1. a)

$$\Sigma_X = \begin{bmatrix} var(X_1) & cov(X_1, X_2) & cov(X_1, X_3) \\ cov(X_2, X_1) & var(X_2) & cov(X_2, X_3) \\ cov(X_3, X_1) & cov(X_3, X_2) & var(X_3) \end{bmatrix}$$

Since covariance is symmetric,

$$\Sigma_X = \begin{bmatrix} var(X_1) & cov(X_1, X_2) & cov(X_1, X_3) \\ cov(X_1, X_2) & var(X_2) & cov(X_2, X_3) \\ cov(X_1, X_3) & cov(X_2, X_3) & var(X_3) \end{bmatrix}$$

Therefore, Σ_X is symmetric and the statement is false.

b) This statement is false – if Σ_X is singular, there is no joint pdf.

c) $B_2 \mid B_1 = 1 \sim N(1, 2-1)$

Therefore,

$E[B_2 \mid B_1 = 1] = 1 \neq 2$, so the statement is false

d) One property of $S_X(f)$ is that $S_X(-f) = S_X(f)$

$S_X(-f) = 1 - f + \sin(-f) \neq 1 + f + \sin(f)$, so this statement is false

e)

$$\Sigma_X = \begin{bmatrix} var(N_1) & cov(N_1, N_2) & cov(N_1, N_3) \\ cov(N_2, N_1) & var(N_2) & cov(N_2, N_3) \\ cov(N_3, N_1) & cov(N_3, N_2) & var(N_3) \end{bmatrix}$$

Since $cov(N_{t_1}, N_{t_2}) = \lambda \min(t_1, t_2) = \min(t_1, t_2)$

and $N_t \sim \text{Poisson}(\lambda t) \Rightarrow Var(N_t) = \lambda t = t$

$$\Sigma_X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Thus, the statement is true

2. a) $E[W] = E[E[W \mid N]]$

$= E[pN]$

$= p\lambda$

b) $V[W] = V[E[W \mid N]] + E[V[W \mid N]]$

$= V[pN] + E[Np(1-p)]$

$= p^2\lambda + \lambda p(1-p)$

3. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$g(u,v) = \frac{f\left(A^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right)}{|\det A|}$$

$$= \frac{f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)}{2}$$

4. $Y = AX + b$

$Y \sim N(A m_x + b,\ A \Sigma_x A^T)$

By spectral theorem,

$\Sigma_x = Q \Lambda Q^T$

Let $A = Q^T \Rightarrow \Sigma_Y = Q^T \Sigma_x Q = \Lambda$

Thus, the components of y are uncorrelated

$Q^T m_x + b = 0$

$b = -Q^T m_x$

5. a) $E[T] = E[\text{Exp}(2)] = \frac{1}{2}$ week

b) $\mathbb{P}(N_{t+2} - N_t = 0) = e^{-2(2)} \frac{(2\cdot 2)^0}{1} = e^{-4}$

c) $E[N_{52}] = 2(52) = 104$ attacks

d) Let X = the number of weeks out of 52 weeks with at least 7 attacks

X_i = indicator of if week i had at least 7 attacks

$X = \sum_{i=1}^{52} X_i$, each X_i is independent

$$\begin{aligned}
\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0) \\
&= 1 - \mathbb{P}(X_i = 0)^{52} \\
&= 1 - \mathbb{P}(S_7 \geq 1)^{52} \qquad S_7 \sim \text{Gamma}(7, \lambda) \\
&= 1 - (1 - \mathbb{P}(S_7 < 1))^{52} \\
&= 1 - (1 - F(1))^{52}
\end{aligned}$$

6. $R_X(t_1, t_2) = E[Y_{t_1} Y_{t_2}]$
$= E[e^{B_{t_1}} e^{B_{t_2}}]$

Case 1: $t_1 \le t_2$

$R_X(t_1,t_2) = E[e^{2B_{t_1}} e^{B_{t_2} - B_{t_1}}]$ $\quad B_{t_1}$ and $B_{t_2} - B_{t_1}$ are independent

$= E[e^{2B_{t_1}}] E[e^{B_{t_2} - B_{t_1}}]$

$= E\left[e^{2\sqrt{t_1} \frac{B_{t_1}}{\sqrt{t_1}}}\right] E\left[e^{\sqrt{t_2 - t_1} \frac{B_{t_2} - B_{t_1}}{\sqrt{t_2 - t_1}}}\right]$

$B_{t_1} \sim N(0, t_1) \Rightarrow \frac{B_{t_1}}{\sqrt{t_1}} \sim N(0,1)$

$B_{t_2} - B_{t_1} \sim N(0, t_2 - t_1) \Rightarrow \frac{B_{t_2} - B_{t_1}}{\sqrt{t_2 - t_1}} \sim N(0,1)$

$= E\left[e^{2\sqrt{t_1} Z}\right] E\left[e^{\sqrt{t_2 - t_1} Z}\right]$

$= M(2\sqrt{t_1}) M(\sqrt{t_2 - t_1})$

$= \exp(2t_1) \exp(\frac{1}{2}(t_2 - t_1))$

$= \exp\left(\frac{4t_1}{2} + \frac{t_2 - t_1}{2}\right)$

$= \exp\left(\frac{3t_1 + t_2}{2}\right)$

Case 2: $t_1 > t_2$

$R_X(t_1, t_2) = E[e^{2B_{t_2}} e^{B_{t_1} - B_{t_2}}]$

$= E[e^{2B_{t_2}}] E[e^{B_{t_1} - B_{t_2}}]$

$= E\left[e^{2\sqrt{t_2} \frac{B_{t_2}}{\sqrt{t_2}}}\right] E\left[e^{\sqrt{t_1 - t_2} \frac{B_{t_1} - B_{t_2}}{\sqrt{t_1 - t_2}}}\right]$

$= E\left[e^{2\sqrt{t_2} Z}\right] E\left[e^{\sqrt{t_1 - t_2} Z}\right]$

$= M(2\sqrt{t_2}) M(\sqrt{t_1 - t_2})$

$= \exp(2t_2) \exp(\frac{1}{2}(t_1 - t_2))$

$= \exp\left(\frac{3t_2 + t_1}{2}\right)$

To generalize both cases,

$$R_X(t_1, t_2) = \exp\left(\frac{t_1 + t_2}{2} + \min(t_1, t_2)\right)$$

7. a) $\mu_X(t) = E[X_t] = E[X_0(-1)^{N_t}]$

$= E[X_0]\,E[(-1)^{N_t}]$

$= [(1)\frac{1}{2} + (-1)\frac{1}{2}]\,E[(-1)^{N_t}]$

$= 0$

b) $R_X(t_1, t_2) = E[X_{t_1} X_{t_2}]$

$= E[X_0(-1)^{N_{t_1}} X_0(-1)^{N_{t_2}}]$

$= E[X_0^2]\,E[(-1)^{N_{t_1}+N_{t_2}}]$

$= (1^2(\frac{1}{2}) + (-1)^2(\frac{1}{2}))\,E[(-1)^{N_{t_1}+N_{t_2}}]$

$= E[(-1)^{N_{t_1}+N_{t_2}}]$

Case 1: $t_1 \leq t_2$

$R_X(t_1, t_2) = E[(-1)^{2N_{t_1} - N_{t_1} + N_{t_2}}]$

$= E[(-1)^{2N_{t_1}}]\,E[(-1)^{N_{t_2}-N_{t_1}}]$

$= E[(-1)^{N_{t_2}-N_{t_1}}]$

Let $Y = N_{t_2} - N_{t_1} \sim \text{Poisson}(\lambda(t_2 - t_1))$

$$= \sum_{y=0}^{\infty} (-1)^y \frac{[\lambda(t_2-t_1)]^y e^{-\lambda(t_2-t_1)}}{y!}$$

$$= e^{-\lambda(t_2-t_1)} \sum_{y=0}^{\infty} \frac{[-\lambda(t_2-t_1)]^y}{y!}$$

$$= e^{-\lambda(t_2-t_1)}\,e^{-\lambda(t_2-t_1)}$$

$$= e^{-2\lambda(t_2-t_1)}$$

Case 2: $t_1 > t_2$

Using the same procedure but swapping t_1 with t_2,

$R_X(t_1, t_2) = e^{-2\lambda(t_1-t_2)}$

To generalize $R_X(t_1, t_2)$ to both cases,

$R_X(t_1, t_2) = e^{-2\lambda|t_1-t_2|}$

c) Since $\mu_X(t) = 0$ is constant and $R_X(t_1, t_2) = e^{-2\lambda|t_1-t_2|}$ depends only on the difference $t_1 - t_2$, $\{X_t, t \geq 0\}$ is WSS

8. $S_X(f) = e^{-f^2/2}$

$$E[X_t^2] = \int_{-\infty}^{\infty} e^{-f^2/2}\, df$$

$= \sqrt{2\pi}$ (from calculator)

9. done