1\. a) $E[W_2] = E[\max(0, T_1 - 30)]$

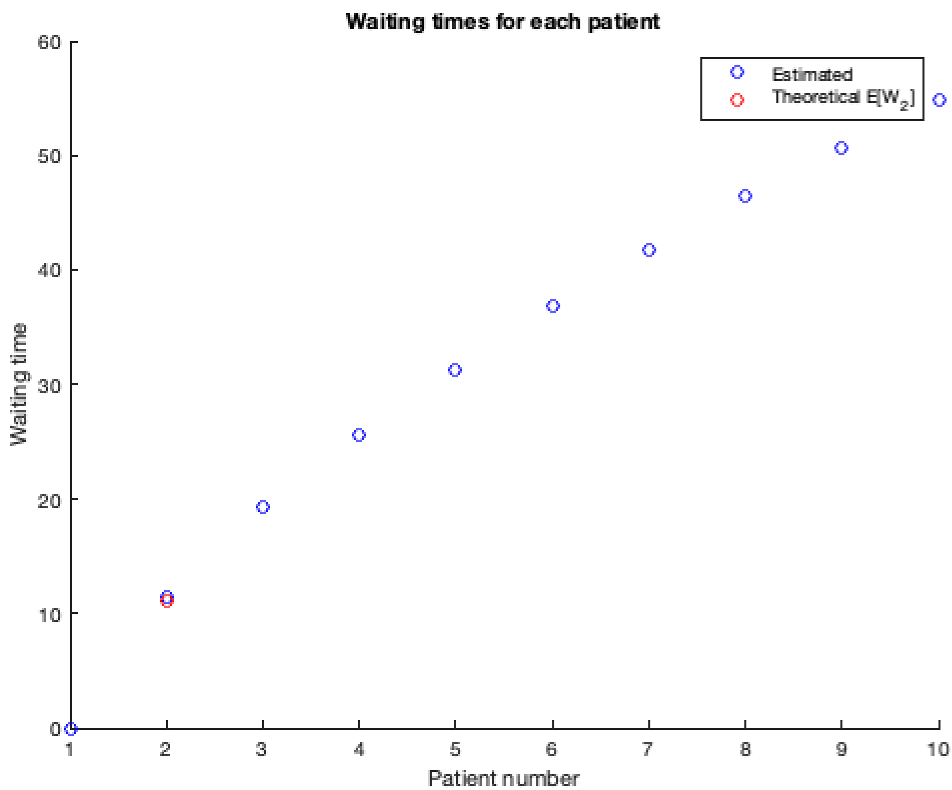
$$= \int_{\Delta t}^{\infty} t \, \mathbb{P}(T_1 = t) \, dt = \int_{\Delta t}^{\infty} (t - \Delta t) \frac{1}{\Delta t} e^{-t/\Delta t} dt = \frac{30}{e}$$

b) See code

c) See code

```
N = 10^4;
n = 10;
delta_t = 30;
W_2_theoretical = 30 / exp(1);

W = zeros(n, 1);
for i=1:N
    time = 0;
    for j=1:n
        appointment_time = delta_t * (j - 1);
        time = max(time, appointment_time);
        W(j) = W(j) + time - appointment_time;

        T = exprnd(delta_t);
        time = time + T;
    end
end

W = W / N;
disp(W);

hold on
scatter(1:n, W, 'b');
scatter(2, W_2_theoretical, 'r');
legend('Estimated', 'Theoretical E[W_2]');
title('Waiting times for each patient');
xlabel('Patient number');
ylabel('Waiting time');
hold off
```

```
                         0
          11.4758346385871
          19.3529656204703
           25.609319392826
          31.2867758474249
          36.8643891265663
          41.7048149289427
          46.4554314710373
          50.7093955342013
          54.8064254991106
```



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```
N = 10^4;
n = 10;
delta_t = 30;

W = zeros(n, 1);
for i=1:N
    time = 0;
    for j=1:n
        appointment_time = delta_t * (j - 1);
        time = max(time, appointment_time);
        W(j) = W(j) + time - appointment_time;

        T = exprnd(delta_t);
        time = time + T;
    end
end

W = W / N;
disp(W(10));

% trying 20 patients with delta_t = 15
n = 20;
delta_t = 15;

W = zeros(n, 1);
for i=1:N
    time = 0;
    for j=1:n
        appointment_time = delta_t * (j - 1);
        time = max(time, appointment_time);
        W(j) = W(j) + time - appointment_time;

        T = exprnd(delta_t);
        time = time + T;
    end
end

W = W / N;
disp(W(20));

% The estimation of the wait time for the 10th patient with delta_t = 30
% minutes is around 53 minutes, whereas the estimation of the wait time for
the 20th
% patient with delta_t = 15 minutes is around 43 minutes. Therefore, it is
% better to be the 20th patient with delta_t = 15 minutes.
```

54.2653792275777

42.1085009720776

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2. a) $\mathbb{P}(N_{t_2} - N_{t_1} = 0) = \mathbb{P}(N_{t_1+(t_2-t_1)} - N_{t_1} = 0)$

$$= e^{-\lambda(t_2-t_1)} \frac{(\lambda(t_2-t_1))^0}{1}$$

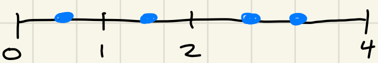
$$= e^{-\lambda(t_2-t_1)}$$

b) $\mathbb{P}(N_1 - N_0 = 1) = e^{-\lambda(1)} \frac{(\lambda(1))^1}{1} = \lambda e^{-\lambda}$

Since each interval has length one, each interval has the same probability of one event occurring since Poisson processes are stationary. Each interval is also independent, so the probability that there is one event in each interval is $(\lambda e^{-\lambda})^n = \lambda^n e^{-\lambda n}$

c) Split into 3 cases:

case 1.



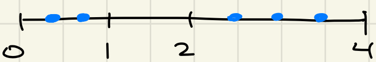
$$\mathbb{P}(\text{case } 1) = \mathbb{P}(N_1 - N_0 = 1)\,\mathbb{P}(N_2 - N_1 = 1)\,\mathbb{P}(N_4 - N_2 = 2)$$

$$= \lambda e^{-\lambda} \cdot \lambda e^{-\lambda} \cdot e^{-\lambda(2)} \frac{(2\lambda)^2}{2}$$

$$= \lambda^2 e^{-2\lambda} \cdot e^{-2\lambda} \cdot 2\lambda^2$$

$$= 2\lambda^4 e^{-4\lambda}$$

case 2.



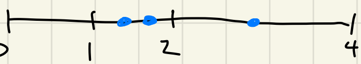
$$\mathbb{P}(\text{case } 2) = \mathbb{P}(N_1 - N_0 = 2)\,\mathbb{P}(N_2 - N_1 = 0)\,\mathbb{P}(N_4 - N_2 = 3)$$

$$= e^{-\lambda} \frac{\lambda^2}{2} \cdot e^{-\lambda} \cdot e^{-2\lambda} \frac{(2\lambda)^3}{6}$$

$$= e^{-4\lambda} \cdot \frac{\lambda^2}{2} \cdot \frac{4\lambda^3}{3}$$

$$= \frac{2}{3} \lambda^5 e^{-4\lambda}$$

case 3.



$$\mathbb{P}(\text{case } 3) = \mathbb{P}(N_1 - N_0 = 0)\,\mathbb{P}(N_2 - N_1 = 2)\,\mathbb{P}(N_4 - N_2 = 1)$$

$$= e^{-\lambda} \cdot e^{-\lambda} \frac{\lambda^2}{2} \cdot e^{-2\lambda} \frac{(2\lambda)}{1}$$

$$= \lambda^3 e^{-4\lambda}$$

$$\mathbb{P}(\text{case } 1) + \mathbb{P}(\text{case } 2) + \mathbb{P}(\text{case } 3) = 2\lambda^4 e^{-4\lambda} + \frac{2}{3}\lambda^5 e^{-4\lambda} + \lambda^3 e^{-4\lambda}$$

$$= \lambda^3 e^{-4\lambda}\left(1 + 2\lambda + \frac{2}{3}\lambda^2\right)$$

3. $F_{T_1 | N_t}(x) = \mathbb{P}(T_1 \leq x \mid N_t = 1) = \mathbb{P}(N_x - N_0 = 1 \mid N_t - N_0 = 1)$

$$= \frac{\mathbb{P}(N_x - N_0 = 1, N_t - N_x = 0)}{\mathbb{P}(N_t - N_0 = 1)}$$

$$= \frac{e^{-\lambda x} \frac{(\lambda x)}{1} \cdot e^{-\lambda(t-x)}}{e^{-\lambda t} \frac{(\lambda t)}{1}}$$

$$= \frac{(\lambda x) e^{-\lambda x - \lambda t + \lambda x}}{\lambda t \, e^{-\lambda t}}$$

$$= \frac{x}{t}$$

only applicable for $0 \leq x \leq t$

Thus, T_1 given $N_t = 1$ follows uniform distribution $U[0, t]$

4. a) $\mathbb{P}(N_\tau - N_0 = 0) = e^{-\lambda\tau}$

b) Let S denote the waiting time

Let T denote the time until the next car passes

$$E[S] = 0 \cdot \mathbb{P}(T > \tau) + \mathbb{P}(T \le \tau)(E[S] + E[T \mid T \le \tau])$$

$$E[S] = \mathbb{P}(T \le \tau) E[S] + \mathbb{P}(T \le \tau) E[T \mid T \le \tau]$$

$$E[S](1 - \mathbb{P}(T \le \tau)) = \mathbb{P}(T \le \tau) E[T \mid T \le \tau]$$

$$E[S] = \frac{\mathbb{P}(T \le \tau) E[T \mid T \le \tau]}{1 - \mathbb{P}(T \le \tau)}$$ since $T \sim \mathrm{Exp}(\lambda)$,

$$= \frac{\mathbb{P}(T \le \tau) E[T \mid T \le \tau]}{1 - (1 - e^{-\lambda\tau})}$$

$$= \frac{\mathbb{P}(T \le \tau) E[T \mid T \le \tau]}{e^{-\lambda\tau}}$$

$$= \frac{1}{e^{-\lambda\tau}} \int_0^\tau t\, \mathbb{P}(T = t)\, dt$$ since $T \sim \mathrm{Exp}(\lambda)$,

$$= \frac{1}{e^{-\lambda\tau}} \int_0^\tau t \lambda e^{-\lambda t}\, dt$$

$$= \frac{1}{e^{-\lambda\tau}} \frac{1 - e^{-\lambda\tau}(\lambda\tau + 1)}{\lambda}$$

$$= \frac{1}{e^{-\lambda\tau}} \frac{1 - \lambda\tau e^{-\lambda\tau} - e^{-\lambda\tau}}{\lambda}$$

$$= \frac{1}{\lambda}\left(e^{\lambda\tau} - \lambda\tau - 1\right)$$

5. a) $$\mathbb{P}(\text{winning}) = \mathbb{P}(N_{\tau^*} - N_{\tau} = 1)$$
$$= e^{-\lambda(\tau^* - \tau)} \frac{\lambda(\tau^* - \tau)}{1}$$
$$= \lambda(\tau^* - \tau) e^{-\lambda(\tau^* - \tau)}$$

b) $$\frac{d}{d\tau} \lambda(\tau^* - \tau) e^{-\lambda(\tau^* - \tau)} = 0$$
$$\lambda(\tau^* - \tau) e^{-\lambda(\tau^* - \tau)}(\lambda) + e^{-\lambda(\tau^* - \tau)}(-\lambda) = 0$$
$$\lambda^2(\tau^* - \tau) e^{-\lambda(\tau^* - \tau)} = \lambda e^{-\lambda(\tau^* - \tau)}$$
$$\lambda^2(\tau^* - \tau) = \lambda$$
$$\lambda(\lambda\tau^* - \lambda\tau - 1) = 0$$
$$\lambda\tau^* - \lambda\tau - 1 = 0$$
$$\lambda\tau = \lambda\tau^* - 1$$
$$\tau_{opt} = \tau^* - \frac{1}{\lambda}$$

c) $$\mathbb{P}(N_{\tau^*} - N_{\tau_{opt}} = 1) = \lambda\left(\tau^* - \left(\tau^* - \frac{1}{\lambda}\right)\right) e^{-\lambda\left(\tau^* - \left(\tau^* - \frac{1}{\lambda}\right)\right)}$$
$$= \lambda\left(\frac{1}{\lambda}\right) e^{-\lambda\left(\frac{1}{\lambda}\right)}$$
$$= \frac{1}{e}$$

d) $$E[N_{\tau^*}] = E[N_{\tau_{opt}}] + 1 \cdot \mathbb{P}(N_{\tau^*} - N_{\tau_{opt}} \geq 1)$$
$$= \lambda\tau_{opt} + (1 - \mathbb{P}(N_{\tau^*} - N_{\tau_{opt}} = 0))$$
$$= \lambda\tau_{opt} + 1 - e^{-\lambda(\tau^* - \tau_{opt})}$$
$$= \lambda\left(\tau^* - \frac{1}{\lambda}\right) + 1 - e^{-\lambda\left(\tau^* - \tau^* + \frac{1}{\lambda}\right)}$$
$$= \lambda\tau^* - e^{-1}$$
$$= \lambda\tau^* - \frac{1}{e}$$