mc^2} \quad (19)$$

$$= \Delta\phi_0\sqrt{qE_0T\beta_s^3\gamma_s^3\lambda\sin(-\phi_s)/2\pi mc^2} \quad (20)$$

If w_0 is given $\Delta\phi_0$ follows from (20) *and vice versa*. Putting $w_0 = \epsilon_w/\Delta\phi_0$ into (20) we get

$$\Delta\phi_0 = \sqrt{\epsilon_w / \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda \sin(-\phi_s) / 2\pi m c^2}} \quad (21)$$

and from (21) we get finally

$$\gamma_0 = \epsilon_w / \Delta\phi_0^2 = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda \sin(-\phi_s) / 2\pi m c^2} \quad (22)$$

and

$$\beta_0 = 1/\gamma_0 \quad (23)$$

NOTE: the two twiss parameters γ_0 and β_0 are completely defined by the emittance ϵ_w and the cavity field E_0 , rf-phase ϕ_s , rf-wavelength λ and particle impuls $\sim \gamma\beta$.

4 Appendix

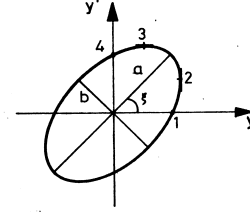
4.1 SIMULAC Variables

Table 1: variable names

$\epsilon_w = \text{emitw}$	$\Delta\phi = \text{Dphi}$	$\Delta\phi_0 = \text{Dphi0}$	$w = w$	$w_0 = w0$
$\epsilon_W = \text{emitW}$	$\Delta z = z$	$\Delta W = \text{DW}$	$\Delta p/p = \text{Dp2p}$	$\Delta p/p_0 = \text{Dp2p0}$
$\epsilon_z = \text{emitz}$	$\beta_z = \text{betaz}$	$\gamma_z = \text{gammaz}$	$\alpha_z = \text{alphaz}$	$\lambda = \text{lamb}$
$Ez_{avg} = \text{EzAvg}$	$Ez_{peak} = \text{EzPeak}$	$\phi_+ = \text{phi_1}$	$\phi_- = \text{phi_2}$	$\psi = \text{psi}$
$\gamma = \text{gamma}$	$\gamma\beta = \text{gb}$	$\beta = \text{beta}$	$E_0T = \text{E0T}$	$mc^3 = \text{m0c3}$
$mc^2 = \text{m0c2}$	$\epsilon_{xi} = \text{emitx_i}$	$\epsilon_{yi} = \text{emity_i}$	$\epsilon_{zi} = \text{emitz_i}$	$\beta_{xi} = \text{betax_i}$
$\beta_{yi} = \text{betay_i}$	$\alpha_{xi} = \text{alfax_i}$	$\alpha_{yi} = \text{alfay_i}$	$\gamma_{xi} = \text{gamax_i}$	$\gamma_{yi} = \text{gamay_i}$
$\omega = \text{omg}$	$\phi = \text{phi}$	$\phi_s = \text{phis}$		

4.2 Relations Between Ellipse and Twiss Parameters

5.4 Geometrical properties of the ellipse



	$\alpha, \beta, \gamma, \epsilon$	c_1, c_2, c_3, c_4	L, S, ϵ
	$\beta\gamma - \alpha^2 = 1$ $H = 1/2(\beta + \gamma)$	$\epsilon = c_1c_4 - c_2c_3$ $H = 1/2(c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$	$H = \frac{1}{2L}(L^2 + S^2 + 1)$
y_1	$\sqrt{\epsilon/\gamma}$	$\epsilon/\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
y_2	$\sqrt{\epsilon\beta}$	$\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L} \sqrt{S^2 + L^2}$
y_3	$-\alpha\sqrt{\epsilon/\beta}$	$(c_1c_3 + c_2c_4)/\sqrt{c_1^2 + c_2^2}$	$S\sqrt{\epsilon/L} / \sqrt{S^2 + L^2}$
y_4	$-\alpha\sqrt{\epsilon/\gamma}$	$(c_1c_3 + c_2c_4)/\sqrt{c_3^2 + c_4^2}$	$S\sqrt{\epsilon/L}$
y_5	$\sqrt{\epsilon\gamma}$	$\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
y_6	$\sqrt{\epsilon/\beta}$	$\epsilon/\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L} \sqrt{S^2 + L^2}$
a	$\sqrt{\epsilon/2} (\sqrt{H+1} + \sqrt{H-1})$		
b	$\sqrt{2\epsilon} / (\sqrt{H+1} + \sqrt{H-1}) = \sqrt{\epsilon/2} (\sqrt{H+1} - \sqrt{H-1})$		
$a/b > 1$	$H + \sqrt{H^2 - 1}$		
$\tan \xi$	$[-\alpha(H + \sqrt{H^2 - 1})] / [\beta(H + \sqrt{H^2 - 1}) - 1]$	$[c_2 + c_3(H + \sqrt{H^2 - 1})] / [c_1(H + \sqrt{H^2 - 1}) - c_4]$	$S / [L(H + \sqrt{H^2 - 1}) - 1]$
$\sin 2\xi$	$-\alpha/\sqrt{H^2 - 1}$	$(c_1c_3 + c_2c_4)/\epsilon\sqrt{H^2 - 1}$	$S/L\sqrt{H^2 - 1}$
$\cos 2\xi$	$(\beta - \gamma)/2\sqrt{H^2 - 1}$	$(c_1^2 + c_2^2 - c_3^2 - c_4^2)/2\epsilon\sqrt{H^2 - 1}$	$(L^2 + S^2 - 1)/2L\sqrt{H^2 - 1}$
$\tan 2\xi$	$-2\alpha/(\beta - \gamma)$	$2(c_1c_3 + c_2c_4)/(c_1^2 + c_2^2 - c_3^2 - c_4^2)$	$2S/(L^2 + S^2 - 1)$