Differentially Private Algorithms for Synthetic Power System Datasets (Online Appendix: Reformulation of Problem (7))

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The full formulation of the bilevel post-processing optimization of the TCO algorithm takes the form:

$$
\begin{aligned}
\underset{\overline{\varphi}}{\text{minimize}} \quad & \textstyle\sum_{\tau=1}^{t} \left\|\overline{\mathcal{C}}_{\tau} - \mathcal{C}_{k^{\tau}}(\overline{\varphi})\right\| + \left\|\overline{\varphi} - \overline{\varphi}^{t-1}\right\| && \text{(1a)}\\
\text{subject to} \quad & \mathcal{C}_{k^{\tau}}(\overline{\varphi}) = \underset{p_{k^{\tau}} \in \mathcal{P}_{k^{\tau}}}{\text{minimize}} \quad c_{k^{\tau}}^{\top} p_{k^{\tau}} && \text{(1b)}\\
& \qquad\qquad \text{subject to} \quad \mathbb{1}^{\top}(p_{k^{\tau}} - d_{k^{\tau}}) = 0, && \text{(1c)}\\
& \qquad\qquad\qquad\qquad\quad \left\|F(p_{k^{\tau}} - d_{k^{\tau}})\right\|_{1} \leqslant \overline{\varphi}, && \text{(1d)}\\
& \qquad\qquad\qquad\qquad\quad \forall \tau = 1, \dots t,
\end{aligned}
$$

where the upper-level problem (1a) optimizes dataset $\overline{\varphi}$ in response to the OPF costs $\mathcal{C}_{k^{\tau}}(\overline{\varphi})$ provided by t worst-case DC-OPF problems (1b)–(1d) in the lower level. This problem falls within the class of bilevel optimization problems, with a limited computational tractability. To obtain its tractable reformulation, we replace each lower-level problem with its Karush–Kuhn–Tucker conditions (KKTs). Let $\underline{\gamma}_{k^{\tau}}, \overline{\gamma}_{k^{\tau}} \in \mathbb{R}^{n}$ be the dual variables associated with generation limit constraints in set $\mathcal{P}_{k^{\tau}}$, and let $\lambda_{k^{\tau}} \in \mathbb{R}^{n}$ and $\underline{\mu}_{k^{\tau}}, \overline{\mu}_{k^{\tau}} \in \mathbb{R}^{e}$ be the dual variables associated with power balance and flow limit constraints (1c) and (1d), respectively. The KKTs of the k^{τ}-th problem include primal feasibility $p_{k^{\tau}} \in \mathcal{P}_{k^{\tau}}$, (1c) and (1d), dual feasibility

$$
\underline{\gamma}_{k^{\tau}}, \overline{\gamma}_{k^{\tau}} \geqslant \mathbb{0}, \quad \underline{\mu}_{k^{\tau}}, \overline{\mu}_{k^{\tau}} \geqslant \mathbb{0}, \tag{2a}
$$

stationarity conditions

$$
\mathbb{1} \cdot \lambda_{k^{\tau}} - F^{\top}\overline{\mu}_{k^{\tau}} + F^{\top}\underline{\mu}_{k^{\tau}} - \overline{\gamma}_{k^{\tau}} + \underline{\gamma}_{k^{\tau}} - c_{k^{\tau}} = \mathbb{0}, \tag{2b}
$$

and complementarity slackness

$$
\begin{aligned}
\overline{\mu}_{k^{\tau}} \circ (\overline{\varphi}_{k^{\tau}} - F(p_{k^{\tau}} - d_{k^{\tau}})) = \mathbb{0}, \quad & \overline{\gamma}_{k^{\tau}} \circ (\overline{p}_{k^{\tau}} - p_{k^{\tau}}) = \mathbb{0},\\
\underline{\mu}_{k^{\tau}} \circ (F(p_{k^{\tau}} - d_{k^{\tau}}) + \overline{\varphi}_{k^{\tau}}) = \mathbb{0}, \quad & \underline{\gamma}_{k^{\tau}} \circ (p_{k^{\tau}} - \underline{p}_{k^{\tau}}) = \mathbb{0}.
\end{aligned} \tag{2c}
$$

Although the latter conditions are non-convex, they can be replaced with equivalent mixed-integer constraints using the Special Ordered Set of Type 1 (SOS1) variables. For example, for the 1st condition in (2c) we have:

$$
\{\overline{\mu}_{k^{\tau}}, \overline{\varphi}_{k^{\tau}} - F(p_{k^{\tau}} - d_{k^{\tau}})\} \in \text{SOS1}, \tag{3}
$$

where the dual variable and the slack of the power flow limit constraint are enforced to be a pair of SOS1 variables (element-wise), i.e., only one of them can take a non-zero value, and the other one being zero.

We now arrive at the single-level tractable reformulation of bilevel problem (1) of the form:

$$
\begin{aligned}
\underset{\overline{\varphi}, p, \underline{\mu}, \overline{\mu}, \underline{\gamma}, \overline{\gamma}}{\text{minimize}} \quad & \textstyle\sum_{\tau=1}^{t} \left\|\overline{\mathcal{C}}_{\tau} - c_{k^{\tau}}^{\top} p_{k^{\tau}}\right\| + \left\|\overline{\varphi} - \overline{\varphi}^{t-1}\right\| && \text{(4a)}\\
\text{subject to} \quad & \underline{p}_{k^{\tau}} \leqslant p_{k^{\tau}} \leqslant \overline{p}_{k^{\tau}}, && \text{(4b)}\\
& \mathbb{1}^{\top}(p_{k^{\tau}} - d_{k^{\tau}}) = 0, && \text{(4c)}\\
& \left\|F(p_{k^{\tau}} - d_{k^{\tau}})\right\|_{1} \leqslant \overline{\varphi}, && \text{(4d)}\\
& \underline{\gamma}_{k^{\tau}}, \overline{\gamma}_{k^{\tau}}, \underline{\mu}_{k^{\tau}}, \overline{\mu}_{k^{\tau}} \geqslant \mathbb{0}, && \text{(4e)}\\
& \mathbb{1} \cdot \lambda_{k^{\tau}} - F^{\top}(\overline{\mu}_{k^{\tau}} - \underline{\mu}_{k^{\tau}}) - \overline{\gamma}_{k^{\tau}} + \underline{\gamma}_{k^{\tau}} - c_{k^{\tau}} = \mathbb{0}, && \text{(4f)}\\
& \{\overline{\mu}_{k^{\tau}}, \overline{\varphi}_{k^{\tau}} - F(p_{k^{\tau}} - d_{k^{\tau}})\} \in \text{SOS1}, && \text{(4g)}\\
& \{\underline{\mu}_{k^{\tau}}, F(p_{k^{\tau}} - d_{k^{\tau}}) + \overline{\varphi}_{k^{\tau}}\} \in \text{SOS1}, && \text{(4h)}\\
& \{\overline{\gamma}_{k^{\tau}}, \overline{p}_{k^{\tau}} - p_{k^{\tau}}\} \in \text{SOS1}, && \text{(4i)}\\
& \{\underline{\gamma}_{k^{\tau}}, p_{k^{\tau}} - \underline{p}_{k^{\tau}}\} \in \text{SOS1}, && \text{(4j)}\\
& \forall \tau = 1, \dots t.
\end{aligned}
$$

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