1\. a) $E[X] = \frac{1+2}{2} = \frac{3}{2}$

$E[Y] = E[E[Y|X=x]] = E[\frac{1}{1/x}] = E[x] = \frac{3}{2}$

$E[Z] = E[E[Z|X=x, Y=y]] = E[x] = \frac{3}{2}$

$$m = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

$V[X] = \frac{(2-1)^2}{12} = \frac{1}{12}$

$V[Y] = E[V[Y|X]] + V[E[Y|X]]$

$= E[\frac{1}{(1/x)^2}] + V[X] = E[x^2] + V[X]$

$= \int_1^2 x^2 f_X(x) dx + V[X]$

$= \int_1^2 x^2 dx + \frac{1}{12}$

$= \frac{x^3}{3} \Big|_1^2 + \frac{1}{12}$

$= \frac{8}{3} - \frac{1}{3} + \frac{1}{12} = \frac{7}{3} + \frac{1}{12} = \frac{29}{12}$

$V[Z] = E[V[Z|X,Y]] + V[E[Z|X,Y]]$

$= E[V[Z|X]] + V[E[Z|X]]$

$= E[1] + V[X]$

$= \frac{13}{12}$

$cov(X,Y) = E[XY] - E[X]E[Y]$

$= E[E[XY|X=x]] - E[X]E[Y]$

$= E[X^2] - E[X]E[Y]$

$= \frac{7}{3} - \frac{3}{2}\frac{3}{2} = \frac{7}{3} - \frac{9}{4} = \frac{1}{12}$

$cov(X,Z) = E[XZ] - E[X]E[Z]$

$= E[E[XZ|X=x]] - E[X]E[Z]$

$= E[X^2] - E[X]E[Z]$

$= \frac{7}{3} - \frac{3}{2}\frac{3}{2} = \frac{1}{12}$

$cov(Y,Z) = E[YZ] - E[Y]E[Z]$

$= E[E[YZ|X=x]] - E[Y]E[Z]$

$= \frac{7}{3} - \frac{3}{2}\frac{3}{2} = \frac{1}{12}$

$$\Sigma = \begin{bmatrix} V[X] & cov(X,Y) & cov(X,Z) \\ cov(Y,X) & V[Y] & cov(Y,Z) \\ cov(Z,X) & cov(Z,Y) & V[Z] \end{bmatrix} = \begin{bmatrix} 1/12 & 1/12 & 1/12 \\ 1/12 & 29/12 & 1/12 \\ 1/12 & 1/12 & 13/12 \end{bmatrix}$$

b) See code

```
sigma = [1/12 1/12 1/12; 1/12 29/12 1/12; 1/12 1/12 13/12];
[V, D] = eig(sigma);
Q_t = transpose(V);
disp(Q_t);
```

```
    0.9963   -0.0326   -0.0796
    0.0772   -0.0670    0.9948
    0.0378    0.9972    0.0643
```

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2. $Y_1 = \sqrt{-2\log X_1}\cos(2\pi X_2)$

$Y_2 = \sqrt{-2\log X_1}\sin(2\pi X_2)$

$Y = (Y_1, Y_2)^T$

$f_Y(y) = f_X(H(y))\,|\det J_H(y)|$

$f_X(x) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{1-0}\cdot\frac{1}{1-0} = 1$

$$f_Y(y) = |\det J_H(y)| = \begin{vmatrix} \frac{\partial h_1}{\partial Y_1} & \frac{\partial h_1}{\partial Y_2} \\ \frac{\partial h_2}{\partial Y_1} & \frac{\partial h_2}{\partial Y_2} \end{vmatrix}$$

$$Y_1^2 + Y_2^2 = (-2\log X_1)\cos^2(2\pi X_2) + (-2\log X_1)\sin^2(2\pi X_2)$$
$$= (-2\log X_1)(\cos^2(2\pi X_2) + \sin^2(2\pi X_2))$$
$$= -2\log X_1$$

$$-\frac{Y_1^2 + Y_2^2}{2} = \log X_1 \quad\Rightarrow\quad X_1 = e^{-\frac{Y_1^2+Y_2^2}{2}}$$

$$\frac{Y_2}{Y_1} = \frac{\sin(2\pi X_2)}{\cos(2\pi X_2)} = \tan(2\pi X_2)$$

$$X_2 = \frac{1}{2\pi}\tan^{-1}(Y_2/Y_1)$$

Let $h_1 = X_1$, $h_2 = X_2$

$$\frac{\partial h_1}{\partial Y_1} = e^{-\frac{Y_1^2+Y_2^2}{2}}(-Y_1) \qquad \frac{\partial h_1}{\partial Y_2} = e^{-\frac{Y_1^2+Y_2^2}{2}}(-Y_2)$$

$$\frac{\partial h_2}{\partial Y_1} = \frac{1}{2\pi}\frac{1}{1+(Y_2/Y_1)^2}Y_2(-Y_1^{-2}) = \frac{1}{2\pi}\frac{-Y_2}{(1+Y_2^2/Y_1^2)Y_1^2} = -\frac{Y_2}{2\pi(Y_1^2+Y_2^2)}$$

$$\frac{\partial h_2}{\partial Y_2} = \frac{1}{2\pi}\frac{1}{1+(Y_2/Y_1)^2}\frac{1}{Y_1} = \frac{1}{2\pi}\frac{1}{1+Y_2^2/Y_1^2}\cdot\frac{1}{Y_1} = \frac{1}{2\pi}\frac{Y_1^2}{Y_1^2+Y_2^2}\frac{1}{Y_1}$$
$$= \frac{Y_1}{2\pi(Y_1^2+Y_2^2)}$$

$$f_Y(y) = \left| -Y_1 e^{-\frac{Y_1^2+Y_2^2}{2}}\frac{Y_1}{2\pi(Y_1^2+Y_2^2)} - (-Y_2)e^{-\frac{Y_1^2+Y_2^2}{2}}\frac{-Y_2}{2\pi(Y_1^2+Y_2^2)}\right|$$
$$= \left|\frac{e^{-\frac{Y_1^2+Y_2^2}{2}}}{2\pi(Y_1^2+Y_2^2)}(-Y_1^2 - Y_2^2)\right| = \left|-\frac{1}{2\pi}e^{-\frac{Y_1^2+Y_2^2}{2}}\right|$$
$$= \frac{1}{2\pi}e^{-\frac{Y_1^2+Y_2^2}{2}}$$

$$f_{Y_1}(y) = \int_{-\infty}^{\infty} f_Y(y)\, dY_2 = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{Y_1^2+Y_2^2}{2}} dY_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_1^2}{2}}$$

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_Y(y)\, dY_1 = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{Y_1^2+Y_2^2}{2}} dY_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_2^2}{2}}$$

$$f_Y(y) = \frac{1}{2\pi} e^{-\frac{Y_1^2+Y_2^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_1^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_2^2}{2}} = f_{Y_1}(y)\, f_{Y_2}(y)$$

Therefore, Y_1 and Y_2 are independent

b) $$f_{Y_1}(y) = \int_{-\infty}^{\infty} f_Y(y)\, dY_2 = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{Y_1^2+Y_2^2}{2}} dY_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_1^2}{2}}$$

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_Y(y)\, dY_1 = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{Y_1^2+Y_2^2}{2}} dY_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_2^2}{2}}$$

$Y_1 \sim N(0,1)$ and $Y_2 \sim N(0,1)$

3. a) $g(Y) = \Sigma_{XY} \Sigma_Y^{-1} (Y - \mu_Y) + \mu_X$

$$\Sigma_{XY} = \Sigma_{X,GX+W} = \Sigma_{X,GX} + \Sigma_{X,W} = \Sigma_{X,GX}$$
$$= E[X(GX)^T] - E[X]E[GX]^T$$
$$= (E[XX^T] - E[X]E[X]^T)G^T$$
$$= \Sigma_X G^T$$

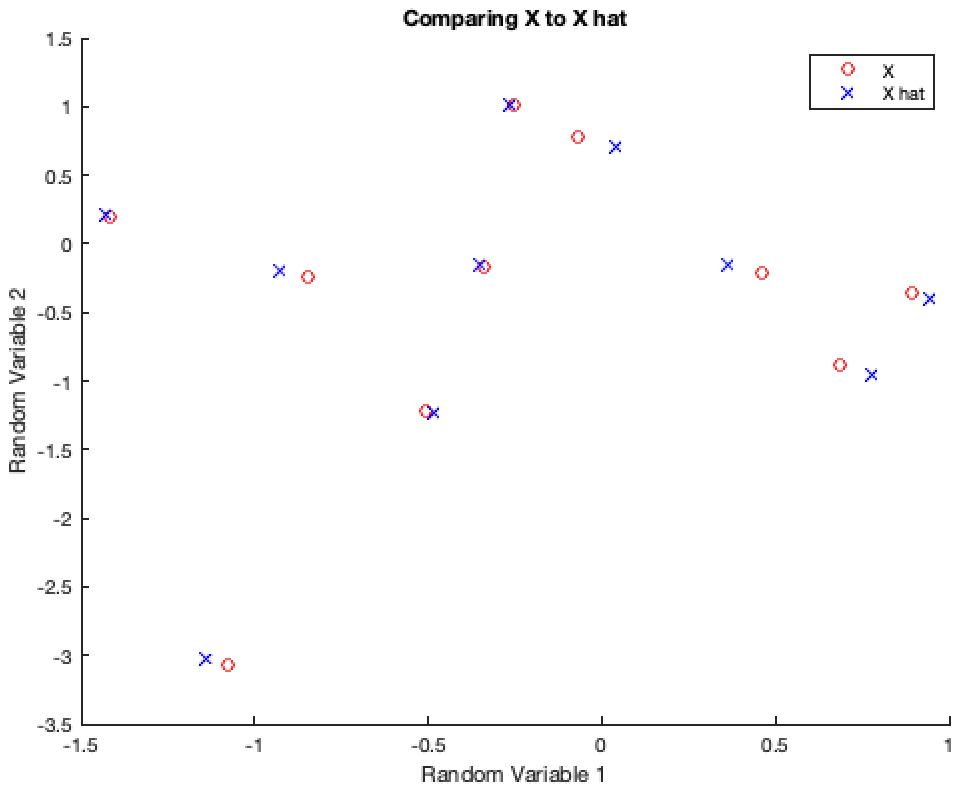
$$\Sigma_Y = \Sigma_{GX+W} = \Sigma_{GX} + \Sigma_{GX,W} + \Sigma_{W,GX} + \Sigma_W$$
$$\Sigma_{GX} = G\Sigma_X G^T$$
$$\Sigma_{GX,W} = E[GXW^T] - E[GX]E[W]^T$$
$$= (E[XW^T] - E[X]E[W]^T)G$$
$$= \Sigma_{XW} G = 0$$
$$\Sigma_{W,GX} = \Sigma_{GX,W}^T = 0$$
$$\Sigma_Y = G\Sigma_X G^T + \Sigma_W$$

$$\mu_Y = E[Y] = E[GX + W]$$
$$= E[GX] = GE[X]$$
$$= G\mu_X$$

$$g(Y) = \Sigma_X G^T (G\Sigma_X G^T + \Sigma_W)^{-1} (Y - G\mu_X) + \mu_X$$

b) See code

```
n = 10;
G = [1 2; 3, 4];
G_T = G';
e = 0.03;
X = zeros(0, 2);
X_hat = zeros(0, 2);

sigma_X = [1 0; 0 1];
sigma_W = [e^2 0; 0 e^2];
miu_X = [0; 0];

for i = 1:n
     x = normrnd(0, 1, 2, 1);
     w = mvnrnd([0; 0], sigma_W)';
     y = G * x + w;

     x_hat = sigma_X * G_T * (G * sigma_X * G_T + sigma_W)^(-1) * (y - G *
miu_X) + miu_X;

     X_hat = [X_hat; x_hat'];
     X = [X; x'];
end

hold on
scatter(X(:, 1), X(:, 2), 'red');
scatter(X_hat(:, 1), X_hat(:, 2), 'blue', 'x');
legend('X', 'X hat');
title('Comparing X to X hat');
xlabel('Random Variable 1');
ylabel('Random Variable 2');
hold off
```



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4. a) $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ 3X_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} [X_1] + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Since X can be expressed as $X = AZ + \mu$ where $Z \sim N(0, I)$,

X is Gaussian since its components are jointly distributed

b) $\mathrm{cov}(X) = CC^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

c) $\det(\Sigma_X) = 9 - 3 \cdot 3 = 0$

Σ_X is singular, so X does not have a density

5. a) If X is Gaussian, it can be expressed as

$X = BZ + \mu$, where $Z = (z_1, \dots, z_m)^T$ and $z_1 \cdots z_m \overset{iid}{\sim} N(0,1)$

Then,

$Y = A(BZ + \mu) = ABZ + A\mu$

Let $C = AB$ and $\mu_Y = A\mu$

Then,

$Y = CZ + \mu_Y$

Since $Z = (z_1, \dots, z_m)^T$ and $z_1 \cdots z_m \overset{iid}{\sim} N(0,1)$, Y's components are jointly normally distributed, and Y is normal, proving the claim.

b) We can disprove the claim by providing a counterexample:

Let X be a non-Gaussian vector:

$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ where $X_1 \sim N(0,1)$ and $X_2 \not\sim N(0,1)$

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Then,

$Y = AX = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1 \end{bmatrix}$

Since Y's components follow $\sim N(0,1)$, they are jointly normally distributed, so Y is Gaussian. However, X is not Gaussian, so the claim is disproved.