

# On Longitudinal Emittance

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## 1 Overview

The standard formula for an upright ellipse in phase-space  $\Delta\phi \otimes w$  is:

$$\frac{\Delta\phi^2}{\Delta\phi_0^2} + \frac{w^2}{w_0^2} = 1 \quad (1)$$

with  $\Delta\phi = \phi - \phi_s$  and  $w \equiv \delta\gamma = \Delta W/mc^2$ .  $\phi_s$  being the synchronous phase,  $mc^2$  the rest energy,  $W$  the total energy and  $\gamma$  the Lorentz factor. It has the emittance

$$\epsilon_w = |\Delta\phi_0 w_0| \quad (2)$$

and units [rad]. The ellipse intersects the  $\Delta\phi$ -axis at  $\Delta\phi_0$  and the  $w$ -axis at  $w_0$ . The intersection with the  $w$ -axis determines the  $\beta$ -function by the relation  $\beta_w = \epsilon_w/w_0^2$ . Its units are [rad].

Let's change to new coordinates, for instance the pair of canonical variables  $z \otimes \Delta p/p$ , as it is used internally in the *TRACE 3-D* program. The transformation from old to new coordinates is:  $z = -\kappa\Delta\phi = -\frac{\beta\lambda}{2\pi}\Delta\phi$  and  $\Delta p/p = \tau w = \gamma/(\gamma^2 - 1)w = (\gamma\beta^2)^{-1}w$ . This gives the modified ellipse equation:

$$\frac{z^2}{(\kappa\Delta\phi_0)^2} + \frac{(\Delta p/p)^2}{(\tau w_0)^2} = 1 \quad (3)$$

which has the transformed emittance

$$\epsilon_z = \kappa|\Delta\phi_0| * \tau|w_0| = \kappa\tau\epsilon_w = \frac{\beta\lambda}{2\pi}\gamma/(\gamma^2 - 1)\epsilon_w = \frac{\lambda}{2\pi\gamma\beta}\epsilon_w, \quad (4)$$

with units [ $m \times rad$ ]. Again the  $\beta$ -function is given by

$$\beta_z = \epsilon_z/(\Delta p/p)_0^2 = \kappa\tau\epsilon_w/(\tau w_0)^2 = \kappa/\tau \times \beta_w = \frac{\beta\lambda}{2\pi} \frac{\gamma^2 - 1}{\gamma} \beta_w, \quad (5)$$

with units [ $m$ ].

For the  $\Delta\phi \otimes \Delta W$  phase space, because  $\Delta W = mc^2 w$ , we have  $\kappa = 1$  and  $\tau = mc^2$ . So that

$$\epsilon_W = mc^2 \epsilon_w \quad [rad \times eV] \quad (6)$$

$$\beta_W = 1/mc^2 \beta_w \quad [rad/eV] \quad (7)$$

Finally for the  $\Delta z \otimes \Delta W$  phase space we get the emittance

$$\epsilon_{zW} = \frac{\beta\lambda}{2\pi} mc^2 \epsilon_w \quad [m \times eV] \quad (8)$$

$$\beta_{zW} = \frac{\beta\lambda}{2\pi} \frac{1}{mc^2} \beta_w \quad [m/eV] \quad (9)$$

The ESS conceptual design report uses the  $z \otimes z'$  phase space, i.e. the emittance  $\epsilon_{zz'}$ . Since  $\delta\gamma = w = \beta^2\gamma^3\delta\beta/\beta = \beta^2\gamma^3z'$  and  $\Delta\phi = \frac{2\pi}{\beta\lambda}z$  we have:

$$\epsilon_{zz'} = \frac{\lambda}{2\pi\beta\gamma^3} \epsilon_w \quad (10)$$

with units  $[m \times rad]$ .

Instead of longitudinal position some people use arrival time. For the  $\Delta t \otimes \Delta W$  phase space we use  $\Delta t = -(\beta c)^{-1}z = (\beta c)^{-1}\frac{\beta\lambda}{2\pi}\Delta\phi$  and get

$$\epsilon_{tW} = \frac{\lambda}{2\pi c} mc^2 \epsilon_w \quad [sec \times eV] \quad (11)$$

## 2 Full Treatment

The ellipse in normal form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (12)$$

defines the emittance  $\epsilon$  as:

$$\epsilon = |a * b| \quad (13)$$

Changing scales of  $x, y$  coordinates:  $x = x'/\kappa$  and  $y = y'/\tau$  and inserting into normal form:

$$\frac{x'^2}{(a\kappa)^2} + \frac{y'^2}{(b\tau)^2} = 1 \quad (14)$$

gives scaled emittance  $\epsilon'$

$$\epsilon' = |(a\kappa) * (b\tau)| = \kappa\tau\epsilon \quad (15)$$

Let  $\beta_x = x_0^2/\epsilon$  then  $\beta_x = (x'/\kappa)^2/(\epsilon'/\kappa\tau) = \frac{\tau}{\kappa}\beta'_x$ , we get

$$\beta'_x = \frac{\kappa}{\tau}\beta_x \quad (16)$$

In phase space  $\Delta\phi$ ,  $z$  and  $\Delta t$  are usually used as abscissa and  $w$ ,  $\Delta W$ ,  $\Delta p/p$  and  $z'$  as ordinates. We use  $\kappa$  to connect abscissa and  $\tau$  to connect different ordinates. Six different combinations of abscissa can be made and 11 combinations for ordinates. Their corresponding  $\kappa$ - and  $\tau$ -values are assembled in the following tables.

<b><math>\kappa</math>-values</b>			
wanted↓ in terms of→	$\Delta\phi$ [rad]	$z$ [m]	$\Delta t$ [sec]
$\Delta\phi$ [rad]	1	$2\pi/\beta\lambda$	$2\pi\beta c/\beta\lambda$
$z$ [m]	$\beta\lambda/2\pi$	1	$\beta c$
$\Delta t$ [sec]	$\beta\lambda/(2\pi\beta c)$	$1/\beta c$	1

<b><math>\tau</math>-values</b>				
wanted↓ in terms of →	$\delta\gamma = w$	$\Delta W$ [eV]	$\Delta p/p$	$z'$ [rad]
$\delta\gamma = w$	1	$1/(mc^2)$	$\gamma\beta^2$	$\gamma(\gamma\beta)^2$
$\Delta W$ [eV]	$mc^2$	1	$mc^2\gamma\beta^2$	$mc^2\gamma^3\beta^2$
$\Delta p/p$	$(\gamma\beta^2)^{-1}$	$(mc^2\gamma\beta^2)^{-1}$	1	$\gamma^2$
$z'$ [rad]	$\gamma^{-1}(\gamma\beta)^{-2}$	$(mc^2\gamma^3\beta^2)^{-1}$	$\gamma^{-2}$	1
with $W = mc^2(\gamma - 1)$ as kinetic energy.				

Example: Phase space  $z \otimes \Delta W$  in terms of  $\Delta\phi \otimes \delta\gamma$ :  $\kappa = \beta\lambda/2\pi$ ,  $\tau = mc^2$ .

$$\epsilon_{zW} = \kappa\tau\epsilon_w = (\beta\lambda/2\pi)mc^2\epsilon_w. \quad (17)$$

$$\beta_{zW} = \frac{\kappa}{\tau}\beta_w = \frac{\beta\lambda/2\pi}{mc^2}\beta_w. \quad (18)$$

More interesting details about emittance definitions, normalized and unnormalized, and their units can be found in the UserManual of the *TRACEWIN* program.

### 3 Twiss Parameter Values

To simplify we assume the twiss parameter  $\alpha = 0$ . The twiss parameter  $\gamma$  then reduces to  $1/\beta$  and only two free parameters  $\epsilon$  and  $\beta$  remain to describe the ellipse in phase space completely.

For small amplitude longitudinal oscillations the separatrix intersects the w-axis at  $w_0$  and is given by

$$w_0 = \frac{\Delta W}{mc^2} = \sqrt{2qE_0LT\beta_s^3\gamma_s^3\lambda\phi_s^2\sin(-\phi_s)/\pi mc^2} \quad (19)$$

With  $w_0$  and  $\Delta\phi_0$  the maximal emittance on the separatrix is given.

$$\epsilon_w = w_0 * \Delta\phi_0 \quad (20)$$

and from (20) we get finally  $\gamma_0 = \epsilon_w/\Delta\phi_0^2$  and  $\beta_0 = 1/\gamma_0$ .

NOTE: the two twiss parameters  $\gamma_0$  and  $\beta_0$  are completely defined by the emittance  $\epsilon_w$ , the cavity field  $E_0$ , rf-phase  $\phi_s$ , rf-wavelength  $\lambda$  and particle impuls  $\sim \gamma\beta$ .

## 4 Appendix

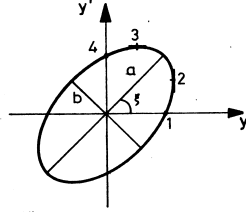
### 4.1 SIMULAC Variables

Table 1: **Variable Names**

$\epsilon_w = \text{emitw}$	$\Delta\phi = \text{Dphi}$	$\Delta\phi_0 = \text{Dphi0}$	$w = w$	$w_0 = w0$
$\epsilon_W = \text{emitW}$	$\Delta z = z$	$\Delta W = \text{DW}$	$\Delta p/p = \text{Dp2p}$	$\Delta p/p_0 = \text{Dp2p0}$
$\epsilon_z = \text{emitz}$	$\beta_z = \text{betaz}$	$\gamma_z = \text{gammaz}$	$\alpha_z = \text{alphaz}$	$\lambda = \text{lamb}$
$Ez_{avg} = \text{EzAvg}$	$Ez_{peak} = \text{EzPeak}$	$\phi_+ = \text{phi\_1}$	$\phi_- = \text{phi\_2}$	$\psi = \text{psi}$
$\gamma = \text{gamma}$	$\gamma\beta = \text{gb}$	$\beta = \text{beta}$	$E_0T = \text{E0T}$	$E_0LT = \text{E0LT}$
$mc^2 = \text{m0c2}$	$mc^3 = \text{m0c3}$	$\epsilon_{xi} = \text{emitx\_i}$	$\epsilon_{yi} = \text{emity\_i}$	$\epsilon_{zi} = \text{emitz\_i}$
$\beta_{xi} = \text{betax\_i}$	$\beta_{yi} = \text{betay\_i}$	$\alpha_{xi} = \text{alfax\_i}$	$\alpha_{yi} = \text{alfay\_i}$	$\gamma_{xi} = \text{gamax\_i}$
$\gamma_{yi} = \text{gamay\_i}$	$\omega = \text{omg}$	$\phi = \text{phi}$	$\phi_s = \text{phisoll}$	$\Delta W/W = \text{DT2T}$
$W = \text{Tkin}$				

## 4.2 Relations Between Ellipse and Twiss Parameters

### 5.4 Geometrical properties of the ellipse



	$\alpha, \beta, \gamma, \epsilon$	$c_1, c_2, c_3, c_4$	$L, S, \epsilon$
	$\beta\gamma - \alpha^2 = 1$ $\Pi = \frac{1}{2}(\beta + \gamma)$	$\epsilon = c_1c_4 - c_2c_3$ $\Pi = \frac{1}{2}(c_1^2 + c_2^2 + c_3^2 + c_4^2)/\epsilon$	$H = \frac{1}{2L}(L^2 + S^2 + 1)$
$y_1$	$\sqrt{\epsilon/\gamma}$	$\epsilon/\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
$y_2$	$\sqrt{\epsilon\beta}$	$\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L} \sqrt{S^2 + L^2}$
$y_3$	$-\alpha/\sqrt{\epsilon\beta}$	$(c_1c_3 + c_2c_4)/\sqrt{c_1^2 + c_2^2}$	$S\sqrt{\epsilon/L} \sqrt{S^2 + L^2}$
$y_4$	$-\alpha/\sqrt{\epsilon\gamma}$	$(c_1c_3 + c_2c_4)/\sqrt{c_3^2 + c_4^2}$	$S\sqrt{\epsilon/L}$
$y_5$	$\sqrt{\epsilon\gamma}$	$\sqrt{c_3^2 + c_4^2}$	$\sqrt{\epsilon/L}$
$y_6$	$\sqrt{\epsilon/\beta}$	$\epsilon/\sqrt{c_1^2 + c_2^2}$	$\sqrt{\epsilon/L} \sqrt{S^2 + L^2}$
a	$\sqrt{\epsilon/2} (\sqrt{\Pi+1} + \sqrt{\Pi-1})$		
b	$\sqrt{2\epsilon}/(\sqrt{\Pi+1} + \sqrt{\Pi-1}) = \sqrt{\epsilon/2} (\sqrt{\Pi+1} - \sqrt{\Pi-1})$		
a/b > 1	$\Pi + \sqrt{\Pi^2 - 1}$		
$\tan \xi$	$[-\alpha(H + \sqrt{\Pi^2 - 1})]/[\beta(H + \sqrt{\Pi^2 - 1}) - 1]$	$[c_2 + c_3(H + \sqrt{\Pi^2 - 1})]/[c_1(\Pi + \sqrt{\Pi^2 - 1}) - c_4]$	$S/[L(H + \sqrt{\Pi^2 - 1}) - 1]$
$\sin 2\xi$	$-\alpha/\sqrt{\Pi^2 - 1}$	$(c_1c_3 + c_2c_4)/\epsilon\sqrt{\Pi^2 - 1}$	$S/L\sqrt{\Pi^2 - 1}$
$\cos 2\xi$	$(\beta - \gamma)/2\sqrt{\Pi^2 - 1}$	$(c_1^2 + c_2^2 - c_3^2 - c_4^2)/2\epsilon\sqrt{\Pi^2 - 1}$	$(L^2 + S^2 - 1)/2L\sqrt{\Pi^2 - 1}$
$\tan 2\xi$	$-2\alpha/(\beta - \gamma)$	$2(c_1c_3 + c_2c_4)/(c_1^2 + c_2^2 - c_3^2 - c_4^2)$	$2S/(L^2 + S^2 - 1)$

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