The Distribution of Wealth and Monetary Policy

July 1, 2021

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Abstract

This paper presents a tractable heterogenous agent new keynesian model featuring a buffer stock income process, sticky wages and sticky prices, and heterogeneity in discount factors to generate a large aggregate MPC.

Keywords Precautionary saving, Heterogeneous Agents, Monetary Policy, permanent income hypothesis

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Thanks to Chris Carroll

1 Introduction

The representative agent's inadequacy in matching the empirical marginal propensity to consume (MPC) has led to a burgeoning literature on heterogeneity. Models with heterogeneous agents have been able to generate empirically consistent MPCs while uncovering new channels through which fiscal and monetary policy can influence macroeconomic aggregates. To generate the large MPCs found in the data, the majority of the literature on heterogenous agents have implemented a two asset structure where households invest into liquid and illiquid assets. With this portfolio structure, the model can be calibrated to ensure a significant proportion of households hold high levels of illiquid assets and low levels of liquid assets. These illiquid households are unable to effectively smooth income shocks and will therefore allow the model to feature a large aggregate MPC. Such an asset structure, however, significantly increases the computational complexity when solving the model. In comparison, Carroll, Slacalek, Tokuoka, and White (2017) demonstrate that a modest degree of heterogeneity in discount factors can generate a large aggregate MPC without introducing an additional asset to the household's portfolio choice thereby providing a much more tractable method in generating a large aggregate MPC. In this paper, we extend the model in Carroll, Slacalek, Tokuoka, and White (2017) by adding nominal rigidities. In particular, the purpose of this paper is to present a computationally tractable heterogenous agent new keynesian model featuring heterogeneity in discount factors to generate a large aggregate MPC. To further clarify, the claim is not to state illiquid assets should never be included to the portfolio decision. There are certainly instances where a model featuring heterogeneity necessitates such an asset structure (for example Bayer, Lütticke, Pham-Dao, and Tjaden (2019)). The emphasis is on the case when a large aggregate MPC is desired while illiquidity is relevant. Under this scenario, it is computationally advantageous to allow heterogeneity in the discount factor instead of including an illiquid asset.

2 The Model

The model integrates the household specification of Carroll, Slacalek, Tokuoka, and White (2017) with the production, labor market, finance, and government specification of Auclert, Rognlie, and Straub (2020).

2.1 Households

There is a continuum of households of mass 1 distributed on the unit interval and indexed by i. Households are ex-ante heterogeneous in their discount factors and subject to idiosyncratic income shocks. Each household faces the following problem:

$$\max_{\{\mathbf{c}_{it+s}\}_{s=0}^{\infty}} \mathrm{E_t} \left[\sum_{s=0}^{\infty} (\not{D}\beta_i)^{t+s} U\left(\mathbf{c}_{it+s}, n_{it+s}\right) \right] \tag{1}$$

subject to

$$\mathbf{a}_{it} = \mathbf{m}_{it} - \mathbf{c}_{it}$$
$$\mathbf{a}_{it} + \mathbf{c}_{it} = \mathbf{z}_{it} + (1 + r_t^a)\mathbf{a}_{it-1}$$
$$\mathbf{a}_{it} \geq 0$$

where $U(\mathbf{c}_{it}, n_{it}) = \frac{\mathbf{c}_{it}^{1-\rho}}{1-\rho} - \varphi \mathbf{p}_{it} \frac{n_{it}^{1+v}}{1+v}$ and β_i is the discount factor of household i. $\mathbf{m}_{it}$ denotes household i's market resources at time t to be expended on consumption or invested at a mutual fund. $\mathbf{c}_{it}$ is the level of consumption and $\mathbf{a}_{it}$ is the value of household i's shares at the mutual fund during period t where the mutual fund's return is r_{t+1}^a. $\mathbf{m}_{it}$ is determined by labor income, $\mathbf{z}_{it}$, and the gross return on assets from the last period, $(1 + r_t^a)\mathbf{a}_{it-1}$. $\not{D}$ is the probability of death. Death is included in our model to ensure permanent income, $\mathbf{p}$, and thus wealth, has a limiting distribution. Labor supply of household i at time t is denoted by n_{it}. Given the formulation of sticky wages described in section 2.4, labor supply is an aggregate state variable and therefore consumption serves as the sole control variable in the dynamic problem.

$$\mathbf{z}_{it} = \mathbf{p}_{it}\xi_{it}$$
$$\mathbf{p}_{it+1} = \mathbf{p}_{it}\psi_{it+1}$$

Labor income is subject to permanent and transitory idiosyncratic shocks. In particular, household i's labor income is composed of a permanent component, $\mathbf{p}_{it}$ indicating the level of permanent income and a transitory component, ξ_{it}, indicating the transitory income shock received by household i at time t. $\mathbf{p}_{it}$ is subject to permanent income shocks ψ_{it+1} where ψ_{it} is iid mean one lognormal with standard deviation σ_ψ, $\forall t$.

The transitory component follows

$$\xi_{it} = \begin{cases} u & \text{with probability } \mho \\ \theta_{it}(1 - \tau_t) \int_0^1 w_{gt} n_{igt} \, dg & \text{with probability } 1 - \mho \end{cases}$$

where u are unemployment benefits, τ_t is the tax rate , w_{gt} is the real wage for labor type g, n_{igt} is the labor supply for labor type g and θ_t is an iid mean-one lognormal with standard deviation σ_θ. The probability of receiving an unemployment shock in a given period where households forego their after-tax labor income and instead receive unemployment benefits is denoted by $\mho$.

Given the formulation of sticky wages described in section 2.4, the transitory component simplifies to

$$\xi_{it} = \begin{cases} u & \text{with probability } \mho \\ \theta_{it}(1-\tau_t)\frac{w_t N_t}{(1-\mho)} & \text{with probability } 1-\mho \end{cases} \tag{2}$$

where the real wage w_t and is labor supply N_t enter as aggregate state variables. For details on the derivation, see Appendix B.

2.2 Financial Intermediary

The financial intermediary performs a mutual fund activity where it collects assets from households and invests them into government bonds B_t, stocks v_{jt}, and nominal reserves M_t at the central bank.

In particular, at the end of period t, the assets collected from households A_t must be invested into shares v_{jt} of firm j at price q^s_{jt} , government bonds B_t at price q^b_t and nominal reserves M_t.

$$A_t = \frac{M_t}{P_t} + q^b_t B_t + \int_0^1 q^s_{jt} v_{jt}\, dj \tag{3}$$

where A_t is the dollar value of the mutual fund's assets at the end of period t and v_{jt} is the portfolio share of firm j stocks with $\int_0^1 v_{jt}\, dj = 1$.

The mutual fund's return in the next period is then

$$(1 + r^a_{t+1}) = \frac{B_t + \int_0^1 (q^s_{jt+1} + D_{jt+1}) v_{jt}\, dj + (1+i_t)\frac{M_t}{P_{t+1}}}{A_t}$$

where D_{jt+1} are dividends of firm j and i_t is the nominal interest rate on nominal reserves.

The mutual fund is risk neutral and looks to maximize its expected return

$$\max_{\{B_t, M_t, v_{jt}\}} \mathrm{E}_t \left[1 + r^a_{t+1}\right] = \mathrm{E}\left[\frac{B_t + \int_0^1 (q^s_{jt+1} + D_{jt+1}) v_{jt}\, dj + (1+i_t)\frac{M_t}{P_{t+1}}}{\frac{M_t}{P_t} + q^b_t B_t + \int_0^1 q^s_{jt} v_{jt}\, dj}\right]$$

The first order conditions lead to the no arbitrage equations:

$$\mathrm{E}_t\left[1+r_{t+1}^a\right]=\frac{1}{q_t^b}=\frac{\mathrm{E}_t\left[q_{jt+1}^s+D_{jt+1}\right]}{q_{jt}^s}=(1+i_t)\mathrm{E}_t\left[\frac{P_t}{P_{t+1}}\right]\equiv 1+r_t \tag{4}$$

where r_t is defined to be the real interest rate in period t. In equilibrium ,we will assume $M_t = 0$

2.3 Goods Market

There is a continuum of monopolistically competitive intermediate good producers indexed by $j \in [0,1]$ who produce intermediate goods Y_{jt} to be sold to a final good producer at price P_{jt}. Using intermediate goods Y_{jt} for $j \in [0,1]$, the final good producer produces a final good Y_t to be sold to households at price P_t.

2.3.1 Final Good Producer

A perfectly competitive final good producer purchases intermediate goods Y_{jt} from intermediate good producers at price P_{jt} and produces a final good Y_t according to a CES production Function.

$$Y_t=\left(\int_0^1 Y_{jt}^{\frac{\epsilon_p-1}{\epsilon_p}}\,dj\right)^{\frac{\epsilon_p}{\epsilon_p-1}}$$

where ϵ_p is the elasticity of substitution.

Given P_{jt} , the price of intermediate good j , the final good producer maximizes his profit

$$\max_{Y_{jt}} P_t\left(\int_0^1 Y_{jt}^{\frac{\epsilon_p-1}{\epsilon_p}}\,dj\right)^{\frac{\epsilon_p}{\epsilon_p-1}}-\int_0^1 P_{jt}Y_{jt},\ dj$$

The first order condition leads to demand for good j

$$Y_{jt}=\left(\frac{P_{jt}}{P_t}\right)^{-\epsilon_p}Y_t \tag{5}$$

and the price index

$$P_t = \left(\int_0^1 P_{jt}^{1-\epsilon_p} \, dj \right)^{\frac{1}{1-\epsilon_p}} \tag{6}$$

2.3.2 Intermediate Good Producer

Intermediate goods producers employ labor and produce according to a Cobb Douglas Production function.

$$Y_{jt} = Z_t N_{jt}$$

where $log(Z_t) = \rho_Z log(Z_{t-1}) + \epsilon_Z$

Prices are sticky a la Calvo where intermediate good producers may reset their price with probability $1 - \lambda_p$ each period. Each intermediate firm j chooses P_{jt} to maximize its dividend D_{jt} and stock price q_{jt}^s

$$\max_{\{P_{jt}\}} \overbrace{\frac{(P_{jt} - MC_t)Y_{jt}}{P_t}}^{=D_{jt}} + q_{jt}^s \left(P_{jt} \right)$$

where $q_{jt}^s \left(P_{jt} \right) = \frac{\mathrm{E}_t\left[q_{jt+1}^s + D_{jt+1}(P_{jt}) \right]}{1+r_t}$

The problem above can be restated as

$$\max_{\{P_{jt}\}} \mathrm{E}_t \left[\sum_{s=0}^{\infty} (\lambda_P)^s M_{t,t+s} \left[\frac{(P_{jt} - MC_{t+s})Y_{jt+s}}{P_{t+s}} \right] \right]$$

subject to

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\epsilon_p} Y_t$$

where $M_{t,t+s} = \prod_{k=t}^{t+s-1} \frac{1}{1+r_k}$ is the stochastic discount factor and $MC_t = \frac{W_t}{A_t}$ is the marginal cost of firm j.

This problem leads to the Phillips Curve in our model

$$\pi_t^p = \frac{\mathrm{E}_t[\pi_{t+1}^p]}{1+r^*} + \lambda(\mu_t^p - \mu^p) \tag{7}$$

where r^* is the natural rate of interest in the steady state, $\lambda = \frac{(1-\lambda_p)(1-\frac{\lambda_p}{1+r^*})}{\lambda_p}$, $\mu_t^p = log(P_t) - log(W_t) + log(Z_t)$ and $\mu^p = \frac{\epsilon_p}{1-\epsilon_p}$

2.4 Labor Market

Given the probability of receiving an unemployment shock is $\mho$, we assume, without loss of generality, households $i \in [\mho, 1]$ are 'employed' with positive labor supply and households $i \in [0, \mho]$ are 'unemployed' and provide no labor.

$$n_{it} = \begin{cases} k & \text{if } i \in [\mho, 1] \\ 0 & \text{if } i \in [0, \mho] \end{cases}$$

where $k > 0$.

There is a continuum of monopolistically competitive labor unions indexed by $g \in [0, 1]$ that collects labor $n_{igt} > 0$ from employed households $i \in [\mho, 1]$. Labor supply of household $i \in [\mho, 1]$ is then

$$n_{it} = \int_0^1 n_{igt}\, dg$$

Following Auclert, Rognlie, and Straub (2020), we assume each labor union g demands the same level of labor supply from each household. Denote n_{gt} the level of labor demanded from each household by labor union g at time t, that is, assume $n_{igt} = n_{gt}$. This assumption will imply labor income heterogeneity to be solely the consequence of permanent and transitory income shocks. The total level of labor collected from employed households by labor union g is then

$$(1 - \mho)n_{gt} = \int_{\mho}^1 n_{igt}\, di$$

Each period t, labor unions sell their labor collected from households to a competitive labor packer who demands labor N_{gt} at price W_{gt} from union g .

Thus, in equilibrium,

$$N_{gt} = (1 - \mho) n_{gt} = \int_{\mho}^{1} n_{igt} \, di$$

2.4.1 Competitive Labor Packer

A perfectly competitive labor packer purchases labor N_{gt} from labor unions $g \in [0, 1]$ and produces N_t using constant elasticity of substitution technology be sold to firms at price W_t

$$N_t = \left(\int_0^1 N_{gt}^{\frac{\epsilon_w - 1}{\epsilon_w}} \, dg \right)^{\frac{\epsilon_w}{\epsilon_w - 1}}$$

The labor packer's maximizes its profit with respect the menu of wages W_{gt} for $g \in [0, 1]$

$$\max_{n_{jgt}} W_t \left(\int_0^1 N_{jgt}^{\frac{\epsilon_w - 1}{\epsilon_w}} \, dg \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - \int_0^1 W_{gt} N_{jgt} \, dj$$

The first order condition leads to the labor packer's demand for labor type g

$$N_{gt} = \left(\frac{W_{gt}}{W_t} \right)^{-\epsilon_w} N_t \tag{8}$$

and wage index follows

$$W_t = \left(\int_0^1 W_{gt}^{1 - \epsilon_w} \, dg \right)^{\frac{1}{1 - \epsilon_w}} \tag{9}$$

2.4.2 Labor Unions

Labor Union g sets its wage to maximize the expected aggregate lifetime utility of all employed households. Wages are sticky a la Calvo where unions may adjust its wage with probability λ_w.

$$\max_{\{W_{gt}\}} \mathrm{E_t} \left[\sum_{s=0}^{\infty} (\bar{\beta} \; \not{D} \lambda_w)^s \int_{\mho}^{1} U \left(c_{it+s}(W_{gt+s}), n_{it+s} \right) di) \right]$$

where $\bar{\beta} = \int_{\mho}^{1} \beta_i \, di$

subject to the following three constraints

$$N_{gt} = \left(\frac{W_{gt}}{W_t} \right)^{-\epsilon_w} N_t$$

$$N_{gt} = (1 - \mho) n_{gt} = \int_{\mho}^{1} n_{igt} \, di$$

$$W_t = \left(\int_0^1 W_{gt}^{1-\epsilon_w} \, dg \right)^{\frac{1}{1-\epsilon_w}}$$

The wage Phillips Curve follows from the first order condition

$$\pi_t^w = \bar{\beta} \; \not{D} \mathrm{E}_t \left[\pi_{t+1}^w \right] + \frac{(1 - \lambda_w)}{\lambda_w} (1 - \bar{\beta} \; \not{D} \lambda_w)(\mu^w - \mu_t^w) \tag{10}$$

where μ^w is the optimal wage markup and the wage markup is defined as

$\mu_t^w = log \left(\frac{W_t}{P_t} \right) - log\,(1 - \tau) - mrs_t$

$mrs_t = log \left(- \frac{\int_0^1 U_n(c_{it}, n_{it}) \ di}{\int_0^1 p_{it} \theta_{it} U_c(c_{it}, n_{it}) \ di} \right)$

2.5 Government Policy

2.5.1 Fiscal Policy

The government funds its purchases and unemployment insurance payments by issuing debt and taxing labor income. In particular, it follows

$$B_{t-1} + G + u\mho = q_t^b B_t + \tau \int_0^1 \int_0^1 w_{igt} n_{igt} \, dg \, di$$

which simplifies to

$$B_{t-1} + G + u\mho = q_t^b B_t + \tau w_t N_t$$

where G are government expenditures.

2.5.2 Monetary Policy

The central bank follows the taylor rule:

$$i_t = r^* + \phi_\pi \pi_t^p + \phi_y (Y_t - Y_{ss}) + \epsilon_t^m$$

where ϕ_π is the Taylor rule coefficient for inflation, ϕ_y is the Taylor rule coeffficient for the output gap, r^* is the steady state interest rate, Y_{ss} is the steady state level of output, $\epsilon_t^m = \rho_v \epsilon_{t-1}^m + \varepsilon_t$ are innovations to the taylor rule.

2.6 Equilibrium

An equilibrium in this economy is a sequence of:

- Policy Functions $(c_{it}(m))_{t=0}^{\infty}$

- Prices $\left(r_t, r_{t+1}^a, i_t, q_t^s, q_t^b, w_t, \pi_t^p, \pi_t^w\right)_{t=0}^{\infty}$

- Aggregates $(C_t, Y_t, N_t, D_t, A_t, B_t)_{t=0}^{\infty}$

Such that:

$(c_{it}(m))_{t=0}^{\infty}$ solves the household's maximization problem given $(w_t, N_t, r_t^a)_{t=0}^{\infty}$.

The Mutual fund, final goods producer, intermediate goods producers, labor packer, and labor unions maximize their objective function.

The government budget constraint holds.

The nominal interest rate is set according to the central bank's Taylor rule.

Markets clear:

$$A_t = q_t^b B_t + q_t^s = \int_0^1 \mathbf{p}_{it} \left(m_{it} - c_{it}(m_{it})\right) \, di$$

$$Y_t = C_t + G$$

where $C_t \equiv \int_0^1 \mathbf{p}_{it} c_{it}(m_{it})\, di$

3 Computation Methodology

3.1 Household's Problem

We solve the household's problem by iterating over the first order condition and applying the endogenous gridpoints method developed by Carroll (2006). The computational efficiency for obtaining the consumption policies is greatly improved by dividing the variables of the household's problem by the level of permanent income $\mathbf{p}_{it}$, leaving normalized market resources $m_{it} = \frac{\mathbf{m}_{it}}{\mathbf{p}_{it}}$ as the sole state variable (See Appendix A.1 for details).

3.2 General Equilibrium Steady State

To compute the steady state of the model, we begin with a guess of the mean of the uniformly distributed discount factors to target a predetermined interest rate. We then solve the household's problem and retriveve the steady state consumption policy. Using the steady state consumption policy, we then simulate the model forward and compute the aggregate level of consumption $C_t = \int_0^1 \mathbf{p}_{it} c_{it}(m_{it})\, di$ and the aggregate level of assets $A_t = \int_0^1 \mathbf{p}_{it}\left(m_{it} - c_{it}(m_{it})\right)$ for the terminal period of the simulation. If either the goods or asset market does not clear given the computed aggregate levels of consumption and assets, we input a new guess of the mean of the discount factors and solve and simulate the model again. We repeat this process until the goods and asset markets clear.

3.3 General Equilibrium Impulse Responses

To compute the impulse responses of the model, we follow the sequence space jacobian method of Auclert, Bardóczy, Rognlie, and Straub (2019) . The method synthesizes ideas developed from Boppart, Krusell, and Mitman (2018) by defining the model in sequence space and from Reiter (2009) by linearizing around the steady state. In particular, we define the model as a system of equations in sequence space and then linearize the system around the steady state to solve for the impulse responses to a perfect foresight "MIT shock", that is, an unanticipated shock whose subsequent path is known to all agents. In a linearized system to first order, aggregate certainty equivalence holds Simon (1956) Theil (1957) and therefore the perfect foresight responses to an unexpected shock are the same responses to the shocks of the full stochastic model. For

Table 1 Household Calibration

Calibrated Parameters			
Description	Parameter	Value	Source/Target
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Real Interest Rate	r	$1.05^{.25} - 1$	Conventional
Mean Discount Factor	β	0.9773	$r = 1.05^{.25} - 1$
Disutility of Labor Coefficient	φ	.883	N = 1.22
Probability of Death	$\not{D}$	0.00625	40 Year Work Life
Tax Rate	τ	0.165	Arbitrary
Frisch	$\frac{1}{v}$	.5	Conventional
Unemployment Benefits	u	0.095	Arbitrary
Probability of Unemployment	$\mho$	0.05	CSTW(2017)
Std Dev of Log Permanent Shock	σ_{ψ}	0.06	CSTW(2017)
Std Dev of Log Transitory Shock	σ_{θ}	0.2	CSTW(2017)

further computational details, see Appendix A.2 .

4 Calibration

Our model is calibrated to a quarterly frequency. Calibration of household parameters are presented in table 1 while calibration of economy parameters can be found in table 2. There are five discount factors uniformly distributed among agents in the economy with a mean of .9773 and spread of .0049. The discount factors were adjusted to target a 5 percent annualized real interest rate. Similarly the disutility of labor is set to .883 to target a labor supply of 1.22 . The target of a 5 percent real interest rate and a labor supply level of 1.22 are completely arbitrary. The coefficient of risk aversion is set to 2 and the frisch elasticity of labor supply is set to 1/2, in line with Chetty(2012). The probability of death is set to .99375 to ensure the average working lifespanc of households is $\frac{1}{(1-.99375)} = 160$ periods, or equivalently, 40 years. The standard deviations of transitory and permanent shocks follows from Carroll, Slacalek, Tokuoka, and White (2017). The rest of household parameters are all arbitrarily chosen. With regards to the economy calibration, the calvo parameters for price stickiness is set to .85 and the wage stickiness is set .8 both of which lie in a conventional range. The steady

Table 2 Economy Calibration

Calibrated Parameters			
Description	Parameter	Value	Source/Target
Calvo Price Stickiness	λ_p	.85	Conventional
Calvo Wage Stickiness	λ_w	.8	Conventional
Steady State Price Markup	μ_p	1.012	Arbitrary
Steady State Price Markup	μ_w	1.05	Arbitrary
Government Spending	G	0.19	Arbitrary
Steady State Bond Supply	B	0.5	Arbitrary
Taylor Rule Inflation Coefficient	ϕ_π	0	
Taylor Rule Output Gap Coefficient	ϕ_y	0	
Assets to Output Ratio	$\frac{A}{Y}$	1.4	
Government Bond to Output Ratio	$\frac{B}{Y}$	0.4	

price markup is set to 1.012 implying a 1.2 percent markup while the wage markup has been set 1.05 impying a 5 percent wage markup. The taylor rule coefficients have been initially set to zero implying a fixed nominal rate to provide transparency in understanding the impulse responses produced from the model and assess its accuracy without exterior pressures within the model. The taylor coefficients will be modified later when discussing results. The rest of the parameters are arbitrarily chosen.

5 Results

In this section, we analyze the impulse responses produced from our model for a monetary contraction and a productivity shock.

5.1 Monetary Policy Shock

The impulse responses to the monetary policy shock can be found in figure 1. The monetary policy shock is a 100 basis points increase in the nominal rate and follows an AR(1) with a coefficient of .5. As expected, the rise in the nominal rate leads to a rise in the real interest rate as prices cannot flexibly adjust to clear the goods market. The rise in the real rate leads to a fall in consumption due to the intertemporal substitution channel causing output to fall as well. This fall in output necessitates a fall in labor demand raising the wage markup ($\frac{W_t}{P_t} - mrs_t$) and thus inducing downward pressure on nominal wages. Downward pressure on nominal wages in turn leads to downward pressure on prices as firm markups rise. Given that prices are stickier than wages and given the downward pressures on both nominal prices and nominal wages, the real wage must fall. The fall in consumption is further amplified from income effects as labor and real wages both fall. This amplification through the income channel is particularly significant in heterogenous agent new keynesian models as the large aggregate MPC induces consumption to react strongly to the rise in labor and real wages.

5.2 Productivity Shock

The impulse responses to the productivity shock can be found in figure 2. The productivity shock is a 1 percent increase in Z_t following an AR(1) with a coefficient of .9. The rise in productivity raises both output and firm markups, inducing downward pressure on prices which inturn induces downward pressure on nominal wages. However, given the goods market must clear, consumption must rise causing upward pressure on the nominal wage since the economy marginal rate of substitution is a function of the average marginal utility of consumption. The net effect will see nominal wages rise leading to upward pressure on prices. The net effect of all these different pressures will see the real wage rise and therefore amplify the consumption response signficantly due to the large aggregate MPC in the model. In addition, to the fixed nominal interest rate, inflation will see the real interest rate fall further amplifying consumption. This amplification of consumption will lead to rises in output and labor due to upward pressure on labor demand.

6 Issues and Extensions

6.1 Issues

The most prominent issue I have encountered thus far is determining whether the impulse responses are accurate in regards to the specification of the model. Producing the impulses responses is an exercise in linear algebra and vector calculus. Because the jacobians of the model were constructed without automatic differentiation, the accuracy of the impulse responses produced is uncertain as the smallest algebraic mistake may render them incorrect. Despite this uncertainty, the impulse responses of the model do mirror the responses produced from dynare under certain calibrations. In particular, as long as prices are calibrated to be at least slightly stickier than wages, the impulse responses from a monetary policy shock behave in exactly the same manner as those produced from dynare. In addition, the impulse responses from a productivity shock seem essentially no different to those produced from dynare. Problems arise whenever wages are calibrated to be stickier than prices. When wages are stickier than prices, for instance if the parameter for wage stickiness is set to .85 and the that of price stickiness to .8, then the impulse responses to a monetary contraction invert across the x axis.

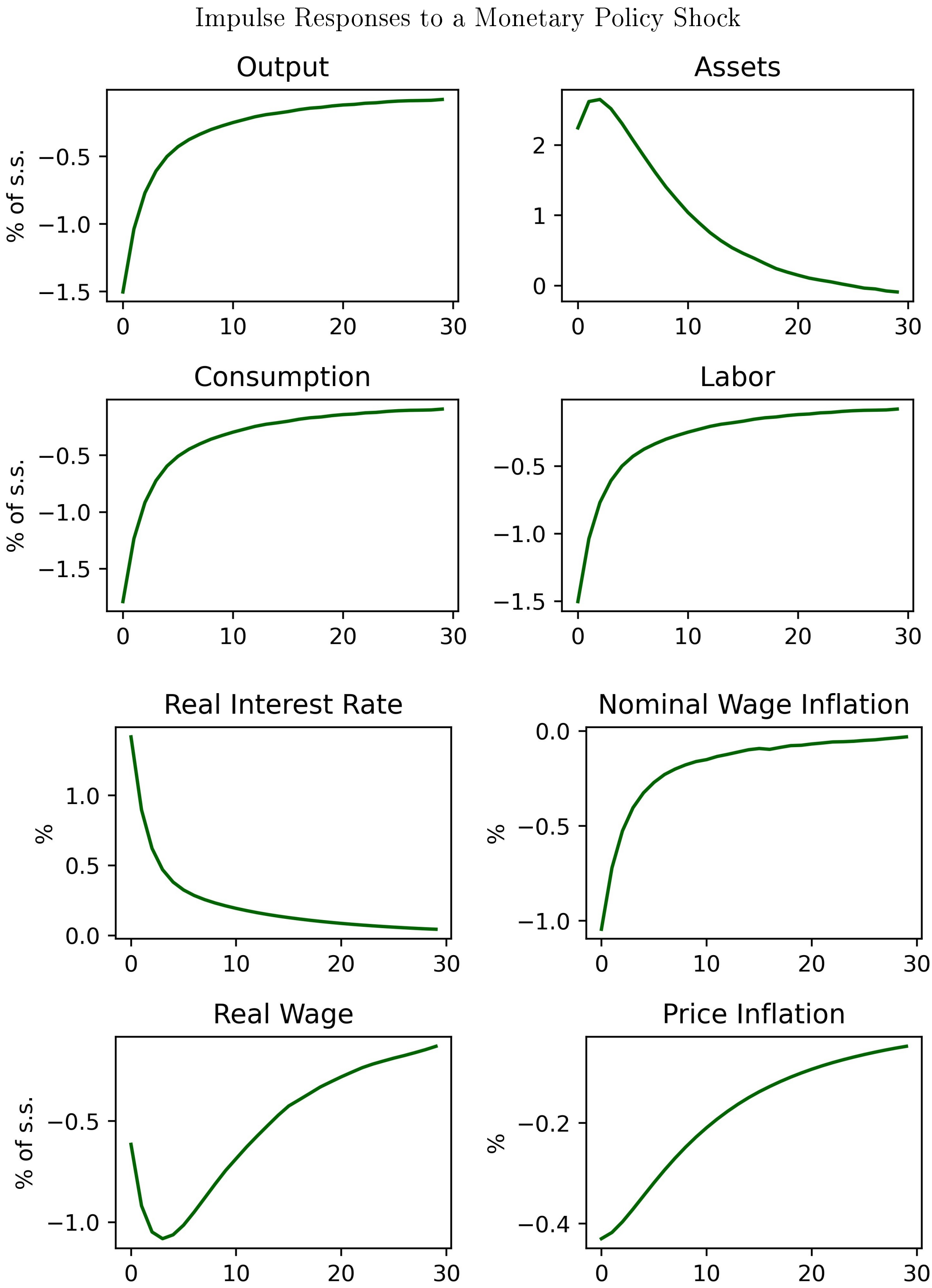

Figure 1 Impulse Response to One Percent Increase in Nominal Rate

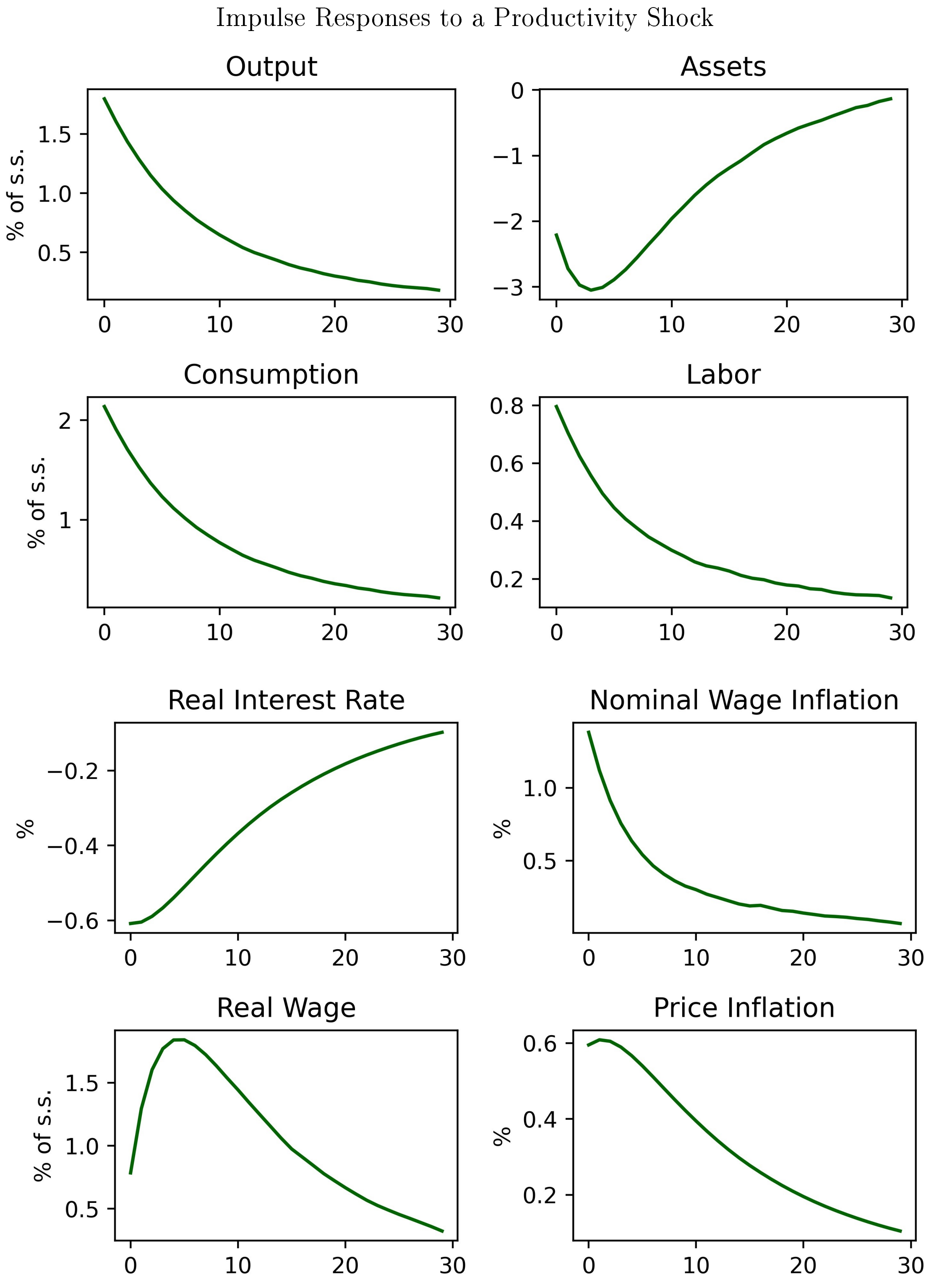

Figure 2 Impulse Response to One Percent Increase in Z_t

That is, with the rise in the nominal rate, the results see a rise in consumption, output, labor, inflation and the real wage. More interestingly, the magnitudes of the impulse responses are exactly the same as those seen in figure 1 except that they are in the wrong direction. This is much cause for concern regarding the accuracy of the code that produces the impulse responses as they do not survive under calibrations where wages are stickier than prices. It is possible, however, that these inverted responses may be accurate to the model due to the strong responsiveness of consumption present in the model. For instance, when the inverse frish elasticity is set to a large value (from v =2 to v=7), rendering labor to be significantly less sensitive to changes in wages, the impulse responses do not invert anymore (consumption, output, labor, real wages all fall after a rise in the nominal rate). This is evidence that perhaps the consumption response is so strong that the slightest change in calibration could lead to large amplification effects that can invert the impulse response. My current hypothesis is that when prices are slightly more flexible than wages, the real wage may perhaps rise as an increase in the nominal rate induces downward pressure on both prices and wages and if the fall in prices is larger than the fall in wages, the real wage may rise inducing a strong consumption response which amplifies and lead aggregate variables such as labor supply, and output to rise. To test this hypothesis, I plan on recomputing the consumption jacobians , calibrating households to have a smaller aggregate MPC to attenuate consumption response to a rise in wages and labor.To ensure the impulse responses are indeed accurate to the model, I am currently writing code that can automatically compute the jacobians of the model, reducing the likelihood of algebraic mistakes. In addition, I plan on reframing the system that defines the model to include the asset market equilibrium equation and exclude the goods market clearing equation. In theory, it is only necessary to include one of the clearing equations in the system to solve the model and the results of the model should not depend on which equation is chosen. If my impulse responses do not differ under the goods market equation and the asset market clearing condition, then I can be confident of the accuracy of the impulse responses.

This inverting error may also be due to inaccuracies when computing the consumption jacobians used to form the impulse responses. To solve for the impulse responses, the consumption jacobians with respect to the wage, labor supply and interest rate must all be computed. To compute these jacobians, one sided numerical differentiation is implemented by calculating the change in aggregate consumption over 200 periods with respect to a small change in the wage, labor or interest rate at a particular time period t. To obtain accurate changes in consumption, it is best that the change in the relevant variable be very small (around 10e-4). Computing the consumption jacobians to a change in the wage or a change in labor are quite inaccurate due to a lack of computing power. Figure 3 plots wage consumption jacobians at two different levels of change in the wage at periods $t = 0$ and $t = 50$. In the graph, dx indicates the change in the wage. The blue and orange graphs plot the difference in aggregate consumption and its steady state level divided by a change in the wage at periods $t = 0$ and $t = 50$. The figure demonstrates that a larger change in the wage leads to smoother, yet less

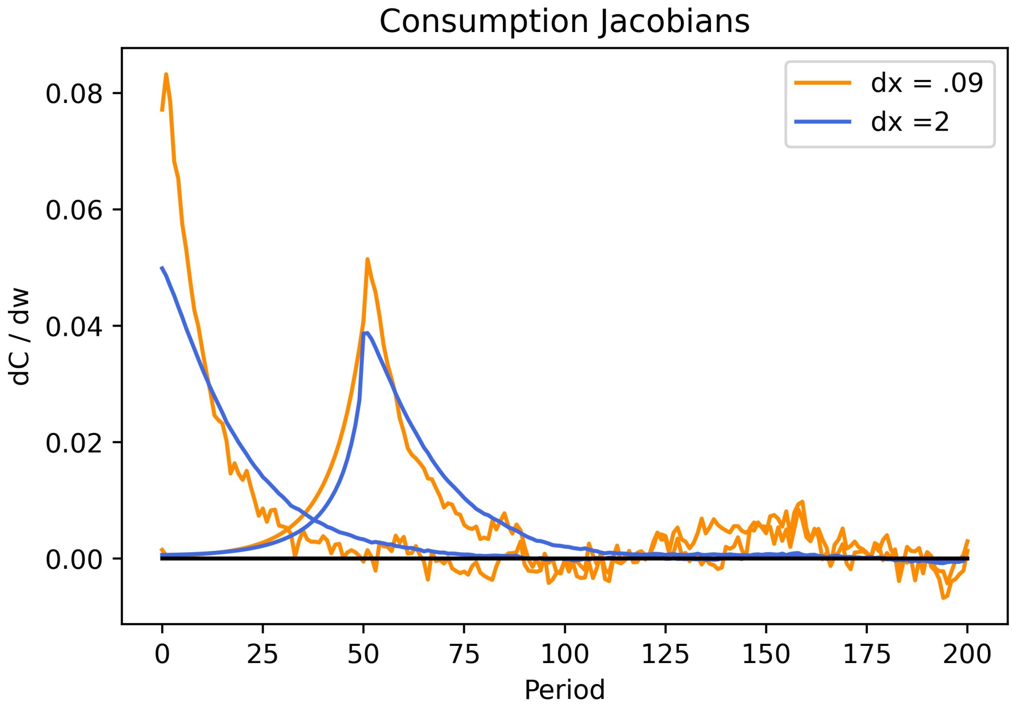

Figure 3 Consumption Jacobians to a "dx" change in the wage.

accurate, jacobians. In particular, notice when the change in the wage is set to $dx = .09$, the responses are larger (orange plot) while changes in the wage set to $dx = 2$ lead to smoother yet attenuated responses (blue plot). To make the jacobians both accurate and smooth, one would require to simulate the model with a very large number of agents leading the jacobians to be extremely computationally costly objects to compute. When computing the impulse responses in figure 1 and 2, the change in the wage and labor was set to $dx = 2$ while the change in the interest rate was set to $dx = .0001$. I believe this may also provide an alternative reason for why the impulse responses invert when wages are made stickier than prices because, in the code, the impulse responses do not invert if the size of the wage and labor consumption jacobians are scaled by a factor greater than one. Similarly, if the consumption jacobian to the interest rate are scaled by a factor less than one then the inversion error disappears as well. This is consistent with the inaccuracies presented in figure 3 as the consumption jacobians to a $dx = 2$ change in the wage are smaller than their true value.

6.2 Extensions

Once the impulse responses have been accurately solved for, it would be interesting to augment the model to allow for uncertainty shocks. Uncertainty shocks can be defined as a shock to the variance of either permanent or transitory income. In addition, comparing the impulse responses of an economy with heterogeneity in discount factors to an economy without such heterogeneity would solidify the argument that a modest amount of heterogeneity can generate a large aggregate MPC.

Appendices

A Computational Details

A.1 Household Bellman Equation

Household i's dynamic program is

$$V(\mathbf{m}_{it}, \mathbf{p}_{it}) = \max_{\{\mathbf{c}_{it}\}} \frac{\mathbf{c}_{it}^{1-\rho}}{1-\rho} - \varphi \mathbf{p}_{it} \frac{n_{it}^{1+v}}{1+v} + \beta_i \not{D} \mathrm{E}_t[V(\mathbf{m}_{it+1}, \mathbf{p}_{it+1})]$$

subject to

$$\begin{aligned}
\mathbf{m}_{it} &= \mathbf{z}_{it} + (1 + r_t^a)\mathbf{a}_{it-1} \\
\mathbf{c}_{it} + \mathbf{a}_{it} &= \mathbf{z}_{it} + (1 + r_t^a)\mathbf{a}_{it-1} \\
\mathbf{a}_{it} &\geq 0
\end{aligned}$$

This can be normalized to

$$V(m_{it}) = \max_{\{c_{it}\}} \frac{c_{it}^{1-\rho}}{1-\rho} - \varphi \frac{n_{it}^{1+v}}{1+v} + \beta_i \not{D} \mathrm{E}_t[\psi_{it+1}^{1-\rho} V(m_{it+1})]$$

subject to

$$\begin{aligned}
m_{it} &= \xi_{it} + (1 + r_t^a)\frac{a_{it-1}}{\psi_{it}} \\
c_{it} + a_{it} &= \xi_{it} + (1 + r_t^a)\frac{a_{it-1}}{\psi_{it}} \\
a_{it} &\geq 0
\end{aligned}$$

Here non boldface variables are normalized by permanent income p_{it}. e.g. $x_{it} = \frac{\mathbf{x_{it}}}{\mathbf{p}_{it}}$

First order condition for consumption:

$$c_{it}^{-\rho} - \beta_i \not{D} \mathrm{E}_t \left[(1 + r_{t+1}^a)\psi_{it+1}^{-\rho} V'(m_{it+1})\right] = 0$$

A.2 Model as System

The model can be defined by the system of equations below. The solution of the model must see this system be equal to a vector of zeros for periods $t = 0, 1, 2, 3, ...$ The system is defined over the sequence space of all endogenous variables $\mathbf{U}$ and exogenous variables $\mathbf{Z}$ of the model.

$$
H_t(\mathbf{U}, \mathbf{Z}) = \begin{pmatrix}
Y_t - Z_t N_t \\
B_{t-1} - q_t^b B_t + u\mho + G - \tau w_t N_t \\
i_t - r^* - \phi \pi_t^p - \phi_y (Y_t - Y_{ss}) - v_t \\
\pi_t^p - \frac{\pi_{t+1}^p}{1+r^*} + \lambda(\mu_t^p - \mu_p) \\
\pi_t^w - \bar{\beta} \not{D} \pi_{t+1}^w - (\frac{1-\lambda_w}{\lambda_w})(1 - \bar{\beta} \not{D} \lambda_w)(\mu^w - \mu_t^w) \\
1 + r_t - \frac{1+i_t}{1+\pi_{t+1}^p} \\
1 + r_{t+1}^a - \frac{q_{t+1}^s + D_{t+1}}{q_t^s} \\
r_t - r_{t+1}^a \\
\frac{w_t}{w_{t-1}} - \frac{\Pi_t^w}{\Pi_t^p} \\
\mathcal{C}_t(\{r_s^a, w_s, N_s\}_{s=0}^{s=T}) - Y_t + G
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad t = 0, 1, 2, 3, ...
$$

where

$\mathcal{C}_t(\{r_s^a, w_s, N_s\}_{s=0}^{s=T}) = \int_0^1 \mathbf{p}_{it} c_{it}(m_{it})\, di$

$c_{it}(m_{it})$ is the steady state normalized consumption policy for household i in period t.

$\mathbf{U} = \big(Y_t, N_t, D_t, B_t, w_t, \pi_t^p, \pi_t^w, r_t, r_{t+1}^a, i_t, q_t^s, q_t^s\big)_{t=0}^{t=T}$

$\mathbf{Z} = (Z_t, v_t)_{t=0}^{t=T}$

Note the asset market clearing condition $\mathcal{A}_t(\{r_s^a, w_s, N_s\}_{s=0}^{s=T}) = q_t^b B_t + q^s$ is not included in the system as the goods market clearing condition holds if and only if the asset market clearing condition holds. On the same note, the government budget constraint nor the equation for the stock price is included in the reduced system as both appear in the asset clearing equation.

A.2.1 Reduced System

The previous system of a little more than a dozen endogenous variables can be reduced to a system of three endogenous variables.

Endogenous Variables are $(r_t, w_t, N_t)_{t=0}^{T}$

Exogenous Variables are $(Z_t, v_t)_{t=0}^{T}$

The reduced system:

$$H_t(\mathbf{U}, \mathbf{Z}) = \begin{pmatrix} \mathcal{H}_{t,1} \\ \\ \mathcal{H}_{t,2} \\ \\ \mathcal{H}_{t,3} \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad t = 0, 1, 2, ..., T \tag{11}$$

where

$$\mathcal{H}_{t,1} = \mathcal{C}_t \left(\{ r_s^a, w_s, N_s \}_{s=0}^{s=T} \right) - Z_t N_t + G$$

$$\mathcal{H}_{t,2} = log(w_t) - log(w_{t-1}) + \left(\tfrac{1-\lambda_w}{\lambda_w} (1 - \not{D} \lambda_w) \textstyle\sum_{k=0}^{\infty} \not{D}^k (\mu_{t+k}^w - \mu^w) \right) - \left(\lambda \textstyle\sum_{k=0}^{\infty} \tfrac{1}{(1+r^*)^k} (\mu_{t+k}^p - \mu^p) \right)$$

$$\mathcal{H}_{t,3} = (1 + r_t) \left(1 + -\lambda \textstyle\sum_{k=1}^{\infty} \tfrac{1}{(1+r^*)^k} (\mu_{t+k}^p - \mu^p) \right)$$
$$- \left(1 + r^* + \phi \left(-\lambda \textstyle\sum_{k=0}^{\infty} \tfrac{1}{(1+r^*)^k} (\mu_{t+k}^p - \mu^p) \right) + \phi_y (Z_t N_t - Y_{ss}) + v_t \right)$$

$$\mu_t^p = log(\tfrac{1}{w_t}) + log(Z_t)$$

$$\mu_t^w = log(w_t) + log(1 - \tau_t) - mrs_t$$

$$mrs_t = log\left(-\frac{\int_0^1 U_n(\mathbf{c}_{it},n_{it})\ di}{\int_0^1 \mathbf{p}_{it}\theta_{it}U_c(\mathbf{c}_{it},n_{it})\ di}\right) = log\left(\frac{\int_0^1 \varphi\mathbf{p}_{it}n_{it}^{v}\ di}{\int_0^1 \mathbf{p}_{it}\theta_{it}\mathbf{c}_{it}^{-\rho}\ di}\right) = log\left(\frac{\int_0^1 \varphi\mathbf{p}_{it}\left(\frac{N_t}{1-\mho}\right)^{v} di}{\int_0^1 \mathbf{p}_{it}\theta_{it}\mathbf{c}_{it}^{-\rho}\ di}\right)$$
$$= log\left(\varphi\left(\frac{N_t}{1-\mho}\right)^{v}\right) + log\left(\int_0^1 \mathbf{p}_{it}\,di\right) - log\left(\int_0^1 \mathbf{p}_{it}\theta_{it}\mathbf{c}_{it}^{-\rho}\ di\right)$$

and the arguments of the function H_t are

$\mathbf{U} = (r_0, r_1, ...r_T, w_0, w_1, ..., w_T, N_0, N_1, ..., N_T)$

$\mathbf{Z} = (Z_0, Z_1, ...Z_T, \upsilon_0, ..., \upsilon_T)$

A.3 Jacobian of System

A.3.1 Implicit Function Theorem

By applying the implicit function theorem to the reduced system, we can obtain the endogenous responses of the real interest rate, wage, and labor supply given some exogenous shock

$$d\mathbf{U} = -\mathbf{H_U}^{-1}\mathbf{H_Z}d\mathbf{Z}$$

where $d\mathbf{U}$ are the endogenous responses of the real interest rate, the real wage, and labor supply to an exogenous shock $d\mathbf{Z}$.

Specifically, notation is defined below

$$d\mathbf{U} = (dr_0, dr_1, ...dr_T, dw_0, dw_1, ..., dw_T, dN_0, dN_1, ..., dN_T)$$

$$d\mathbf{Z} = (dZ_0, dZ_1, ...dZ_T, d\upsilon_0, ...d\upsilon_T)$$

$$\mathbf{H_U} = \begin{pmatrix} H_{\mathbf{U},1} \\ \\ H_{\mathbf{U},2} \\ \\ H_{\mathbf{U},3} \\ \end{pmatrix} \qquad \mathbf{H_Z} = \begin{pmatrix} H_{\mathbf{Z},1} \\ \\ H_{\mathbf{Z},2} \\ \\ H_{\mathbf{Z},3} \\ \end{pmatrix}$$

$$H_{\mathbf{U},i} = \begin{pmatrix} \frac{\partial \mathcal{H}_{0,i}}{\partial r_0} & \dots & \frac{\partial \mathcal{H}_{0,i}}{\partial r_T} & \frac{\partial \mathcal{H}_{0,i}}{\partial w_0} & \dots & \frac{\partial \mathcal{H}_{0,i}}{\partial w_T} & \frac{\partial \mathcal{H}_{0,i}}{\partial N_0} & \dots & \frac{\partial \mathcal{H}_{0,i}}{\partial N_T} \\ \frac{\partial \mathcal{H}_{1,i}}{\partial r_0} & \dots & \frac{\partial \mathcal{H}_{1,i}}{\partial r_T} & \frac{\partial \mathcal{H}_{1,i}}{\partial w_0} & \dots & \frac{\partial \mathcal{H}_{1,i}}{\partial w_T} & \frac{\partial \mathcal{H}_{1,i}}{\partial N_0} & \dots & \frac{\partial \mathcal{H}_{1,i}}{\partial N_T} \\ . \\ \\ . \\ \\ . \\ \\ \frac{\partial \mathcal{H}_{T,i}}{\partial r_0} & \dots & \frac{\partial \mathcal{H}_{T,i}}{\partial r_T} & \frac{\partial \mathcal{H}_{T,i}}{\partial w_0} & \dots & \frac{\partial \mathcal{H}_{T,i}}{\partial w_T} & \frac{\partial \mathcal{H}_{T,i}}{\partial N_0} & \dots & \frac{\partial \mathcal{H}_{T,i}}{\partial N_T} \\ \end{pmatrix}$$

$$H_{\mathbf{Z},t} = \begin{pmatrix} \frac{\partial \mathcal{H}_{0,i}}{\partial Z_0} & \dots & \frac{\partial \mathcal{H}_{0,i}}{\partial Z_T} & \frac{\partial \mathcal{H}_{0,i}}{\partial v_0} & \dots & \frac{\partial \mathcal{H}_{0,i}}{\partial v_T} \\ \frac{\partial \mathcal{H}_{1,i}}{\partial Z_0} & \dots & \frac{\partial \mathcal{H}_{1,i}}{\partial Z_T} & \frac{\partial \mathcal{H}_{1,i}}{\partial v_0} & \dots & \frac{\partial \mathcal{H}_{1,i}}{\partial v_T} \\ . \\ . \\ . \\ \frac{\partial \mathcal{H}_{T,i}}{\partial Z_0} & \dots & \frac{\partial \mathcal{H}_{T,i}}{\partial Z_T} & \frac{\partial \mathcal{H}_{T,i}}{\partial v_0} & \dots & \frac{\partial \mathcal{H}_{T,i}}{\partial v_T} \end{pmatrix}$$

Or equivalently

$$\mathbf{H_U} = \begin{pmatrix} H_{\mathbf{U},0} \\ H_{\mathbf{U},1} \\ . \\ . \\ . \\ H_{\mathbf{U},T} \end{pmatrix} \qquad \mathbf{H_Z} = \begin{pmatrix} H_{\mathbf{Z},0} \\ H_{\mathbf{Z},1} \\ . \\ . \\ . \\ H_{\mathbf{Z},T} \end{pmatrix}$$

$$H_{\mathbf{U},t} = \begin{pmatrix} \frac{\partial \mathcal{H}_{t,1}}{\partial r_0} & \cdots & \frac{\partial \mathcal{H}_{t,1}}{\partial r_T} & \frac{\partial \mathcal{H}_{t,1}}{\partial w_0} & \cdots & \frac{\partial \mathcal{H}_{t,1}}{\partial w_T} & \frac{\partial \mathcal{H}_{t,1}}{\partial N_0} & \cdots & \frac{\partial \mathcal{H}_{t,1}}{\partial N_T} \\ \frac{\partial \mathcal{H}_{t,2}}{\partial r_0} & \cdots & \frac{\partial \mathcal{H}_{t,2}}{\partial r_T} & \frac{\partial \mathcal{H}_{t,2}}{\partial w_0} & \cdots & \frac{\partial \mathcal{H}_{t,2}}{\partial w_T} & \frac{\partial \mathcal{H}_{t,2}}{\partial N_0} & \cdots & \frac{\partial \mathcal{H}_{t,2}}{\partial N_T} \\ \frac{\partial \mathcal{H}_{t,3}}{\partial r_0} & \cdots & \frac{\partial \mathcal{H}_{t,3}}{\partial r_T} & \frac{\partial \mathcal{H}_{t,3}}{\partial w_0} & \cdots & \frac{\partial \mathcal{H}_{t,3}}{\partial w_T} & \frac{\partial \mathcal{H}_{t,3}}{\partial N_0} & \cdots & \frac{\partial \mathcal{H}_{t,3}}{\partial N_T} \end{pmatrix}$$

$$H_{\mathbf{Z},t} = \begin{pmatrix} \frac{\partial \mathcal{H}_{t,1}}{\partial Z_0} & \cdots & \frac{\partial \mathcal{H}_{t,1}}{\partial Z_T} & \frac{\partial \mathcal{H}_{t,1}}{\partial v_0} & \cdots & \frac{\partial \mathcal{H}_{t,1}}{\partial v_T} \\ \frac{\partial \mathcal{H}_{t,2}}{\partial Z_0} & \cdots & \frac{\partial \mathcal{H}_{t,2}}{\partial Z_T} & \frac{\partial \mathcal{H}_{t,2}}{\partial v_0} & \cdots & \frac{\partial \mathcal{H}_{t,2}}{\partial v_T} \\ \frac{\partial \mathcal{H}_{t,3}}{\partial Z_0} & \cdots & \frac{\partial \mathcal{H}_{t,3}}{\partial Z_T} & \frac{\partial \mathcal{H}_{t,3}}{\partial v_0} & \cdots & \frac{\partial \mathcal{H}_{t,3}}{\partial v_T} \end{pmatrix}$$

A.3.2 Obtaining All Other Responses

To obtain all other responses given the endogenous response $\mathbf{dU}$, we compute the jacobians of the endogenous variable with respect to the relevant endogenous variables that have already been solved for. For example, to compute the response of consumption to some exogenous shock $d\mathbf{Z}$, we sum the jacobians of consumption with respect to the real interest rate, wage and labor supply

$$d\mathbf{C} = \mathcal{J}^{\mathcal{C},r} d\mathbf{r} + \mathcal{J}^{\mathcal{C},w} d\mathbf{w} + \mathcal{J}^{\mathcal{C},N} d\mathbf{N}$$

where $\mathcal{J}^{\mathcal{C},w}$ is the jacobian of aggregate consumption $\mathcal{C}_t(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})$ with respect to the wage.

To be clear ,

$$d\mathbf{w} = (dw_1, dw_2, ..., dw_T)'$$

$$
\mathcal{J}^{\mathcal{C},w} = \begin{pmatrix} \frac{\partial \mathcal{C}_0(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_0} & \frac{\partial \mathcal{C}_0(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_1} & \dots & \frac{\partial \mathcal{C}_0(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_T} \\ \frac{\partial \mathcal{C}_1(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_0} & \frac{\partial \mathcal{C}_1(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_1} & \dots & \frac{\partial \mathcal{C}_1(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_T} \\ . & & & \\ . & & & \\ . & & & \\ \frac{\partial \mathcal{C}_T(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_0} & \frac{\partial \mathcal{C}_T(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_1} & \dots & \frac{\partial \mathcal{C}_T(\{r_s^a, w_s, N_s\}_{s=0}^{s=T})}{\partial w_T} \end{pmatrix}
$$

B Additional Derivations

An employed household's transitory income is

$$
\theta_{it}(1-\tau)\int_0^1 \frac{W_{gt}}{P_t} n_{igt}\, dg
$$

where W_{gt} denotes the nominal wage for labor type g and P_t the price of the final good.

Now notice since $n_{igt} = \frac{N_{gt}}{1-\mho}$ must hold in equilibrium,

$$
\theta_{it}(1-\tau)\int_0^1 \frac{W_{gt}}{P_t}\frac{N_{gt}}{1-\mho}\, dg
$$

Then substituting in the demand for labor type g, $N_{gt} = \left(\frac{W_{gt}}{W_t}\right)^{-\epsilon_w} N_t$

$$
\theta_{it}(1-\tau)\int_0^1 \frac{W_{gt}}{P_t}\frac{\left(\frac{W_{gt}}{W_t}\right)^{-\epsilon_w} N_t}{1-\mho}\, dg
$$

and imposing the definition of the wage index , $W_t = \left(\int_0^1 W_{gt}^{1-\epsilon_w}\, dg\right)^{\frac{1}{1-\epsilon_w}}$

$$
\theta_{it}(1-\tau)\frac{N_t}{(1-\mho)W_t^{-\epsilon_w} P_t} W_t^{1-\epsilon_w}
$$

defining the real wage $w_t = \frac{W_t}{P_t}$ leads to the expression in the paper

$$\theta_{it}(1-\tau)\frac{w_t N_t}{(1-\mho)}$$

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