Toolbox: Dimensional Analysis

In this toolbox, we will learn one of the most basic, powerful, and near-universally-applicable techniques in the physicists' (and therefore astrophysicists') arsenal. What a(n astro)physicist calls "dimensional analysis" is different from what you may have done in chemistry (which I would call "unit conversions" or something instead). The outline for the material in this packet is as follows:

- Dimensions *vs.* Units
- Applications:
  - Newton's Law of Universal Gravitation: Surface gravity
  - The period of a mass on a Hookeian spring
  - The period of a pendulum
  - Kepler III and central mass
  - Kepler II and orbital speed (circular orbit)

The hard sciences are founded on the ability to make measurements. How well we can make measurements depends on technology, and the types of measurement are varied, but they can all be organized in terms of the **dimensions** of what is being measured. Some examples of measurements or estimates of magnitudes we make every day are

- Distance measurements (distance in space between students during passing)
- Time measurements (distance in time between now and lunch)
- Mass measurements (usually called weight measurements until your physics teacher gets all cranky about force *vs.* mass)
- Force measurements (How hard should I push on this heavy door to get it open?)
- Speed measurements (How fast is the bus driving?)
- Temperature measurements (Why is it always hot/cold in the planetarium?)

1. Make a list of five more such quantities that you often estimate or measure.

BEGIN ANSWER

- volume
  - (mass) density
- rotational
  - speed
  - inertia
  - torque
- energy
  - luminosity
  - power

END ANSWER

The list can go on for a very long time, but most of the concepts on the resulting list are expressible in terms of more fundamental ones. For example, speed measurements are expressible in terms of distances in space as they change in time. Force measurements are expressible in terms of masses, distances, and times (the unit of force is $\frac{kg \cdot m}{s^2}$).

Dimensions and units

To organize our list, we introduce the concept of **dimensions**. These can be fundamental such as the dimensions of mass M, length L, and time T, or composite such as the dimensions of speed which are $\frac{L}{T} = L \cdot T^{-1}$. According to the previous parenthetical remark, forces have dimensions of $\frac{M \cdot L}{T^2} = M \cdot L \cdot T^{-2}$. In order to talk about dimensions, we will use the following notation:

> *For a quantity X, the dimensions of X will be denoted by $[X]$. So, for example, if F denotes a force, then $[F] = M\ L\ T^{-2}$.*

2. What are the dimensions of distance, velocity, acceleration, and (linear) momentum?

- $[\text{distance}] = L$
- $[\text{velocity}] = \frac{L}{T}$
- $[\text{acceleration}] = \frac{L}{T^2}$
- $[\text{(linear) momentum}] = \frac{M \cdot L}{T}$

Dimensions are a fixed property of a quantity, but the conventions we use to measure them are varied. These latter conventions are called **units**. For example, yards are a unit of measure of length. Meters and kilometers are competing units for the same dimension.

3. Give three units for each of the three fundamental dimensions.

BEGIN ANSWER

- mass:
  - kilograms
  - grams
  - slugs
- length:
  - meters
  - centimeters
  - yards
- time
  - seconds
  - minutes
  - years

END ANSWER

Conventional agreements on which units to use are called "systems of units (of measurement)". The metric system is an example, as is the Imperial system of units. In this country, we are still clinging to the "United States customary (system of) units". In science education, the standard is called the International System of units, or SI units from the French *Système international d'unités*.

To do dimensional analysis, you can use the dimensions of length, time, and mass, or--equivalently--you can fix a system of units and use those. However, please make an effort to distinguish a **unit** from a **dimension**.

4. In class we will talk about Newton's Law of Universal gravitation. This says that two objects with masses m_1 and m_2 separated by a distance r attract each other with a force given by

$$F = G_N \, \frac{m_1 \, m_2}{r^2} \tag{1}$$

> In this formula, G_N is a constant that determines the strength of the gravitational force called **Newton's constant**. Find the dimensions $[G_N]$ of this constant. What are the SI units of G_N?

BEGIN ANSWER

We know that $[m_1] = M = [m_2]$ and that $[r] = L$, so we need to figure out $[F]$. We could be lazy and look it up, or remember that it was already mentioned somewhere above, but then we wouldn't learn anything other than a fact. Instead, we realize after a modicum of reflection that we already know from Newton's second law that $F = ma$. Again $[m] = M$ and acceleration it the rate of change of speed. Speed has dimensions $\frac{L}{T}$ so the rate of change of this has dimensions $\frac{L}{T^2}$, as we already worked out above. Putting together $F = ma$ and the rule (1) above, we find

$$[ma] = [F] = [G] \frac{[m_1]}{[m_2]} [r^2]$$

or

$$M \frac{L}{T^2} = [G]$$

Solving for $[G] = M \frac{L}{T^2} \frac{L^2}{M^2}$ and simplifying, we find

$$[G] = \frac{L^3}{M \cdot T^2}$$

This is an important result that will come in handy throughout this course.

END ANSWER

5. We learn in middle school that objects in free-fall near the surface of the

Earth accelerate at a rate of $g \approx 9.8\ m/s^2$. This is called the **surface gravity**.

a) Use dimensional analysis to guess a formula for g in terms of Newton's constant G_N, the mass $M_\oplus$ of the Earth, and the radius $R_\oplus$ of the Earth. (Hint: In problem 4 you should have found that $[G_N] = M^{-1}\, L^3\, T^{-2}$. Multiply by appropriate powers of mass and radius to make a combination with dimensions of acceleration.)

b) Newton's constant has a value of $G_N = 6.67 \times 10^{-11}\ m^3\ kg^{-1} s^{-2}$. If $R_\oplus = 6400\ km$ and $M_\oplus = 6.0 \times 10^{24}\ kg$, what do you get for the surface gravity g?

BEGIN ANSWER

a) We want an acceleration $[g] = \frac{L}{T^2}$, which has no mass in it. To get rid of the M^{-1} in $[G]$, therefore, we must multiply by the mass of the Earth $M_\oplus$:

$$[GM_\oplus] = \frac{L^3}{T^2}$$

We want to get rid of two of the powers of L, but we have only the radius of the Earth to work with. $[R_\oplus] = L$ so $[R_\oplus^2] = L^2$. So if we divide our intermediate result by $R_\oplus^2$, we get

$$\left[\frac{GM_\oplus}{R_\oplus^2}\right] = \frac{L}{T^2}$$

which is an acceleration.

It is tempting to conclude that

$$g \stackrel{?}{=} \frac{GM_\oplus}{R_\oplus^2}$$

We return to this question below.

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In [5]: from aa_tools.astroc import G,M_earth, R_earth, sf
g=G*M_earth/R_earth**2
print("b) Assuming our formula for g, we find a value of g =",round(g,
```

```
b) Assuming our formula for g, we find a value of g = 9.7 m/s^2.
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END ANSWER

In the previous problem, you should have found that the combination $G_N M_\oplus R_\oplus^{-2}$ has the correct dimensions to be an acceleration. From this we conclude that the acceleration due to gravity at the surface of the Earth g must be proportional to this combination. We write this as

$$g \propto \frac{G_N M_\oplus}{R_\oplus^2}$$

Now in general if $Y \propto X$ for any variables or combinations of variables X and Y, then this means that there is a quantity c independent of the variables in X and Y such that $Y = cX$. Going back to our gravitational case, this means that there is a real, dimensionless number c such that $g = c\, G_N M_\oplus R_\oplus^{-2}$. In class we will *derive* this formula by combining Newton's second law with his law of universal gravitation to find that, in this case, $c = 1$.

This is the generic situation: You can often fix the overall form of a physical relation by dimensional analysis (and some basic physical intuition) alone, but you will be left with a proportionality constant that is a pure number like 1, $\frac{1}{2}$, 2π, $$, *et cetera*. You cannot determine this number from dimensional analysis, because it is dimensionless. If you want to find the value of the proportionality constant, you will either have to measure it, or learn how to derive your formula from first principles.

6. A spring is called Hookean if the force F you must exert on the spring to stretch it is proportional to the length Δx by which you stretch it. In formulas $F \propto \Delta x$. (This will be true of most springs provided you don't stretch them too much.) As before, this means that there is some constant Hooke called k independent of F and Δx such that

$$F = k\, \Delta x \tag{2}$$

a. Find the dimensions $[k]$ of the spring constant k.

b. If you attach a mass m to the spring, stretch it, and release, the mass will bounce back and forth as it stretches/compresses the spring in a motion called "simple harmonic oscillation". Using dimensional analysis, find a formula for the period P of this oscillation. Express it as an equality with the constant of proportionality denoted by c.

BEGIN ANSWER

a) We already worked out that $[F] = M \cdot L \cdot T^{-2}$ from Newton II. The RHS of Hooke's law has dimensions $[k\Delta x] = [k][\Delta x] = [k]L$. Setting them equal and solving for

$$[k] = \frac{M}{T^2}$$

b) If we have a mass m then we now have a new dimension $[m] = M$ to play with. The period P of a motion is the time it takes to go through one full cycle of the simple harmonic motion. Thus $[P] = T$, so we need to make something with dimensions of time. The first step is to get the M out of $[k]$, which we can do by dividing by m. Even better, we take the reciprocal of this ratio to find

$$\left[\frac{m}{k}\right] = T^2$$

If we had something with dimensions of time, we could remove one of the factors of T. A little reflection, however, should result in the realization that the *square root* of our ratio has the correct dimensions already!

$$\left[\sqrt{\frac{m}{k}}\right] = T$$

Thus we may conclude that $P \propto \sqrt{\frac{m}{k}}$ for the period of the oscillation, so

$$P = c\sqrt{\frac{m}{k}}$$

for some constant $c > 0$.

END ANSWER

> N.B: In the previous problem, it was actually a *good* thing that there was no other parameter τ (this is the Greek lower-case letter tau) with the dimensions of time in the problem. For suppose there had been. Then the parameter $\xi = \sqrt{\frac{m}{k\tau}}$ would have been *dimensionless* $[\xi] = 1$. This would have invalidated our conclusion that P was proportional to $\sqrt{\frac{m}{k}}$, because $f(\xi)\sqrt{\frac{m}{k}}$ would have the dimensions of time *for any function f of ξ!*

7. Consider a pendulum consisting of a bob of mass m on a very light rod of length ℓ. When the pendulum is displaced by a small angle from its resting position and released, it swings back and forth in a regular motion called "simple harmonic motion" because the bob is pulled down by our surface gravity $g \approx 9.8\ m/s^2$. The time it takes to swing and come back is called the **period** of the oscillation. It is denoted by P.

   a) Using dimensional analysis and the variables given, write a formula for the period. Express your answer as an equality with the constant of proportionality denoted by c.

   b) What physical characteristic of the pendulum is irrelevant to the period?

BEGIN ANSWER

a) This time, we have the strength of the gravity field $[g] = L \cdot T^{-2}$, but this does not have any powers of mass in it. Thus, we only need a length $[\ell] = L$ to reason by complete analogy to problem 6. Thus, we find

$$\left[\sqrt{\frac{\ell}{g}}\right] = T$$

Thus, we conclude that

$$P = c\sqrt{\frac{\ell}{g}}$$

for some positive constant c (not necissarily with the same value as in problem 6).

b) The mass m of the bob: Counterintiutively, small masses and large masses will have the same oscillation periods![^1]

END ANSWER

In the previous problems, you found formulae for the periods of a mass on a spring and a simple pendulum. Dimensional analysis tells you the general form of the period, but not the coefficient.[^2] In the two cases above, the coefficient can be worked out from calculus to be $c = 2\pi \approx 6.3$. This should serve as a warning that the coefficient is not always 1 or close to 1.

8. Recall that Kepler's Third Law states that for any planet orbiting the Sun, the ratio of the square of the period to the cube of the semi-major axis is the same number as for any other planet: $\frac{P^2}{a^3} = K$ for some constant K independent of the planet.

   a. Find the dimensions of K.

   b. Write a formula for K in terms of Newton's constant and the mass $M_\odot$ of the Sun.

   c. In class, we will show that the proportionality constant in your formula is $c = 4\pi$. Solve your equation for the period and note that it depends only on the size of the orbit and the mass of the Sun.

BEGIN ANSWER

a) $[P] = T$ and $[a] = L$, so $[K] = \frac{T^2}{L^3}$. Why does this combination look familiar...

b) ... oh yeah! That's what it was: $[GM_\odot] = \frac{L^3}{T^2}$, because the mass cancels the one in G! Therefore $K \propto \frac{1}{GM_\odot}$, so $P^2 \propto \frac{a^3}{GM_\odot}$, there is some positive constant c such that

$$P = c\sqrt{\frac{a^3}{GM_\odot}}$$

c) We are being told that $c = 4\pi$, so

$$P = 4\pi\sqrt{\frac{a^3}{GM_\odot}}$$

The RHS depends only on a and $M_\odot$. Showing it from Newton's laws would not only explain why Kepler's Third Law is true, it would also explain that the constant K is essentially just the reciprocal of the solar mass (dressed up with Newton's universal constant). Furthermore, it also would explain why it holds for systems orbiting other bodies, and it gives a rock-solid way to use the orbital parameters to determine the mass of the central object.

END ANSWER

9. Kepler's Second Law states that the area swept out per unit time by a line connecting the Sun to an orbiting object is a constant.

   a) Argue that this implies that the speed of an orbiting body can only be constant if the orbit is circular.

   b) For a circular orbit of radius r, argue that that speed v_{orbit} of the orbiter is proportional to $\sqrt{\frac{GM}{r}}$ where M is the mass of the **principal** body (*i.e.* the one being orbited).

   c) In class, we will derive that the constant of proportionality is $c = 1$. If the International Space Station (ISS) is orbiting at an altitude of $400\ km$, use the values in [problem 5]{.underline}](#a3tjawmibco6) to show that the speed of the ISS is about $7700\ m/s$.

   d) How long does it take the ISS to orbit once?

BEGIN ANSWER

a) Suppose the orbit has non-zero eccentricity. Then near near perihelion the distance from the orbiting object to the Sun is strictly smaller than at any other point in its orbit, and near aphelion it is larger than at any other point. If the orbital speed is constant than in one fixed unit of time, the wedge traced out near perihelion will have to be strictly small than for that same unit of time for any other wedge with the difference being the largest near aphelion. But this cannot be since the areas are the same. Thus the speed must be highest near perihelion and lowest near aphelion. Thus we have a contradiction so the starting assumption of non-zero eccentricity is false, and the orbit is a circle.

b) The only things at hand are the mass of the Sun M, that of the orbiting object m, and the radius r of the circular orbit. Suppose for the moment that m is irrelevant to the problem. (This is true according to Kepler since the that combination of periods and orbital radii were the same for all the planets. Since all of them have (widely) different values of m, it means that the dynamics do not depend on these ms.) Since this is a gravitational problem, the solution must include Newton's universal constant G. We want a speed, which has no mass in it, so again we consider the combination GM with dimensions of $\frac{L^3}{T^2}$. The orbital speed has dimensions $[v_o] = \frac{L}{T}$, so if we rid ourselves of one power of L, at least we can make $v_o^2 \propto \frac{GM}{r}$ with our ingredients. Thus we conclude that, up to a positive constant c,

$$v_o = c\sqrt{\frac{GM}{r}}$$

as least *modulo* all our assumptions.

Let us revisit these assumptions. One was that the result should depend on G. Indeed, if $G \to 0$ there is no gravitational force according to equation (1), and it makes sense that the orbital contribution to the speed would go to 0.

The other assumption, that m is irrelevant is harder to argue because with m in the story $\xi = \frac{m}{M}$ is a dimensionless parameter. Thus, there is no *a priori* reason that c couldn't have been a whole function $f(\xi)$. One way to argue against this point is to notice that $M \gg m$ and so ξ is very small. To first approximation, any correction would be proportional to *at best* order $\xi = \frac{m}{M} \ll 1$.

Of course, this argument fails if m gets so large that it becomes comperable to M, and actually this does happen! In fact, as we will cover in class, Newton pointed out that Kepler's first law is false because planets have a non-neglibible gravitaional pull on the Sun so *the Sun-planet system actually orbit each other*! This is especially apparent in binary star systems in which m and M are comperable, so we will have to learn how to fix Kepler's Laws for these cases.

In [13]:

```
from aa_tools.astroc import G, M_earth, R_earth, km, sf
from math import sqrt,pi

h=400*km
r=R_earth+h
v_ISS = sqrt( G*M_earth/r )

print("At that altitude, the radius of the orbit it r =",sf(r),"m.")
print("This gives an orbital speed of v_orbit = ",round(v_ISS),"m/s.")

C=2*pi*r
P=C/v_ISS

print("The circumference of this orbit is C = ",sf(C),"m, which at thi
print("This is a period of about", round(P/60),"minutes.")
```

```
At that altitude, the radius of the orbit it r = 6.8e+06 m.
This gives an orbital speed of v_orbit =  7656 m/s.
The circumference of this orbit is C =  4.3e+07 m, which at this speed
takes P =  5.6e+03 seconds.
This is a period of about 93 minutes.
```

END ANSWER

[^1]: Actually, this analysis does not hold up under closer inspection. The reason is that in this case there actually *is* another dimensionless parameter, namely the angle from which you released the bob to start the pendulum. Since radians are not dimensionful: π radians is the ratio of the circumference C of a disk to its diameter D. Both have the dimensions of length, so $\left[\frac{C}{D}\right] = 1$. Because of this, the actual period of a pendulum is $P = f(\theta)\sqrt{\ell \mathfrak{g}}$ for a very complicated function $f(\theta)$. (See this Wikipedia page for more.)

[^2]: Actually, the situation is often worse: In the case of the pendulum, there is a hidden dimensionless variable which is the starting angle. Since angles have no physical dimension, the period could depend on an arbitrary function of this variable. In fact, it does so for arbitrarily large starting angles (I can show you the messy function for this), but it turns out that at small angles this dependence goes away. For this reason, the small-angle pendulum is also called a "simple" pendulum. Ask me if you want to know more.