Multi-Stage Linear Decision Rules for Stochastic Control of Natural Gas Networks with Linepack (Online Appendix)

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I. MULTI-STAGE OPTIMAL POWER FLOW PROBLEM FORMULATION

We model power system as a network with N nodes and Λ transmission lines connecting those nodes. At any stage t of the control horizon, power generation from conventional and renewable units must satisfy L loads collected in vector $d_t \in \mathbb{R}_+^L$ and allocated in the network according to incidence matrix $M_d \in \mathbb{R}^{N \times L}$. Towards this goal, the power system operator dispatches P conventional and R renewable generation units, which are allocated in the network according to incidence matrices $M_g \in \mathbb{R}^{N \times P}$ and $M_r \in \mathbb{R}^{N \times R}$, respectively, while meeting the minimum/maximum generation limits $\underline{g}/\overline{g} \in \mathbb{R}_+^P$ of conventional generators and power flow limits $\overline{f} \in \mathbb{R}^\Lambda$. The cost of conventional generation is modeled using a linear function with coefficients collected in vector $c \in \mathbb{R}_+^P$. The power flows are modelled using the *DC* power flow approximation and the matrix of power transfer distribution factors $F \in \mathbb{R}^{N \times \Lambda}$. Given the stochastic renewable in-feed $\tilde{r}_t(\zeta^t)$ from Section II-C, the multi-stage optimal power flow problem is addressed as the following stochastic program:

$$\underset{\tilde{g}_t}{\text{minimize}} \quad \mathbb{E}_{\mathbb{P}_\zeta}\left[\sum_{t=1}^T c^\top \tilde{g}_t(\zeta^t)\right] \tag{1a}$$

$$\text{subject to:} \quad \mathbb{P}_{\zeta^t}\left[\mathbb{1}^\top(M_g\tilde{g}_t(\zeta^t) + M_r\tilde{r}_t(\zeta^t) - M_dd_t) = 0\right] = 1, \tag{1b}$$

$$\mathbb{P}_{\zeta^t}\left[\begin{array}{l} -\overline{f} \leqslant F(M_g\tilde{g}_t(\zeta^t) + M_r\tilde{r}_t(\zeta^t) - M_dd_t) \leqslant \overline{f}, \\ \underline{g} \leqslant \tilde{g}_t(\zeta^t) \leqslant \overline{g}, \end{array}\right] \geqslant 1-\varepsilon_t, \quad \forall t \in \mathcal{T}, \tag{1c}$$

where the objective function to be minimized is the expected generation costs, subject to two sets of probabilistic constraints. Constraint (1b) is enforced to preserve the power balance at each time stage with probability 1, and constraint (1c) is enforced to satisfy the power flow and generation limits jointly at every time stage with probability at least $1 - \varepsilon_t$, where ε_t is a small parameter chosen by the power system operator. Defining generator response as the following multi-stage linear decision rule:

$$\tilde{g}_t(\zeta^t) = G_tS_t\zeta, \quad \forall t \in \mathcal{T}, \tag{2}$$

with matrix $G \in \mathbb{R}^{N \times k^t}$ of variables, and taking the same reformulation steps as for the gas network optimization problem in Section III-A, this chance-constrained program reformulates into a computationally tractable SOCP program:

$$\underset{G_t, y_t^f, x_t^f}{\text{minimize}} \quad \sum_{t=1}^T c^\top G_tS_t\widehat{\mu} \tag{3a}$$

$$\text{subject to:} \quad \mathbb{1}^\top(M_gG_t + M_r\Omega_t - M_d\begin{bmatrix} d_t & \mathbb{0}_{L \times k^t-1} \end{bmatrix}) = \mathbb{0}, \tag{3b}$$

$$\sqrt[-2]{\overline{\varepsilon}_t}\left\|\begin{array}{c} \widehat{F}[(M_gG_t + M_r\Omega_t - M_d\begin{bmatrix} d_t & \mathbb{0}_{L \times k^t-1} \end{bmatrix})S_t]_\ell^\top \\ y_{t\ell}^f \end{array}\right\| \leqslant \overline{f}_\ell - x_{t\ell}^f, \tag{3c}$$

$$\sqrt[-2]{\overline{\varepsilon}_t}\left\|\begin{array}{c} \widehat{F}[G_tS_t]_i^\top \\ y_{ti}^g \end{array}\right\| \leqslant \tfrac{1}{2}(\overline{g}_i - \underline{g}_i) - x_{ti}^g, \tag{3d}$$

$$\left|[(M_gG_t + M_r\Omega_t - M_d\begin{bmatrix} d_t & \mathbb{0}_{L \times k^t-1} \end{bmatrix})S_t]_\ell\widehat{\mu}\right| \leqslant y_{t\ell}^f + x_{t\ell}^f, \tag{3e}$$

$$\left|[G_tS_t]_i\widehat{\mu} - \tfrac{1}{2}(\overline{g}_i - \underline{g}_i)\right| \leqslant y_{ti}^g + x_{ti}^g, \tag{3f}$$

$$\overline{f}_\ell \geqslant x_{t\ell}^f \geqslant 0, \ y_{t\ell}^f \geqslant 0, \tag{3g}$$

$$\tfrac{1}{2}(\overline{g}_i - \underline{g}_i) \geqslant x_{ti}^g \geqslant 0, \ y_{ti}^g \geqslant 0, \tag{3h}$$

$$\forall t = 1,\ldots,T, \ \forall i = 1,\ldots,N, \ \forall \ell = 1,\ldots,\Lambda, \tag{3i}$$

where $\widehat{\mu}$ is the mean vector of renewable power deviations and $\widehat{F}$ is the Cholesky decomposition of covariance matrix $\widehat{\Sigma}$ of renewable power deviations. Here, linear constraint (3b) is the reformulation of (1b), and the set of conic and linear constraints (3c)–(3h) is a distributionally robust reformulation of the joint chance constraint (1c) on power flows and generation limits, where $\bar{\varepsilon}_t$ is the individual constraint violation probability. The power system operator optimizes variable G of generator response (2) using program (3) and then converts this information into the stochastic gas extraction as explained in Section II-C.

II. Network Topology Optimization Problem

To obtain a MISOCP reformulation of the network topology optimization problem from Section III-B3, conider the following steps. First, the linearized gas flow equation

$$\tilde{\phi}_{tc}(\zeta^t) = w_{0tc} + W_{1tc}\tilde{\varrho}_t(\zeta^t) + W_{2tc}\tilde{\kappa}_t(\zeta^t), \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}, \tag{4a}$$

is formulated in multi-stage linear decision rules as

$$F_{tc}S_t\zeta = w_{0tc} + W_{1tc}P_tS_t\zeta + W_{2tc}K_tS_t\zeta, \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}, \tag{4b}$$

where $F_{tc} \in \mathbb{R}^{E \times k^t}$ is the finite matrix of flow variable coefficients subject to optimization. Then, taking the path outlined in Section III-A2, this reformulates into the following set of linear constraints

$$\left(F_{tc} - \begin{bmatrix} w_{0tc} & \mathbb{0}_{E \times (k^t - 1)} \end{bmatrix} - W_{1tc}P_t - W_{2tc}K_t\right)S_t = \mathbb{0}, \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}. \tag{4c}$$

The stochastic reference node pressure equality in (10c) reformulates similarly as before:

$$\tilde{\varrho}_{rt}(\zeta^t) = \sum_{c=1}^{C} v_c \mathring{\varrho}_{rtc} \iff \left([P_t]_r - \sum_{c=1}^{C} v_c \begin{bmatrix} \mathring{\varrho}_{rtc} & \mathbb{0}_{1 \times (k^t-1)} \end{bmatrix}\right)S_t = \mathbb{0}, \quad \forall t \in \mathcal{T}. \tag{5}$$

Then, the stochastic gas flow constraint in (10c) is first reformulated as a set of linear equations:

$$\tilde{\varphi}_t(\zeta^t) = \sum_{c=1}^{C} v_c \tilde{\phi}_{tc}(\zeta^t)$$

$$\iff \Phi_t S_t \zeta = \sum_{c=1}^{C} v_c F_{tc} S_t \zeta$$

$$\overset{Z_{tc}=v_c F_{tc}}{\iff} \Phi_t S_t \zeta = \sum_{c=1}^{C} Z_{tc} S_t \zeta$$

$$\iff (\Phi_t - \sum_{c=1}^{C} Z_{tc})S_t = \mathbb{0}, \quad \forall t \in \mathcal{T}, \tag{6a}$$

where variable $Z_{tc} \in \mathbb{R}^{E \times k^t}$ substitutes the product of binary variable v_c and continuous variable F_{tc}. This bilinear term is next reformulated using the standard Big-M approach:

$$Z_{tc}^{ij} = v_c F_{tc}^{ij} \iff \begin{cases} \underline{M} \leqslant Z_{tc}^{ij} \leqslant \overline{M} \\ \underline{M}v_c \leqslant Z_{tc}^{ij} \leqslant \overline{M}v_c \\ F_{tc}^{ij} - (1-v_c)\overline{M} \leqslant Z_{tc}^{ij} \leqslant F_{tc}^{ij} - (1-v_c)\underline{M} \end{cases} \quad \forall t \in \mathcal{T}, \forall c \in \mathcal{C}, \forall i = 1, \dots, E, \forall j = 1, \dots, k^t, \tag{6b}$$

where Z_{tc}^{ij} (and F_{tc}^{ij}) denotes the entry of matrix Z_{tc} (and F_{tc}) at position (i,j), and $\overline{M}$ and $\underline{M}$ are sufficiently large positive and negative constants, respectively. Then, we obtain the following tractable MISOCP reformulation:

$$\underset{\Theta_t, P_t, K_t, \Psi_t, \Phi_t^{(\cdot)}, Z_{tc}, F_{tc}, x_t^{(\cdot)}, y_t^{(\cdot)}, v_c}{\text{minimize}}$$

$$\sum_{t=1}^{T} \left(c_1^\top \Theta_t S_t \widehat{\mu} + \text{Tr}\left[\Theta_t^\top \text{dg}[c_2]\Theta_t S_t \left(\widehat{\Sigma} + \widehat{\mu}\widehat{\mu}^\top\right)S_t^\top\right] + \alpha^\varrho \sum_{t=2}^{T} \text{Tr}\left[\left(\tilde{P}_t - \tilde{P}_{t-1}\right)\widehat{\Sigma}\left(\tilde{P}_t - \tilde{P}_{t-1}\right)^\top\right]\right) \tag{7a}$$

subject to:

$$\left(A^+\Phi_t^+ + A^-\Phi_t^- - \Theta_t S_t + BK_t + \Delta_t\right)S_t = \mathbb{0} \tag{7b}$$

$$\left(F_{tc} - \begin{bmatrix} w_{0tc} & \mathbb{0}_{E \times (k^t-1)} \end{bmatrix} - W_{1tc}P_t - W_{2tc}K_t\right)S_t = \mathbb{0}, \quad \forall c \in \mathcal{C} \tag{7c}$$

$$\left([P_t]_r - \sum_{c=1}^{C} v_c\begin{bmatrix} \mathring{\varrho}_{rtc} & \mathbb{0}_{1 \times (k^t-1)} \end{bmatrix}\right)S_t = \mathbb{0} \tag{7d}$$

$$(\Phi_t - \sum_{c=1}^{C} Z_{tc})S_t = \mathbb{0} \tag{7e}$$

$$\left(\Phi_t - \tfrac{1}{2}\Phi_t^+ - \tfrac{1}{2}\Phi_t^-\right)S_t = \mathbb{0} \tag{7f}$$

$$\begin{cases} \underline{M} \leqslant Z_{tc}^{ij} \leqslant \overline{M} \\ \underline{M}v_c \leqslant Z_{tc}^{ij} \leqslant \overline{M}v_c \\ F_{tc}^{ij} - (1-v_c)\overline{M} \leqslant Z_{tc}^{ij} \leqslant F_{tc}^{ij} - (1-v_c)\underline{M} \end{cases} \quad \forall c \in \mathcal{C}, \forall i = 1, \ldots, E, \forall j = 1, \ldots, k^t \tag{7g}$$

$$\sum_{c=1}^{C} v_c = 1, \quad v_c \in \{0,1\} \tag{7h}$$

$$\left(\Psi_t - \tfrac{1}{2}\mathrm{dg}[s]\left(K_t + |A|^\top P_t\right)\right)S_t = \mathbb{0} \tag{7i}$$

$$\left(\Psi_t S_t - \Phi_t^+ + \Phi_t^-\right)S_t - \Psi_{(t-1)}S_{(t-1)} = \mathbb{0} \tag{7j}$$

$$\sqrt{\tfrac{1-\overline{\varepsilon}_t}{\overline{\varepsilon}_t}}\left\|\widehat{F}[\Phi_t S_t]_\ell^\top\right\| \leqslant [\Phi_t S_t \widehat{\mu}]_\ell \; {}^* \tag{7k}$$

$$\sqrt{\tfrac{1-\overline{\varepsilon}_t}{\overline{\varepsilon}_t}}\left\|\widehat{F}[\Psi_t S_t]_\ell^\top\right\| \leqslant [\Psi_t S_t \widehat{\mu} - \psi_0]_\ell \tag{7l}$$

$$\sqrt[-2]{\overline{\varepsilon}}\left\|\begin{matrix}\widehat{F}[\Theta_t S_t]_n^\top \\ y_{tn}^\vartheta\end{matrix}\right\| \leqslant \tfrac{1}{2}\left(\overline{\vartheta}_n - \underline{\vartheta}_n\right) - x_{tn}^\vartheta \tag{7m}$$

$$\sqrt[-2]{\overline{\varepsilon}}\left\|\begin{matrix}\widehat{F}[K_t S_t]_\ell^\top \\ y_{t\ell}^\kappa\end{matrix}\right\| \leqslant \tfrac{1}{2}\left(\overline{\kappa}_\ell - \underline{\kappa}_\ell\right) - x_{t\ell}^\kappa \tag{7n}$$

$$\sqrt[-2]{\overline{\varepsilon}}\left\|\begin{matrix}\widehat{F}[P_t S_t]_n^\top \\ y_{tn}^\varrho\end{matrix}\right\| \leqslant \tfrac{1}{2}\left(\overline{\varrho}_n - \underline{\varrho}_n\right) - x_{tn}^\varrho \tag{7o}$$

$$\left|[\Theta_t S_t]_n \widehat{\mu} - \tfrac{1}{2}\left(\overline{\vartheta}_n - \underline{\vartheta}_n\right)\right| \leqslant y_{tn}^\vartheta + x_{tn}^\vartheta \tag{7p}$$

$$\left|[K_t S_t]_\ell \widehat{\mu} - \tfrac{1}{2}\left(\overline{\kappa}_\ell - \underline{\kappa}_\ell\right)\right| \leqslant y_{t\ell}^\kappa + x_{t\ell}^\kappa \tag{7q}$$

$$\left|[P_t S_t]_n \widehat{\mu} - \tfrac{1}{2}\left(\overline{\varrho}_n - \underline{\varrho}_n\right)\right| \leqslant y_{tn}^\varrho + x_{tn}^\varrho \tag{7r}$$

$$\tfrac{1}{2}\left(\overline{\vartheta}_n - \underline{\vartheta}_n\right) \geqslant x_{tn}^\vartheta \geqslant 0, \; y_{tn}^\vartheta \geqslant 0 \tag{7s}$$

$$\tfrac{1}{2}\left(\overline{\kappa}_\ell - \underline{\kappa}_\ell\right) \geqslant x_{t\ell}^\kappa \geqslant 0, \; y_{t\ell}^\kappa \geqslant 0 \tag{7t}$$

$$\tfrac{1}{2}\left(\overline{\varrho}_n - \underline{\varrho}_n\right) \geqslant x_{tn}^\varrho \geqslant 0, \; y_{tn}^\varrho \geqslant 0 \tag{7u}$$

$$\forall t \in \mathcal{T}, \; \forall n \in \mathcal{N}, \; \forall \ell \in \mathcal{E}, \; {}^*\forall \ell \in \mathcal{E}_a \tag{7v}$$

which optimizes the trade-off between the expected cost and pressure variability measure, by choosing only one binary variable (topology) from $v_1, \ldots, v_C$, and by optimizing the multi-stage linear decision rules under the chosen topology.