Convex Hull Formulations for Linear Modeling of Energy Storage Systems

David Pozo, *Senior Member, IEEE*

***Abstract*—In this letter, two formulations of the linear convex hull of an energy storage system (ESS) are presented. The convex hulls are constructed from the standard parameters of an ESS, namely the charging and discharging power rate capacities, charging and discharging efficiencies, and energy capacity limits. These formulations provide the tightest linear approximation of an ESS model that takes into account the complementarity of charging and discharging modes.**

***Index Terms*—Energy storage systems, battery units, convex hull, complementarity constraint.**

I. Introduction

The non-simultaneous charging and discharging modeling of energy storage systems (ESS), usually involves using binary variables. It is an either-or condition. Thus, the ESS formulations that capture such complementarity are non-convex. This can impose burdens and impracticalities in real-world scenarios, prompting efforts to identify conditions that could remove the need for binary variables and constraints that represent complementarity between charging and discharging states. However, it is important to note that these conditions cannot be universally applied to all problems; instead, they are closely tied to the specific application of the problem itself.

The main contribution of this letter is to provide two linear convex hull formulations for the operation of ESS that consider charging and discharging mode complementarity. It has been demonstrated that the explicit construction of the ESS convex hull is possible since all of its vertices can be derived analytically beforehand, relying on the standard limits of the ESS model. A collection of observations that constitute the main takeaways of this letter are presented.

II. Standard ESS Formulation With Complementarity

ESS operation can be modeled in the standard power-energy domain. Three sets of variables for every period of operation, namely, charging power p_t^c, discharging power p_t^d, and state of energy (SoE) e_t, are typically used[1]. In addition, ESS models may consider the complementarity of charging and discharging power, therefore, no simultaneous charging and discharging can happen. This implies additional binary variables or any binary indicator construct, that capture the disjunctive nature of the charging/discharging ESS operation.

Model 1 ESS with complementary (**ESS-MILP**)

$$\mathcal{P}^{0\text{-}1} = \Bigg\{ e_t, p_t^c, p_t^d \in \mathbb{R}_{\geq 0}, z_t, y_t \in \{0,1\} \;\Bigg|$$

$$e_{t+1} = e_t + \eta_c p_t^c \Delta - \frac{1}{\eta_d} p_t^d \Delta \quad (1a)$$

$$0 \leq p_t^c \leq \overline{P}^c z_t \quad (1b)$$

$$0 \leq p_t^d \leq \overline{P}^d y_t \quad (1c)$$

$$\underline{E} \leq e_t \leq \overline{E} \quad (1d)$$

$$z_t + y_t \leq 1 \Bigg\} \quad (1e)$$

Model (1) represents one of the simplest instances of an ESS model. It accounts for the backlog energy in the ESS (1a), charging and discharging power capacity rates (1b) and (1c), and the minimum and maximum SoE limits (1d). For the sake of simplicity, initial and final energy storage levels are omitted. Δ represents the time resolution, e.g. $\Delta = 1$ for one-hour period resolution. Non-simultaneous charging and discharging is modeled using binary variables z_t and y_t to represent the charging mode ($z_t = 1, y_t = 0$), the discharging mode ($z_t = 0, y_t = 1$), and idle mode ($z_t = 0, y_t = 0$) (1e). Due to the presence of binary variables, this formulation leads to a set of mixed-integer linear expressions defined here by the non-convex set $\mathcal{P}^{0\text{-}1}$.

The feasible operation region of **ESS-MILP** can be easily represented for a particular period t in a three-dimensional (3D) space of variables p_t^c, p_t^d, and e_t. Fig. 1 illustrates a 3D representation of the **ESS-MILP** feasible space in t, as well as projections onto the 2D (p_t^c, e_t)-space when ESS is in charging mode (in pink color), and (p_t^d, e_t)-space when ESS is in discharging mode (in green color).

The charging capacity p_t^c is always limited by the power capacity rate $\overline{P}^c$. However, when the energy stored in the ESS is close to the total capacity (i.e., large e_t), tighter limits apply to the charging capacity rate p_t^c. In this case, the remaining energy capacity for fully charging the ESS, i.e. $(\overline{E} - e_t)/\eta^c$, limits the charging power (segment C_3-C_4). Similarly, we can derive limits

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The author resides in Puebla del Príncipe 13342, Spain (e-mail: davidpozocamara@gmail.com).

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[1]Without loss of generality, *charging and discharging power* and *state of energy* are used as the three main variables representing an ESS that resembles battery technologies. Other domains, such as hydro, gas storage, and power-to-X among others, can use different terms.

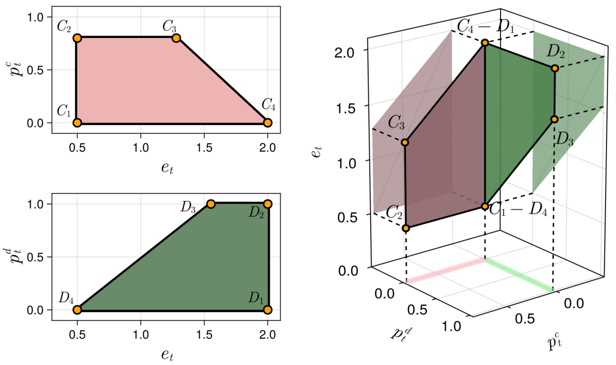

Fig. 1. Feasible region of the ESS model **ESS-MILP** (1) in period t, $\mathcal{P}_t^{0\text{-}1}$. *Top-left:* Projection onto the (p_t^c, e_t)-space, i.e., $z_t = 1$ and $y_t = 0$. *Bottom-left:* Projection onto the (p_t^d, e_t)-space, i.e., $z_t = 0$ and $y_t = 1$. *Right:* 3D representation on the (p_t^c, p_t^d, e_t)-space and projections. The EES parameter values used in this example: $\Delta = 1, \overline{E} = 2, \underline{E} = 0.5, \overline{P}^c = 0.8, \overline{P}^d = 1, \eta_d = 0.95$, and $\eta^c = 0.9$.

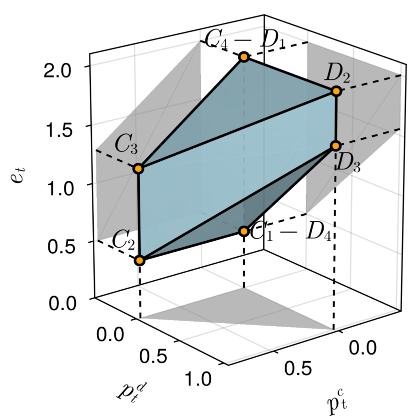

Fig. 2. Convex hull of the **ESS-MILP** operation region in blue color and projections in gray color. ESS parameters are the same as the example from Fig. 1.

for the discharging capacity rates p_t^d when the ESS is almost emptied.

Observation 1. Preliminaries: The ESS feasible operating region in charging and discharging mode only are nonempty polytopes in the (p_t^c, p_t^d, e_t)-space.

In other words, the charging-only mode feasible operation in the (p_t^c, p_t^d, e_t)-space is a linear and nonempty convex region. It is represented in pink color in Fig. 1. The same applies to the discharging-only mode of feasible operation (Fig. 1, region in green color).

III. Convex Hull Formulation

Based on *Observation 1*, it is possible to use the so-celebrated Balas's findings to formulate the convex hull of a union of two polyhedra [1]. It is represented in Fig. 2. Interestingly, the vertices of the ESS feasible operating region can be explicitly defined.

Observation 2. V-representation convex hull: The convex hull of the **ESS-MILP** *can be explicitly built by using a linear convex combination of the vertices of the ESS feasible region.*

Depicted in Fig. 2 and listed in matrix $\mathbf{V}$, the six vertices of the feasible space of ESS operation can be used for building the linear convex hull of the **ESS-MILP** . The six vertices have an explicit definition dependent on the standard ESS parameters (capacity rates, efficiencies, and storing capacity)[2]. Observe that vertices, $\mathbf{V}$, are not dependent on the time period t.

$$\mathbf{V} = \begin{array}{c} \begin{array}{ccc} \hat{\mathbf{p}}^{\mathbf{c}} & \hat{\mathbf{p}}^{\mathbf{d}} & \hat{\mathbf{e}} \end{array} \\ \begin{bmatrix} 0 & 0 & \underline{E} \\ \overline{P}^c & 0 & \underline{E} \\ \overline{P}^c & 0 & \overline{E} - \overline{P}^c \Delta \eta^c \\ 0 & 0 & \overline{E} \\ 0 & \overline{P}^d & \overline{E} \\ 0 & \overline{P}^d & \underline{E} + \overline{P}^d \Delta / \eta^d \end{bmatrix} \end{array} \begin{array}{l} C_1 - D_4 \\ C_2 \\ C_3 \\ C_4 - D_1 \\ D_2 \\ D_3 \end{array}$$

Model 2 V-Representation Convex Hull (**VCH-LP**)

$$\mathcal{P}^{\text{VCH}} = \Big\{ e_t, p_t^c, p_t^d, \lambda_{tk} \in \mathbb{R}_{\geq 0} \;\Big|$$

$$e_{t+1} = e_t + \eta_c p_t^c \Delta - \frac{1}{\eta_d} p_t^d \Delta \tag{2a}$$

$$p_t^c = \sum_k \hat{p}_k^c \lambda_{tk} \tag{2b}$$

$$p_t^d = \sum_k \hat{p}_k^d \lambda_{tk} \tag{2c}$$

$$e_t = \sum_k \hat{e}_k \lambda_{tk} \tag{2d}$$

$$\sum_k \lambda_{tk} = 1 \quad \Big\} \tag{2e}$$

The V-representation of the convex hull **VCH-LP** is given by the polyhedron $\mathcal{P}^{\text{VCH}}$ defined in (2). A non-negative weighting variable λ_{tk} is used for describing the convex hull in the (p_t^c, p_t^d, e_t)-space as a convex linear combination of all vertices. We use k for indexing each of the vertices and the hat symbol for indicating their actual coordinates/values of the vertices.

Observation 3. H-representation convex hull: The convex hull of the **ESS-MILP** *can be explicitly built by the half-spaces employing information of the vertices of the ESS feasible region.*

Similarly to the **VCH-LP**, we make use of the vertices and some algebra to formulate the hyperplane equations that build the **ESS-MILP** convex hull. The H-representation of the convex hull **HCH-LP** is defined by (3).

Observation 4. HCH-LP model correspondence: There are some equivalences between the proposed convex hull formulations and the existing literature.

i) The set (3a)–(3d) is exactly the same as one of the simplest and most used ideal and generic linear ESS

[2]This is true for ESS of more than 1-hour duration, i.e., the majority of ESS with application to power systems. The general version of the vertices coordinates, $\mathbf{V}$, is given in the online appendix [2].

Model 3 H-Representation Convex Hull (**HCH-LP**)

$$\mathcal{P}^{\mathrm{HCH}} = \Bigg\{ e_t, p_t^c, p_t^d \in \mathbb{R}_{\geq 0} \;\Bigg|$$

$$e_{t+1} = e_t + \eta_c p_t^c \Delta - \frac{1}{\eta_d} p_t^d \Delta \tag{3a}$$

$$p_t^c \leq \overline{P}^c \tag{3b}$$

$$p_t^d \leq \overline{P}^d \tag{3c}$$

$$\underline{E} \leq e_t \leq \overline{E} \tag{3d}$$

$$\frac{p_t^c}{\overline{P}_t^c} + \frac{p_t^d}{\overline{P}_t^d} \leq 1 \tag{3e}$$

$$e_t \leq \overline{E} - \eta_c p_t^c \Delta \tag{3f}$$

$$e_t \geq \underline{E} + \frac{1}{\eta_d} p_t^d \Delta \quad \Bigg\} \tag{3g}$$

models. It captures the energy inventory (3a); charging and discharging capacity fixed rates, (3b) and (3c), and storage limits (3d).

ii) Both, the **VCH-LP** (2) and **HCH-LP** (3), are equivalent. The V- and H-representation of a convex hull are proved to be equivalent by the Minkowski-Weyl theorem [3]. Thus, either formulation should lead to the same outcomes.

iii) The set (3a)–(3e) is equivalent to relaxing the binary variables in the **ESS-MILP** formulation, i.e., $\{(z_t, y_t) \in \mathbb{R} \mid 0 \leq z_t \leq 1, 0 \leq y_t \leq 1\}$.

IV. Numerical Experiments

The two proposed convex hull formulations (**HCH-LP** and **VCH-LP**) are assessed in this study, alongside the mixed-integer formulation (**ESS-MILP**), and a simplified linear programming ESS model, referred to as **Simp-LP** (*Observation 4 (i)*). The Set-Point Tracking (SPT) problem is utilized for numerical analysis. Proposed in the context of a household with photovoltaic (PV) generation. The SPT strategy aims to schedule the control of a battery's charging and discharging actions to follow a power signal. This signal is equal to a household's power demand, offset by the power generated from rooftop PV modules.

$$\min \sum_{t \in T} \left[\left(p_t^d - p_t^c \right) - p_t^{\mathrm{sig}} \right]^2, \text{ s.t. ESS model} \tag{4}$$

The optimal control of the battery's charge-discharge operations is determined as the output of the (4) equation. The problem is defined within a 24-hour horizon, with hourly intervals, and executed 100 times. Each instance is generated by randomly selecting unique battery parameters and PV profiles. The relevant data is in [2].Table I provides a summary of the results for each ESS formulation. None of the LP models were entirely successful in circumventing this simultaneous charging and discharging (column 2). However, convex hull formulations show a notable reduction (about half of cases) on this issue, in comparison to **Simp-LP** . Importantly, while the convex hull formulations offer the tighter LP approach of the actually feasible region of the ESS, they do not prevent simultaneous charging and discharging, as evidenced in this study. The last column represents the average resolution time for each problem instance. The **ESS-MILP** model proves more challenging to solve due to the binary variables. The **Simp-LP** model provides the fastest solution. Among the two convex formulations, **HCH-LP** outperforms **VCH-LP** in this application context, possibly due to the additional 144 decision variables, λ_{tk}, that **VCH-LP** entails.

TABLE I
Summary of Numerical Experiments

ESS Formulation	$\lvert p_t^c p_t^d \rvert > 10^{-4}$ [%]	Time [ms]
ESS-MILP	0.0	5
HCH-LP	15.5	0.76
VCH-LP	15.5	1.12
Simp-LP	29.0	0.47

V. Conclusion

The discussion in this letter is motivated by the enthusiasm for finding ESS linear formulations. We highlight the importance that open fronts in existing state-of-the-art with regard to relaxing complementarity conditions for ESS modeling. Two convex hull formulations are explicitly built using standard ESS limits. Such convex hulls can provide an additional toolkit for power system modelers. An illustrative example showcases the use of the proposed convex hull formulations.

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