2 Electrostatics

Electrostatics is the study of nonmoving electric charges, electric conductors and dielectrics, and DC potential sources. Most capacitive sensors use simple planar parallel-plate geometry and do not require expertise in electrostatics, but the field is reviewed here for background. A standard text in electrostatics such as Haus and Melcher [1989] can be consulted for more detail. Some applications for capacitive sensors may use nonplanar electrode geometry; these may need more extensive electrostatic field analysis. Analysis tools presented here include closed-form field solutions, electric field sketching, Teledeltos paper simulation and finite element modeling.

2.1 APPROXIMATIONS

Real-world capacitive sensor designs involve moving charges, partially conducting surfaces, and AC potential sources. For an accurate analysis of the fields and currents that make up a capacitive sensor, Maxwell's equations relating electric and magnetic fields, charge density, and current density should be used. But a simplifying approximation which ignores magnetic fields is almost always possible with insignificant loss of accuracy. Systems in which this approximation is reasonable are defined as electroquasistatic [Haus, p.66]:

> Maxwell's equations describe the most intricate electromagnetic wave phenomena. Of course, the analysis of such fields is difficult and not always necessary. Wave phenomena occur on short time scales or at high frequencies that are often of no practical concern. If this is the case the fields may be described by truncated versions of Maxwell's equations applied to relatively long time scales and low frequencies. We will find that a system composed of perfect conductors and free space is electroquasistatic [if] an electromagnetic wave can propagate through a typical dimension of the system in a time that is shorter than times of interest.

Our capacitive sensor applications are almost all small and slow by these measures, and our conductors are all conductive enough so that their time constant is much shorter than our circuit response times, so we can use these simplified versions of Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu_0 \mathbf{H} \approx 0$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\varepsilon_0 \mathbf{E} + \mathbf{J} \approx 0$$

$$\nabla \cdot \varepsilon_0 \mathbf{E} = \rho$$

$$\nabla \cdot \mu_0 \mathbf{H} = 0$$

A given distribution of charge density ρ produces the electric field intensity **E**; the magnetic field intensity **H** is approximated by zero.

Units

The magnetic permeability of vacuum, μ_0, is a fundamental physical constant, defined in SI units as $4\pi \times 10^{-7}$ N/A^2. The electric permittivity of vacuum, ε_0, is defined by μ_0 and c, the speed of light in vacuum, as $\varepsilon_0 = 1/\mu_0 c^2$. As c is defined exactly as 299,792,458 m/s

$$\varepsilon_0 = 8.8541878 \cdot 10^{-12} \quad \text{F/m}$$

Any dielectric material has an electric permittivity which is higher than vacuum; it is measured as the relative dielectric constant ε_r, with ε_r in the range of 2–10 for most dry solid materials and often much higher for liquids.

2.2 CHARGES AND FIELDS

With the simplified equations above, electrostatic analysis reduces to the discovery of the electric field produced by various charge distributions in systems of materials with various dielectric constants.

2.2.1 Coulomb's law

Two small charged conductors in a dielectric with charges of Q_1 and Q_2 coulombs, separated by r meters, exert a force in newtons

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r r^2} \qquad 2.1$$

The force is along the line connecting the charges and will try to bring the charges together if the sign of their charge is opposite.

The coulomb is a large quantity of charge. It is the charge transported by a 1 A current in 1 s; as an electron has a charge of 1.60206×10^{-19} C, a coulomb is about 6×10^{18} electrons. The force between two 1 C charges spaced at 1 mm is 9×10^{15} N, about 30 times the weight of the earth. But electrostatic forces can often be ignored in practical systems, as the charge is usually very much smaller than a coulomb.

With V volts applied to a parallel plate capacitor of plate area A square meters and spacing d meters, the energy stored in the capacitor is

$$E = \frac{1}{2}CV^2 = \frac{1}{2}\varepsilon_0\varepsilon_r\frac{A}{d} \cdot V^2 \qquad 2.2$$

and the force in newtons is then the partial derivative of energy vs. plate spacing

$$F = \frac{\partial E}{\partial d} = \frac{1}{2}\frac{C}{d}V^2 = \frac{-\varepsilon_0\varepsilon_r}{2}\frac{A}{d^2} \cdot V^2 = -4.427 \cdot 10^{-12}\varepsilon_r\frac{AV^2}{d^2} \qquad 2.3$$

Transverse forces for simple plate geometrys are small, and can be made insignificant with overlapped plates; for some interdigitated structures these forces may be significant and can be calculated using the partial derivative of energy with transverse motion.

For a large air-dielectric capacitor charged to 1 V DC and composed of two 1 m square plates at 1 mm spacing, the force between the plates is attractive at 4.427×10^{-6} N. This force may be troublesome in some sensitive applications. AC operation does not offer a solution to unwanted electrostatic force as both positive and negative half cycles are attractive, but the small force does not affect most capacitive sensor designs, and it can be balanced to zero by use of the preferred three-electrode capacitive sensor geometry. It is exploited in the silicon-based accelerometer of Chapter 15.

Electric field

Two charged conducting plates illustrate the concept of electric field (Figure 2.1).

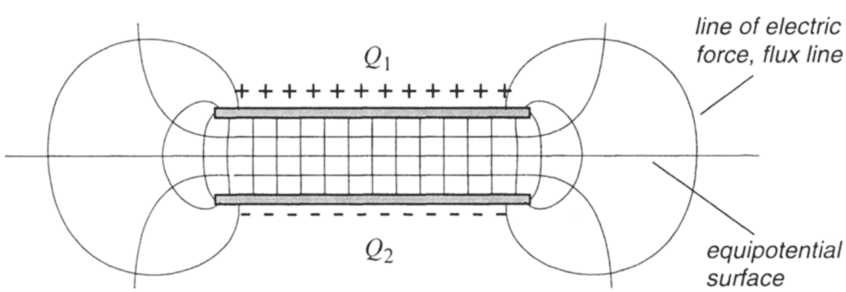


Figure 2.1 Electric field for parallel plates

Voltage gradient

If the plates are arbitrarily assigned a voltage, then a scalar potential V, between 0 and 1 V, can be assigned to the voltage at any point in space. Surfaces where the voltage is the same are equipotential surfaces. The electric field **E,** a vector quantity, is the gradient of the voltage V and is defined as

$$\mathbf{E} = -\frac{dV}{d\mathbf{n}} = -\operatorname{grad} V = -\nabla V \quad 1\ \text{V/m} \tag{2.4}$$

where **n** is a differential element perpendicular to the equipotential surface at that point. In the sketch of parallel-plate fields (Figure 2.1), surfaces of constant V are equipotential surfaces and lines in the direction of maximum electric field are lines of force. The voltage along any path between two points a and b can be calculated as

$$V_{ab} = \int_a^b -\mathbf{E} \cdot d\mathbf{n} \quad \text{V} \tag{2.5}$$

In the linear region near the center of the parallel plates (Figure 2.1) the electric field is constant and perpendicular to the plates; eq. 2.5 produces $V_{12} = E\,d$ where d is the plate spacing.

2.2.2 Conduction and displacement current

When an electric field is produced in any material, a current flows. This current is the sum of a conduction current density $\mathbf{J_c}$ and a displacement current density $\mathbf{J_d}$. These terms specify current density in amperes/meter^2.

For a metallic conductor, conduction current is produced by movement of electrons; for electrolytes the current is due to migration of ions. The current density for conduction current is

$$\mathbf{J_c} = \sigma\mathbf{E} \quad \text{A/m}^2 \tag{2.6}$$

where σ is the conductivity in mho/meter.

For a good high resistance dielectric, conduction current is near zero and charge is transferred by a reorientation of polar molecules causing displacement current. Highly polar molecules such as water can transfer more charge than less polar substances or vacuum, and will have a higher dielectric constant ε_r. Also, displacement current is produced by charges accumulating on nearby electrodes under the influence of an applied voltage until the repulsive force of like charges balances the applied voltage. The definition of displacement current is

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0 \varepsilon_r \mathbf{E}) \quad \text{A/m}^2 \tag{2.7}$$

with a direction chosen to be the same as the direction of the **E** vector. The displacement current is transient in the case where a DC or a static field is applied, and alternating for AC fields. If a DC voltage is imposed on a system of conductors and dielectrics, displacement current flows briefly to distribute charges until Laplace's equation (eq. 2.10) is satisfied. With an AC field, the alternating displacement current continues to flow with a magnitude proportional to the time rate of change of the electric field.

Gauss' law

The total flow of charge due to displacement current through a surface is found by integrating **D** over that surface

$$\psi = \int_{surf} \mathbf{D}ds \quad \mathrm{C} \qquad 2.8$$

with ds an elementary area and **D** the flux density normal to ds. Gauss' law states that the total displacement or electric flux through any closed surface which encloses charges is equal to the amount of charge enclosed.

The displacement charge ψ is the total of the charges on the electrodes and the charge displaced in the polar molecules of a dielectric in the electric field. Highly polar molecules like water, with a dielectric constant of 80, act as though a positive charge is concentrated at one end of each molecule and a negative charge at the other. As an electric field is imposed, the molecules align themselves with the field, and the sum of the charge displaced during this alignment is the dielectric charge displacement. Note that in a system with finite charges the conduction current density $\mathbf{J_c}$ can be zero for a perfect insulator, but the minimum value of the displacement current density $\mathbf{J_d}$ must be nonzero due to the nonzero dielectric constant of vacuum.

If the excitation voltage is sinusoidal and ε_r and ε_0 are constants, **D** will have a cosine waveform and the displacement current can be found by

$$i = \frac{d\psi}{dt} \quad \mathrm{A}$$

Poisson's equation

The relationship between charge density ρ and displacement current **D** is Poisson's equation

$$\mathrm{div}\mathbf{D} = \rho \quad \mathrm{C/m^3} \qquad 2.9$$

div **D**, the divergence of **D**, the net outward flux of **D** per unit volume, is equal to the charge, enclosed by the volume.

Laplace's equation

In a homogenous isotropic medium with ε_r constant and scalar, and with no free charge, Poisson's equation can be rewritten

$$\mathrm{div}\varepsilon_0\mathbf{E} = \varepsilon_0\mathrm{div}\mathbf{E} = \rho = 0 \qquad 2.10$$

This version of Poisson's equation for charge-free regions is Laplace's equation. This equation is important in electromagnetic field theory. In rectangular coordinates, Laplace's equation is

$$\nabla^2 V = \frac{\partial}{\partial x^2}V + \frac{\partial}{\partial y^2}V + \frac{\partial}{\partial z^2}V = 0 \qquad 2.11$$

Much of electrostatics is occupied with finding solutions to this equation, or its equivalent in polar or cylindrical coordinates.

> The solution of problems in electrostatics is to find a potential distribution that will satisfy Laplace's equation with given electrode geometry and electrode voltages. Generally, the potential distribution in the interelectrode space and the charge distribution on the electrodes are not known. The charges on the electrodes will distribute themselves so that the conductors become equipotential surfaces and so that Laplace's equation is satisfied in the interelectrode space [Jordan, 1950, p. 48].

A solution of Laplace's (or Poisson's) equation produced the three-dimensional field line plot which was sketched in Figure 2.1. Unfortunately, analytic solution of Laplace's equation is only possible for some simple cases.

Solutions exist for some two-dimensional problems, or for three-dimensional problems which are extruded two-dimensional shapes. Heerens [1986] has published solutions for cylindrical and toroidal electrode configurations with rectangular cross sections which can be extended to many other geometries used in capacitive sensors.

2.2.3 Induced charge

When a positive test charge is brought near a conductor, free electrons in the conductor are attracted to the surface near the charge, and for a floating conductor, holes, or positive charges, are repelled to the opposite surface. With a grounded conductor, the holes flow through the connection to ground and the electrode has a net negative charge. The charges come to an equilibrium in which the repulsive force of the surface electrons is balanced by the attraction of the surface electrons to the test charge.

Electric fields inside a conductor are usually negligible if current flow is small, so the surface of the conductor is an equipotential surface. An electric field outside the conductor but near its surface has equipotential surfaces which parallel the conductor and lines of flux which intersect the conductor at right angles. The magnitude of the conductor's surface charge is equal to the flux density in the adjacent dielectric, $\sigma = |\mathbf{D}|$.

The effect of induced charge is seen in applications such as capacitive proximity detection, as the far-field effects of a capacitive sense electrode must also include the contribution to the E-field of the charge the electrode induces on nearby floating conductors.

2.2.4 Superposition

As for any linear isotropic system, the principle of superposition can be applied to electric field analysis. The electric field of a number of charges can be calculated as the vector sum of the field due to each individual charge. Also, the field in a system of charged conductors can be determined by assuming all conductors are discharged except one, calculating the resultant field and repeating the process with each conductor and calculating the vector sum. Superposition is a very useful and powerful technique for simplifying a complex problem into many simple problems.

2.2.5 Charge images

The distribution of charge on conductors can be determined, often with considerable difficulty, by calculating the electric field distribution. Lord Kelvin suggested a simple graphical method. A charge $+q$ in a dielectric near a conducting plane produces a charge density of opposite sign on the nearby surface of the plane. The electric field produced in the dielectric is the same as if the charge density on the surface of the plane was replaced by a single charge $-q$ inside the plane at a symmetrical location. The charge image is similar to an optical image in a mirror (Figure 2.2).

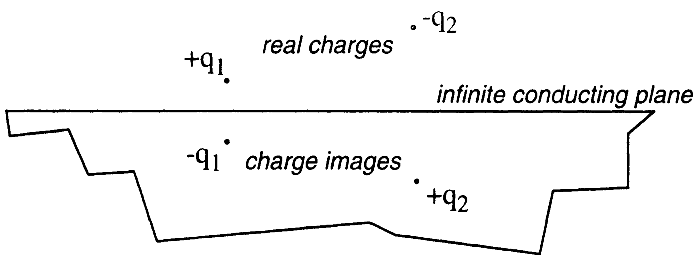


Figure 2.2 Charge images

Lord Kelvin's result can also be derived by looking at the field lines around two charges of opposite polarity and noticing that the line *SS* is, by symmetry, an equipotential surface which can be replaced by a conducting surface. See Figure 2.3.

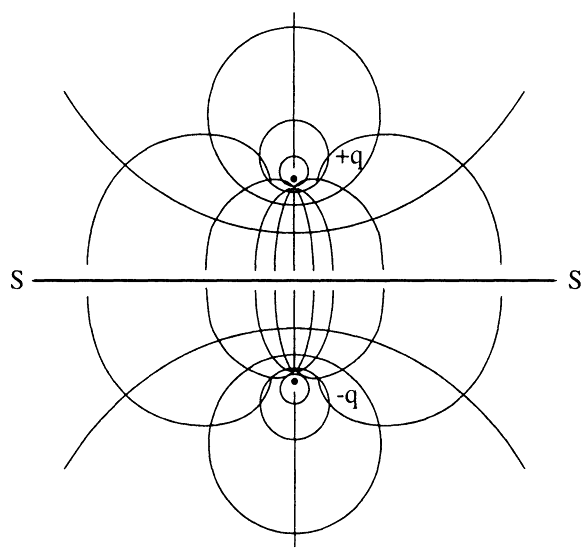


Figure 2.3 Fields of opposite charges

Then no change in the field structure on the $+q$ side of S is seen if the charge at $-q$ is replaced with the induced surface charge on the conductor *SS* and $-q$ is removed.

Maxwell's capacitor

Maxwell studied a capacitor built with two parallel plates of area A and two partially conducting dielectrics of thickness L, dielectric constant ε, and conductivity σ (Figure 2.4).

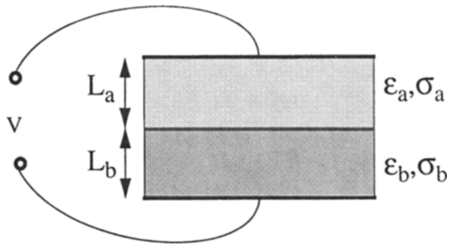


Figure 2.4 Maxwell's capacitor

The terminal behavior and the electric fields can be analyzed by application of Laplace's equation. The diligent student will find that a sudden application of a voltage V will produce an electric field which is initially divided between the two regions in proportion to the thickness L of the region and its dielectric constant ε, but due to the finite conductivity it will redivide over time in the ratio of the thickness and the conductivity σ. The lazy student will notice that the boundary between dielectrics is an equipotential surface which can be replaced by a conductor, and that the single capacitor can then be dissected into two capacitors with capacitance $\varepsilon_a \varepsilon_0 A/L_a$ and $\varepsilon_b \varepsilon_0 A/L_b$. Then the equivalent circuit is drawn (Figure 2.5)

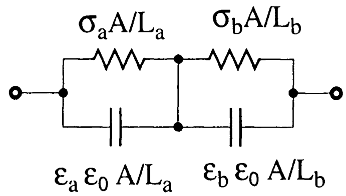


Figure 2.5 Maxwell's capacitor, equivalent circuit

and the circuit is quickly solved by elementary circuit theory, or, for the truly lazy student, SPICE.

3 CAPACITANCE

For the parallel plate geometry (Figure 2.1), a voltage V can be applied to the plates to produce a total flux ψ. Then, the amount of flux in coulombs which is produced by V volts is

$$\psi = Q = CV \tag{2.12}$$

The new symbol C is the capacitance of the plates in coulombs/volt.

Capacitance is calculated by evaluating

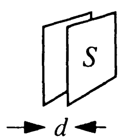
$$C = \frac{\varepsilon_0 \varepsilon_r \Psi}{\int \mathbf{D} \cdot dl} = \frac{\Psi}{\int \mathbf{E} \cdot dl}$$

with dl an elementary length along a flux line of displacement current. This integral gives the capacitance of an elementary volume surrounding the flux line, and must be repeated for all flux lines emanating from one of the plates and terminating in the other plate. For two-electrode systems, all flux lines which emanate from one plate will terminate on the other, but with multiple electrodes this is generally not true.

2.3.1 Calculating Capacitance

Parallel plates

As an example of calculation of capacitance, Gauss' law can be applied to a surface surrounding one of the parallel plates in Figure 2.1. If the surface is correctly chosen and the fringing flux lines at the edge of the plates are ignored, the total charge Q inside the surface is equal to the total displacement flux D times the area of the surface S, resulting in



$$C = \frac{\varepsilon_0 \varepsilon_r S}{d} = 8.854 \times 10^{-12} \times \frac{\varepsilon_r S}{d} \qquad 2.13$$

where

C = capacitance, farads

$\varepsilon_0 = 8.854 \times 10^{-12}$

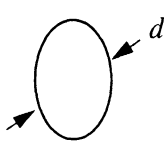
ε_r = relative dielectric constant, 1 for vacuum

S = area, square meters

d = spacing, meters

Disk

The simplest configuration is a single thin plate with a diameter of d meters. This has a well-defined capacitance to a ground at infinity



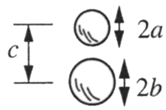
$$C = 35.4 \times 10^{-12} \varepsilon_r \times d \qquad 2.14$$

Sphere

$$C = 55.6 \times 10^{-12} \varepsilon_r \times d \qquad 2.15$$

Two spheres

The capacitance in farads between two spheres of radius a and b meters and separation c is approximately [Walker, p. 83]

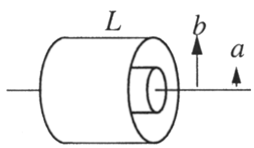


$$C \approx \frac{4\pi\varepsilon_0\varepsilon_r}{\frac{a+b}{ab} - \frac{1}{c}} \qquad 2.16$$

The approximation is good if a and b are much less than $2c$. Note that with this geometry and the single disk above, capacitance scales directly with size. For the extruded geometries below, capacitance scales directly with the length and is independent of the cross-section size.

Concentric cylinders

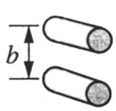
Another geometry which results in a flux distribution which can be easily evaluated is two concentric cylinders. The capacitance (farads) between two concentric cylinders of length L and radius a and b meters is



$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(b/a)} L \qquad 2.17$$

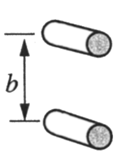
Parallel cylinders

For cylinders of length L meters and radius a meters separated by b meters, the capacitance in farads is



$$C = \frac{\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b + \sqrt{b^2 - 4a^2}}{2a}\right)} L \qquad 2.18$$

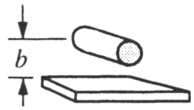
If $b >> a$,



$$C \approx \frac{\pi\varepsilon_0\varepsilon_r}{\ln\frac{b}{a}} L \qquad 2.19$$

Cylinder and plane

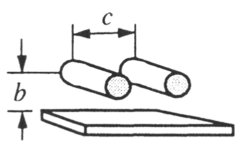
A cylinder with length L and radius a located b meters above an infinite plane [Hayt, p. 118] is



$$C = \frac{2\pi\varepsilon_0\varepsilon_r L}{\ln\left[\frac{(b + \sqrt{b^2 - a^2})}{a}\right]} = \frac{2\pi\varepsilon_0\varepsilon_r L}{\operatorname{acosh}(b/a)} \qquad 2.20$$

Two cylinders and plane

The mutual capacitance (farads) between two cylinders of length L and radius a meters is reduced by the proximity of a ground plane [Walker, p. 39]



$$C_m \approx \frac{\pi\varepsilon_0\varepsilon_r L \cdot \ln\left[1 + \frac{2b}{c}\right]}{\left[\ln\left(\frac{2b}{a}\right)\right]^2} \qquad 2.21$$

The approximation is good if $2b >> a$. A graph (Figure 2.6) of mutual capacitance C_m in pF vs. b in cm, with $L = 1$ m, $a = 0.5$ mm and $c = 2$ cm, and with $c = 20$ cm shows a 20-80x increase of coupling capacitance as the ground plane is moved away from the conductors.

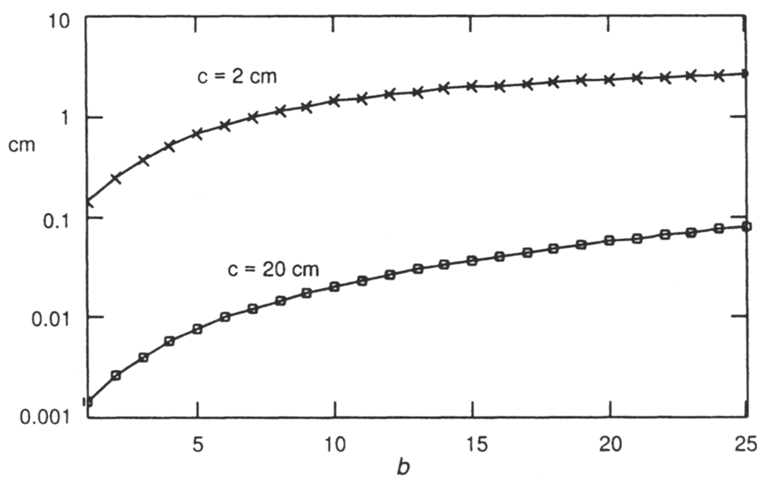
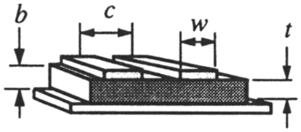


Figure 2.6 Two cylinders and plane, graph of mutual capacitance vs. distance to ground plane

Two strips and plane

A geometry seen in printed circuit boards is two rectangular conductors separated from a grounded plane by a dielectric. For this case, the mutual capacitance in farads between the conductors [Walker, p. 51] of length L, width w, thickness t, and spacing c

meters over an infinite ground substrate covered with a dielectric of thickness t meters with a dielectric constant ε_r is approximately



$$C_m \approx \frac{\pi\varepsilon_0\varepsilon_{r(\text{eff})}L}{\left[\ln\left(\frac{\pi(c-w)}{w+t}+1\right)\right]} \qquad 2.22$$

The effective dielectric constant $\varepsilon_{r(\text{eff})}$ is approximately 1 if $c >> b$, or if $c \approx b$, $\varepsilon_{r(\text{eff})} \approx (1+\varepsilon_r)/2$. With the strips' different widths w_1 and w_2, the equation becomes [Walker, p. 52]

$$C_m \approx \frac{55.6\varepsilon_{r(\text{eff})}L}{\ln\left[\pi^2c^2\left(\frac{1}{w_1+t}\right)\left(\frac{1}{w_2+t}\right)\right]} \quad \text{F, m} \qquad 2.23$$

2.3.2 Multielectrode capacitors

Most discrete capacitors used in electronics are two-terminal devices, while most air-spaced capacitors used for sensors have three or more terminals, with the added electrodes acting as shields or guards to control fringing flux, reduce unwanted stray capacitance, or shield against unwanted pickup of external electric fields.

One use of a three-electrode capacitor (Figure 2.7) is in building accurate reference capacitors of small value [Moon, 1948, pp. 497–507]

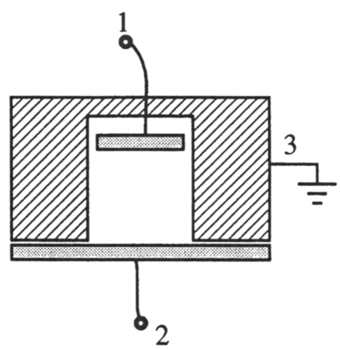


Figure 2.7 Small value reference capacitor

Reference capacitors of this construction were built by the National Bureau of Standards in 1948. A capacitor of 0.0001 pF has an accuracy of 2%, and a capacitor of 0.1 pF has an accuracy of 0.1% [Stout, pp. 288–289]. Electrode 3, the shield, or guard, electrode, acts in this case to shield the sensed electrode (1) from extraneous fields and to divert most of the field lines from the driven electrode (2) so only a small percent of the displacement current reaches the sensed electrode.

In general, the capacitance of a pair of electrodes which are in proximity to other electrodes can be shown, for arbitrary shapes, as in Figure 2.8.

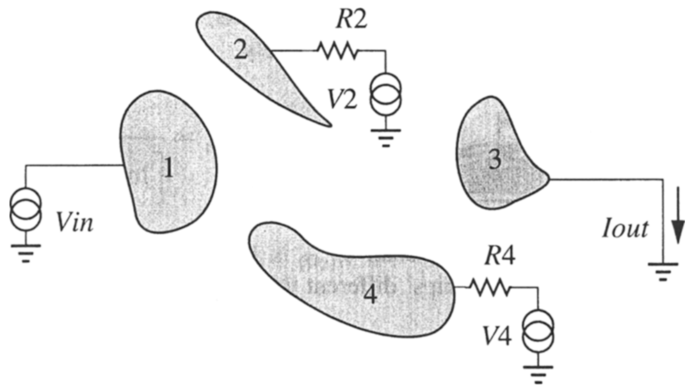


Figure 2.8 Four arbitrary electrodes

The capacitance between, for example, electrode 1 and electrode 3 is defined by calculating or measuring the difference of charge Q produced on electrode 3 by an exciting voltage Vin impressed on electrode 1. If electrodes 2 and 4 are connected to excitation voltages, they will make a contribution to the charge on 3 which is neglected when measuring the capacitance between 1 and 3. The shape of electrodes 2 and 4 and their impedance to ground will have an effect on C_{13}, but the potential is unimportant. Even though a potential change will produce a totally different electric field configuration, the field configuration can be ignored for calculation of capacitance, as the principle of superposition applies; a complete field solution is sufficient but not necessary. If $R2$ and $R4$ are zero and $V2$ and $V4$ are zero, the charge in coulombs can be measured directly by applying a V volt step to Vin and integrating current flow in amps

$$Q_3 = \int Ioutdt$$

Then capacitance in farads is calculated using

$$C_{13} = \frac{Q_3}{Vin}$$

Or, when electrodes 2 and 4 are nonzero

$$C_{13} = \frac{\partial Q_3}{\partial Vin}$$

When $R2$ and $R4$ are a high impedance relative to the capacitive impedances involved, C_{31} is a higher value than if $R2$ and $R4$ are low. Electrodes 2 and 4 act as shields with $R2$ and $R4$ low, intercepting most of the flux between 1 and 3 and returning its current to ground, considerably decreasing C_{13}. With $R2$ and $R4$ high, these electrodes increase C_{13} over the free air value. With linear media, $C_{13} = C_{31}$.

Solving a multiple-electrode system with arbitrary impedances is extraordinarily tedious using the principles of electrostatics. Electrostatic fields are difficult to solve, even approximately. But the problem can often be reduced to an equivalent circuit and handled easily by using approximations and superposition and elementary circuit theory. The four-electrode equivalent circuit, for example, is shown in Figure 2.9.

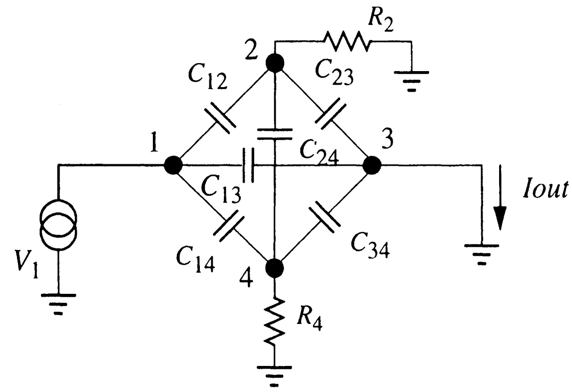


Figure 2.9 Four-electrode circuit

If the time constant of $R2$ with $C12$ and $C23$ and the time constant of $R4$ with $C14$ and $C34$ are small with respect to the excitation frequency, $R2$ and $R4$ may be replaced by short circuits. Then the circuit reduces to that shown in Figure 2.10.

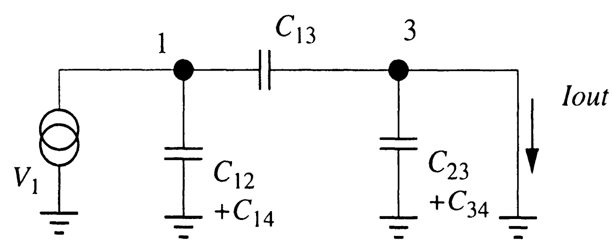


Figure 2.10 Reduced four-electrode circuit

Since a capacitor shunting a low impedance voltage source or a low impedance current measurement can be neglected, the circuit can be further reduced to that shown in Figure 2.11.

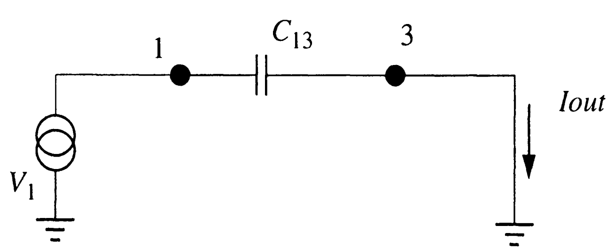


Figure 2.11 Further reduced four-electrode circuit

If possible, it is easier to convert a problem in electrostatics to a problem in circuit theory, where more effective tools are available and SPICE simulations can be used. This type of analysis will be used to understand the effects of guard and shield electrodes with capacitive sensors.

ANALYTICAL SOLUTIONS

Aside from the easy symmetric cases previously discussed, many other useful electrode configurations have been solved analytically. Some of these solutions are shown in this section.

2.4.1 Effect of Gap Width

Small gaps

For a geometry where adjacent electrodes are separated by a small insulating gap (Figure 2.12)

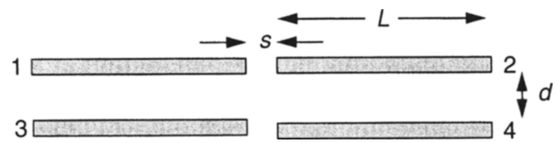


Figure 2.12 Small gap electrodes

Heerens [1986, p. 902] has given an exact solution for the change in capacitance $C' = \delta \cdot C$ between electrodes 1 and 4 or 2 and 3 as a function of gap width

$$\delta = e^{\frac{-\pi s}{d}} \qquad 2.24$$

Then δ can be plotted against s, with $d = 1$ cm (Figure 2.13).

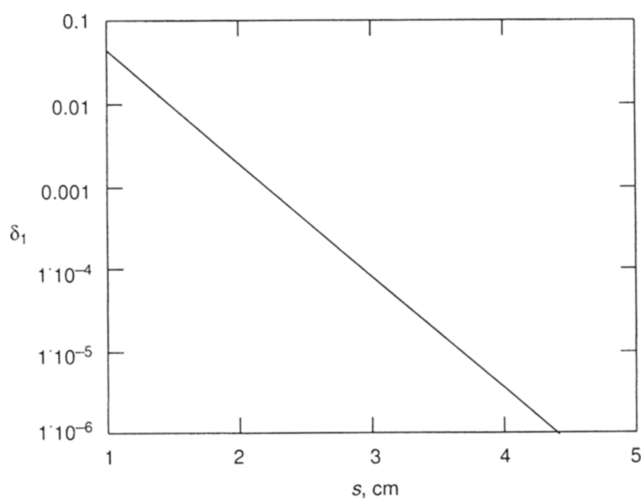


Figure 2.13 Small gap, eq. 2.24

From this plot we see that a physical gap s which is less than 1/5 of the separation d of the electrodes can be considered to be infinitely thin, with an error of less than 10^{-6}. The gap thickness has much less effect on C_{13} and C_{24} than on C_{14} and C_{23}. The electrodes can be considered to have an infinitesimal gap in the center of the actual gap. This rule of thumb, where features less than 1/5 of the plate spacing can be ignored, shows the degree of precision needed to produce accurate capacitive sensors.

2.4.2 Planar Geometries

Overlapping parallel plates

The mutual capacitance of overlapping parallel plates with this geometry (Figure 2.14)

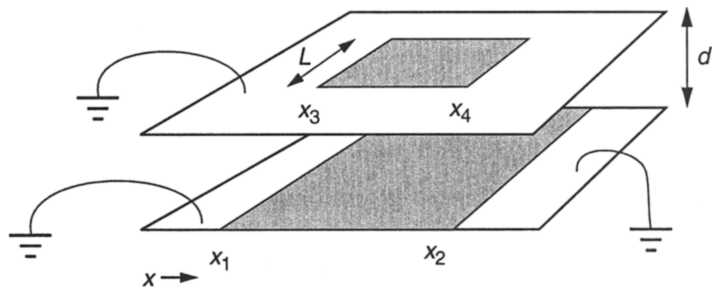


Figure 2.14 Overlapping plates

with unshaded areas separated by a narrow gap and grounded, and the length of the lower electrode infinite, is described [Heerens, 1983, p. 3] as

$$C = \frac{\varepsilon_0 \varepsilon_r L}{\pi} \ln \frac{\cosh\left[\frac{\pi}{2d}(x_4 - x_1)\right]\cosh\left[\frac{\pi}{2d}(x_3 - x_2)\right]}{\cosh\left[\frac{\pi}{2d}(x_3 - x_1)\right]\cosh\left[\frac{\pi}{2d}(x_4 - x_2)\right]} \quad \text{F, m} \qquad 2.25$$

This equation will be accurate to better than 1 ppm if the length of the lower electrode overlaps the top electrode by more than $5d$ and the gaps between the ground areas and the electrodes are less than 1/5 d. Choosing the following values to illustrate the function, we can evaluate capacitance vs. spacing

$\varepsilon_0 = 8.854 \times 10^{-12}$	$x_1 = 0.000$
$\varepsilon_r = 1$	$x_2 = 0.003$
$d = 0$ to 0.005	$x_3 = 0.001$
$L = 0.025$	$x_4 = 0.002$

With these parameters, this curve of capacitance vs. spacing shows an approximately exponential decline which becomes nonlinear near zero spacing (Figure 2.15).

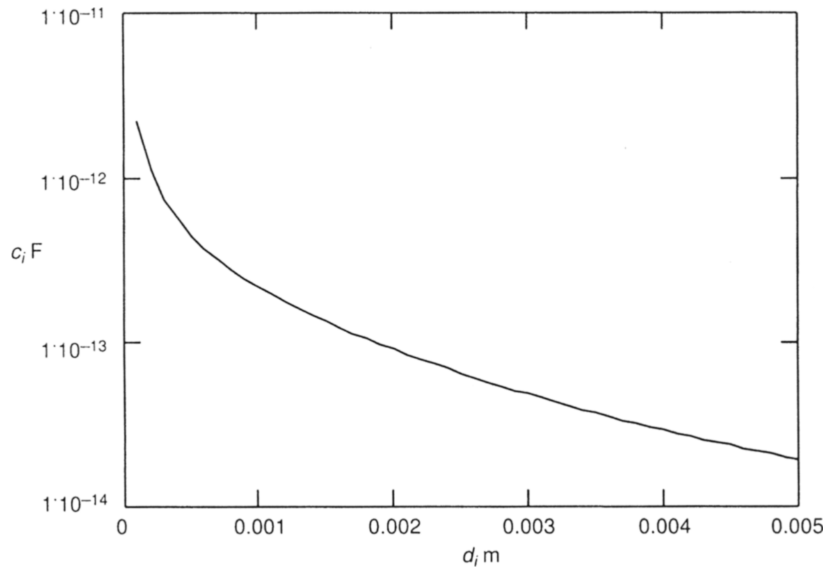


Figure 2.15 Overlapping parallel plate capacitance

Coplanar plates

Two plates in the same plane (Figure 2.16), surrounded again by ground with $L_1 \gg L_2$, have a mutual capacitance [Heerens, 1983, p. 8] which is given by

$$C = \frac{\varepsilon_0 \varepsilon_r L_2}{\pi} \ln \frac{(s + b_1)(s + b_2)}{s(s + b_1 + b_2)} \qquad 2.26$$

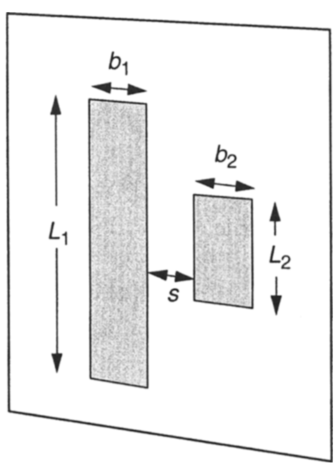


Figure 2.16 Coplanar plates

Coplanar plates with shield

Overlapping parallel plates with the geometry shown in Figure 2.17, with narrow gaps and with the ground planes infinite in extent, or at least five times the d dimension larger than the electrodes, are represented by this equation [Heerens, 1986, p. 901]

$$C = \frac{\varepsilon_0 \varepsilon_r L}{\pi} \ln \frac{\sinh\left[\frac{\pi}{2d}(x_1 - x_3)\right] \sinh\left[\frac{\pi}{2d}(x_2 - x_4)\right]}{\sinh\left[\frac{\pi}{2d}(x_2 - x_3)\right] \sinh\left[\frac{\pi}{2d}(x_1 - x_4)\right]} \qquad 2.27$$

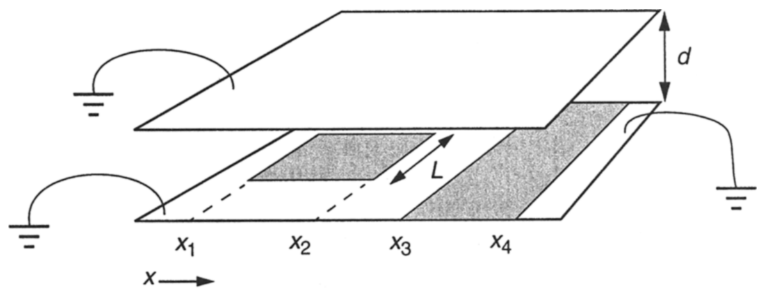


Figure 2.17 Overlapping plates

2.4.3 Cylindrical Geometry

Cocylindrical plates

Two square plates surrounded by ground and mapped to the inside surface of a cylinder [Heerens, 1983] have a capacitance which is independent of the cylinder radius, as with all extruded shapes (Figure 2.18).

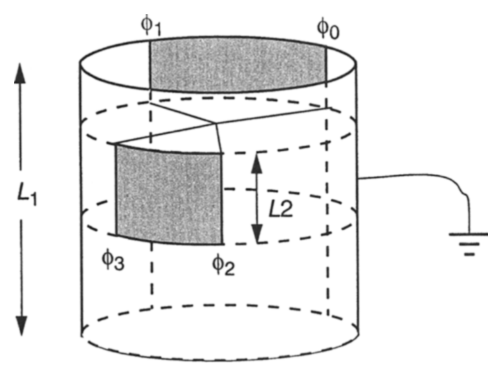


Figure 2.18 Cocylindrical plates

If L_1 is long compared to the length of the shorter plate, L_2, the mutual capacitance of the plates is given by

$$C = \frac{\varepsilon_0 \varepsilon_r L_2}{\pi} \ln \frac{\sin\left(\frac{\phi_3 - \phi_0}{2}\right) \sin\left(\frac{\phi_2 - \phi_1}{2}\right)}{\sin\left(\frac{\phi_2 - \phi_0}{2}\right) \sin\left(\frac{\phi_3 - \phi_1}{2}\right)} \qquad 2.28$$

APPROXIMATE SOLUTIONS

For most capacitive sensor designs, fringe capacitance and stray capacitance can be ignored or approximated without much trouble, but if maximum accuracy is needed, or if problems are encountered with capacitive crosstalk or strays, it is useful to have an analytical method as shown above to evaluate the capacitance of various electrode configurations. Usually it is inconvenient to measure the actual fringe or stray capacitance values, as the strays associated with the measuring equipment are much larger than the strays you are trying to measure. Calculating the strays is possible only for simple geometry with spatial symmetry in a given coordinate system. But an approximate solution is generally adequate; three options that give approximate solutions are field line sketches, Teledeltos™ paper, and finite element analysis.

2.5.1 Sketching field lines

Electric force lines terminate at right angles to conductors. Equipotential surfaces cross the force lines at right angles and tend to parallel conductive surfaces. Starting with a sketch of the conductors and their voltages, Poisson's equation can be solved graphically by trial and error in two dimensions by following these restrictions. Only one set of field lines (or a trivial translation) is produced by a given configuration of conductors. If an additional restriction is followed, that the four-sided shapes formed by the lines are square (or as square as they can get), the field magnitude will be proportional to the closeness of the lines. Field line sketching has been extended to an art by Hayt and others. A simple two-dimensional field sketch is shown in Figure 2.19.

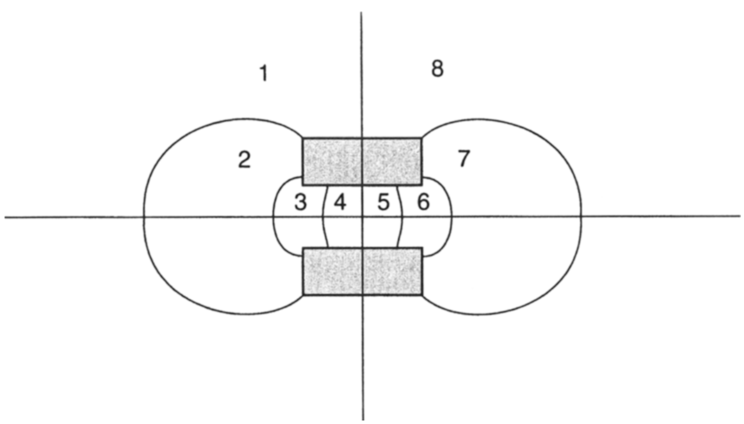


Figure 2.19 Two-dimensional field sketch

Note that some areas like 4 and 5 are nearly square, while areas 1 and 2 are very distorted. The distorted areas can be further subdivided for more precision. After the sketch is finished, block counting is used to estimate capacitance. In the sketch of Figure 2.19 the electrodes are separated by eight blocks sideways (blocks 1–8) and two blocks lengthwise. This has the same capacitance as a parallel-plate capacitor (Figure 2.20) with no fringing fields and a width-to-spacing ratio of 4:1

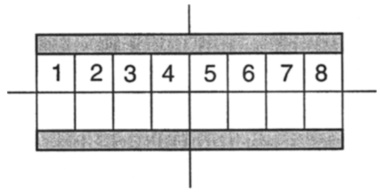


Figure 2.20 Equivalent parallel-plate capacitor

The capacitance is then calculated from eq. 2.14

$$C = \frac{\varepsilon_0 \varepsilon_r S}{d} = 8.854 \times 10^{-12} \times \frac{\varepsilon_r DL}{d} \qquad \text{F, m}$$

where D/d is the electrode width-to-separation ratio and L is the length. With an air-dielectric capacitor having $D/d = 4$, $C = 4 \times 8.854 \times L$ pF, where L is the length in meters.

The example above is a section of a three-dimensional shape which is extruded into the third dimension, and end effects are ignored for calculation of C. Field line sketching techniques can also be extended for nonextruded three-dimensional electrodes with cubical shapes replacing the squares.

2.5.2 Teledeltos paper

Teledeltos™ paper is a black resistive paper constructed with a thin carbon coating on ordinary paper backing. It can be used to plot two-dimensional field lines without trial and error. The geometry of the conductors is painted in silver paint and excited by a DC voltage, and a voltmeter is used to determine the equipotential surfaces, or an ohmmeter is used to determine capacitance. Teledeltos paper is also useful to solve for the resistance of two-dimensional shapes such as thin-film resistors used in integrated circuits.

Ordering the supplies

Bob Pease [1994] has contributed instructions on how to obtain this useful but elusive material:

> Simply buy an International Money Order for 44.00 pounds sterling. This pays for everything, including the paper, tax, packing, plus shipping, air freight to anywhere in the USA. (Unfortunately, the fee for the money order will be about \$30, but this is an acceptable expense, if you are warned). Send this money order to Mr. David Eatwell at the address [below]. This will soon get you a roll 29 in. wide by 45 ft. long, about 6 kilohms per square, Grade SC20.
>
> Or if you send a money order for 36.50 pounds sterling, you can get a roll 18 in. × 59 ft., tax and air shipping included. Either way, the price per square foot is the

> same, about 5 cents per square foot, fairly reasonable, as most experiments take only 1/2 or 1 or 2 square feet.
>
> Mr. David Eatwell, Sensitized Coatings, Bergen Way, North Lynn Industrial Estate, King's Lynn, Norfolk, England PE30 2JL

Your local post office mails the International Money Order and your letter to a U.S. facility in Memphis where it is processed and mailed to England. The normal turnaround time for this service is four to six weeks (the post office does not mail international money orders in Express Mail).

Mr. Pease suggests a two-component silver-loaded epoxy to paint the conductors. One-half oz can be purchased for about $15.00 from Planned Products, 303 Potrero St. Suite 53, Santa Cruz, CA 95060, (408) 459-8088.

Measuring capacitance

After collecting these supplies, the electrode shapes are painted on the Teledeltos paper, and fine copper wires are painted to the electrodes and connected to a voltage source or a resistance meter. The reciprocal of the resistance between electrodes is a measure of capacitance. With paper which has a resistance of 6 kΩ/square, a resistance between two electrodes of 1.5 kΩ would imply four squares in parallel. This is extended to three dimensions by extruding the electrode shapes into the paper, and noticing that the capacitance of parallel plates is proportional to A/d from eq. 2.13. Then the capacitance of the three-dimensional shape is $D/d \times t$, where D/d is the width/spacing ratio and t is the thickness dimension in meters. With four squares in parallel D/d is 4, and with the thickness t meters, the capacitance is $4t \times 0.556 \times 10^{-9}\,\varepsilon_r$. For the electrode pattern illustrated in Figure 2.21, the capacitance C is

$$C = 0.556 \cdot 10^{-9} \varepsilon_r \cdot \frac{1.5k}{6k} \qquad \text{pF/m}$$

in air, independent of the scale of the electrodes if the cross-section dimensions are scaled together and the thickness dimension is constant.

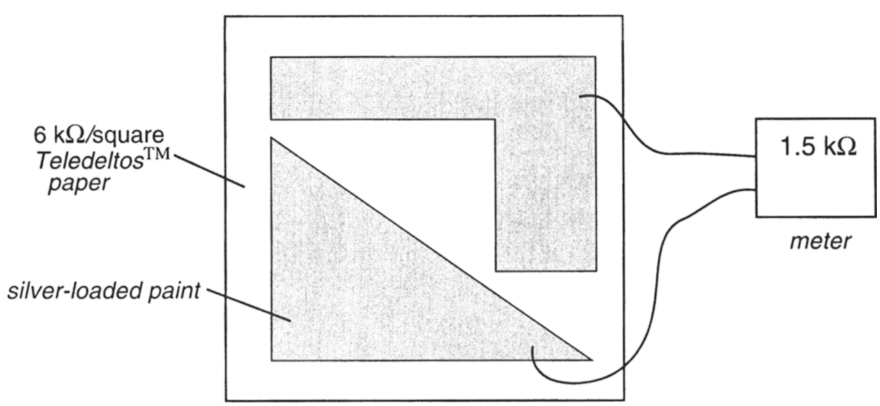


Figure 2.21 Teledeltos™ paper

For field plotting, a 10 V DC supply is connected to the electrodes, and a DC voltmeter with high input impedance (10 MΩ will be fine) is used to plot equipotentials. Multiple electrodes can be simulated, and the shapes can be altered with a pair of scissors as needed to trim designs.

Three-dimensional field plotting has been done using an electrolyte tank. Salt water makes a good electrolyte. Two- and three-dimensional field plotting can also be done by computer, using finite element methods. See the next section, "Finite Element Analysis."

2.6 FINITE ELEMENT ANALYSIS

Since Poisson's law has no direct solution except for some symmetric cases, approximate methods are used. Field line sketching was used by early experimenters, but this is more of an art than a science and it is a cut-and-paste approximation. Teledeltos™ paper (see Section 2.5.2, "Teledeltos paper") offers a more direct solution, but it does not work for three-dimensional problems and it also requires some patience with conductive paint and scissors. Also, due to the tolerance of the paper's resistivity, Teledeltos™ provides a solution accurate only to 5% or so.

A more recent science, finite element analysis, has been used for a variety of problems which can be represented by fields which vary smoothly in an area or volume, and which have no direct solution. FEA was first applied to stress analysis in civil engineering and mechanical engineering, and is now also used for static electric and magnetic field solutions, as well as for dynamic fields and traveling wave solutions.

FEA divides an area into a number of polygons, usually triangles, although squares are sometimes used. Then the field inside a triangle is assumed to be represented by a low-order polynomial, and the coefficients of the polynomial are chosen to match the boundary conditions of the neighbor polygons by a method similar to cubic spline curve fitting or polygon surface rendering. The accuracy of fit is calculated, and in areas where the fit is poorer than a preset constant the polygons are subdivided and the process is repeated. For three-dimensional analysis, the polygons are replaced by cubes or tetrahedrons. A short overview of FEA methods for capacitive sensor design is found in Bonse et al. [1995]. This reference shows FEA error compared to an analytic solution to be less than 0.18%.

FEA is also used by researchers in microwave technology. One approach is to draw a two-dimensional electrode pattern using a shareware geometric drawing package called PATRAN, and pass it to a shareware FEA solver. These programs can be acquired over Internet from an anonymous ftp site, rle-vlsi.mit.edu, at RLE, the Research Laboratory for Electronics at MIT. Ftp "fastcap" from the pub directory.

More convenient FEA software tools integrate drawing and solution packages. MCS/EMAS is available from MacNeal-Schwendler Corp., Los Angeles, CA (213)258-9111. Another, Maxwell, is available from Ansoft, Inc., Four Station Square, Suite 660, Pittsburgh, PA 15219, (412)261-3200. Its features are:

- Integrated modeling, solving, and postprocessing
- Handles electric and magnetic fields
- Solves for fields, energy, forces, capacitances, coupling, etc.
- Runs on Unix workstations, or PCs with Windows or Windows NT

- 2D package about $2500, 3D about $20,000
- 2D package can be configured for x-y or r-θ coordinate systems
- Parametric analysis option available
- Many different dielectric and conductive materials supported

Maxwell was used to produce the following field charts. The error criterion was set at 0.1%, and the typical solution took 2–15 min on a Pentium 90 processor.

2.6.1 FEA plot

A simple electrode shape demonstrates the steps in FEA analysis. This shape represents a two-dimensional cross section, extruded into the third dimension to a depth of 1 m. The first step is to enter the electrode shape (Figure 2.22).

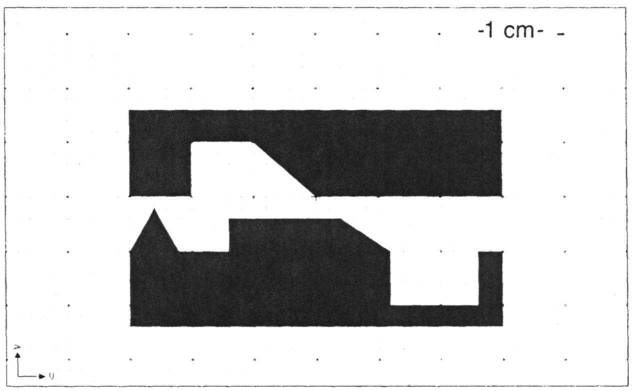


Figure 2.22 FEA electrode shape

Next, the electrodes and the background are assigned material properties. In this case, aluminum was used for electrodes and air for the background. Then the desired error criteria are entered and the project is solved, with the solver adding and subdividing triangles until the requested error bound is reached (Figure 2.23).

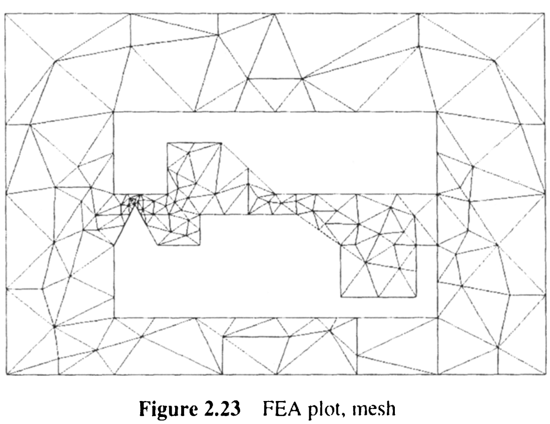


Figure 2.23 FEA plot, mesh

This shows the triangle mesh which was needed to solve the electric field to an accuracy of 0.1%. Usually between 200 and 2000 separate triangles are needed to achieve this level of accuracy. Note the concentration of small triangles near electrode points where the field is changing rapidly.

Equipotentials can then be plotted (Figure 2.24).

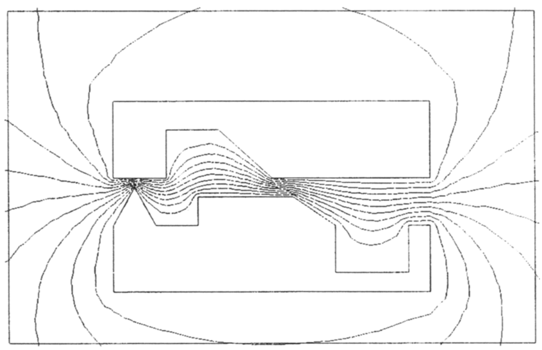


Figure 2.24 FEA equipotential

For this example, the lower electrode was assigned a voltage of 100 V and the upper electrode was assigned 0 V. The equipotentials show the constant-voltage field lines.

With an air dielectric, the interelectrode capacitance was calculated for a 1 m length as 1.003×10^{-10} F. Using eq. 2.13, this capacitance is equal to a 6 cm wide, 1 m long parallel plate capacitor with 0.53 cm spacing.

2.6.2 Fringe fields

FEA plots help to show the effect of fringe fields on the capacity of simple two-plate capacitors. Figure 2.25 shows a thick-plate air-dielectric parallel-plate capacitor, 1 m long with a 1 × 6 cm gap. If the parallel plate formula eq. 2.13 is applied, the calculated capacitance is $\varepsilon_0\varepsilon_r A/s$, or $8.854 \times 10^{-12} \times 0.06 / 0.01$ F/m, which evaluates to 53.1 pF.

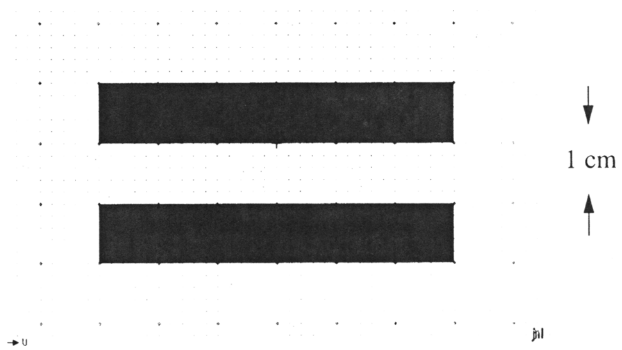


Figure 2.25 Thick plate capacitor, electrodes

The actual capacitance as calculated by FEA is 71.7 pF, 35% larger than the capacitance without considering fringe effects. The absolute capacitance difference due to fringe fields, 18.6 pF, will stay about the same as the gap is decreased or the area of the plates is increased, so it will be a much smaller percentage of the total capacitance for close-spaced geometries.

Beveling the plate edges or using a thinner plate will reduce the fringe capacitance (Figure 2.26).

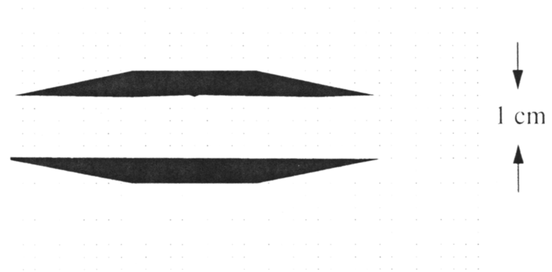


Figure 2.26 Beveled plate capacitor electrodes

Here, FEA calculates 67.5 pF, so the value of the fringe capacitance has decreased to 14.4 pF for 1 m.

Surround with ground

With electrode systems which have a large area relative to the gap, fringe fields will be 1–5% or less and can usually be neglected. If large gaps must be accommodated, surrounding the plates with ground reduces the fringe flux. Surrounding the plates with ground can be done as shown in Figures 2.27 and 2.28.

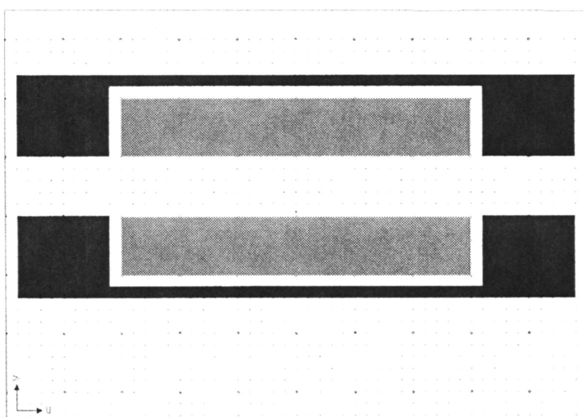


Figure 2.27 Ground shield, electrode configuration

FEA indicates that the capacitance is now reduced to 50.9 pF, indicating a negative fringe capacitance of -2 pF.

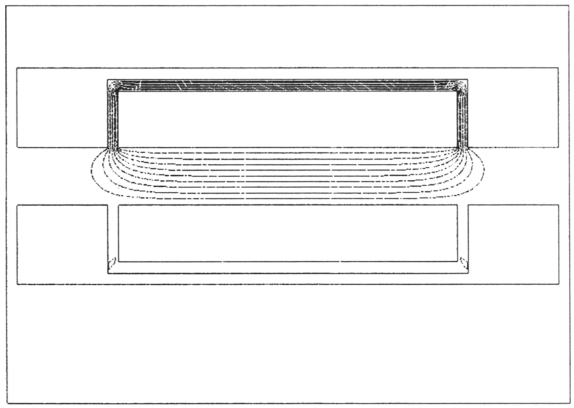


Figure 2.28 Ground shield, equipotentials

2.6.3 Crosstalk

One problem which impairs the performance of two-plate motion sensing capacitors is crosstalk, or unwanted capacitive coupling between, for example, an electrode drive plate and a pickup plate. If this crosstalk is constant it can be canceled in the electronic circuit, but with moving electrodes this is usually not possible. Crosstalk can cause an area-variation motion sensor to falsely indicate transverse motion components; it will give an erroneous indication that a plate has moved transversely when only plate spacing has changed. Luckily, crosstalk diminishes quickly with plate separation, as is shown in this FEA analysis (Figure 2.29).

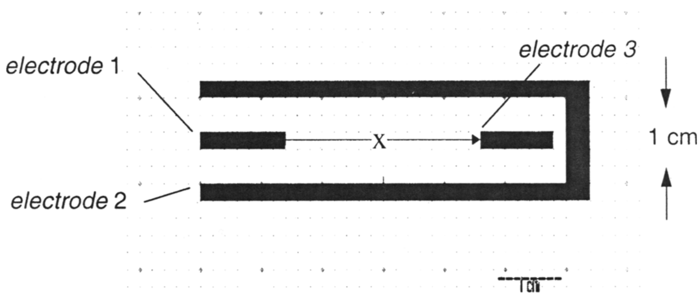


Figure 2.29 Crosstalk electrode configuration

The test signal is applied to electrode 1, and its coupling to electrode 3 is analyzed with electrode 2 grounded. Coupling is defined as the capacitance between 1 and 3 as a percent of the capacitance of 1 to (2 + 3). Parametric analysis shows the variation of coupling with the x dimension increasing from 1 mm to 32 mm (Figure 2.30).

The rapid falloff of coupling with distance is typical. A logarithmic plot shows another typical characteristic, an approximate straight line plot on log-lin coordinates (Figure 2.31).

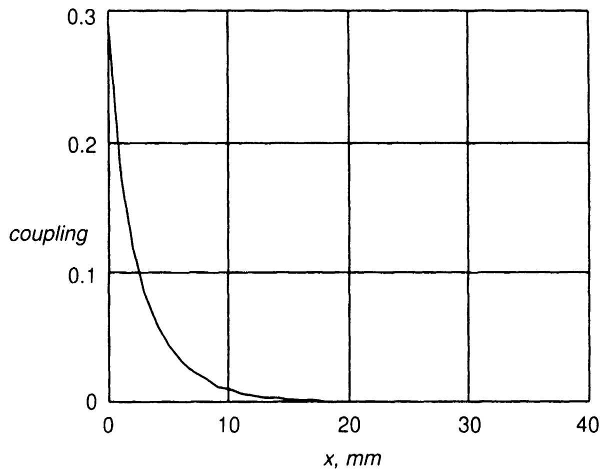


Figure 2.30 Crosstalk, linear parametric plot

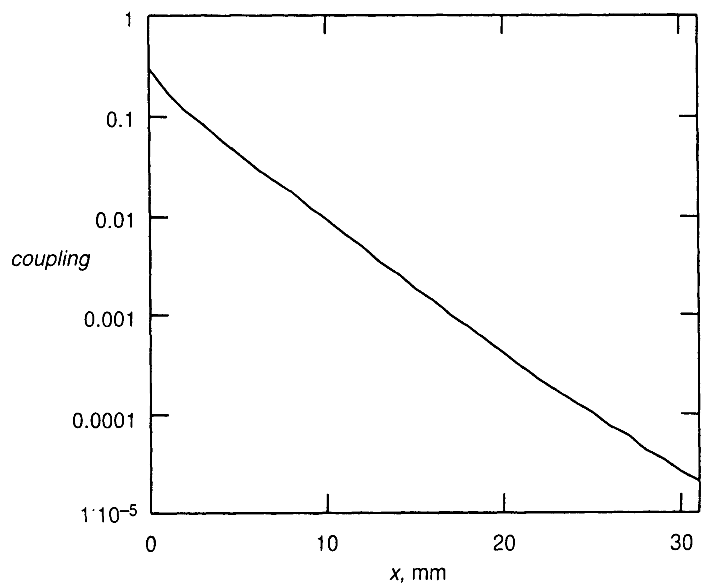


Figure 2.31 Crosstalk, log-linear parametric plot

Another plot, called a line plot (Figure 2.32), shows the variation of electric field along the line marked "x" (Figure 2.29) with the x dimension at 32 mm. Electrode 1 has a test voltage of 100 V applied; electrodes 2 and 3 are grounded.

The crosstalk coupling worsens if the ground does not surround the electrodes. In Figure 2.33 the top shield is removed.

With the top shield missing, crosstalk increases and the curve of coupling vs. distance falls off more slowly with separation. This is shown by adding the one-side curve to the previous two-side shield plot (Figure 2.34).

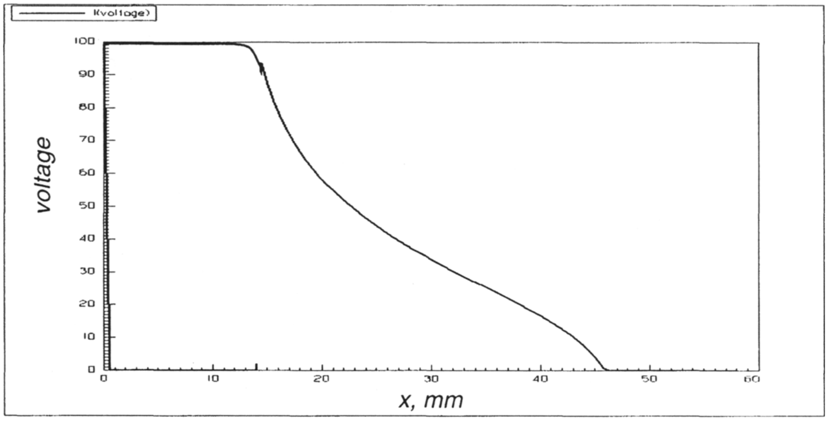


Figure 2.32 Crosstalk, line plot

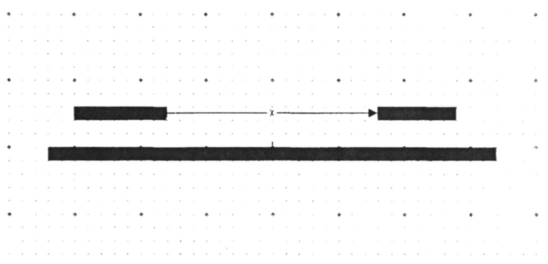


Figure 2.33 Crosstalk, one side shield, electrode configuration

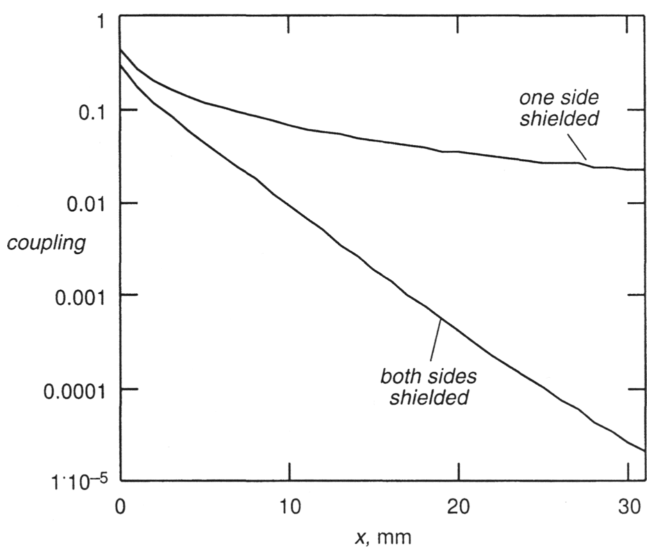


Figure 2.34 Crosstalk, one side shield, parametric plot

2.6.4 Moving shield

If a two-plate moving electrode structure for motion detection is replaced by a three-plate moving-shield structure (Figure 2.35), the capacitance can be approximately first-order insensitive to spacing change. FEA modeling can show how good this approximation is.

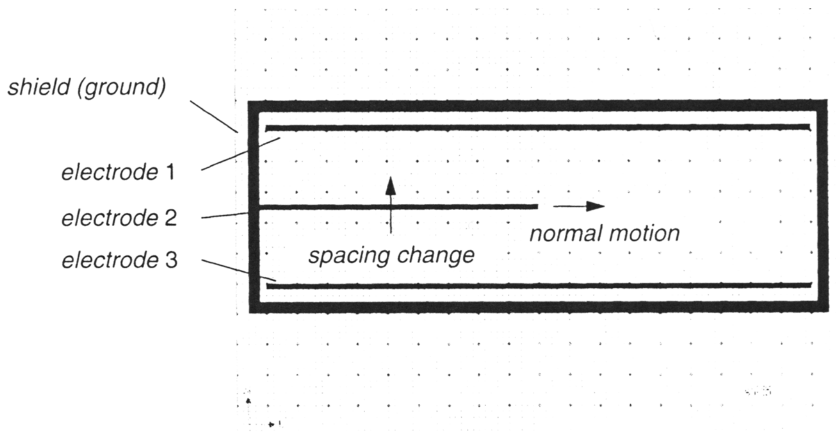


Figure 2.35 Moving shield initial electrode configuration

This represents a cross section of a shielded three-electrode structure used to measure the change of length of electrode 2 by measuring changes in C_{13}. As ground electrode 2 lengthens in the x axis from left to right, the capacitance between electrode 1 and 3 will be linearly decreased. To check how sensitive this structure is to spacing changes, we move electrode 2 in the y axis while leaving its x position unchanged. The final electrode position is shown in Figure 2.36.

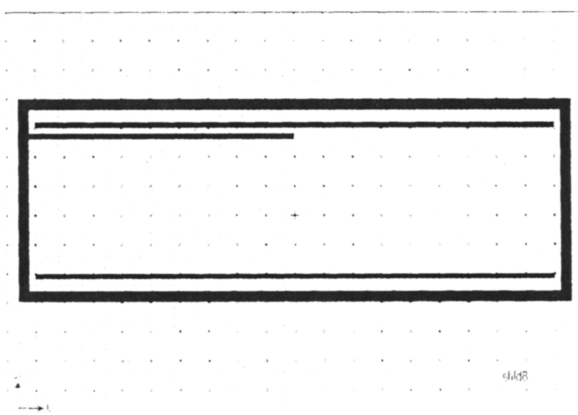


Figure 2.36 Moving shield, final electrode configuration

Figure 2.37 shows that this shape has about 6 mm change in apparent x position as only the electrode spacing is changed. The electrode configuration tested is quite short, however, and has a relatively large gap. If these results are extended to a configuration with a gap which is 1% of the electrode length, the variation of apparent position with spacing will decrease by a factor of 30.

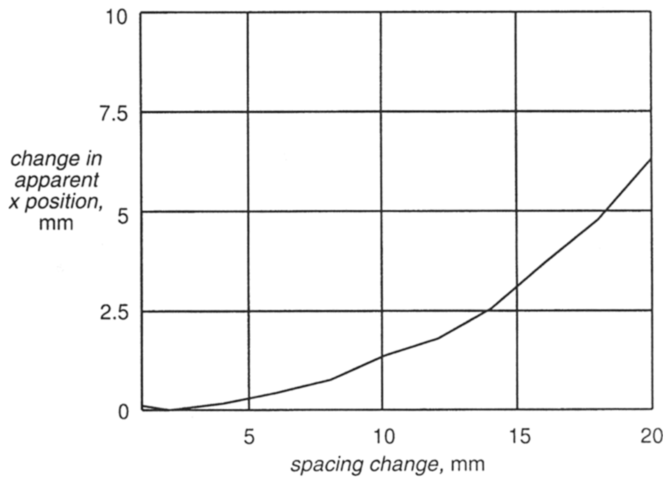


Figure 2.37 Moving shield sensitivity to spacing change

2.6.5 Proximity detector

A possible electrode configuration for proximity detection is analyzed in Figures 2.38–2.40 with FEA. This analysis make use of the r-θ coordinate system as a substitute for true three-dimensional analysis.

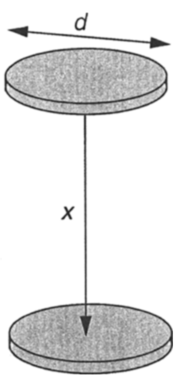


Figure 2.38 Monopole proximity detector, electrode configuration

The change in capacitive coupling between the electrodes with diameter d of 2 cm and a change of spacing x is shown in Figure 2.40.

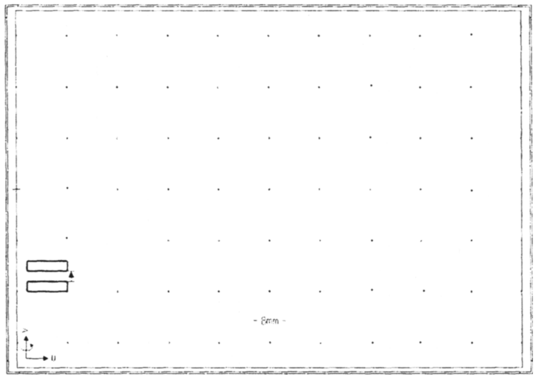


Figure 2.39 Monopole proximity detector, electrode configuration

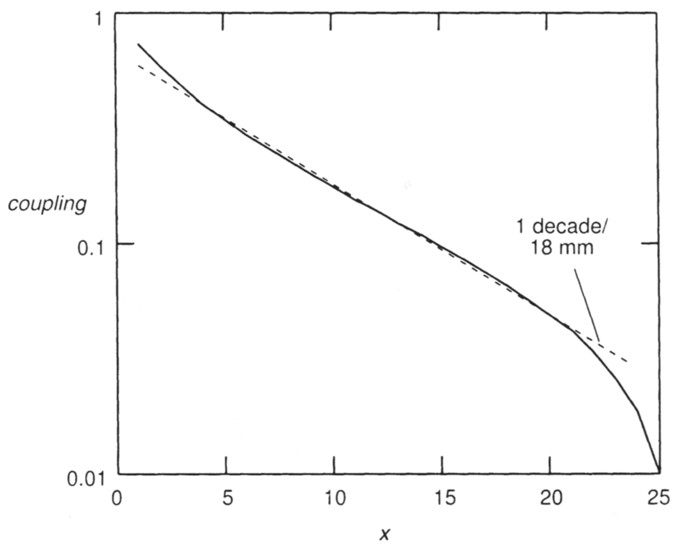


Figure 2.40 Monopole proximity detector, parametric plot, log

As x increases, the distance between electrodes increases and the capacitance falls off as the log of spacing, except as the top electrode approaches the grounded shield the falloff becomes more pronounced. The left boundary of the shielding box is the center of the r-θ coordinate system and does not function as a shield.