1. a) $V[X] = E[X^2] - (E[X])^2$

variance is always non-negative, so $V[X] \geq 0$ and

$E[X^2] - (E[X])^2 \geq 0$

$E[X^2] \geq (E[X])^2$

So the statement is true

b) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(\emptyset)}{P(B)} = 0$

We are given that A has positive probability, so $P(A) > 0$.

$P(A \mid B) = 0 \neq P(A)$, so A and B are not independent and the statement is false

c) $E[X] = E[Y] = 0$

$V[XY] = E[(XY)^2] - (E[XY])^2$ since $X \perp\!\!\!\perp Y$,

$= E[X^2Y^2] - (E[X]E[Y])^2$ since $E[X] = E[Y] = 0$,

$= E[X^2Y^2]$

$= E[X^2]E[Y^2]$

$= (E[X^2] - (E[X])^2)(E[Y^2] - (E[Y])^2)$

$= V[X]V[Y]$

$= \sigma_x^2 \sigma_y^2$

The statement is true

d) $E[E[X \mid Y]] = E[X]$ $\qquad E[E[Y \mid X]] = E[Y]$

It is not true that $E[X] = E[Y]$ for any two random variables X and Y, so $E[E[X \mid Y]] \neq E[E[Y \mid X]]$ and the statement is false

e) $P(a \leq X \leq b) = \int_a^b f(x)dx$

$P(a \leq X \leq a) = \int_a^a f(x)dx = 0$

So the statement is true

2. $$\mathbb{P}(\text{spam} \mid \text{free}) = \frac{\mathbb{P}(\text{free} \mid \text{spam})\,\mathbb{P}(\text{spam})}{\mathbb{P}(\text{free})}$$

$$= \frac{0.9 \cdot 0.4}{0.9(0.4) + 0.05(0.5) + 0.05(0.1)}$$

$$= 0.923$$

3. Let Y_n be the number of left jumps after n jumps. Since a left jump happens w/ probability p, we have that $Y_n \sim Bin(n, p)$

The number of right jumps after n jumps is $n - Y_n$, so
X_n = right jumps − left jumps = $n - Y_n - Y_n = n - 2Y_n$

$$E[X_n] = E[n - 2Y_n]$$ taking out constants,
$$= n - 2E[Y_n]$$ since $Y_n \sim Bin(n, p)$,
$$= n - 2np$$
$$= n(1-2p)$$

$$V[X_n] = V[n - 2Y_n]$$ taking out constants,
$$= V[-2Y_n]$$
$$= 4V[Y_n]$$ since $Y_n \sim Bin(n, p)$,
$$= 4np(1-p)$$

4. $$f_{X,Y}(x,y) = \frac{e^{-x/y}e^{-y}}{y} \qquad 0 < x < \infty \qquad 0 < y < \infty$$

$$E[X \mid Y=y] = \int_0^\infty x f_{X|Y}(x|y)\,dx$$

$$= \int_0^\infty x \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}\,dx$$

$$= \frac{e^{-y}}{y}\int_0^\infty e^{-x/y}\,dx$$

$$= \frac{e^{-y}}{y} \cdot -ye^{-x/y}\Big|_0^\infty = -e^{-y}e^{-x/y}\Big|_0^\infty$$

$$= \left(-e^{-y}(0)\right) - \left(-e^{-y}\right) = e^{-y}$$

$$E[X \mid Y=y] = \int_0^\infty x \frac{e^{-x/y}e^{-y}}{y} \cdot \frac{1}{e^{-y}}\,dx = \frac{1}{y}\int_0^\infty x e^{-x/y}\,dx$$

$u = x \qquad dv = e^{-x/y}dx$

$du = dx \qquad v = -ye^{-x/y}$

$$= \frac{1}{y}\left[-xye^{-x/y} - \int -ye^{-x/y}\,dx\right]$$

$$= \frac{1}{y}\left[-xye^{-x/y} + y\int e^{-x/y}\,dx\right]$$

$$= \frac{1}{y}\left[-xye^{-x/y}\Big|_0^\infty + y(-ye^{-x/y})\Big|_0^\infty\right]$$

$$= \frac{1}{y}\left[0 + (y^2)\right] = y$$

5. $M_{X,Y}(s,t) = \exp(1 + s + s^2 - \cos t + s^{2023} t)$

$$M_{X,Y}(s,0) = E[\exp(sX + y(0))]$$
$$= E[\exp(sX)]$$
$$= M_X(s)$$

$$M_X(s) = M_{X,Y}(s,0) = \exp(1 + s + s^2 - 1)$$
$$= \exp(s + s^2)$$
$$= \exp((1)s + 2s^2/2) \Rightarrow \mu = 1, \ \sigma^2 = 2$$

Therefore, $X \sim N(1,2)$

6. $\mathbb{P}(2 < X < 8) = \mathbb{P}(-3 < X-5 < 3)$

$= \mathbb{P}(|X-5| < 3)$

$= \mathbb{P}(|X-m| < 3)$

$= 1 - \mathbb{P}(|X-m| \geq 3)$

$\geq 1 - \frac{\sigma^2}{3^2}$ by Chebyshev's Inequality

$= 1 - \frac{3}{9} = \frac{2}{3}$

$\mathbb{P}(2 < X < 8) \geq \frac{2}{3}$

7. $P\left(\sum_{i=1}^{n} T_i > 2E\left[\sum_{i=1}^{n} T_i\right]\right) = P\left(\sum_{i=1}^{n} T_i > 2\sum_{i=1}^{n} E[T_i]\right)$

$= P(n\overline{T}_n > 2n\mu) = P(\overline{T}_n > 2\mu) = 1 - P(\overline{T}_n \le 2\mu)$

$= 1 - P(\overline{T}_n - \mu \le \mu)$

$= 1 - P\left(\frac{\overline{T}_n - \mu}{\sigma/\sqrt{n}} \le \frac{\mu}{\sigma\sqrt{n}}\right)$

By CLT, $= 1 - \Phi\left(\frac{\mu}{\sigma\sqrt{n}}\right)$

$= \Phi\left(-\frac{\mu}{\sigma/\sqrt{n}}\right)$

8. a) $E[X] = E\left[E[X \mid \text{raininess}]\right]$

$$= E[X \mid \text{rainy}]\,\mathbb{P}(\text{rainy}) + E[X \mid \text{dry}]\,\mathbb{P}(\text{dry})$$

$$= 9(0.1) + 3(0.9)$$

$$= 3.6$$

b) $\mathbb{P}(X=0) = \mathbb{P}(X=0 \mid \text{rainy})\,\mathbb{P}(\text{rainy}) + \mathbb{P}(X=0 \mid \text{dry})\,\mathbb{P}(\text{dry})$

$$= e^{-\lambda_r}\frac{\lambda_r^0}{0!}(0.1) + e^{-\lambda_d}\frac{\lambda_d^0}{0!}(0.9)$$

$$= e^{-9}(0.1) + e^{-3}(0.9)$$

$$= 0.0448$$

c) $V[X] = E[X^2] - (E[X])^2$

$$= E[X^2 \mid \text{rainy}]\,\mathbb{P}(\text{rainy}) + E[X^2 \mid \text{dry}]\,\mathbb{P}(\text{dry}) - 3.6^2$$

$$= (\lambda_r^2 + \lambda_r)(0.1) + (\lambda_d^2 + \lambda_d)(0.9) - 3.6^2$$

$$= (90)(0.1) + (12)(0.9) - 3.6^2$$

$$= 6.84$$